First results for a mathematical theory of possibilistic Markov processes

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Abstract

We provide basic results for the development of a theory of possibilistic Markov processes. We define and study possibilistic Markov processes and possibilistic Markov chains, and derive a possibilistic analogon of the Chapman-Kolmogorov equation. We also show how possibilistic Markov processes can be constructed using one-step transition possibilities.

1 INTRODUCTION

Stochastic processes which possess the Markov property are very important from a practical point of view. Moreover, they are much simpler to deal with theoretically than stochastic processes in general.

In a previous paper [7], we addressed the measure-theoretic foundations for a theory of *possibilistic processes*, that is, processes for which the available information about their behaviour is modelled by a possibility measure rather than by a probability measure. In this paper, we give a measure-theoretic treatment of a special case: possibilistic Markov processes. For an interesting, but less formal and less general account of possibilistic Markov processes, with a discussion of their meaning and possible applications, we refer to [8].

In sections 2–4 we provide the introductory material and pointers to the literature, which should help readers understand the ensuing discussion. In section 5, we define possibilistic Markov processes and discuss a number of their properties. In section 6, we indicate how possibilistic Markov processes can be constructed from an initial possibility distribution and a collection of one-step transition possibilities.

2 PRELIMINARY NOTIONS

By $(L, \leq)$ we mean a complete lattice with smallest element 0_L and greatest element 1_L, and we assume that $0_L \neq 1_L$.

A subset $\mathcal{R}$ of the power class $\wp(X)$ of a nonempty set X is called a *τ-field* or *ample field* [4, 9] on X iff it is closed under arbitrary unions and under complementation. The *atom* of $\mathcal{R}$ containing the element x of X is defined as $[\,x\,]_\mathcal{R} = \bigcap\{A \mid A \in \mathcal{R} \text{ and } x \in A\}$. Furthermore, for any subset A of X, $A \in \mathcal{R} \Leftrightarrow A = \bigcup_{x\in A} [\,x\,]_\mathcal{R}$. The couple $(X, \mathcal{R})$ is called a *τ-space*.

If $\mathcal{A}_1 \subseteq \wp(X_1)$ and $\mathcal{A}_2 \subseteq \wp(X_2)$, where X_1 and X_2 are nonempty sets, then a $X_1 - X_2$-mapping f is called *$\mathcal{A}_1 - \mathcal{A}_2$-measurable* iff $(\forall B \in \mathcal{A}_2)(f^{-1}(B) \in \mathcal{A}_1)$.

Let $(X_t \mid t \in T)$ be a family of nonempty sets with nonempty index set T. Let $(\mathcal{R}_t \mid t \in T)$ be a family of corresponding ample fields, i.e., $\mathcal{R}_t$ is an ample field on X_t, $t \in T$. For simplicity, the *Cartesian product* of the family $(X_t \mid t \in T)$ of sets is denoted by X_T. It is the set of all $T - \bigcup_{t\in T} X_t$-mappings x such that $(\forall t \in T)(x(t) \in X_t)$.

For any $t \in T$, $\mathbf{pr}_{T,t}$ is the t-th projection mapping from X_T onto X_t, given by $\mathbf{pr}_{T,t}(x) = x(t)$, $x \in X_T$. For any nonempty subset T' of T, $\mathbf{pr}_{T,T'}$ is the $X_T - X_{T'}$-mapping such that for any $x \in X_T$, $\mathbf{pr}_{T,T'}(x) = x|_{T'}$ is the restriction of the mapping x to T'.

Furthermore, if f_t is a mapping from a set X to X_t for all $t \in T$, then the *product mapping* is the unique $X - X_T$ mapping f_T such that $f_t = \mathbf{pr}_{T,t} \circ f_T$ for all $t \in T$.

Finally, the *product τ-field* [7] of the family $(\mathcal{R}_t \mid t \in T)$ on X_T is denoted by $\mathcal{R}_T$, and is the smallest τ-field $\mathcal{H}$ on X_T such that $\mathbf{pr}_{T,t}$ is a $\mathcal{H} - \mathcal{R}_t$-measurable mapping for all $t \in T$. $(X_T, \mathcal{R}_T)$ is called the *product τ-space* of the family $((X_t, \mathcal{R}_t) \mid t \in T)$ of τ-spaces.

The following notions and results can be found in [1, 2, 3]. A t-norm $\mathcal{T}$ on $(L, \leq)$ is *completely distribu-*

tive w.r.t. supremum iff for any $\lambda \in L$ and any family $(\mu_j \mid j \in J)$ of elements of L: $\mathcal{T}(\lambda, \sup_{j\in J} \mu_j) = \sup_{j\in J} \mathcal{T}(\lambda, \mu_j)$. In this case, the structure $(L, \leq, \mathcal{T})$ is called a *complete lattice with t-norm.*

Let λ and μ be elements of L and let $\mathcal{T}$ be a t-norm on $(L, \leq)$. The *residual* $\lambda \ \triangle_{\mathcal{T}} \ \mu$ for $\mathcal{T}$ of λ by μ is defined as $\lambda \ \triangle_{\mathcal{T}} \ \mu = \sup\{\nu \mid \nu \in L \text{ and } \mathcal{T}(\nu, \mu) \leq \lambda\}$.

An element α of L is called an *inverse* for $\mathcal{T}$ of λ w.r.t. μ iff $\mathcal{T}(\alpha, \lambda) = \mu$. Furthermore, $\mathcal{T}$ is called *weakly invertible* iff for any λ and μ in L with $\mu \leq \lambda$, there exists an inverse for $\mathcal{T}$ of λ w.r.t. μ. If $(L, \leq, \mathcal{T})$ is a complete lattice with t-norm, then $\mathcal{T}$ is weakly invertible iff for any λ and μ in L with $\mu \leq \lambda$, $\mathcal{T}(\mu \ \triangle_{\mathcal{T}} \ \lambda, \lambda) = \mu$.

Furthermore, for a τ-space $(X, \mathcal{R})$, $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$ is the set of the $X-L$-mappings which are $\mathcal{R}-\wp(L)$-measurable, or simply, *$\mathcal{R}$-measurable.* A $X - L$-mapping is $\mathcal{R}$-measurable iff it is constant on the atoms of $\mathcal{R}$.

A $\mathcal{R} - L$-mapping Π is a $(L, \leq)$-*possibility measure* [1, 4] on $(X, \mathcal{R})$ iff $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$ for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$. The triple $(X, \mathcal{R}, \Pi)$ is called a $(L, \leq)$-*possibility space.* Furthermore, every $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$ possesses a unique distribution $\pi \in \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$, given by $\pi(x) = \Pi([\,x\,]_{\mathcal{R}})$, $x \in X$.

Assume that $(L, \leq, \mathcal{T})$ is complete lattice with t-norm and that Π is a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π. Then, for any element A of $\mathcal{R}$ and any $\mathcal{R}$-measurable $X - L$-mapping h, the $(L, \leq, \mathcal{T})$-*possibility integral* of h on A w.r.t. Π can be written as [5]:

$$(\mathcal{P})\!\!\int_A h \mathrm{d}\Pi = \sup_{x\in A} \mathcal{T}(h(x), \pi(x)).$$

Finally, in the rest of this paper, we shall also use the following notations. Let $T_n = \{0, \ldots, n\}$ for any $n \in \mathbb{N}$, and $T_{n,m} = \{n, \ldots, m\}$ for any couple $(n, m) \in \mathbb{N}^2$ such that $n \leq m$. Furthermore, if A is a subset of a nonempty set X, then we write $A \Subset X$ iff A is a finite subset of X. $(X, \mathcal{R})$ denotes an arbitrary, but fixed τ-space and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

3 CONDITIONAL POSSIBILITY

In this section, we briefly discuss De Cooman's [2] notion of conditional possibility of possibilistic variables, which we shall use in section 5 to define possibilistic Markov families and processes.

If $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ is a $(L, \leq)$-possibility space and $(X, \mathcal{R})$ is a τ-space, then a $\Omega - X$-mapping f is a *possibilistic variable* in $(X, \mathcal{R})$ iff f is a $\mathcal{R}_\Omega - \mathcal{R}$-measurable mapping [1]. The τ-space $(X, \mathcal{R})$ in which the possibilistic variable takes its values, is called the *sample space* of f, and $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ is called the *basic space* of f. In this paper, and unless explicitly stated to the contrary, all possibilistic variables will be assumed to have the $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as their basic space. Finally, if π_Ω represents the distribution of Π_Ω and f is a possibilistic variable in $(X, \mathcal{R})$, then the $X - L$-mapping π_f, which is given by $\pi_f(x) = \sup_{\omega\in f^{-1}([\,x\,]_{\mathcal{R}})} \pi_\Omega(\omega)$, $x \in X$, is called the *possibility distribution* of f.

Let us recall the notion of almost everywhere equality [1].

Definition 1 *Let $\mathcal{T}$ be a t-norm on $(L, \leq)$, let $E \in \mathcal{R}$ and $(h_1, h_2) \in \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)^2$. When for any $x \in E$*

$$\mathcal{T}(h_1(x), \pi(x)) = \mathcal{T}(h_2(x), \pi(x)),$$

we say that h_1 and h_2 are $(\Pi, \mathcal{T})$-equal almost everywhere on E, and we write $h_1 \overset{(\Pi,\mathcal{T},E)}{=} h_2$. Instead of $h_1 \overset{(\Pi,\mathcal{T},X)}{=} h_2$, we also write $h_1 \overset{(\Pi,\mathcal{T})}{=} h_2$ and say that h_1 and h_2 are almost everywhere $(\Pi, \mathcal{T})$-equal, or equivalently, that they are $(\Pi, \mathcal{T})$-equivalent.

It is clear that for any $E \in \mathcal{R}$, $\overset{(\Pi,\mathcal{T},E)}{=}$ is an equivalence relation on $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$ [1].

Assume that f_1 and f_2 are possibilistic variables in the τ-spaces $(X_1, \mathcal{R}_1)$ and $(X_2, \mathcal{R}_2)$ respectively, with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$. If we provide the Cartesian product $X_1 \times X_2$ with the product τ-field $\mathcal{R}_1 \times \mathcal{R}_2$ [7, 9], then the $\Omega - X_1 \times X_2$-mapping (f_1, f_2) is also a possibilistic variable in $(X_1 \times X_2, \mathcal{R}_1 \times \mathcal{R}_2)$ [2]. Furthermore, if $(L, \leq, \mathcal{T})$ is a complete lattice with weakly invertible t-norm, then it can be proven [2] that for any $A_1 \in \mathcal{R}_1$ there exists a $\mathcal{R}_2$-measurable X_2-L-mapping h satisfying the integral equation

$$(\forall A_2 \in \mathcal{R}_2)\left((\mathcal{P})\!\!\int_{A_2} h \mathrm{d}\Pi_{f_2} = \Pi_{(f_1,f_2)}(A_1 \times A_2)\right), \tag{1}$$

the solutions of which are unique in the sense of $(\Pi_{f_2}, \mathcal{T})$-equivalence. Using this result, *the conditional possibility of possibilistic variables* can be defined as follows [2].

Definition 2 *Let $(L, \leq, \mathcal{T})$ be a complete lattice with weakly invertible t-norm. Let A_1 be an element of $\mathcal{R}_1$. Then an element of the equivalence class of solutions of the integral equation (1) is represented by $\Pi_{f_1|f_2}(A_1 \mid \cdot)$. For any x_2 in X_2, $\Pi_{f_1|f_2}(A_1 \mid x_2)$ is called the conditional $(L, \leq, \mathcal{T})$-possibility that f_1 takes a value in A_1 given that f_2 takes a value in $[\,x_2\,]_{\mathcal{R}_2}$. For any x_1 in X_1, $\Pi_{f_1|f_2}([\,x_1\,]_{\mathcal{R}_1} \mid x_2)$ is also written as $\pi_{f_1|f_2}(x_1 \mid x_2)$.*

It should be stressed that conditional possibilities in this approach are only defined uniquely up to almost everywhere equality. As a result, it is only meaningful to express the equality of conditional possibilities as an almost everywhere equality.

By means of residuals, the following formulas for conditional possibilities can be derived [2].

Proposition 3 *Let $(L, \leq, \mathcal{T})$ be a complete lattice with weakly invertible t-norm. Let A_1 be an element of $\mathcal{R}_1$ and let x_1 be an element of X_1, then*

$$\Pi_{f_1|f_2}(A_1 \mid \cdot) \overset{(\Pi_{f_2},\mathcal{T})}{=} \Pi_{(f_1,f_2)}(A_1 \times [\,\cdot\,]_{\mathcal{R}_2}) \;\;\triangle_{\mathcal{T}}\;\; \pi_{f_2}(\cdot)$$

and

$$\pi_{f_1|f_2}(x_1 \mid \cdot) \overset{(\Pi_{f_2},\mathcal{T})}{=} \pi_{(f_1,f_2)}(x_1, \cdot) \;\;\triangle_{\mathcal{T}}\;\; \pi_{f_2}(\cdot).$$

4 REMARKS ON FAMILIES OF POSSIBILISTIC VARIABLES

In this section, we generalize our notion of a consistent family of distributions [7]. Furthermore, we indicate how generally a τ-space $(X, \mathcal{R})$ can be transported by a mapping to a set Y. In particular, we obtain some generalizations of results of De Cooman [2], and we prove that the family $(\pi_{f_{T'}} \mid \emptyset \subset T' \Subset T)$ of possibility distributions associated with a family $(f_t \mid t \in T)$ of possibilistic variables is consistent.

Throughout this section, let $((X_t, \mathcal{R}_t) \mid t \in T)$ be a family of τ-spaces with nonempty index set T. First of all, let us recall the notions of family of possibilistic variables and possibilistic process [7].

Definition 4 *Let T be a nonempty set. A family $(f_t \mid t \in T)$, such that f_t is a possibilistic variable in the τ-space $(X_t, \mathcal{R}_t)$, $t \in T$, is called a family of possibilistic variables in $((X_t, \mathcal{R}_t) \mid t \in T)$. If $X_t = X$ and $\mathcal{R}_t = \mathcal{R}$, $t \in T$, then $(f_t \mid t \in T)$ is called a possibilistic process in $(X, \mathcal{R})$. Families of possibilistic variables and possibilistic processes whose index set is countable, are respectively called discrete families of possibilistic variables and discrete possibilistic processes.*

The following relation between the measurability of a product mapping and the measurability of its components can be proven.

Proposition 5 *Let $(X, \mathcal{R})$ be a τ-space and let T' be a nonempty subset of T. Assume f_t is a $X - X_t$-mapping for any $t \in T'$, then $f_{T'}$ is a $\mathcal{R} - \mathcal{R}_{T'}$-measurable mapping iff f_t is a $\mathcal{R} - \mathcal{R}_t$-measurable mapping for all $t \in T'$.*

Corollary 6 *Assume f_t is a $\Omega - X_t$-mapping for any $t \in T$ and let T' be a nonempty subset of T. $f_{T'}$ is a possibilistic variable in $(X_{T'}, \mathcal{R}_{T'})$ iff f_t is a possibilistic variable in $(X_t, \mathcal{R}_t)$ for all $t \in T'$.*

Let us now generalize the notion of a consistent family of distributions [7]. Let $\pi_{T'}$ be the distribution of a $(L, \leq)$-possibility measure $\Pi_{T'}$ on $(X_{T'}, \mathcal{R}_{T'})$ for any nonempty, finite subset T' of T. Assume $A_t \in \mathcal{R}_t$ for all $t \in T$.

Definition 7 *The family $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is called consistent w.r.t. $(A_t \mid t \in T)$ iff for any two nonempty, finite subsets T_1 and T_2 of T such that $\emptyset \subset T_1 \subseteq T_2 \Subset T$, and for any $x \in A_{T_1}$:*

$$\pi_{T_1}(x) = \sup_{y \in A_{T_2}, \mathbf{pr}_{T_2,T_1}(y) = x} \pi_{T_2}(y).$$

Furthermore, $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is called consistent iff $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent w.r.t. $(X_t \mid t \in T)$.

For the remainder of this section, assume that $(f_t \mid t \in T)$ is a family of possibilistic variables in $((X_t, \mathcal{R}_t) \mid t \in T)$.

By Corollary 6, we know that, for any nonempty subset T' of T, $f_{T'}$ is a possibilistic variable in $(X_{T'}, \mathcal{R}_{T'})$ with possibility distribution $\pi_{f_{T'}}$, which is the distribution of the $(L, \leq)$-possibility measure $\Pi_{f_{T'}}$ on $(X_{T'}, \mathcal{R}_{T'})$. We now want to prove that the family $(\pi_{f_{T'}} \mid \emptyset \subset T' \Subset T)$ of possibility distributions is consistent. To this end, we can use the following proposition, which indicates how a τ-space $(X, \mathcal{R})$ can be transported by a mapping h to a set Y.

Proposition 8 *Consider a τ-space $(X, \mathcal{R})$ and a mapping h from the set X to a set Y. Then $\mathcal{R}^h = \{E \mid E \in \wp(Y) \text{ and } h^{-1}(E) \in \mathcal{R}\}$ is the greatest τ-field $\mathcal{H}$ on Y for which h is $\mathcal{R} - \mathcal{H}$-measurable. If f is a $\Omega - X$-mapping, let $f^h = h \circ f$. If f is a possibilistic variable in $(X, \mathcal{R})$, then f^h is a possibilistic variable in $(Y, \mathcal{R}^h)$ such that for any $A \in \mathcal{R}^h$:*

$$\Pi_{f^h}(A) = \Pi_f(h^{-1}(A)),$$

and for any $y \in Y$

$$\pi_{f^h}(y) = \Pi_f(h^{-1}([\,y\,]_{\mathcal{R}^h})).$$

Furthermore, assume that h is a bijection. Then $\mathcal{R}^h = \{h(A) \mid A \in \mathcal{R}\}$, and for any $x \in X$, $[\,h(x)\,]_{\mathcal{R}^h} = h([\,x\,]_{\mathcal{R}})$. Furthermore, f is a possibilistic variable in $(X, \mathcal{R})$ iff f^h is a possibilistic variable in $(Y, \mathcal{R}^h)$. Finally, if f is a possibilistic variable in $(X, \mathcal{R})$, then the possibility distribution π_{f^h} of f^h is given by $\pi_f \circ h^{-1}$.

So, if the mapping h is a bijection, we can identify $(X, \mathcal{R})$ with $(Y, \mathcal{R}^h)$, and there is a one-one relation between the possibilistic variables in $(X, \mathcal{R})$ and those in $(Y, \mathcal{R}^h)$.

The following result generalizes some findings of De Cooman [2].

Proposition 9 *Let* $\emptyset \subset T_2 \subseteq T_1 \subseteq T$*, then* $\mathcal{R}_{T_1}^{\mathbf{pr}_{T_1,T_2}} = \mathcal{R}_{T_2}$ *and* $\mathbf{pr}_{T_1,T_2}([\,x\,]_{\mathcal{R}_{T_1}}) = [\,x|_{T_2}\,]_{\mathcal{R}_{T_2}}$ *for any* $x \in X_{T_1}$*. Furthermore,* $f_{T_1}^{\mathbf{pr}_{T_1,T_2}} = f_{T_2}$*. Hence,* $\Pi_{f_{T_2}}(A) = \Pi_{f_{T_1}}(\mathbf{pr}_{T_1,T_2}^{-1}(A))$ *for any* $A \in \mathcal{R}_{T_2}$*, and* $\pi_{f_{T_2}}(x) = \sup_{\mathbf{pr}_{T_1,T_2}(y)=x} \pi_{f_{T_1}}(y)$ *for any* $x \in X_{T_2}$*.*

Corollary 10 $(\pi_{f_{T'}} \mid \emptyset \subset T' \Subset T)$ *is a consistent family of distributions.*

Now, assume T_1 and T_2 are nonempty subsets of T such that $T_1 \cap T_2 = \emptyset$. Then $(\mathbf{pr}_{T_1 \cup T_2, T_1}, \mathbf{pr}_{T_1 \cup T_2, T_2})$ is a bijection from $X_{T_1 \cup T_2}$ to $X_{T_1} \times X_{T_2}$. Using Propositions 8 and 9, we can identify $(X_{T_1 \cup T_2}, \mathcal{R}_{T_1 \cup T_2})$ with $(X_{T_1} \times X_{T_2}, \mathcal{R}_{T_1} \times \mathcal{R}_{T_2})$, and the possibilistic variable (f_{T_1}, f_{T_2}) in $(X_{T_1} \times X_{T_2}, \mathcal{R}_{T_1} \times \mathcal{R}_{T_2})$ with the possibilistic variable $f_{T_1 \cup T_2}$ in $(X_{T_1 \cup T_2}, \mathcal{R}_{T_1 \cup T_2})$, whose possibility distribution is given by $\pi_{f_{T_1 \cup T_2}}(x) = \pi_{(f_{T_1}, f_{T_2})}(x|_{T_1}, x|_{T_2})$ for any $x \in X_{T_1 \cup T_2}$.

Next, consider an element t of T. Then we can similarly identify $(X_{\{t\}}, \mathcal{R}_{\{t\}})$ with $(X_t, \mathcal{R}_t)$, and the possibilistic variable $f_{\{t\}}$ in $(X_{\{t\}}, \mathcal{R}_{\{t\}})$ with the given possibilistic variable f_t in $(X_t, \mathcal{R}_t)$, such that $\pi_{f_{\{t\}}}(x) = \pi_{f_t}(x(t))$ for any $x \in X_{\{t\}}$.

As a result, if T' is a nonempty subset of T such that $t \notin T'$, we can identify the τ-spaces $(X_t \times X_{T'}, \mathcal{R}_t \times \mathcal{R}_{T'})$ and $(X_{T'} \times X_t, \mathcal{R}_{T'} \times \mathcal{R}_t)$ with $(X_{T' \cup \{t\}}, \mathcal{R}_{T' \cup \{t\}})$, and the possibilistic variables $(f_t, f_{T'})$ and $(f_{T'}, f_t)$ in $(X_t \times X_{T'}, \mathcal{R}_t \times \mathcal{R}_{T'})$ and $(X_{T'} \times X_t, \mathcal{R}_{T'} \times \mathcal{R}_t)$ respectively with the possibilistic variable $f_{T' \cup \{t\}}$ in $(X_{T' \cup \{t\}}, \mathcal{R}_{T' \cup \{t\}})$, such that $\pi_{f_{T' \cup \{t\}}}(x) = \pi_{(f_t, f_{T'})}(x(t), x|_{T'}) = \pi_{(f_{T'}, f_t)}(x|_{T'}, x(t))$ for any $x \in X_{T' \cup \{t\}}$.

Finally, assume T' is a nonempty, finite subset of T with $n \in \mathbb{N}^*$ elements. Let $T' = \{t_1, \ldots, t_n\}$, then we can identify $(\times_{i=1}^n X_{t_i}, \times_{i=1}^n \mathcal{R}_{t_i})$ with $(X_{T'}, \mathcal{R}_{T'})$, and the possibilistic variable $(f_{t_1}, \ldots, f_{t_n})$ in $(\times_{i=1}^n X_{t_i}, \times_{i=1}^n \mathcal{R}_{t_i})$ with the possibilistic variable $f_{T'}$ in $(X_{T'}, \mathcal{R}_{T'})$, such that $\pi_{(f_{t_1}, \ldots, f_{t_n})}(x(t_1), \ldots, x(t_n)) = \pi_{f_{T'}}(x)$ for any $x \in X_{T'}$.

In what follows, we shall often make use of these identifications without explicitly mentioning them.

5 POSSIBILISTIC MARKOV FAMILIES AND PROCESSES

In this section, we define a possibilistic Markov family as a family of possibilistic variables, indexed by a partially ordered set, and satisfying a possibilistic version of the well-known Markov condition [6]. In particular, if this family is a possibilistic process, the possibilistic Markov family is called a possibilistic Markov process. We prove that a possibilistic Markov family satisfies a possibilistic analogon of the Chapman-Kolmogorov equation [6]. Furthermore, we derive an alternative formulation of the definition, which tells us that the property of being a possibilistic Markov family is invariant under order-reversal of the partially ordered index set. Finally, we derive an alternative characterization for discrete possibilistic Markov families.

We assume that $(T, \leq)$ is a nonempty, partially ordered set, that $(L, \leq, \mathcal{T})$ is a complete lattice with a weakly invertible t-norm, that $(f_t \mid t \in (T, \leq))$ is a family of possibilistic variables in the family of τ-spaces $((X_t, \mathcal{R}_t) \mid t \in T)$, and that $A_t \in \mathcal{R}_t$ for all $t \in T$.

Inspired by probability theory [6], we now define a possibilistic Markov family as follows.

Definition 11 $(f_t \mid t \in (T, \leq))$ *is called a possibilistic Markov family in* $((X_t, \mathcal{R}_t) \mid t \in T)$ *w.r.t.* $(A_t \mid t \in T)$ *iff for any nonempty, finite chain* T' *in* $(T, \leq)$ *and any element* $t \in T$ *such that* $t > \sup T'$*:*

$$\pi_{f_t|f_{T'}}(x \mid \cdot) \overset{(\Pi_{f_{T'}}, \mathcal{T}, A_{T'})}{=} \pi_{f_t|f_{\sup T'}}(x \mid \cdot) \circ \mathbf{pr}_{T', \sup T'},$$

for any $x \in A_t$*.* $(f_t \mid t \in (T, \leq))$ *is called a possibilistic Markov family in* $((X_t, \mathcal{R}_t) \mid t \in T)$ *iff* $(f_t \mid t \in (T, \leq))$ *is a possibilistic Markov family in* $((X_t, \mathcal{R}_t) \mid t \in T)$ *w.r.t.* $(X_t \mid t \in T)$*. Assume* $X_t = X, \mathcal{R}_t = \mathcal{R}$ *and* $A_t = A \in \mathcal{R}$ *for all* $t \in T$*. If* $(f_t \mid t \in (T, \leq))$ *is a possibilistic Markov family in* $((X_t, \mathcal{R}_t) \mid t \in T)$ *w.r.t.* $(A_t \mid t \in T)$*, then* $(f_t \mid t \in (T, \leq))$ *is also called a possibilistic Markov process in* $(X, \mathcal{R})$ *w.r.t. the set* A*, and is called a possibilistic Markov process in* $(X, \mathcal{R})$ *if* $A = X$*. When* $(X, \mathcal{R})$ *has a countable number of atoms, possibilistic Markov processes in* $(X, \mathcal{R})$ *are also called possibilistic Markov chains in* $(X, \mathcal{R})$*.*

We can derive the following properties from this definition.

Proposition 12 *Let* $(f_t \mid t \in (T, \leq))$ *be a possibilistic Markov family in* $((X_t, \mathcal{R}_t) \mid t \in T)$ *w.r.t.* $(A_t \mid t \in T)$ *and let* T_1 *be a nonempty subset of* T*. Then the subfamily* $(f_t \mid t \in (T_1, \leq))$ *of* $(f_t \mid t \in (T, \leq))$ *is a possibilistic Markov family in* $((X_t, \mathcal{R}_t) \mid t \in T_1)$ *w.r.t.* $(A_t \mid t \in T_1)$*.*

Corollary 13 *A subfamily of a possibilistic Markov family (process) is also a possibilistic Markov family (process).*

Now, we want to prove that possibilistic Markov families satisfy a possibilistic analogon of the Chapman-Kolmogorov equation. We use the following proposition.

Proposition 14 *Assume $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ w.r.t. $(A_t \mid t \in T)$, such that $(\pi_{f_{T'}} \mid \emptyset \subset T' \Subset T)$ is consistent w.r.t. $(A_t \mid t \in T)$. For any $(t_0, t_1, t_2) \in T^3$ such that $t_0 < t_1 < t_2$ and for any $x \in A_{t_2}$:*

$$\pi_{f_{t_2}|f_{t_0}}(x \mid \cdot) \overset{(\Pi_{f_{t_0}}, \mathcal{T}, A_{t_0})}{=} \sup_{y \in A_{t_1}} \mathcal{T}(\pi_{f_{t_2}|f_{t_1}}(x \mid y), \pi_{f_{t_1}|f_{t_0}}(y \mid \cdot)).$$

Corollary 15 *Assume $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family. For any $(t_0, t_1, t_2) \in T^3$ such that $t_0 < t_1 < t_2$ and for any $x \in X_{t_2}$:*

$$\pi_{f_{t_2}|f_{t_0}}(x \mid \cdot) \overset{(\Pi_{f_{t_0}}, \mathcal{T})}{\underline{\underline{\quad}}} \sup_{y \in X_{t_1}} \mathcal{T}(\pi_{f_{t_2}|f_{t_1}}(x \mid y), \pi_{f_{t_1}|f_{t_0}}(y \mid \cdot)).$$

So, a possibilistic Markov family satisfies a true analogon of the Chapman-Kolmogorov equation [6].

In the following proposition, we obtain a formula for the possibility distribution $\pi_{f_{T'}}$, where T' is a finite chain in $(T, \leq)$. Use is made of the conditional possibility distributions $\pi_{f_{t_1}|f_{t_2}}(\cdot \mid \cdot)$ where t_1 covers (immediately follows) t_2 in the chain T'.

Proposition 16 *Assume T' is a finite chain in $(T, \leq)$, say $T' = \{t_0, \ldots, t_n\}$, where $n \in \mathbb{N}^*$ and $t_i < t_{i+1}$ for any $i \in \{0, \ldots, n-1\}$. If $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ w.r.t. $(A_t \mid t \in T)$, then for any $x \in A_{T'}$:*

$$\pi_{f_{T'}}(x) = \mathcal{T}(\pi_{f_{t_0}}(x(t_0)), \mathcal{T}_{i=0}^{n-1} \pi_{f_{t_{i+1}}|f_{t_i}}(x(t_{i+1}) \mid x(t_i))).$$

Using this formula we obtain an alternative characterization for a possibilistic Markov family.

Theorem 17 *Assume $(\pi_{f_{T'}} \mid \emptyset \subset T' \Subset T)$ is consistent w.r.t. $(A_t \mid t \in T)$. Then, $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ w.r.t. $(A_t \mid t \in T)$ iff*

$$\pi_{f_{S' \cup T'}|f_t}(x \mid \cdot) \overset{(\Pi_{f_t}, \mathcal{T}, A_t)}{=} \mathcal{T}(\pi_{f_{S'}|f_t}(x|_{S'} \mid \cdot), \pi_{f_{T'}|f_t}(x|_{T'} \mid \cdot))$$

for any $x \in A_{S' \cup T'}$, where S' and T' are finite chains in $(T, \leq)$ and $t \in T$ such that $\sup S' < t < \inf T'$.

Corollary 18 *The family of possibilistic variables $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ iff*

$$\pi_{f_{S' \cup T'}|f_t}(x \mid \cdot) \overset{(\Pi_{f_t}, \mathcal{T})}{\underline{\underline{\quad}}} \mathcal{T}(\pi_{f_{S'}|f_t}(x|_{S'} \mid \cdot), \pi_{f_{T'}|f_t}(x|_{T'} \mid \cdot))$$

for any $x \in X_{S' \cup T'}$, where S' and T' are finite chains in $(T, \leq)$ and $t \in T$ such that $\sup S' < t < \inf T'$.

If $(T, \leq)$ can be interpreted as time set, this alternative characterization 'tells' us that the possibilistic variables $f_{S'}$ and $f_{T'}$ are possibilistically independent [3] given that f_t takes a value in a fixed atom of $\mathcal{R}_t$, or equivalently, that past and future are possibilistically independent conditional on the present. From Theorem 17 we immediately derive the following result.

Corollary 19 *Assume $(\pi_{f_{T'}} \mid \emptyset \subset T' \Subset T)$ is consistent w.r.t. $(A_t \mid t \in T)$. Then, $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ w.r.t. $(A_t \mid t \in T)$ iff $(f_t \mid t \in (T, \geq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ w.r.t. $(A_t \mid t \in T)$. In particular, $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ iff $(f_t \mid t \in (T, \geq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$.*

If $(T, \leq)$ can be interpreted as a time set, this means that the property of being a possibilistic Markov family is invariant under time reversal.

Let us now look at the special case of discrete families of possibilistic variables. In particular, let $T = T_n$ where $n \in \mathbb{N}^*$ or $T = \mathbb{N}$. Let $\leq$ be the natural linear ordering of natural numbers.

Then, the following proposition gives another characterization of discrete possibilistic Markov families.

Proposition 20 *$(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t, \mathcal{R}_t) \mid t \in T)$ w.r.t. $(A_t \mid t \in T)$ iff for any $n \in T$ such that $n+1 \in T$, and for any $x \in A_{n+1}$:*

$$\pi_{f_{n+1}|f_{T_n}}(x \mid \cdot) \overset{(\Pi_{f_{T_n}}, \mathcal{T}, A_{T_n})}{=} \pi_{f_{n+1}|f_n}(x \mid \cdot) \circ \mathbf{pr}_{T_n, n}.$$

6 FROM A CONSISTENT FAMILY OF DISTRIBUTIONS TO A POSSIBILISTIC MARKOV FAMILY

In this section, we show how we can turn a consistent family of distributions into a possibilistic Markov family.

We assume that $(L, \leq)$ is a complete chain and that $\mathcal{T}$ is a weakly invertible t-norm on $(L, \leq)$ which is completely distributive w.r.t. supremum. We also assume that T is a countable chain and therefore essentially $T = \mathbb{N}$ or $T = T_n$ where $n \in \mathbb{N}$. Let $\leq$ be the natural linear ordering of natural numbers. Also, let $((X_t, \mathcal{R}_t) \mid t \in T)$ be a family of τ-spaces.

Let $t \in T$. If $\mathcal{R}_t$ has a finite number of atoms, let $X_t^* = X_t$ and $\mathcal{R}_t^* = \mathcal{R}_t$. If $\mathcal{R}_t$ has an infinite number of atoms, let $X_t^* = X_t \cup \{\infty_t\}$, where $\infty_t \notin X_t$ and $\mathcal{R}_t^*$ is the τ-field on X_t^* whose atoms are $\{\infty_t\}$ and the atoms of $\mathcal{R}_t$. Then $(X_t^*, \mathcal{R}_t^*)$ is called the **-extension* of $(X_t, \mathcal{R}_t)$ [7].

For any $n \in T$ such that $n+1 \in T$ let ${}_n\pi$ be an element of $\mathcal{G}_{(L,\leq)}^{\mathcal{R}_n \times \mathcal{R}_{n+1}}(X_n \times X_{n+1})$ such that

$$\sup_{v \in X_{n+1}} {}_n\pi(u, v) = 1_L$$

for any $u \in X_n$. For notational simplicity, we also denote ${}_n\pi_{uv} = {}_n\pi(u, v)$, which represents the transition possibility at time n from a state in $[\,u\,]_{\mathcal{R}_n}$ to a state in $[\,v\,]_{\mathcal{R}_{n+1}}$ at time $n+1$, or equivalently, the *one-step transition possibility* at time n from a state in $[\,u\,]_{\mathcal{R}_n}$ to a state in $[\,v\,]_{\mathcal{R}_{n+1}}$.

Furthermore, let $(\pi_u \mid u \in X_0) \subseteq L$, where π_u is the possibility that the system is in $[\,u\,]_{\mathcal{R}_0}$ at time 0, or equivalently, the *initial possibility* that the system is in $[\,u\,]_{\mathcal{R}_0}$.

We also need the following notations. Assume $n \in T$ and $k \in \mathbb{N}^*$ such that $n + k \in T$ and let $(u, v) \in X_n \times X_{n+k}$. If $k \geq 2$, let

$${}_n\pi^{(k)}(u, v) = \sup_{\substack{y \in X_{T_{n,n+k}} \\ y(n)=u, y(n+k)=v}} \mathcal{T}_{i=n}^{n+k-1} {}_i\pi_{y(i)y(i+1)},$$

otherwise let

$${}_n\pi^{(1)}(u, v) = {}_n\pi_{uv}.$$

Then, ${}_n\pi^{(k)}(u, v)$ is in fact the *k-step transition possibility at time n* from a state in $[\,u\,]_{\mathcal{R}_n}$ to a state in $[\,v\,]_{\mathcal{R}_{n+k}}$. In this way, we have defined a $\mathcal{R}_n \times \mathcal{R}_{n+k}$-measurable $X_n \times X_{n+k} - L$-mapping ${}_n\pi^{(k)}$.

Given the one-step transition possibilities and the initial possibilities, and inspired by Proposition 16, we define for any $n \in T \setminus \{0\}$ and $x \in X_{T_n}$:

$$\pi_{T_n}(x) = \mathcal{T}(\pi_{x(0)}, \mathcal{T}_{i=0}^{n-1} {}_i\pi_{x(i)x(i+1)}).$$

For any nonempty, finite subset T' of T, and for any $x \in X_{T'}$, let

$$\pi_{T'}(x) = \sup_{\mathbf{pr}_{T_{\sup T'}, T'}(y)=x} \pi_{T_{\sup T'}}(y).$$

Furthermore, let $\pi_{T_0}(x) = \pi_{x(0)}$ for any $x \in X_{T_0}$.

Assume T' is a nonempty, finite subset of T. Since $\pi_{T'} \in \mathcal{G}_{(L,\leq)}^{\mathcal{R}_{T'}}(X_{T'})$, it follows that there is a unique $(L, \leq)$-possibility measure $\Pi_{T'}$ on $(X_{T'}, \mathcal{R}_{T'})$ with $\pi_{T'}$ as its distribution. Because $\mathcal{T}$ is completely distributive w.r.t. supremum, $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is a consistent family of distributions. This enables us to use the following theorem [7].

Theorem 21 *Let $(L, \leq)$ be a complete chain and let $((X_t, \mathcal{R}_t) \mid t \in T)$ be a family of τ-spaces with nonempty index set T. Let $\pi_{T'}$ be the distribution of a $(L, \leq)$-possibility measure $\Pi_{T'}$ on $(X_{T'}, \mathcal{R}_{T'})$ for any nonempty, finite subset T' of T, and assume that $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent. Then, there exist a $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ and a family $(f_t \mid t \in T)$ of possibilistic variables in an associated family $((X_t^*, \mathcal{R}_t^*) \mid t \in T)$ of *-extensions of $((X_t, \mathcal{R}_t) \mid t \in T)$, with $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as basic space, such that $\pi_{f_{T'}}|_{X_{T'}} = \pi_{T'}$ and $\Pi_{f_{T'}}(X_{T'}^*) = \Pi_{T'}(X_{T'})$, for any nonempty, finite subset T' of T. If T is finite, then there exist a $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ and a family $(f_t \mid t \in T)$ of possibilistic variables in the family $((X_t, \mathcal{R}_t) \mid t \in T)$ with $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as basic space, such that $\pi_{T'}$ is the possibility distribution of $f_{T'}$ for any nonempty subset T' of T.*

So, if $T = T_n$ where $n \in \mathbb{N}$, there exists a family $(f_t \mid t \in (T, \leq))$ of possibilistic variables in $((X_t, \mathcal{R}_t) \mid t \in T)$ such that $\pi_{f_{T'}} = \pi_{T'}$ for any nonempty subset T' of T.

In case $T = \mathbb{N}$, let $(X_t^*, \mathcal{R}_t^*)$ be a *-extension [7] of $(X_t, \mathcal{R}_t)$ for any $t \in T$. According to Theorem 21, we can find a family $(f_t \mid t \in (T, \leq))$ of possibilistic variables in $((X_t^*, \mathcal{R}_t^*) \mid t \in T)$ such that $\pi_{f_{T'}}|_{X_{T'}} = \pi_{T'}$ and $\Pi_{f_{T'}}(X_{T'}^*) = \Pi_{T'}(X_{T'})$ for any nonempty, finite subset T' of T. It easily follows from the above remarks that $(f_t \mid t \in (T, \leq))$ is also consistent w.r.t. $(X_t \mid t \in T)$.

Furthermore, let

$${}_n\widetilde{\pi}^{(k)}(u, v) = {}_n\pi^{(k)}(u, v)$$

if $(u, v) \in X_n \times X_{n+k}$, and

$${}_n\widetilde{\pi}^{(k)}(u, v) = 0_L$$

if T is infinite and $(u, v) \in (X_n^* \times X_{n+k}^*) \setminus (X_n \times X_{n+k})$. Then, we obtain the following result.

Proposition 22 *Let $t \in T$ and let T' be a nonempty, finite subset of T, such that $\sup T' < t$. Then for any $x \in X_t$:*

$$\pi_{f_t|f_{T'}}(x \mid \cdot) \overset{(\Pi_{f_{T'}}, \mathcal{T}, X_{T'})}{=} {}_{\sup T'}\widetilde{\pi}^{(t-\sup T')}(\cdot, x) \circ \mathbf{pr}_{T', \sup T'}.$$

As a special case, we find the following relation between the transition possibilities and the conditional possibilities, formed with the elements of $(f_t \mid t \in (T, \leq))$.

Corollary 23 *Assume $n \in T$ and $k \in \mathbb{N}^*$ such that $n+k \in T$ and let $x \in X_{n+k}$, then*

$$\pi_{f_{n+k}|f_n}(x \mid \cdot) \overset{(\Pi_{f_n}, \mathcal{T}, X_n)}{=} {}_n\widetilde{\pi}^{(k)}(\cdot, x).$$

Using Proposition 22, we can now prove that $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family in $((X_t^*, \mathcal{R}_t^*) \mid t \in T)$ w.r.t. $(X_t \mid t \in T)$.

Theorem 24 *For any $t \in T$ and for any nonempty, finite subset T' of T such that $\sup T' < t$, it follows that for any $x \in X_t$*

$$\pi_{f_t|f_{T'}}(x \mid \cdot) \overset{(\Pi_{f_{T'}}, \mathcal{T}, X_{T'})}{=} \pi_{f_t|f_{\sup T'}}(x \mid \cdot) \circ \mathbf{pr}_{T', \sup T'}.$$

Corollary 25 *If T is finite or $(X_t, \mathcal{R}_t)$ is compact for any $t \in T$, then $(f_t \mid t \in (T, \leq))$ is a possibilistic Markov family.*

7 CONCLUSION

The results in this paper show that it is indeed possible to give a formal, measure-theoretic basis to a theory of possibilistic Markov processes, which is to a large extent formally analogous to the basic theory of stochastic Markov processes. The possibilistic Chapman-Kolmogorov equation, together with the results in section 6, provide the link of this formal treatment with the discussions found in the literature, notably Joslyn's approach [8], where time-invariant one-step transition possibilities are used to define possibilistic Markov processes.

It should nevertheless be noted that we only make use of one particular definition of conditional possibility and possibilistic independence (for a discussion of other definitions with pointers to the literature, see [2, 3]). More work remains to be done in assessing how and whether other existing definitions may lead to a satisfactory formal theory. Due to limitations of space, the material given here is rather condensed. Moreover, proofs had to be omitted. For a more detailed discussion containing the proofs, we refer to a forthcoming paper.

Acknowledgements

Gert de Cooman is a Postdoctoral Fellow of the Belgian National Fund for Scientific Research (NFWO). He would like to thank the NFWO for partially funding the research reported on in this paper.

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