THE FORMAL ANALOGY BETWEEN POSSIBILITY AND PROBABILITY THEORY

Gert DE COOMAN
Universiteit Gent, Vakgroep Elektrische Energietechniek
Technologiepark 9, 9052 Zwijnaarde, Belgium
E-mail: gert.decooman@rug.ac.be

ABSTRACT

It is well known that the theory of probability can be treated and developed in a consistent and uniform way using the classical theory of measure and integration. Indeed, the Russian scientist Kolmogorov identified probability with normalized classical measures and used the Lebesgue theory of integration to give a logically consistent and unifying account of probability theory. In this paper, we indicate how, in an analogous way, a unified and consistent treatment of possibility theory can be given. Using seminormed fuzzy integrals, a generalization of Sugeno's fuzzy integrals ideally suited for working with our *possibility measures*, we discuss how a theory of possibility can be developed along the same formal lines as the theory of probability.

1. Introduction

Early in this century, the Russian scientist A. N. Kolmogorov[17] showed how the theory of probability could be consistently formulated in a unifying measure- and integral-theoretic framework. As a result of his work, probability became identified with normalized classical measures. At the same time, the *Lebesgue integral* was used to define *expectations* (or mean values). It was shown that the very important notion of *conditional probability* can be introduced using special types of integral equations, and the relationship was studied between these conditional probabilities and the notion of *stochastic independence*, also formulated measure-theoretically. Also, in this measure- and integral-theoretic formulation of probability theory, a *stochastic variable* was formally introduced as a special measurable mapping from a basic space to a sample space, and it was shown that, probabilistically, the behaviour of a *real stochastic variable* can be completely described using its *probability distribution function*, or equivalently, its frequency or density function. For a more detailed account of how this is accomplished, we refer to any standard textbook on the subject[2].

In our doctoral dissertation[4] we have shown that a formally very analogous, measure and integral-theoretic formulation of possibility theory is feasible, and that this formulation sheds an interesting light on the problem of how to define conditioning and independence in a possibilistic framework. A series of three papers on this subject is in preparation[5,6,7]. In this paper, we intend to give a brief survey and discussion of the most important results given there, and the main ideas behind them. A brief summary of the formal analogy between probability theory and our account of possibility theory is given in table 1. It will be used as a *fil rouge* throughout the paper.

PROBABILITY THEORY	POSSIBILITY THEORY
σ-field	ample field
unit interval $[0, 1]$	complete lattice $(L, \leq)$
addition	supremum
multiplication	t-seminorm, t-norm
probability measure (σ-additivity)	possibility measure (supitivity)
Lebesgue integral	seminormed fuzzy integral (= possibility integral)
stochastic variable	possibilistic variable
real stochastic variable	fuzzy variable (= measurable fuzzy set)
probability distribution function	possibility distribution function
expectation of a real stochastic variable	possibility of a fuzzy variable
almost everywhere equality	almost everywhere equality (modified, more general form)
product probability measure	product possibility measure
product Lebesgue integral	product possibility integral
conditional expectation and probability	conditional possibility
stochastic independence	possibilistic independence

Table 1: Overview of the formal analogy between probability and possibility theory

2. Basic notions

Let us start the discussion by introducing a number of basic notions. We shall denote by X an arbitrary *universe*, i.e., a nonempty set.

By $(L, \leq)$ we shall mean a *complete lattice*[3], with bottom 0_L and top 1_L. We shall assume that $0_L \neq 1_L$. A *triangular seminorm* or, shortly, *t-seminorm* P on the complete lattice $(L, \leq)$ is a binary operator on L that is isotonic and satisfies the following boundary condition: $(\forall \lambda \in L)(P(1_L, \lambda) = P(\lambda, 1_L) = \lambda)$. A *triangular norm* or, shortly, *t-norm* T on $(L, \leq)$ is a t-seminorm on $(L, \leq)$ that is furthermore associative and commutative. We shall, unless explicitly stated otherwise, assume further on that the t-seminorm P on $(L, \leq)$ is *completely distributive* w.r.t. supremum, i.e., for any λ in L and any family $(\mu_j \mid j \in J)$ of elements of L: $P(\lambda, \sup_{j \in J} \mu_j) = \sup_{j \in J} P(\lambda, \mu_j)$ and $P(\sup_{j \in J} \mu_j, \lambda) = \sup_{j \in J} P(\mu_j, \lambda)$, and analogously for the t-norm T. We have given a detailed discussion of triangular seminorms and norms defined on complete lattices, and more in general, on bounded partially ordered sets, in a previous paper[10].

An *ample field*[9,19] $\mathcal{R}$ on a universe X is a set of subsets of X that is closed under *arbitrary* unions and intersections, and under complementation in X. Ample fields are immediate generalizations of power sets. The *atom* of $\mathcal{R}$ containing the element x of X will be denoted by $[x]_\mathcal{R}$ and is defined by: $[x]_\mathcal{R} = \bigcap\{A \mid x \in A \text{ and } A \in \mathcal{R}\}$. Atoms of ample fields can be interpreted as generalizations of singletons. A subset E of X will be called $\mathcal{R}$-*measurable* iff $E \in \mathcal{R}$. A measurable set is also called an *event*.

With any subset A of a universe X, we can associate its *characteristic* $X - L$ *mapping* χ_A, with $\chi_A(x) = 1_L$, $x \in A$, and $\chi_A(x) = 0_L$, $x \in \text{co}A$. An arbitrary $X - L$ mapping will be called a $(L, \leq)$-*fuzzy set*[14] (or simply fuzzy set) in X. It is an obvious generalization of a characteristic $X - L$-mapping. The set of the $(L, \leq)$-fuzzy sets in X will be denoted by $\mathcal{F}_{(L,\leq)}(X)$. We shall frequently denote the constant $X - \{\lambda\}$-mapping, $\lambda \in L$, by $\underline{\lambda}$. We shall also make use of the partial order relation $\sqsubseteq$ on $\mathcal{F}_{(L,\leq)}(X)$: for any h_1 and h_2 in $\mathcal{F}_{(L,\leq)}(X)$, $h_1 \sqsubseteq h_2 \Leftrightarrow (\forall x \in X)(h_1(x) \leq h_2(x))$. A $X - L$ mapping h will be called $\mathcal{R}$-*measurable* iff it is constant on the atoms of $\mathcal{R}$. A $\mathcal{R}$-measurable $X - L$ mapping is also called a $(L, \leq)$-*fuzzy variable* – or fuzzy variable – in $(X, \mathcal{R})$. A fuzzy variable can be considered as a 'fuzzification' of a measurable set, and will sometimes be called a *fuzzy event*. The set of the $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$ is denoted by $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$.

3. Possibility measures and possibility integrals

Let us also define the basic tools we shall be working with. A $(L, \leq)$-*possibility measure*[4,9,11,13] – or simply possibility measure – Π on $(X, \mathcal{R})$ is a complete join-morphism between the complete lattices $(\mathcal{R}, \subseteq)$ and $(L, \leq)$, i.e., for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$. In other words, where probability measures show additivity, possibility measures could be called '*supitive*' mappings, because the role of addition is in possibility theory taken over by supremum.[a] The definition immediately implies that $\Pi(\emptyset) = 0_L$. The structure $(X, \mathcal{R}, \Pi)$ will be called a $(L, \leq)$-*possibility space*. Π is called *normal* iff $\Pi(X) = 1_L$. An $\mathcal{R}$-measurable $X - L$-mapping π such that for any A in $\mathcal{R}$: $\Pi(A) = \sup_{x\in A} \pi(x)$, is called a *distribution* of Π. Such a distribution is *unique*, and satisfies $\pi(x) = \Pi([x]_\mathcal{R})$, $x \in X$.

A $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ is a generalization towards a more general domain and codomain of Zadeh's possibility measures[23], which are, in our terminology, $([0, 1], \leq)$-possibility measures on $(X, \wp(X))$, where $\wp(X)$ is the power set of X. Its introduction can be justified as follows. Using a complete lattice as a codomain allows us to model potential *incomparability* of possibilities, and to associate possibility measures with the $(L, \leq)$-fuzzy sets introduced by Goguen[14], in order to represent more general forms of linguistic uncertainty[8,23]. Why we define our possibility on ample fields, rather than on power sets, needs a more involved justification. Actually, we could call a mapping from an arbitrary subset $\mathcal{A}$ of $\wp(X)$ to L a $(L, \leq)$-possibility

[a]Note that the idempotency of both union and supremum make the well-known condition of mutual disjointness in the definition of additivity redundant in the supitivity definition.

measure iff it is *extendable* to a $(L, \leq)$-possibility measure with domain $\wp(X)$. It has been shown elsewhere[1] that this extendability is equivalent with the extendability to a $(L, \leq)$-possibility measure with domain $\tau(\mathcal{A})$, the smallest ample field that includes $\mathcal{A}$. On the one hand, as long as we look only at the elements of $\mathcal{A}$, it does not matter whether we consider an extension to $\tau(\mathcal{A})$ or to $\wp(X)$. But, on the other hand, specifying a possibility measure on $\tau(\mathcal{A})$ generally involves less information than specifying one on $\wp(X)$, since, for one thing, the atoms of $\wp(X)$ constitute a refinement of the atoms of $\tau(\mathcal{A})$. So, at least in principle, we want to be able to be as nonspecific as possible, and introduce possibility measures on ample fields and not just on power sets.

In a recent article about possibility and necessity integrals[11] we argued that a generalization of Sugeno's fuzzy integral, the seminormed fuzzy integral, is ideally suited for combination with our $(L, \leq)$-possibility measures, in very much the same way as the Lebesgue integral is a perfect match for a probability measure. Let us repeat here the most important points of this argumentation. For a start, whenever the above-mentioned integral is associated with a possibility measure, we shall call it a *possibility integral*. For any $\mathcal{R}$-measurable set A and any $\mathcal{R}$-measurable $X-L$-mapping h, the $(L, \leq, P)$-*possibility integral* of h on A (w.r.t. Π) can be written as:

$$(P)\,⨏_A h \mathrm{d}\Pi = \sup_{x \in A} P(h(x), \pi(x)).$$

We find for the characteristic $X-L$-mapping χ_A of the $\mathcal{R}$-measurable set A that $(P)⨏_X \chi_A \mathrm{d}\Pi = \Pi(A)$. If we define the mapping $\Pi_P\colon \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) \to L$ by $\Pi_P(h) = (P)⨏_X h \mathrm{d}\Pi$ we find that $\Pi_P(\chi_A) = \Pi(A)$, which means that Π_P can be considered as an '*extension*' of Π from $\mathcal{R}$ to $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. Furthermore, for any family $(h_j \mid j \in J)$ of fuzzy variables in $(X, \mathcal{R})$, $\Pi_P(\sup_{j\in J} h_j) = \sup_{j\in J} \Pi_P(h_j)$, which means that Π_P is a complete join-morphism between the complete lattices $(\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X), \sqsubseteq)$ and $(L, \leq)$, and therefore behaves essentially like a possibility measure. The $(L, \leq, P)$-possibility integral allows us in other words to 'extend' the domain of the possibility measure Π from $\mathcal{R}$ to $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. For an arbitrary h in $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$, $\Pi_P(h)$ is called the $(L, \leq, P)$-*possibility* – or simply possibility – of h.

We see that the seminormed fuzzy integral shows a number of very interesting properties when associated with a possibility measure. It is shown elsewhere[4,5,13] that this integral can be used to introduce product possibility measures. For the product possibility integrals associated with these product measures, a Fubini-like theorem can be proven. Other interesting results in classical integration theory, such as the theorem of the Jacobian, or the Radon-Nykodim theorem, also have a possibilistic counterpart in the possibilistic integration theory. We shall see other nice properties further on when we talk about conditional possibility. This is precisely what makes us defend the view that the seminormed fuzzy integral is *the* integral for possibility measures, just as the Lebesgue integral is *the* integral for probability measures.

At the same time, it should be noted that the *t-seminorm* P which appears in this integral plays a role that is similar to that of the *product operator* in the Lebesgue integral (for more details, we refer to the above-mentioned paper on possibility and necessity integrals[11]). In the later sections on conditional possibility and possibilistic independence, this fact will become even more apparent.

4. Possibilistic and fuzzy variables

A variable may be informally defined as an *abstract object that can assume values in a certain universe*. This notion of a variable is used for instance by Zadeh[23], Hisdal[15] and Nguyen[18] in the context of fuzzy set theory and possibility theory. In probability theory, this notion appears in the form of a *stochastic variable*, formally a measurable mapping from a basic space to a sample space. For real stochastic variables, for which the sample space is the set of the reals, distribution functions can be defined that completely characterize their stochastic behaviour[2].

A *possibilistic variable* can be defined analogously. We consider a *basic space* Ω, provided with an ample field $\mathcal{R}_\Omega$ of subsets of Ω. We assume the existence of a $(L, \leq)$-possibility measure Π_Ω on $(\Omega, \mathcal{R}_\Omega)$. At the same time we consider a *sample space* X provided with an ample field $\mathcal{R}$.

Definition 1 *A $\Omega - X$-mapping ξ that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, i.e., for which for any A in $\mathcal{R}$, $\xi^{-1}(A) \in \mathcal{R}_\Omega$, is called a possibilistic variable in $(X, \mathcal{R})$.*

This provides a generalization and formalization of Zadeh's more intuitive notion of a fuzzy variable. We prefer to use the name '*possibilistic* variable', however, because this notion is of central importance in possibility theory, and is only indirectly related with fuzzy sets. Possibilistic variables are the possibilistic counterparts of the stochastic variables in probability theory. It should therefore come as no surprise that these possibilistic variables play a part in possibility theory that is to a high extent comparable to the one played in probability theory by stochastic variables. For one thing, they allow us to formally study the notions of conditional possibility and possibilistic independence, as will be briefly explained further on.

Whereas in probability theory, distributions functions can essentially be defined only for real stochastic variables, it turns out that every possibilistic variable has a distribution function.[b]

Using Π_Ω, the $(L, \leq)$-possibility that the possibilistic variable ξ assumes a value in the element A of $\mathcal{R}$ can be expressed as $\Pi_\Omega(\xi^{-1}(A))$. If we define the $\mathcal{R} - L$-mapping Π_ξ as follows, for any A in $\mathcal{R}$,

$$\Pi_\xi(A) = \Pi_\Omega(\xi^{-1}(A)),$$

[b]This is mainly due to the fact that every possibility measure has a distribution, i.e., can be defined using point information rather than set information, which is certainly not the case for every probability measure.

then Π_ξ is clearly a $(L,\leq)$-possibility measure on $(X,\mathcal{R})$, that will be called the *possibility distribution*[c] *(measure)* of ξ. The distribution π_ξ of Π_ξ is then given by

$$\pi_\xi(x) = \sup_{\xi(\omega)\in[x]_\mathcal{R}} \pi_\Omega(\omega),$$

for any x in X. Following Zadeh's nomenclature[23], π_ξ will be called the *possibility distribution function* of the possibilistic variable ξ. Furthermore, for any A in $\mathcal{R}$, $\Pi_\xi(A) = \sup_{x\in A}\pi_\xi(x)$. Remark that Π_ξ is normal if and only if Π_Ω is.

In probability theory, a very prominent class of variables are the *real* stochastic variables, for which a large body of results have been derived[2]: using Lebesgue integration theory, it is possible for these real variables to introduce mean values or expectations, higher order moments, characteristic functions, etc. If we take a closer look, we see that the deeper reason for this plethora of results is that real stochastic variables assume values in the set of the reals, and so do the probability measures used to describe their behaviour. This makes it possible for those variables to appear in the integrand of the Lebesgue integral associated with these measures.

If we want to look for a possibilistic formal counterpart of real stochastic variables, it is clear that we must look for those possibilistic variables that take values in essentially the same set as the possibility measures used to describe their behaviour. Only then shall we be able to make these variables appear in the integrand of possibility integrals associated with these possibility measures, and closely follow the analogy with probability theory. That is the most important reason why we single out *fuzzy variables* here, why we have given them a special name, and dedicate the rest of this section to their study.

In section 2 we introduced fuzzy variables as measurable *fuzzy* sets, i.e., fuzzifications of measurable sets (or events). Of course, these fuzzy variables are special instances of possibilistic *variables*. This, in our opinion, justifies our use of the name 'fuzzy variables'. As mentioned before, it should on the other hand be noted that Zadeh[23] uses the name 'fuzzy variable' more generally for what we have been calling here a possibilistic variable. Since, however, these variables are the possibilistic counterparts of the stochastic variables in probability theory, we prefer to call them possibilistic, and want to reserve the qualification 'fuzzy' for those variables that are also fuzzy sets.

Let us now consider the universe X, provided with an ample field $\mathcal{R}$ and the $(L,\leq)$-possibility measure Π, as a *basic space*. It is easily verified that a $(L,\leq)$-fuzzy variable h in $(X,\mathcal{R})$ is a $\mathcal{R}-\wp(L)$-measurable $X-L$-mapping, that can therefore be formally considered as a *possibilistic variable* in the sample space $(L,\wp(L))$.

Definition 2 *The possibility distribution (measure)* Γ_h *of* h *is the* $(L,\leq)$*-possibility measure on* $(L,\wp(L))$*, defined by, for any* B *in* $\wp(L)$*,* $\Gamma_h(B) = \Pi(h^{-1}(B))$*. The pos-*

[c]It should be noted that there exists an important conceptual difference between the *distribution* of a possibility measure, and the *possibility distribution* (measure) of a possibilistic variable.

sibility distribution function γ_h *of* h *is defined by, for any* λ *in* L, $\gamma_h(\lambda) = \Gamma_h(\{\lambda\}) = \Pi(h^{-1}(\{\lambda\})$. *Remark that* Γ_h *is normal if and only if* Π *is.*

It should be noted that the definition of a possibility distribution function and that of its probabilistic counterpart are somewhat different to the letter: in the definition of a probability distribution function[2] the real intervals $]-\infty, x]$, $x \in \mathbb{R}$, play a prominent part. In the definition of a possibility distribution function, this role is played by the atoms $\{\lambda\}$, $\lambda \in L$, of the ample field $\wp(L)$. This means that our possibility distribution functions bear a closer formal resemblance to the *density* and *frequency functions* in probability theory. Despite this difference, both distribution functions are very similar *in spirit*. The name '*possibility distribution function*' for the $L - L$-mapping γ_h is not just plucked out of the air: this function tells us how the possibility is distributed over the elements of L.

We can of course expect similarities between results about possibility and probability distribution functions. To give a few examples, let us mention a number of interesting results, that clearly are possibilistic counterparts of well-known theorems in probability theory[2].

Theorem 1 *Let* h *be a* $(L, \leq)$*-fuzzy variable in* $(X, \mathcal{R})$ *and let* g *be a* $L-L$*-mapping. Then* $g \circ h$, *or* $g(h)$, *is a* $(L, \leq)$*-fuzzy variable in* $(X, \mathcal{R})$, *and for any* λ *in* L *and* B *in* $\wp(L)$*:*

$$\Pi(h^{-1}(B)) = (P)\int_B \mathrm{d}\Gamma_h$$

and

$$\gamma_{g(h)}(\lambda) = (P)\int_{g^{-1}(\{\lambda\})} \mathrm{d}\Gamma_h \quad \text{and} \quad \Gamma_{g(h)}(B) = (P)\int_{g^{-1}(B)} \mathrm{d}\Gamma_h.$$

Furthermore,

$$\Pi_P(g(h)) = (P)\int_L g\mathrm{d}\Gamma_h.$$

and in particular

$$\Pi_P(h) = (P)\int_L \mathrm{id}_L \mathrm{d}\Gamma_h = \sup_{\lambda \in L} P(\lambda, \gamma_h(\lambda)),$$

where id_L *is the identical permutation of* L.

Let us conclude this section with a short discussion of the interpretation and the significance of these results. Although the formal analogy between real stochastic variables and fuzzy variables is apparent, there is a notable difference between their respective interpretations. Even though a fuzzy variable can formally be considered as a variable that assumes values in L, we have interpreted it as a measurable fuzzy set, the 'fuzzification' of the notion of event. Whereas the Lebesgue integral in probability theory is used to define the mean value or expectation of a real stochastic variable, here the possibility integral is used to extend the notion of possibility, and define the possibility of fuzzy events, as is explained in section 3. Similarly, as we shall presently argue, the results in the theorem above have an interpretation that differs from their formal counterparts in probability theory.

In Zadeh's fuzzy set theory, it is possible to change the meaning of a $([0,1],\leq)$-fuzzy set by taking its composition with transformations of the real unit interval $[0,1]$, called *linguistic hedges*[20,21,22,24]. The $L-L$-mapping g in the theorem above can be considered as a *generalization* of these linguistic hedges. Whereas its probabilistic counterpart is used for the calculation of the moments and the characteristic functions of real stochastic variables, starting from their probability distribution function, the previous theorem enables us to calculate the possibility of a modified fuzzy variable, using its possibility distribution function. As a special case, we may calculate the possibility of a fuzzy variable from its possibility distribution function. In summary, these results tell us that if we want to work with fuzzy variables, their possibility distributions functions (or measures) contain all the information we need, i.e., it is possible to perform all the calculations with these functions, without having to go back to the possibility measure Π defined on $(X,\mathcal{R})$.

5. Almost everywhere equality

Let us now look at the notion of *almost everywhere equality* of fuzzy variables. A similar notion for real-valued functions plays an important part in the classical theory of measure and integration[2]. It will appear from the results below that in order to make this notion as useful in possibility theory as it is in classical measure theory, we cannot adopt a definition that is an immediate extension of the classical one. As it turns out, we need a more specific definition, of which such an immediate extension turns out to be a generalization. The notion discussed here is of crucial importance for the discussion of conditional possibility and possibilistic independence. In what follows, we consider a $(L,\leq)$-possibility space $(X,\mathcal{R},\Pi)$ and denote the distribution of Π by π.

Definition 3 *Let P be a t-seminorm on $(L,\leq)$. For any E in $\mathcal{R}$, we define the following binary relation on $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. For h_1 and h_2 in $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$: $h_1 \overset{(\Pi,P,E)}{=} h_2 \Leftrightarrow (\forall x \in E)(P(h_1(x),\pi(x)) = P(h_2(x),\pi(x)))$. When $h_1 \overset{(\Pi,P,E)}{=} h_2$, we shall say that h_1 and h_2 are (Π,P)-equal almost everywhere on E. When $h_1 \overset{(\Pi,P,X)}{=} h_2$, we shall also write $h_1 \overset{(\Pi,P)}{=} h_2$ and say that h_1 and h_2 are almost everywhere (Π,P)-equal, or equivalently, that they are (Π,P)-equivalent.*

On any complete lattice $(L,\leq)$, there always exist at least two t-seminorms[10]. This means that our definition is generally applicable, and always makes sense. For any E in $\mathcal{R}$, $\overset{(\Pi,P,E)}{=}$ is reflexive, symmetrical and transitive, and is therefore an equivalence relation on $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. In the following proposition we show that some well-known integral-theoretic results from classical measure theory have analogous counterparts in possibility theory. It provides a characterization of our notion of almost everywhere equality in terms of possibility integrals. It also gives a characterization of the (Π, P)-equivalence of fuzzy variables that plays an important role in the discussion of conditional possibility.

Proposition 1 *Let E be a $\mathcal{R}$-measurable set and let h_1 and h_2 be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$. $h_1 \stackrel{(\Pi,P,E)}{=} h_2$ if and only if for any A in $\mathcal{R}$*

$$A \subseteq E \Rightarrow (P)\!\!-\!\!\!\!\!\!\int_A h_1 \mathrm{d}\Pi = (P)\!\!-\!\!\!\!\!\!\int_A h_2 \mathrm{d}\Pi.$$

Furthermore, h_1 and h_2 are (Π, P)-equivalent if and only if for any A in $\mathcal{R}$

$$(P)\!\!-\!\!\!\!\!\!\int_A h_1 \mathrm{d}\Pi = (P)\!\!-\!\!\!\!\!\!\int_A h_2 \mathrm{d}\Pi.$$

Note that, in general, this result would not be valid if we used the immediate extension of the classical definition of almost everywhere equality in stead of our definition. Let us explore the relation between both definitions. Consider two $(L, \leq)$-fuzzy variables h_1 and h_2 in $(X, \mathcal{R})$ and an arbitrary element E of $\mathcal{R}$. Also consider the set $A = \{x \mid x \in E \text{ and } h_1(x) \neq h_2(x)\}$. If we had extended the classical definition of almost everywhere equality towards fuzzy variables, we would have found as a defining condition for the almost everywhere equality on E of h_1 and h_2:

$$\Pi(\{x \mid x \in E \text{ and } h_1(x) \neq h_2(x)\}) = 0_L.$$

How does this relate to our definition? First, assume that $E \neq \emptyset$. Consider an arbitrary x in E. Either x belongs to A, and then of course $\pi(x) = 0_L$, or x belongs to $E \setminus A$, and then $h_1(x) = h_2(x)$. In either case, we find that $P(h_1(x), \pi(x)) = P(h_2(x), \pi(x))$, whence $h_1 \stackrel{(\Pi,P,E)}{=} h_2$. If $E = \emptyset$, the same conclusion is trivially reached. It appears that our definition is implied by an immediate extension of the classical definition. However, it is easily verified that the reverse implication is not necessarily valid. We conclude that an immediate extension of the classical definition of almost everywhere equality would lead to a definition that is *less specific* than ours.

Let us now take one further step and assume that the t-seminorm P on $(L, \leq)$ satisfies the following property:

$$(\forall(\lambda_1, \lambda_2) \in L^2)(\forall \mu \in L)(P(\lambda_1, \mu) = P(\lambda_2, \mu) \Rightarrow (\mu = 0_L \text{ or } \lambda_1 = \lambda_2)).$$

In this case, P is called *resolving on the left*[10]. Furthermore, assume that $h_1 \stackrel{(\Pi,P,E)}{=} h_2$. The formula above then tells us that $\Pi(\{x \mid x \in E \text{ and } h_1(x) \neq h_2(x)\}) = 0_L$. We conclude that in this particular case, our definition coincides with the immediate extension of the classical definition. Note furthermore that the product operator $\times$ on the unit interval $[0, 1]$ is a t-(semi)norm on $([0, 1], \leq)$ that is in particular resolving on the left. Therefore, the reason why we have to use definitions that are more specific that the classical ones, is that we are using operators P which are much more general than the product operator used in classical measure and probability theory.

6. Conditional possibility

We shall now briefly show how conditional possibility can be introduced in a measure- and integral-theoretic framework. We shall only deal here with the conditional possibility of (fuzzy) events. A more detailed treatment, and the discussion of conditional possibility of possibilistic variables in general, will be published elsewhere[6]. As before, we shall denote by $(X, \mathcal{R}, \Pi)$ a $(L, \leq)$-possibility space. The possibility distribution of Π is denoted by π.

In defining conditional possibilities, we allow ourselves to be inspired by the measure- and integral-theoretic introduction of conditional expectations and probabilities as solutions of special integral equations[2]. What we shall do here is *write down possibilistic formal counterparts for these probabilistic integral equations*, and use their solutions to introduce the notion of conditional possibility. As is discussed in significant detail elsewhere[4,6], this approach solves a number of problems involving conditional possibility, extant in the literature.

Whenever we deal with conditional possibility in this paper, we shall assume that the t-seminorm P (or the t-norm T) that appears in our possibility integrals is *weakly left-invertible*[10], which means that, for any λ and μ in L, the equation $P(\nu, \lambda) = \mu$ in ν has a solution as soon as $\mu \leq \lambda$.

In theorem 2 we discuss the existence and unicity of solutions of the integral equation (1), that will lead in definition 4 to the introduction of conditional possibility for fuzzy events.

Theorem 2 *Let h and g be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$. Then there exists a $L - L$-mapping f satisfying*

$$(\forall B \in \wp(L)) \left((P)\!\!\!\!-\!\!\!\!\!\!\int_B f \mathrm{d}\Gamma_g = (P)\!\!\!\!-\!\!\!\!\!\!\int_{g^{-1}(B)} h \mathrm{d}\Pi \right). \tag{1}$$

Any solution f of this integral equation is unique in the sense of (Γ_g, P)-equivalence.

Definition 4 *Let h and g be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$. Any member of the equivalence class of solutions of the integral equation (1) in the $L - L$-mapping f is denoted by $\Pi(h \mid g = \cdot)$. This means that*

$$(\forall B \in \wp(L)) \left((P)\!\!\!\!-\!\!\!\!\!\!\int_B \Pi(h \mid g = \cdot) \mathrm{d}\Gamma_g = (P)\!\!\!\!-\!\!\!\!\!\!\int_{g^{-1}(B)} h \mathrm{d}\Pi \right).$$

For arbitrary λ in L we shall call $\Pi(h \mid g = \lambda)$ the conditional $(L, \leq, P)$-possibility of h, given that g assumes the value λ. For arbitrary A in $\mathcal{R}$ the $L - L$-mapping $\Pi(\chi_A \mid g = \cdot)$ will also be written as $\Pi(A \mid g = \cdot)$. For arbitrary λ in L we shall call

$\Pi(A \mid g = \lambda)$ *the conditional* $(L, \leq, P)$*-possibility of* A *given that* g *assumes the value* λ*. This means that*

$$(\forall B \in \wp(L)) \left((P)\!\!\!-\!\!\!\!\int_B \Pi(A \mid g = \cdot) \mathrm{d}\Gamma_g = \Pi(A \cap g^{-1}(B)) \right).$$

In the following theorem, we show that these conditional possibilities behave in a certain sense as ordinary normal possibility measures. We emphasize that the 'equalities' that appear in this theorem, are (Γ_g, P)-equivalences of $L-L$-mappings, and are in general less stringent than pointwise equalities of $L-L$-mappings.

Theorem 3 *Let* g *be a* $(L, \leq)$*-fuzzy variable in* $(X, \mathcal{R})$*.*

(i) $\Pi(\emptyset \mid g = \cdot) \overset{(\Gamma_g,P)}{=} \underline{0_L}$ *and* $\Pi(X \mid g = \cdot) \overset{(\Gamma_g,P)}{=} \underline{1_L}$*.*

(ii) Let $(h_j \mid j \in J)$ *be a family of elements of* $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$*. Then*

$$\Pi(\sup_{j \in J} h_j \mid g = \cdot) \overset{(\Gamma_g,P)}{=} \sup_{j \in J} \Pi(h_j \mid g = \cdot).$$

(iii) Let $(A_j \mid j \in J)$ *be a family of elements of* $\mathcal{R}$*. Then*

$$\Pi(\bigcup_{j \in J} A_j \mid g = \cdot) \overset{(\Gamma_g,P)}{=} \sup_{j \in J} \Pi(A_j \mid g = \cdot).$$

The conditional possibilities introduced here have a number of important special cases. The first case is singled out in definition 5. In corollary 1 we show that this special case also behaves to a certain extent as a ordinary normal possibility measure and its distribution, respectively.

Definition 5 *Let* h *and* g *be* $(L, \leq)$*-fuzzy variables in* $(X, \mathcal{R})$*. We introduce the following mappings, for arbitrary* λ *in* L *and arbitrary* B *in* $\wp(L)$*:*

$$\gamma_{h|g}(\lambda \mid \cdot) \colon L \to L \colon \mu \mapsto \Pi(h^{-1}(\{\lambda\}) \mid g = \mu)$$

and

$$\Gamma_{h|g}(B \mid \cdot) \colon L \to L \colon \mu \mapsto \Pi(h^{-1}(B) \mid g = \mu).$$

In other words, we have for any μ *in* L *that*

(i) $\gamma_{h|g}(\lambda \mid \mu) = \Pi(h^{-1}(\{\lambda\}) \mid g = \mu)$*, and we shall call this the conditional* $(L, \leq, P)$*-possibility that* h *assumes the value* λ *given that* g *assumes the value* μ*;*

(ii) $\Gamma_{h|g}(B \mid \mu) = \Pi(h^{-1}(B) \mid g = \mu)$*, and we shall call this the conditional* $(L, \leq, P)$*-possibility that* h *assumes a value in* B *given that* g *assumes the value* μ*.*

Corollary 1 *Let h and g be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$.*

(i) $\Gamma_{h|g}(\emptyset \mid \cdot) \overset{(\Gamma_g,P)}{=} \underline{0_L}$ *and* $\Gamma_{h|g}(L \mid \cdot) \overset{(\Gamma_g,P)}{=} \underline{1_L}$.

(ii) For any family $(B_j \mid j \in J)$ of elements of $\wp(L)$,

$$\Gamma_{h|g}(\bigcup_{j\in J} B_j \mid \cdot) \overset{(\Gamma_g,P)}{=} \sup_{j\in J} \Gamma_{h|g}(B_j \mid \cdot).$$

(iii) For any B in $\wp(L)$, $\Gamma_{h|g}(B \mid \cdot) \overset{(\Gamma_g,P)}{=} \sup_{\lambda\in B} \gamma_{h|g}(\lambda \mid \cdot)$.

We should not lose sight of the following: the mappings $\Pi(h^{-1}(A) \mid g = \cdot)$, for A in $\wp(L)$, underlying this special case of conditional possibility, are only determined up to (Γ_g, P)-equivalence. The same must therefore hold for the conditional possibilities introduced in definition 5. In the next theorem, we show that these can also be considered as solutions of a particular integral equation, that is of course a special case of (1). This equation is the following, for any h and g in $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$ and for any A in $\wp(L)$:

$$(\forall B \in \wp(L)) \left((P)\!\!-\!\!\!\!\int_B f \mathrm{d}\Gamma_g = \Gamma_{(h,g)}(A \times B) \right),$$

where, of course, the $L - L$-mapping f is the unknown. We might just as well have defined these conditional possibilities as arbitrary members of the equivalence class of solutions of this integral equation. Both approaches are clearly equivalent.[d]

Theorem 4 *(i) For any A and B in $\wp(L)$:*

$$(P)\!\!-\!\!\!\!\int_B \Gamma_{h|g}(A \mid \cdot)\mathrm{d}\Gamma_g = \Gamma_{(h,g)}(A \times B).$$

(ii) For any λ and μ in L:

$$P(\gamma_{h|g}(\lambda \mid \mu), \gamma_g(\mu)) = \gamma_{(h,g)}(\lambda, \mu).$$

Let us now look at a second special case. Consider arbitrary events D and E in $\mathcal{R}$. The characteristic $X - L$-mappings χ_D and χ_E are of course $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$. It follows from theorem 4(ii) that for any λ and μ in L, the element $\gamma_{\chi_D|\chi_E}(\lambda \mid \mu)$ of L satisfies $P(\gamma_{\chi_D|\chi_E}(\lambda \mid \mu), \gamma_{\chi_E}(\mu)) = \gamma_{(\chi_D,\chi_E)}(\lambda, \mu)$. If we choose $\lambda = \mu = 1_L$, this may, taking into account $\gamma_{\chi_E}(1_L) = \Pi(\chi_E^{-1}(\{1_L\})) = \Pi(E)$ and $\gamma_{(\chi_D,\chi_E)}(1_L, 1_L) = \Pi((\chi_D, \chi_E)^{-1}(\{(1_L, 1_L)\})) = \Pi(D \cap E)$, also be written as

$$P(\gamma_{\chi_D|\chi_E}(1_L \mid 1_L), \Pi(E)) = \Pi(D \cap E). \tag{2}$$

We now deduce the following definition from definition 5.

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[d]It should however be noted that if we follow the second approach, the defining equalities in definition 5 turn into (Γ_g, P)-equivalences.

Definition 6 *Let D and E be arbitrary elements of $\mathcal{R}$. We know that, by definition,*

$$\gamma_{\chi_D|\chi_E}(1_L \mid 1_L) = \Pi(\chi_D^{-1}(\{1_L\}) \mid \chi_E = 1_L) = \Pi(D \mid \chi_E = 1_L).$$

We shall therefore call $\gamma_{\chi_D|\chi_E}(1_L \mid 1_L)$ the conditional $(L, \leq, P)$-possibility of D given E. $\gamma_{\chi_D|\chi_E}(1_L \mid 1_L)$ will also be written as $\Pi(D \mid E)$, and (2) can therefore be rewritten as

$$P(\Pi(D \mid E), \Pi(E)) = \Pi(D \cap E).$$

Again, it should be noted that $\Pi(D \mid E)$ has been defined using the $L-L$-mapping $\Pi(D \mid \chi_E = \cdot)$, that is only determined up to (Γ_{χ_E}, P)-equivalence. Another, completely equivalent, approach[e] would consist in defining $\Pi(D \mid E)$ as an arbitary member of the set of solutions of the equation $P(\nu, \Pi(E)) = \Pi(D \cap E)$ in the element ν of L.

It is a prominent feature of our conditional possibilities that they are not necessarily uniquely defined, but only up to almost everywhere equality. A careful inspection of the literature on probability theory[2] will show that the same holds for conditional probabilities and expectations. However, in probability theory, this implies for instance that conditional expectations are (as stochastic variables) uniquely determined, except on a set with zero probability. Because the notion of almost everywhere equality is somewhat different in the possibilistic case (see the discussion in the previous section) the nondeterminacy may be more apparent for conditional possibilities, but *it has the same origin!* In principle, there is no fundamental difference between conditional possibilities, and conditional probabilities and expectations as far as their indeterminacy is concerned. As in probability theory, this indeterminacy is of no consequence, because conditional possibilities are never used *per se*, but always in combination with 'ordinary' possibilities. As a result, the indeterminacy disappears, as is apparent from the formulas given above. It therefore follows that this indeterminacy is not in principle a bad thing, or something that should necessarily be avoided, or eliminated by imposing extra criteria to ensure uniqueness.

The nondeterminacy of conditional possibilities has another important consequence, that should never be overlooked (as it should not in probability theory either). Whenever, in any definition, we use a conditional possibility, we should allow for the fact that this conditional possibility is *only determined up to almost everywhere equality.* As a consequence, if we write down equalities between conditional possibilities, these cannot be functional pointwise equalities, but must always be the appropriate almost everywhere equalities (or equivalences). Hisdal's failure to appreciate this has led her to distinguish between Zadeh's notion of noninteractivity[23], and her own definition of possibilistic independence[15]. Her notion of possibilistic independence, by the way, makes no sense, precisely because in her definition she uses the pointwise

[e] In this second approach, however, the defining equality in definition 6 turns into a properly chosen form of almost everywhere equality.

equality of conditional and marginal possibilities, and not the proper almost everywhere equality, as she should have. This example will illustrate that our measure- and integral-theoretic approach to possibility theory can solve a number of problems and inconsistencies in the literature. More details and further examples can be found elsewhere[4,6].

7. Possibilistic independence

We conclude this paper with a brief discussion of the possibilistic independence of (fuzzy) events, and its relation with conditional possibility. As has been mentioned before, a more detailed treatment will be published elsewhere[7].

We start our treatment of possibilistic independence with a basic definition, of which all the other independence definitions turn out to be special cases. As before, we consider an basic space Ω, provided with an ample field $\mathcal{R}_\Omega$ and a $(L, \leq)$-possibility measure Π_Ω with distribution π_Ω.

Definition 7 *Consider a nonempty family* $(\mathcal{E}_j \mid j \in J)$ *of subsets of* $\mathcal{R}_\Omega$. *This family is called* (Π_Ω, T)*-independent iff for any* n *in* $\mathbb{N} \setminus \{0\}$*, for arbitrary and different* j_1, $\ldots$, j_n *in* J*, for any* F_k *in* $\mathcal{E}_{j_k}$*, and for any* G_k *in* $\{F_k, \mathrm{co}F_k\}$, $k = 1, \ldots, n$*:*

$$\Pi_\Omega(\bigcap_{k=1}^{n} G_k) = T_{k=1}^{n} \Pi_\Omega(G_k).$$

We shall in this case also say that the event sets $\mathcal{E}_j$, $j \in J$*, are* (Π_Ω, T)*-independent.*

This definition is an immediate formal counterpart of the definition of stochastic independence of event sets in probability theory[16], that is nevertheless explicitly made *invariant under complementation*[4,7,12].

From definition 7 we derive the following definition for the possibilistic independence of possibilistic variables.

Definition 8 *Consider a nonempty family* $(X_j \mid j \in J)$ *of universes. For every* j *in* J *we consider an ample field* $\mathcal{R}_j$ *on* X_j *and a* $\mathcal{R}_\Omega - \mathcal{R}_j$*-measurable* $\Omega - X_j$*-mapping* f_j*, i.e.,* f_j *is a possibilistic variable in* $(X_j, \mathcal{R}_j)$. *We shall call the family* $(f_j \mid j \in J)$ *of possibilistic variables* (Π_Ω, T)*-independent iff the family* $(f_j^{-1}(\mathcal{R}_j) \mid j \in J)$ *of subsets of* $\mathcal{R}_\Omega$ *is* (Π_Ω, T)*-independent. We shall in this case also say that the possibilistic variables* f_j, $j \in J$*, are* (Π_Ω, T)*-independent.*

Fuzzy variables are special possibilistic variables. This observation leads to the following definition for the possibilistic independence of fuzzy events.

Definition 9 *Consider an nonempty family* $(h_j \mid j \in J)$ *of* $(L, \leq)$*-fuzzy variables in* $(\Omega, \mathcal{R}_\Omega)$. *This family is called* (Π_Ω, T)*-independent iff the family* $(h_j \mid j \in J)$ *of possibilistic variables in* $(L, \wp(L))$ *is* (Π_Ω, T)*-independent. We shall in that case also say that the fuzzy variables* h_j, $j \in J$*, are* (Π_Ω, T)*-independent.*

The following theorem gives a criterion for the independence of a finite number of fuzzy variables. The formal resemblance between this result and the well-known formulas for real stochastic variables[2] – the counterparts of our fuzzy variables – is striking.

Theorem 5 *The following statements are equivalent.*

(i) *The* $(L, \leq)$*-fuzzy variables* $h_1, \ldots, h_m$, $m \in \mathbb{N} \setminus \{0\}$, *in* $(\Omega, \mathcal{R}_\Omega)$ *are* (Π_Ω, T)*-independent.*

(ii) *For any* $(B_1, \ldots, B_m)$ *in* $\wp(L)^m$: $\Gamma_{(h_1,\ldots,h_m)}(B_1 \times \ldots \times B_m) = T_{k=1}^m \Gamma_{h_k}(B_k)$.

(iii) *For any* $(\lambda_1, \ldots, \lambda_m)$ *in* L^m: $\gamma_{(h_1,\ldots,h_m)}(\lambda_1, \ldots, \lambda_m) = T_{k=1}^m \gamma_{h_k}(\lambda_k)$.

The next theorem also has a probabilistic counterpart: the expectation value of a product of independent real stochastic variables equals the product of the expectation values of those variables[2].

Theorem 6 *If the* $(L, \leq)$*-fuzzy variables* $h_1, \ldots, h_m$, $m \in \mathbb{N} \setminus \{0\}$, *in* $(\Omega, \mathcal{R}_\Omega)$ *are* (Π_Ω, T)*-independent, we have, with obvious notations,*

$$(\Pi_\Omega)_T \left(T_{k=1}^m h_k\right) = T_{k=1}^m (\Pi_\Omega)_T(h_k).$$

The following results express the relation between conditional possibility[f] and possibilistic independence of fuzzy variables.

Theorem 7 *Let the t-norm* T *be weakly invertible, so that we may rightfully speak of conditional* $(L, \leq, T)$*-possibility. Let* h *and* g *be* $(L, \leq)$*-fuzzy variables in* $(\Omega, \mathcal{R}_\Omega)$. *If* h *and* g *are* (Π_Ω, T)*-independent,*

$$\Pi_\Omega(h \mid g = \cdot) \stackrel{(\Gamma_g, T)}{=} \underline{(\Pi_\Omega)_T(h)}.$$

Theorem 8 *Let the t-norm* T *be weakly invertible, so that we may rightfully speak of conditional* $(L, \leq, T)$*-possibility. Let* h *and* g *be* $(L, \leq)$*-fuzzy variables in* $(\Omega, \mathcal{R}_\Omega)$. *Then* h *and* g *are* (Π_Ω, T)*-independent if and only if*

$$(\forall \lambda \in L)(\gamma_{h|g}(\lambda \mid \cdot) \stackrel{(\Gamma_g, T)}{=} \underline{\gamma_h(\lambda)}),$$

or equivalently, $(\forall (\lambda, \mu) \in L^2)(T(\gamma_{h|g}(\lambda \mid \mu), \gamma_g(\mu)) = T(\gamma_h(\lambda), \gamma_g(\mu))$.

The notion 'fuzzy variable in $(\Omega, \mathcal{R}_\Omega)$' is a generalization of the notion '$\mathcal{R}_\Omega$-measurable subset of Ω'. Any element A of $\mathcal{R}_\Omega$ can be identified with the $(L, \leq)$-fuzzy variable χ_A in $(\Omega, \mathcal{R}_\Omega)$. We exploit this identification in the next definition, in order to introduce the possibilistic independence of measurable sets.

Definition 10 *Consider a family* $(A_j \mid j \in J)$ *of elements of* $\mathcal{R}_\Omega$. *This family is called* (Π_Ω, T)*-independent iff the family* $(\chi_{A_j} \mid j \in J)$ *of* $(L, \leq)$*-fuzzy variables in* $(\Omega, \mathcal{R}_\Omega)$ *is* (Π_Ω, T)*-independent. We shall in this case also say that the elements* A_j, $j \in J$, *of* $\mathcal{R}_\Omega$ *are* (Π_Ω, T)*-independent.*

Theorem 9 *A family* $(A_j \mid j \in J)$ *of elements of* $\mathcal{R}_\Omega$ *is* (Π_Ω, T)*-independent if and only if for any* n *in* $\mathbb{N} \setminus \{0\}$, *for arbitrary and different* $j_1, \ldots, j_n$ *in* J, *for any* F_k *in* $\{A_{j_k}, \mathrm{co}A_{j_k}\}$, $k = 1, \ldots, n$, $\Pi_\Omega(\bigcap_{k=1}^n F_k) = T_{k=1}^n \Pi_\Omega(F_k)$.

[f] These conditional possibilities are of course defined using the t-norm T rather than the t-seminorm P.

In the following proposition we consider the special case of two events. The reader will notice that four conditions appear, in stead of one in the probabilistic case. This is necessary to guarantee that the independence criterion is invariant for the complementation of these events.

Proposition 2 *Two elements O_1 and O_2 of $\mathcal{R}_\Omega$ are (Π_Ω, T)-independent if and only if*

$$\begin{cases} \Pi_\Omega(O_1 \cap O_2) & = T(\Pi_\Omega(O_1), \Pi_\Omega(O_2)) \\ \Pi_\Omega(O_1 \cap \mathrm{co}O_2) & = T(\Pi_\Omega(O_1), \Pi_\Omega(\mathrm{co}O_2)) \\ \Pi_\Omega(\mathrm{co}O_1 \cap O_2) & = T(\Pi_\Omega(\mathrm{co}O_1), \Pi_\Omega(O_2)) \\ \Pi_\Omega(\mathrm{co}O_1 \cap \mathrm{co}O_2) & = T(\Pi_\Omega(\mathrm{co}O_1), \Pi_\Omega(\mathrm{co}O_2)). \end{cases}$$

Furthermore, for any O in $\mathcal{R}_\Omega$, $\emptyset$, O and Ω are (Π_Ω, T)-independent.

8. Conclusion

The results in this paper show that it is possible to construct a measure- and integral-theoretic account of possibility theory, formally along the same general lines as Kolmogorov's account of probability theory.

9. Acknowledgements

Gert de Cooman is a Senior Research Assistant of the Belgian National Fund for Scientific Research (NFSR). He would like to thank the NFSR for funding the research, reported on in this paper.

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