FROM POSSIBILISTIC INFORMATION TO KLEENE'S STRONG MULTI-VALUED LOGICS

Gert DE COOMAN*
Universiteit Gent, Vakgroep Elektrische Energietechniek
Technologiepark 9, 9052 Zwijnaarde, Belgium

1 Possibilistic extension logics

Possibilistic logic in general [7, 8, 9, 10] investigates how possibilistic uncertainty about propositions is propagated when making inferences in a formal logical system. In this paper, we look at a very particular aspect of possibilistic logic: we investigate how, under certain independence assumptions, the introduction of possibilistic uncertainty in classical propositional logic leads to the consideration of special classes of multi-valued logics, with a proper set of truth values and logical functions combining them. First, we show how possibilistic uncertainty about the truth value of a proposition leads to the introduction of possibilistic truth values. Since propositions can be combined into new ones using logical operators, possibilistic uncertainty about the truth values of the original propositions gives rise to possibilistic uncertainty about the truth value of the resulting proposition. Furthermore, we show that in a *limited* number of special cases there is *truth-functionality*, i.e. the possibilistic truth value of the resulting proposition is a function of the possibilistic truth values of the original propositions. This leads to the introduction of possibilistic-logical functions, combining possibilistic truth values. Important classes of such functions, the possibilistic extension logics, result directly from this investigation. Finally, the relation between these logics and Kleene's strong multi-valued systems is established. This paper is intended as a brief summary of the much more detailed account that can be found in [5].

Let us first define the most common notations. By $(L, \leq)$, we denote a complete lattice [1] with top 1 and bottom 0, where we assume that $0 \neq 1$. The meet of $(L, \leq)$ is denoted by $\frown$, its join by $\smile$. By T we denote a triangular norm on $(L, \leq)$ that is completely distributive w.r.t. supremum [6]. We also use the set $\mathcal{T} = \{\mathit{false}, \mathit{true}\}$ of truth values in classical propositional logic. On $\mathcal{T}$, we define the total order relation $\leq = \{(\mathit{false}, \mathit{false}), (\mathit{false}, \mathit{true}), (\mathit{true}, \mathit{true})\}$. $(\mathcal{T}, \leq)$ is a Boolean chain of length 2, with top *true* and bottom *false*. On this chain, we may define as usual the complement $\neg$, called *negation*; the meet $\wedge$, called *conjunction*; the join $\vee$, called *disjunction* and the *implication* $\Rightarrow$.

We also consider a universe X. A $X - L$-mapping h is called sup-*normal* iff $\sup_{x \in X} h(x) = 1$. The set of all $X - L$-mappings is denoted by L^X. With a subset A of X, we may associate its characteristic $X - \mathcal{T}$-mapping χ_A, with, for any x in X: $\chi_A(x) = \mathit{true}$ if $x \in A$ and $\chi_A(x) = \mathit{false}$ if $x \notin A$.

Next, we introduce the notion of a possibilistic extension, which is related to Zadeh's extension principle, but is here only used within a possibilistic context, without reference to fuzzy sets. By X, X_1, ..., X_n and Y we denote arbitrary universes. First of all, with a $X - Y$-mapping φ we can associate a $L^X - L^Y$-mapping $\widetilde{\varphi}$, defined as follows. For any $X - L$-mapping h the $Y - L$-mapping $\widetilde{\varphi}(h)$ is given by, for any y in Y: $\widetilde{\varphi}(h) \cdot y = \sup_{\varphi(x)=y} h(x)$. $\widetilde{\varphi}$ is called the *$(L, \leq)$-possibilistic extension of φ*.

Also, with a $X_1 \times \cdots \times X_n - Y$-mapping φ we can associate a $L^{X_1} \times \cdots \times L^{X_n} - L^Y$-mapping $\widetilde{\varphi}_T$, defined as follows. For any $(h_1, \ldots, h_n)$ in $L^{X_1} \times \cdots \times L^{X_n}$ the $Y - L$-mapping $\widetilde{\varphi}_T(h_1, \ldots, h_n)$ is given by, for any y in Y: $\widetilde{\varphi}_T(h_1, \ldots, h_n) \cdot y = \sup_{\varphi(x_1, \ldots, x_n)=y} T_{k=1}^n h_k(x_k)$. $\widetilde{\varphi}_T$ is called the *$(L, \leq)$-possibilistic T-extension of φ*.

*Postdoctoral Fellow of the Fund for Scientific Research – Flanders (Belgium).

We are now ready to proceed to the main part of the paper. Possibilistic information about the values that a variable ξ may assume in X is represented by its possibility distribution function π_ξ, a sup-normal $X-L$-mapping [2]. When p is a *clear* (non-vague) property, with *extension* $[P_p] = \{ x \mid x \in X \text{ and } x \text{ is } p \}$ of X, this possibilistic information can be transformed into possibilistic information about the truth value $\chi_{[P_p]}(\xi)$ of the *proposition variable* $P_p(\xi) =$ 'ξ is p'. This information can be represented by the possibility distribution function $\pi_{\chi_{[P_p]}(\xi)}$ of the variable $\chi_{[P_p]}(\xi)$ in $\mathcal{T}$. It is easily verified that the sup-normal $\mathcal{T}-L$-mapping $\pi_{\chi_{[P_p]}(\xi)}$ is given by $\pi_{\chi_{[P_p]}(\xi)} = \widetilde{\chi_{[P_p]}}(\pi_\xi)$. Therefore $\pi_{\chi_{[P_p]}(\xi)}(\nu) = \sup_{\chi_{[P_p]}(x)=\nu} \pi_\xi(x)$ is the $(L,\leq)$-possibility that the proposition variable is true ($\nu = \mathit{true}$) or false ($\nu = \mathit{false}$). $\pi_{\chi_{[P_p]}(\xi)}$ can be generally called a $(L,\leq)$-*possibilistic truth value*.

Definition 1 *We call $(L,\leq)$-possibilistic truth value any* sup-*normal $\mathcal{T} - L$-mapping. The set of the $(L,\leq)$-possibilistic truth values is denoted by $\widetilde{\mathcal{T}}$. We also introduce three $(L,\leq)$-possibilistic truth values with a special meaning: $\widetilde{\mathit{false}} = \{(\mathit{true}, 0), (\mathit{false}, 1)\}$, $\widetilde{\mathit{true}} = \{(\mathit{true}, 1), (\mathit{false}, 0)\}$ and $\widetilde{\mathit{unknown}} = \{(\mathit{true}, 1), (\mathit{false}, 1)\}$.*

When a proposition variable 'ξ is p' has the $(L,\leq)$-possibilistic truth value $\widetilde{\mathit{true}}$, this means that it cannot be false, and is therefore *necessarily true*, taking into account the information about the values that ξ may assume in X. An analogous interpretation can be given to $\widetilde{\mathit{false}}$. When the proposition variable 'ξ is p' has the $(L,\leq)$-possibilistic truth value $\widetilde{\mathit{unknown}}$, this means that, taking into account the information about the values that ξ may assume, it is completely possible that the proposition variable is true, and equally possible that it is false. The truth value of this proposition variable is then *completely unknown*, because of insufficient information about the values that ξ may assume in X.

Above, we have shown how possibilistic information about the values a variable ξ may assume in a universe X can be transformed into possibilistic information about the truth value of a proposition about this variable. Next, we observe that, in general, propositions can be combined to form new propositions, using *logical operators*. In this way, a proposition P can be transformed by the *logical negation operator* into a proposition NOT P. By extension, the *proposition function* P_p can be transformed into a proposition function NOT P_p by the pointwise application of the logical negation operator: (NOT $P_p)(x) = \text{NOT}(P_p(x)) =$ 'x is not p', $x \in X$. In a completely similar way, the proposition variable $P_p(\xi) =$ 'ξ is p' is transformed by the logical negation operator into the proposition variable (NOT $P_p)(\xi)$, defined as 'ξ is not p'. Analogously, the proposition variables 'ξ is p' and 'ξ is q' can be transformed into the proposition variable (P_p AND $P_q)(\xi)$, defined as 'ξ is p and ξ is q', using the logical conjunction operator AND, and into the proposition variable (P_p OR $P_q)(\xi)$, defined as 'ξ is p or ξ is q', using the logical disjunction operator OR.

As is well known, classical propositional logic is *truth-functional*, and the behaviour of logical operators can be characterised by classical-logical functions.

Definition 2 *Let n be strictly positive natural number. A $\mathcal{T}^n - \mathcal{T}$-mapping is called a classical-logical function of arity n. The set of classical-logical functions of arbitrary arity is given the notation $\mathcal{L}$.*

The conjunction $\wedge$, the disjunction $\vee$ and the implication $\Rightarrow$, defined on $\mathcal{T}$, are classical-logical functions of arity 2, characterising the truth-functional behaviour of respectively the logical conjunction, disjunction and implication operator in classical propositional logic. The negation $\neg$, defined on $\mathcal{T}$, is a classical-logical function of arity 1, characterising the truth-functional behaviour of the logical negation operator in that logic. More explicitly, the behaviour of the *logical negation operator* is mirrored in the behaviour of $\neg$ in the following sense: $\chi_{[\text{NOT } P_p]}(x) = \neg\chi_{[P_p]}(x)$, $x \in X$, where, of course $[\text{NOT } P_p] = \{ x \mid x \in X \text{ and } x \text{ is not } p \} = \text{co}[P_p]$. In a completely analogous way, the behaviour of the *logical disjunction operator* is mirrored in the behaviour of $\vee$ in the following sense: $\chi_{[P_p \text{ OR } P_q]}(x) = \chi_{[P_p]}(x) \vee \chi_{[P_q]}(x)$, $x \in X$, where $[P_p \text{ OR } P_q] = \{ x \mid x \in X \text{ and } (x \text{ is } p \text{ OR } x \text{ is } q) \} = [P_p] \cup [P_q]$.

Generally, we can start with n clear predicates $p_1, \ldots, p_n$ with extensions $[P_{p_1}], \ldots, [P_{p_n}]$, and with a n-ary logical operator LOP. This logical operator transforms the proposition variables $P_{p_1}(\xi)$, $\ldots$, $P_{p_n}(\xi)$ into the proposition variable $\text{LOP}(P_{p1}, \ldots, P_{p_n})(\xi)$, defined as LOP('$\xi$ is p_1', $\ldots$, 'ξ is p_n'), with extension $[\mathrm{LOP}(P_{p1},\ldots,P_{p_n})]$. The behaviour of LOP is mirrored by a $\mathcal{T}^n - \mathcal{T}$-mapping ϕ, in the following sense, with obvious notations: $\chi_{[\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})]} = \phi \circ (\chi_{[P_{p_1}]},\ldots,\chi_{[P_{p_n}]})$.

What we now want to do is to extend the classical, truth-functional approach: formally consider $\widetilde{\mathcal{T}}$ as a set of (epistemological) truth values, and look at how these truth values can be combined into new ones. After that, we intend to show that at least for some of these combinations, there is a clear and definite link with combinations of propositions through logical operators. In this way, we intend to prove that, in some cases, possibilistic logic is also truth-functional.

Definition 3 *Let n be a strictly positive natural number. A $(\widetilde{\mathcal{T}})^n - \widetilde{\mathcal{T}}$-mapping is called a $(L,\leq)$-possibilistic-logical function of arity n. The set of the $(L,\leq)$-possibilistic-logical functions of arbitrary arity is given the notation $\widetilde{\mathcal{L}}$.*

We can associate a $(L,\leq)$-possibilistic-logical function with every classical-logical function, simply by looking at its $(L,\leq)$-possibilistic T-extension. Of course, this extension must be properly restricted, because we only work with sup-*normal* $\mathcal{T} - L$-mappings as possibilistic truth values.

Definition 4 *Let n be a strictly positive natural number and let ϕ be a classical-logical function of arity n. The $(L,\leq)$-possibilistic-logical T-extension $\widetilde{\phi}_{\ell T}$ of ϕ is defined as the restriction of the $(L,\leq)$-possibilistic T-extension $\widetilde{\phi}_T$ of ϕ to the set $(\widetilde{\mathcal{T}})^n$, i.e. $\widetilde{\phi}_{\ell T} = \widetilde{\phi}_T|(\widetilde{\mathcal{T}})^n$. We call $(L,\leq)$-possibilistic T-extension logic the set $\widetilde{\mathcal{L}}_T = \{\, \widetilde{\phi}_{\ell T} \mid \phi \in \mathcal{L}\,\}$.*

$(L,\leq)$-possibilistic-logical T-extensions are $(L,\leq)$-possibilistic-logical functions: $(\forall \phi \in \mathcal{L})(\widetilde{\phi}_{\ell T} \in \widetilde{\mathcal{L}})$. We now give the rationale for the introduction of extension logics. We do so by addressing the following question: *how, starting with possibilistic information about the values that a variable ξ may assume in X, can we derive the possibilistic truth value of the combined proposition variable* LOP('ξ is p_1', ..., 'ξ is p_n')?

It is easily verified that if the variables $\chi_{[P_{p_1}]}(\xi)$, ..., $\chi_{[P_{p_n}]}(\xi)$ in $\mathcal{T}$ are possibilistically independent [3, 4, 5], the $(L,\leq)$-possibilistic truth value $\pi_{\chi_{[\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})]}(\xi)} = \widetilde{\chi_{[\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})]}}(\pi_\xi)$ of the proposition variable LOP('ξ is p_1', ..., 'ξ is p_n'), is given by $\pi_{\chi_{[\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})]}(\xi)} = \widetilde{\phi}_{\ell T}(\pi_{\chi_{[P_{p_1}]}(\xi)},\ldots,\pi_{\chi_{[P_{p_n}]}(\xi)})$, where $\pi_{\chi_{[P_{p_k}]}(\xi)} = \widetilde{\chi_{[P_{p_k}]}}(\pi_\xi)$ is the $(L,\leq)$-possibilistic truth value of the proposition variable 'ξ is p_k', $k = 1,\ldots,n$. *Indeed, in the case of possibilistic independence, there is truth-functionality for possibilistic logic.*

In the rest of this section, we study the most important properties of some special $(L,\leq)$-possibilistic-logical functions of arity 1 and 2: $\widetilde{\neg}_{\ell T}$, $\widetilde{\wedge}_{\ell T}$, $\widetilde{\vee}_{\ell T}$ and $\widetilde{\Rightarrow}_{\ell T}$. First of all, it will help us if we can find simple expressions for these operators.

Proposition 5 *1. $(\widetilde{\neg}_{\ell T}\, t) \cdot true = t(false)$ and $(\widetilde{\neg}_{\ell T}\, t) \cdot false = t(true)$.*

2. $(t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) \cdot true = T(t_1(true), t_2(true))$ and $(t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) \cdot false = t_1(false) \smile t_2(false))$.

3. $(t_1 \,\widetilde{\vee}_{\ell T}\, t_2) \cdot true = t_1(true) \smile t_2(true)$ and $(t_1 \,\widetilde{\vee}_{\ell T}\, t_2) \cdot false = T(t_1(false), t_2(false))$.

4. $(t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2) \cdot true = t_1(false) \smile t_2(true)$ and $(t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2) \cdot false = T(t_1(true), t_2(false))$.

Let us now give a brief survey of the most important properties of the above-mentioned possibilistic-logical functions. It should be noted that the equalities in these properties are equalities of $(L,\leq)$-possibilistic truth values, and therefore pointwise equalities of $\mathcal{T} - L$-mappings. Also, t, t_1, t_2 and t_3 denote arbitrary elements of $\widetilde{\mathcal{T}}$.

Proposition 6 *1. $t_1 \,\widetilde{\wedge}_{\ell T}\, t_2 = t_2 \,\widetilde{\wedge}_{\ell T}\, t_1$ and $t_1 \,\widetilde{\vee}_{\ell T}\, t_2 = t_2 \,\widetilde{\vee}_{\ell T}\, t_1$. (Commutativity)*

2. $t \,\widetilde{\wedge}_{\ell T}\, \widetilde{true} = t$ and $t \,\widetilde{\vee}_{\ell T}\, \widetilde{false} = t$. (Neutral elements)

3. $t_1 \,\widetilde{\wedge}_{\ell T}\, (t_2 \,\widetilde{\wedge}_{\ell T}\, t_3) = (t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) \,\widetilde{\wedge}_{\ell T}\, t_3$ and $t_1 \,\widetilde{\vee}_{\ell T}\, (t_2 \,\widetilde{\vee}_{\ell T}\, t_3) = (t_1 \,\widetilde{\vee}_{\ell T}\, t_2) \,\widetilde{\vee}_{\ell T}\, t_3$. (Associativity)

4. $\widetilde{\neg}_{\ell T}\,(t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) = (\widetilde{\neg}_{\ell T}\, t_1)\,\widetilde{\vee}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t_2)$ *and* $\widetilde{\neg}_{\ell T}\,(t_1 \,\widetilde{\vee}_{\ell T}\, t_2) = (\widetilde{\neg}_{\ell T}\, t_1)\,\widetilde{\wedge}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t_2)$. *(De Morgan's Laws)*

5. $t \,\widetilde{\wedge}_{\ell T}\, \widetilde{\mathit{false}} = \widetilde{\mathit{false}}$ *and* $t \,\widetilde{\vee}_{\ell T}\, \widetilde{\mathit{true}} = \widetilde{\mathit{true}}$. *(Absorbing elements)*

6. $\widetilde{\neg}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t) = t$. *(Involutivity)*

7. $t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2 = (\widetilde{\neg}_{\ell T}\, t_1)\,\widetilde{\vee}_{\ell T}\, t_2$. *(Implication)*

8. $(t \,\widetilde{\wedge}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t)) \cdot \mathit{true} = T(t(\mathit{true}), t(\mathit{false}))$, $(t \,\widetilde{\wedge}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t)) \cdot \mathit{false} = 1$, $(t \,\widetilde{\vee}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t)) \cdot \mathit{true} = 1$ *and* $(t \,\widetilde{\vee}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t)) \cdot \mathit{false} = T(t(\mathit{true}), t(\mathit{false}))$. *(Complementation)*

9. $t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2 = (\widetilde{\neg}_{\ell T}\, t_2)\,\widetilde{\Rightarrow}_{\ell T}\,(\widetilde{\neg}_{\ell T}\, t_1)$. *(Contrapositive symmetry)*

10. $(\widetilde{\mathit{true}} \,\widetilde{\Rightarrow}_{\ell T}\, t) = t$. *(Neutrality principle)*

11. $t_1 \,\widetilde{\Rightarrow}_{\ell T}\,(t_2 \,\widetilde{\Rightarrow}_{\ell T}\, t_3) = t_2 \,\widetilde{\Rightarrow}_{\ell T}\,(t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_3)$. *(Exchange principle)*

12. $t_1 \,\widetilde{\wedge}_{\ell T}\, t_2 = \widetilde{\mathit{true}} \Leftrightarrow (t_1 = \widetilde{\mathit{true}}$ *and* $t_2 = \widetilde{\mathit{true}})$, $t_1 \,\widetilde{\vee}_{\ell T}\, t_2 = \widetilde{\mathit{false}} \Leftrightarrow (t_1 = \widetilde{\mathit{false}}$ *and* $t_2 = \widetilde{\mathit{false}})$, $\widetilde{\neg}_{\ell T}\, t = \widetilde{\mathit{true}} \Leftrightarrow t = \widetilde{\mathit{false}}$ *and* $\widetilde{\neg}_{\ell T}\, t = \widetilde{\mathit{false}} \Leftrightarrow t = \widetilde{\mathit{true}}$. *(Boundary conditions)*

Note that $\widetilde{\wedge}_{\ell T}$ and $\widetilde{\vee}_{\ell T}$ are idempotent iff T is, i.e. iff $T = \frown$ [6]. Furthermore, $\widetilde{\wedge}_{\ell T}$ and $\widetilde{\vee}_{\ell T}$ are mutually distributive iff T and sup are mutually distributive. This is again only possible if $T = \frown$ [6]. Thus, it appears that the choice $T = \frown$ is a rather special one. In this respect, note also that if we consider the lattice $(\widetilde{\mathcal{T}}, \widetilde{\leq})$, where $\widetilde{\leq}$ is the partial order relation on $\widetilde{\mathcal{T}}$, introduced in the following section, then $\widetilde{\wedge}_{\ell T}$ is a t-norm and $\widetilde{\vee}_{\ell T}$ is a t-conorm [6] on this structure. These operators are dual [6] w.r.t. the negation $\widetilde{\neg}_{\ell T}$ on $(\widetilde{\mathcal{T}}, \widetilde{\leq})$. $\widetilde{\wedge}_{\ell T}$ is the meet and $\widetilde{\vee}_{\ell T}$ the join of the lattice $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ iff $T = \frown$. We therefore devote the next section to the study of this special case.

2 An interesting special case

In this section, we intend to take a closer look at the notions introduced above, in the special case $T = \frown$. This means that we assume that $(L, \leq)$ is a complete Brouwerian lattice [1].

Proposition 7 *$(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is a bounded distributive lattice (as an algebra) with top $\widetilde{\mathit{true}}$ and bottom $\widetilde{\mathit{false}}$. The partial order relation $\widetilde{\leq}$ on $\widetilde{\mathcal{T}}$ that corresponds with this structure satisfies:*

$$(\forall (t_1, t_2) \in (\widetilde{\mathcal{T}})^2) \left(t_1 \,\widetilde{\leq}\, t_2 \Leftrightarrow \left\{ \begin{array}{l} t_1(\mathit{true}) \leq t_2(\mathit{true}) \\ t_1(\mathit{false}) \geq t_2(\mathit{false}) \end{array} \right. \right).$$

Besides the binary operators meet $\widetilde{\wedge}_{\ell\frown}$ and join $\widetilde{\vee}_{\ell\frown}$ of $(\widetilde{\mathcal{T}}, \widetilde{\leq})$, there also exists the unary operator $\widetilde{\neg}_{\ell\frown}$. Its properties are studied in the next proposition, which also establishes the relationship between possibilistic $\frown$-extension logics and a special class of *multi-valued logics* [11]. By a negation operator on a bounded poset, we mean a dual order-automorphism on that structure [6].

Proposition 8 1. *$\widetilde{\neg}_{\ell\frown}$ is an involutive negation operator, but not a complement operator, on $(\widetilde{\mathcal{T}}, \widetilde{\leq})$.*

2. *$(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Morgan algebra [12], i.e. $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is a bounded distributive lattice (as an algebra), with a unary operator $\widetilde{\neg}_{\ell\frown}$ satisfying (i) $\widetilde{\neg}_{\ell\frown}$ is involutive; and (ii) $\widetilde{\wedge}_{\ell\frown}$, $\widetilde{\vee}_{\ell\frown}$ and $\widetilde{\neg}_{\ell\frown}$ satisfy de Morgan's laws.*

3. *$(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Kleene algebra [12], i.e. $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Morgan algebra with furthermore $(\forall (t_1, t_2) \in (\widetilde{\mathcal{T}})^2)(t_1 \,\widetilde{\wedge}_{\ell\frown}\,(\widetilde{\neg}_{\ell\frown}\, t_1) \,\widetilde{\leq}\, t_2 \,\widetilde{\vee}_{\ell\frown}\,(\widetilde{\neg}_{\ell\frown}\, t_2))$.*

The operators $\widetilde{\neg}_{\ell\frown}$, $\widetilde{\wedge}_{\ell\frown}$ and $\widetilde{\vee}_{\ell\frown}$ on $\widetilde{\mathcal{T}}$ therefore satisfy the characteristic properties of the negation, conjunction and disjunction in the *multi-valued strong Kleene logics* with truth domain $(\widetilde{\mathcal{T}}, \leq)$ [11]. For the *implication* we have, taking into account proposition 6, that $t_1 \widetilde{\Rightarrow}_{\ell\frown} t_2 = (\widetilde{\neg}_{\ell\frown} t_1) \widetilde{\vee}_{\ell\frown} t_2$, $t_1, t_2 \in \widetilde{\mathcal{T}}$, which implies that this implication is a typical instance of a *Kleene-Dienes implication* [11].

At the same time, if $(L, \leq)$ is a Boolean chain (of length 2), we recover Kleene's *strong ternary logic*. Let us briefly study the exact relationship between possibilistic $\frown$-extension logics and Kleene's strong ternary logic. We consider a universe X and two clear properties p and q with extensions $[P_p] = \{\, x \mid x \in X \text{ and } x \text{ is } p \,\}$ and $[P_q] = \{\, x \mid x \in X \text{ and } x \text{ is } q \,\}$. Also, we consider a variable ξ in X. Let us assume that we have the following information about the values that ξ may assume in X: ξ must be an element of A, with $A \subseteq X$. This information can be represented in the form of the normal $(\{0,1\}, \leq)$-possibility measure Π_A, with for arbitrary $B \subseteq X$:

$$\Pi_A(B) = \begin{cases} 1 & ; \quad B \cap A \neq \emptyset \\ 0 & ; \quad B \cap A = \emptyset \end{cases}$$

is the possibility that ξ belongs to B. Indeed, if $B \cap A = \emptyset$, then ξ cannot belong to B, since we already know that $\xi \in A$. Remark that the distribution of Π_A, and therefore also the possibility distribution function of ξ, is the characteristic $X - \{0,1\}$-function χ_A of A [2].

Starting from this possibilistic information χ_A, we now ask ourselves what can be deduced about the truth values of the proposition variables 'ξ is p', 'ξ is q' and a few of their combinations. In order to answer this question, we simply apply the theory developed above in the special case $(L, \leq) = (\{0,1\}, \leq)$. The only triangular norm on $(\{0,1\}, \leq)$ is the meet $\frown$ [6], which immediately leads us to the special case discussed in this section. Note that in this particular case $\widetilde{\mathcal{T}} = \{\widetilde{\mathit{false}}, \widetilde{\mathit{unknown}}, \widetilde{\mathit{true}}\}$ and $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ is a chain of length 3, with bottom $\widetilde{\mathit{false}}$, top $\widetilde{\mathit{true}}$ and in between $\widetilde{\mathit{unknown}}$. In this chain, $\widetilde{\wedge}_{\ell\frown}$ is the meet, $\widetilde{\vee}_{\ell\frown}$ is the join, and $\widetilde{\neg}_{\ell\frown}$ is the unique and involutive negation operator. $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Kleene algebra and is as such isomorphic to the corresponding structure of the strong ternary logic introduced by Kleene [11].

It is readily verified that the $(\{0,1\}, \leq)$-possibilistic truth value $t_{P_p} = \widetilde{\chi_{[P_p]}}(\chi_A)$ of the proposition variable 'ξ is p' is determined by $t_{P_p}(\mathit{true}) = \Pi_A([P_p])$ and $t_{P_p}(\mathit{false}) = \Pi_A(\mathrm{co}[P_p])$, where $\chi_{[P_p]}$ is the characteristic $X - \mathcal{T}$-mapping of $[P_p]$. For t_{P_p} there are therefore three possibilities, since $t_{P_p} \in \widetilde{\mathcal{T}}$. We have that $t_{P_p} = \widetilde{\mathit{true}} \Leftrightarrow A \subseteq [P_p]$, or equivalently, iff it is *necessary* that ξ is p. On the other hand, $t_{P_p} = \widetilde{\mathit{false}} \Leftrightarrow A \cap [P_p] = \emptyset$, or equivalently, iff it is *impossible* that ξ is p. Finally, we have that $t_{P_p} = \widetilde{\mathit{unknown}} \Leftrightarrow A \cap [P_p] \neq \emptyset$ and $A \cap \mathrm{co}[P_p] \neq \emptyset$, or equivalently, iff it is possible but not necessary that ξ is p, in other words, it is *uncertain* whether ξ is p.

Let us now turn our attention to the $(\{0,1\}, \leq)$-possibilistic truth value of the proposition variable 'NOT(ξ is p)', or equivalently, $(\text{NOT } P_p)(\xi)$, or 'ξ is not p'. It is obvious that $[\text{NOT } P_p] = \mathrm{co}[P_p]$, whence $t_{\text{NOT } P_p}(\mathit{true}) = t_{P_p}(\mathit{false})$ and $t_{\text{NOT } P_p}(\mathit{false}) = t_{P_p}(\mathit{true})$. We may therefore write, taking into account proposition 5, that $t_{\text{NOT } P_p} = \widetilde{\neg}_{\ell\frown} t_{P_p}$. *We conclude that for the logical negation operator of classical propositional logic, there is always truth-functionality as far as the* $(\{0,1\}, \leq)$*-possibilistic truth values are concerned.*

Let us now investigate the proposition variable 'ξ is p and ξ is q', or equivalently, $(P_p \text{ AND } P_q)(\xi)$, where P_p AND P_q is a proposition function that is the pointwise conjunction of the proposition functions P_p and P_q. It is obvious that $[P_p \text{ AND } P_q] = [P_p] \cap [P_q]$, whence $t_{P_p \text{ AND } P_q}(\mathit{true}) = \Pi_A([P_p] \cap [P_q])$ and, also taking into account proposition 5, $t_{P_p \text{ AND } P_q}(\mathit{false}) = (t_{P_p} \widetilde{\wedge}_{\ell\frown} t_{P_q}) \cdot \mathit{false}$. *Only if*

$$\Pi_A([P_p] \cap [P_q]) = \Pi_A([P_p]) \frown \Pi_A([P_q]) \tag{1}$$

we have, taking into account proposition 5, that $t_{P_p \text{ AND } P_q}(\mathit{true}) = (t_{P_p} \widetilde{\wedge}_{\ell\frown} t_{P_q}) \cdot \mathit{true}$. Only in this case there is *truth-functionality* for the logical conjunction operator in classical propositional logic as far as the possibilistic truth values are concerned, or equivalently, $t_{P_p \text{ AND } P_q} = t_{P_p} \widetilde{\wedge}_{\ell\frown} t_{P_q}$.

Let us now briefly discuss the meaning of (1). It is easily shown that (1) does not hold iff $A \cap [P_p \text{ AND } P_q] = \emptyset$ and at the same time $A \cap [P_p] \neq \emptyset$, $A \cap \mathrm{co}[P_p] \neq \emptyset$, $A \cap [P_q] \neq \emptyset$ and $A \cap \mathrm{co}[P_q] \neq \emptyset$;

in other words, iff it is *uncertain* (i.e. not impossible and not necessary) whether ξ is p and whether ξ is q, and at the same time *impossible* that ξ is p and ξ is q. Indeed, in that case, we have that $t_{P_p \text{ AND } P_q} = \widetilde{\mathit{false}}$, whereas $t_{P_p} \widetilde{\wedge}_{\ell\frown} t_{P_q} = \widetilde{\mathit{unknown}} \, \widetilde{\wedge}_{\ell\frown} \, \widetilde{\mathit{unknown}} = \widetilde{\mathit{unknown}}$. A similar argument can be given for the disjunction. *We conclude that there is not necessarily truth-functionality for the logical disjunction and conjunction operators of classical propositional logic, as far as the* $(\{0,1\},\leq)$*-possibilistic truth values are concerned.* The possibilistic approach therefore only results in a strong ternary Kleene logic if a number of independence properties are satisfied. Indeed, it is shown in [4] that condition (1) is related to the conditions for the possibilistic (or logical) independence of the events $[P_p]$ and $[P_q]$. In some cases these conditions are not satisfied, and the possibilistic approach is therefore *not truth-functional*, and therefore does not lead to a strong ternary Kleene logic. In these cases however, the strong ternary Kleene logic does provide us with a *conservative approximation*, since wherever it goes wrong, it results in the possibilistic truth value $\widetilde{\mathit{unknown}}$, where the possibilistic approach yields the possibilistic truth values $\widetilde{\mathit{true}}$ or $\widetilde{\mathit{false}}$.

3 Conclusion

Possibilistic logic can be described as a set of techniques that enable us to incorporate possibilistic uncertainty in a formal logical system. It turns out that under a number of independence assumptions, possibilistic logic leads to the special case of a possibilistic extension logic. A special subclass of these, the possibilistic-logical $\frown$-extensions, are related with strong multi-valued Kleene logics. Thus, a possibilistic justification is given for the introduction and use of these Kleene systems.

References

[1] G. Birkhoff. *Lattice Theory*, volume XXV. A.M.S. Colloquium Publications, Providence, RI, 1967.

[2] G. de Cooman. Possibility theory I: The measure- and integral-theoretic groundwork. *International Journal of General Systems*, 25:291–323, 1997.

[3] G. de Cooman. Possibility theory II: Conditional possibility. *International Journal of General Systems*, 25:325–351, 1997.

[4] G. de Cooman. Possibility theory III: Possibilistic independence. *International Journal of General Systems*, 25:353–371, 1997.

[5] G. de Cooman. Towards a possibilistic logic. In D. Ruan, editor, *Fuzzy Set Theory and Advanced Mathematical Applications*, pages 89–133. Kluwer Academic, Boston, 1995.

[6] G. de Cooman and E. E. Kerre. Order norms on bounded partially ordered sets. *The Journal of Fuzzy Mathematics*, 2:281–310, 1994.

[7] D. Dubois, J. Lang and H. Prade. Automated reasoning using possibilistic logic: Semantics, belief revision and variable certainty weights. In *Proceedings of the 5th Workshop on Uncertainty in Artificial Intelligence*, pages 81–87, 1989.

[8] D. Dubois, J. Lang and H. Prade. Possibilistic logic. In D. M. Gabbay, C. J. Hogger, J. A. Robinson and D. Nute, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, Vol. 3, pages 439–513. Oxford University Press, 1994.

[9] D. Dubois and H. Prade. Fuzzy sets in approximate reasoning – Part I: Reasoning with possibility distributions. *Fuzzy Sets and Systems*, 40, 1990.

[10] F. Esteva, P. Garcia and L. Godo. Relating and extending semantical approaches to possibilistic reasoning. *International Journal of Approximate Reasoning*, 10:311–344, 1994.

[11] N. Rescher. *Many-Valued Logic*. McGraw-Hill, New York, 1969.

[12] H. J. Skala. On many-valued logics, fuzzy sets, fuzzy logics and their applications. *Fuzzy Sets and Systems*, 1:129–149, 1978.