POSSIBILITY MEASURES AND POSSIBILITY INTEGRALS DEFINED ON A COMPLETE LATTICE

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Abstract. We consider the definition of possibility measures on complete lattices rather than on complete Boolean algebras of sets. We give a necessary and sufficient condition for the extendability of any mapping to such a possibility measure. We also associate two types of integrals with these possibility measures, and discuss some of their more important properties, amongst which a monotone convergence theorem.

1. Introduction

Since possibility measures were introduced by Zadeh [18] in 1978, the measure-theoretic aspects of possibility measures and possibility integrals have been studied by various authors [2, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 16, 17, 19, 20]. The most general definition of a possibility measure that can be distilled from this work is the following: a possibility measure π is a mapping defined on a *complete Boolean algebra* [3] $\mathbb{B}$ and taking values in a complete lattice L, that is furthermore *supremum-preserving*, in that $\pi(\sup_{j\in J} a_j) = \sup_{j\in J} \pi(a_j)$ for any family $\{a_j \mid j \in J\}$ of elements of $\mathbb{B}$.

Most of the time, the complete lattice L is taken to be the real unit interval $[0, 1]$, linearly ordered by the natural ordering of real numbers, and $\mathbb{B}$ is a complete Boolean algebra of sets (also called *ample field* [7, 16]), which is always atomic[1]. In such a context, possibility measures are completely characterised by their *distribution*, that is, by the values they assume on the atoms of their domains.

But sometimes, and most notably in the work of Dubois and Prade on possibilistic logic [10], possibility measures are defined on a Boolean algebra of propositions, and such Boolean algebras need not be atomic.

It is also possible to extend a possibility measure from a domain of sets to a domain of fuzzy sets, using fuzzy integration [4, 8]. In this case, we still have a supremum-preserving mapping, but it is no longer defined on a complete Boolean algebra of sets, but on a complete lattice of fuzzy sets.

It therefore seems useful to investigate to what extent possibility measures can be introduced and studied in a point-free context. This is the basic motivation for this paper. We shall take a closer look at possibility measures whose domain is a complete lattice $\mathbf{L}$, and which take values in a complete lattice L. The smallest elements of $\mathbf{L}$ and L will be denoted by, respectively, $0_{\mathbf{L}}$ and 0_L, and their greatest elements by $1_{\mathbf{L}}$ and 1_L. The ordering relation will be denoted by $\leq$ in both structures.

Date: 16 December 1998.

Key words and phrases. Triangular norm, possibility measure, possibility integral, P-consistency, possibilistic extension.

This paper is accepted for publication in *Fuzzy Sets and Systems*, 1998.

[1] Recall that a complete lattice P with smallest element 0_P is *atomic* if $P\setminus\{0_P\}$ has a nonempty subset of minimal elements, called *atoms* or *points*, such that every element of P is the supremum of all the atoms below it [1].

Definition 1. An L-possibility measure on $\mathbf{L}$ is a mapping $\pi\colon \mathbf{L} \to L$ with the property:

$$\pi(\sup A) = \sup_{a\in A} \pi(a), \quad A \subseteq \mathbf{L}.$$

We shall often omit explicit reference to the range L and talk about possibility measures rather than L-possibility measures.

Notice that since $0_{\mathbf{L}} = \sup\emptyset$ (where the supremum is taken in the complete lattice $\mathbf{L}$) and $0_L = \sup\emptyset$ (where the supremum is now taken in L), any possibility measure π satisfies $\pi(0_{\mathbf{L}}) = 0_L$. Since for any $a \in \mathbf{L}$, $a = \sup\{x \in \mathbf{L} \mid x \leq a\}$, it also follows that

$$\pi(a) = \sup_{x\leq a} \pi(x), \tag{1}$$

which in turn implies that π is *monotone*: $a \leq b$ implies $\pi(a) \leq \pi(b)$.

Assume for a moment that $\mathbf{L}$ is atomic: the set $\mathbf{A}$ of the minimal elements of $\mathbf{L} \setminus \{0_{\mathbf{L}}\}$ is nonempty and for every $b \in \mathbf{L}$, $b = \sup_{a\in\mathbf{A},a\leq b} a$. It then follows from the definition of a possibility measure π that

$$\pi(b) = \sup_{a\in\mathbf{A},a\leq b} \pi(a). \tag{2}$$

This means that π is completely determined by (and of course completely determines) the values it assumes on the set of atoms $\mathbf{A}$. However, in this paper we shall make no such assumption concerning the atomicity of $\mathbf{L}$.

It is tempting to look what happens when we replace $\leq$ by the strict inequality $<$ in (1). Unfortunately, unless $\mathbf{L}$ is a chain, a mapping $\rho\colon \mathbf{L} \to L$ which satisfies $\rho(a) = \sup_{x<a} \rho(x)$ is not necessarily a possibility measure. But there is a closely related condition in terms of the relation $\not\geq$ rather than $<$, which does guarantee that we are working with a possibility measure.

Definition 2. A strict L-possibility measure on $\mathbf{L}$ is a mapping $\pi\colon \mathbf{L} \to L$ with the property:

$$\pi(a) = \sup_{x\not\geq a} \pi(x), \quad a \in \mathbf{L}.$$

Strict possibility measures are indeed special possibility measures: if π is a strict possibility measure on $\mathbf{L}$, then for any $A \subseteq \mathbf{L}$,

$$\pi(\sup A) = \sup_{x\not\geq \sup A} \pi(x) = \sup_{(\exists a\in A)(x\not\geq a)} \pi(x) = \sup_{a\in A}\sup_{x\not\geq a} \pi(x) = \sup_{a\in A} \pi(a),$$

so π is also a possibility measure on $\mathbf{L}$.

In the rest of this paper, we take a closer look at two problems, which will turn out to be essentially related. In Section 2 we investigate under what conditions a mapping defined on a subset Ψ of $\mathbf{L}$ can be extended to a possibility measure on $\mathbf{L}$. In Section 3 we introduce and study two types of integrals (or functionals) that can be associated with possibility measures, and show that these integrals allow us to extend a possibility measure on $\mathbf{L}$ to a possibility measure on $L^{\mathbf{L}}$, the set of all $\mathbf{L} - L$-mappings.

2. Extension to possibility measures

The central problem in this section is the following. Consider a mapping $\mu\colon \Psi \to L$, where $\Psi \subseteq \mathbf{L}$. What conditions must be imposed on μ to guarantee that this mapping has a possibilistic extension to $\mathbf{L}$, i.e. an L-possibility measure on $\mathbf{L}$ that coincides with μ on Ψ? For the special case of possibility measures defined on complete Boolean algebras of sets, this problem has been called the *possibilistic extension problem*. It was first solved for possibility measures taking values in the unit interval by Wang [15, 16, 17]. Boyen *et al.* [2] have extended Wang's results by looking at the possibilistic extension problem for possibility measures taking values in complete chains and lattices. The problem we discuss here is more general in that the domains of our possibility measures are arbitrary complete lattices.

It turns out that there is a special property, called *P-consistency*, that is necessary for possibilistic extendability.

Definition 3. $\mu\colon \Psi \to L$ is called P-consistent if for all $A \subseteq \Psi$ and $a \in \Psi$:

$$a \leq \sup A \Rightarrow \mu(a) \leq \sup_{b \in A} \mu(b).$$

Notice that, equivalently, μ is P-consistent iff for all $a \in \Psi$,

$$\mu(a) \leq \inf_{\substack{A \subseteq \Psi \\ a \leq \sup A}} \sup_{b \in A} \mu(b), \tag{3}$$

The right-hand side of this inequality defines a function on $\mathbf{L}$ that will have an important part in the sequel, see for instance Theorem 7.

Proposition 4. *If a mapping $\mu\colon \Psi \to L$ can be extended to a possibility measure on $\mathbf{L}$, then it is P-consistent.*

Proof. Assume that there exists a possibility measure $\pi\colon \mathbf{L} \to L$ that coincides with μ on Ψ. Then for any $a \in \Psi$ and $A \subseteq \Psi$ such that $a \leq \sup A$,

$$\mu(a) = \pi(a) \leq \pi(\sup A) = \sup_{b \in A} \pi(b) = \sup_{b \in A} \mu(b),$$

so μ is P-consistent. □

It will be useful to consider the set $\Pi(\mathbf{L}, L)$ of all L-possibility measures on $\mathbf{L}$. We shall order this set by the pointwise ordering $\preceq$: if π_1 and π_2 are in $\Pi(\mathbf{L}, L)$ then

$$\pi_1 \preceq \pi_2 \Leftrightarrow (\forall a \in \mathbf{L})(\pi_1(a) \leq \pi_2(a)).$$

The ordered set $(\Pi(\mathbf{L}, L), \preceq)$ is a complete lattice: for any family $\{\pi_j \mid j \in J\} \subseteq \Pi(\mathbf{L}, L)$, its pointwise supremum $\sup_{j \in J} \pi_j$ is again an L-possibility measure on $\mathbf{L}$. It should be noted that this does not necessarily hold for the pointwise infimum of such a family, which means that the infimum in the complete lattice $(\Pi(\mathbf{L}, L), \preceq)$ is not necessarily pointwise. Instead, we have that $\inf_{j \in J} \pi_j$ is the pointwise supremum of all possibility measures π that are pointwise dominated by the pointwise infimum: $\inf_{j \in J} \pi_j = \sup\{\pi \in \Pi(\mathbf{L}, L) \mid (\forall j \in J)(\pi \preceq \pi_j)\}$. The greatest element of $(\Pi(\mathbf{L}, L), \preceq)$ is the mapping π^1 given by

$$\pi^1(a) = \begin{cases} 0_L & \text{if } a = 0_{\mathbf{L}} \\ 1_L & \text{otherwise,} \end{cases}$$

and the smallest element π^o is the constant $\mathbf{L} - \{0_L\}$-mapping.

This tells us that even if we cannot find an L-possibility measure π on $\mathbf{L}$ that coincides with μ on Ψ, we can always find at least one that *is dominated by* μ on Ψ in the sense that $(\forall a \in \Psi)(\pi(a) \leq \mu(a))$, since π^o will always satisfy this condition. The nonempty set

$$\Pi_\mu(\mathbf{L}, L) = \{\pi \in \Pi(\mathbf{L}, L) \mid (\forall a \in \Psi)(\pi(a) \leq \mu(a))\}$$

of all L-possibility measures on $\mathbf{L}$ that are dominated by μ on Ψ has a greatest element π_μ^g, which is the pointwise supremum of all its members: $\pi_\mu^g = \sup \Pi_\mu(\mathbf{L}, L)$. If there are L-possibility measures on $\mathbf{L}$ which extend μ, they will also belong to $\Pi_\mu(\mathbf{L}, L)$ and therefore be dominated by π_μ^g on $\mathbf{L}$. Consequently, μ will have a possibilistic extension to $\mathbf{L}$ if and only if π_μ^g extends μ. The following result is now immediate.

Theorem 5. *If the set* $\Pi'_\mu(\mathbf{L}, L)$ *of all possibilistic extensions of* $\mu\colon \Psi \to L$ *to* $\mathbf{L}$ *is nonempty, it is a complete join-semilattice (closed under arbitrary nonempty pointwise suprema) with greatest element* π^g_μ.

In the rest of this section, we shall restrict our attention to the case that L is a complete chain, and derive a more manageable expression for π^g_μ. We also intend to show that the P-consistency of the mapping $\mu\colon \Psi \to L$ is in this case not only necessary but also sufficient for the existence of a possibility measure on $\mathbf{L}$ that extends μ. The crucial property which allows us to prove this, is a standard result that is commonly called the characterisation of infimum on a chain. In this context, it takes the following form.

Lemma 6. *Let* L *be a complete chain. Let* $A \subseteq L$ *and* $a \in L$ *such that* $\inf A < a$. *Then there is a* $b \in A$ *such that* $b < a$.

Proof. Assume *ex absurdo* that there is no $b \in A$ such that $b < a$. Since L is a (complete) chain, this implies that $(\forall b \in A)(b \geq a)$, whence $\inf A \geq a$. This contradicts $\inf A < a$. □

Theorem 7. *Let* L *be a complete chain. Then the greatest possibility measure* π^g_μ *that is dominated on* Ψ *by the mapping* $\mu\colon \Psi \to L$ *is given by:*

$$\pi^g_\mu(b) = \inf_{\substack{A \subseteq \Psi \\ b \leq \sup A}} \sup_{a \in A} \mu(a), \quad b \in \mathbf{L} \tag{4}$$

Proof. Let π be the mapping defined by the right-hand side of (4), then we have to show that $\pi^g_\mu = \pi$. To see that π is dominated by μ on Ψ, let $A = \{b\}$ in the right-hand side of (4), for $b \in \Psi$. Let us prove that π is a possibility measure. Since π is increasing in that $\pi(c) \leq \pi(d)$ for all c and d in $\mathbf{L}$ such that $c \leq d$, we find that for any $A \subseteq \mathbf{L}$, $\pi(\sup A) \geq \sup_{b \in A} \pi(b)$. To prove the converse inequality, we assume *ex absurdo* that $\pi(\sup A) > \sup_{b \in A} \pi(b)$. There are now two possible cases. In the first case, there exists $\alpha \in L$ such that

$$\pi(\sup A) > \alpha > \sup_{b \in A} \pi(b).$$

Then for every $b \in A$, $\alpha > \pi(b)$ and using the characterisation of infimum on a chain, there exists $A_b \subseteq \Psi$ such that $b \leq \sup A_b$ and $\alpha > \sup_{a \in A_b} \mu(a)$. This implies that $\sup A \leq \sup_{b \in A} \sup A_b = \sup \bigcup_{b \in A} A_b$, and

$$\begin{aligned}
\pi(\sup A) > \alpha &\geq \sup_{b \in A} \sup_{a \in A_b} \mu(a) \\
&= \sup_{a \in \bigcup_{b \in A} A_b} \mu(a) \\
&\geq \inf_{\substack{B \subseteq \Psi \\ \sup A \leq \sup B}} \sup_{a \in B} \mu(a) = \pi(\sup A),
\end{aligned}$$

a contradiction.

In the second case, there exists no α in L such that

$$\pi(\sup A) > \alpha > \sup_{b \in A} \pi(b).$$

Call $\beta = \sup_{b \in A} \pi(b)$, then for all $b \in A$,

$$\beta \geq \inf_{\substack{B \subseteq \Psi \\ b \leq \sup B}} \sup_{a \in B} \mu(a).$$

If there were a $b_o \in A$ such that for every $C \subseteq \Psi$ for which $b_o \leq \sup C$, we had $\sup_{c \in C} \mu(c) > \beta$, whence $\sup_{c \in C} \mu(c) \geq \pi(\sup A)$ from the assumption, it would follow that

$$\inf_{\substack{C\subseteq\Psi\\ b_o\leq\sup C}}\sup_{c\in C}\mu(c)\geq\pi(\sup A)>\beta,$$

a contradiction. Consequently, for every $b\in A$, there is an $A_b\subseteq\Psi$ such that $\sup A_b\geq b$ and $\beta\geq\sup_{a\in A_b}\mu(a)$. Therefore

$$\begin{aligned}\pi(\sup A)>\beta&\geq\sup_{b\in A}\sup_{a\in A_b}\mu(a)\\&=\sup_{a\in\bigcup_{b\in A}A_b}\mu(a)\\&\geq\inf_{\substack{B\subseteq\Psi\\ \sup A\leq\sup B}}\sup_{b\in B}\mu(b)=\pi(\sup A).\end{aligned}$$

a contradiction. Consequently $\pi(\sup A)=\sup_{b\in A}\pi(b)$, and π is a possibility measure. So $\pi\in\Pi_\mu(\mathbf{L},L)$.

To complete the proof, we have to show that π is the greatest element of $\Pi_\mu(\mathbf{L},L)$. Let π' be an arbitrary element of $\Pi_\mu(\mathbf{L},L)$, then we have for any $b\in\mathbf{L}$:

$$\begin{aligned}\pi(b)&=\inf_{\substack{A\subseteq\Psi\\ b\leq\sup A}}\sup_{a\in A}\mu(a)\\&\geq\inf_{\substack{A\subseteq\Psi\\ b\leq\sup A}}\sup_{a\in A}\pi'(a)\\&=\inf_{\substack{A\subseteq\Psi\\ b\leq\sup A}}\pi'(\sup A)\geq\pi'(b)\end{aligned}$$

So $\pi'\preceq\pi$, which completes the proof. □

We are now ready to prove that P-consistency is also a sufficient condition for possibilistic extendability.

Theorem 8. *Let L be a complete chain. If $\mu\colon\Psi\to L$ is P-consistent, then π^g_μ is a possibilistic extension of μ to $\mathbf{L}$. It is the greatest L-possibility measure on $\mathbf{L}$ that coincides with μ on Ψ.*

Proof. It only needs to be proven that π^g_μ dominates μ on Ψ. Let $b\in\Psi$, then taking into account (4) and the P-consistency of μ,

$$\pi^g_\mu(b)=\inf_{\substack{A\subseteq\Psi\\ b\leq\sup A}}\sup_{a\in A}\mu(a)\geq\inf_{\substack{A\subseteq\Psi\\ b\leq\sup A}}\mu(b)=\mu(b).\quad\square$$

Corollary 9. *Let L be a complete chain. Then a mapping $\mu\colon\Psi\to L$ can be extended to a possibility measure on $\mathbf{L}$ if and only if μ is P-consistent.*

3. Two types of possibility integrals

Consider the set $L^{\mathbf{L}}$ of the $\mathbf{L}-L$-mappings. This set can be ordered pointwise: for any f_1 and f_2, $f_1\preceq f_2$ iff for any $a\in\mathbf{L}$, $f_1(a)\leq f_2(a)$. This ordering is an extension of the pointwise ordering on the set $\Pi(\mathbf{L},L)$, defined in the previous section. Note that $(L^{\mathbf{L}},\preceq)$ is a complete lattice, where supremum and infimum are to be taken pointwise.

We assume that the complete lattice $\mathbf{L}$ is provided with an L-possibility measure π. We aim to define a special type of L-possibility measure on the complete lattice $(L^{\mathbf{L}},\preceq)$ through a procedure related to Sugeno's fuzzy integration. We want this new possibility measure to be an extension of the original possibility measure π. For this purpose, we need a way to embed the ordered structure $\mathbf{L}$ into $L^{\mathbf{L}}$.

We shall give two order-embeddings of $\mathbf{L}$ into $L^{\mathbf{L}}$, each with different properties. They will both be used to define a different kind of integral. They are denoted by

$Z\colon \mathbf{L} \to L^{\mathbf{L}}$ and $\Xi\colon \mathbf{L} \to L^{\mathbf{L}}$ respectively, and defined as follows: for any a in $\mathbf{L}$, $Z(a) = \zeta_a$ and $\Xi(a) = \xi_a$ where

$$\zeta_a(x) = \begin{cases} 1_L & \text{if } x \leq a \\ 0_L & \text{if } x \not\leq a \end{cases} \quad \text{and} \quad \xi_a(x) = \begin{cases} 1_L & \text{if } x \not\geq a \\ 0_L & \text{if } x \geq a. \end{cases}$$

It should be noted that $a \leq b \Leftrightarrow Z(a) \preceq Z(b) \Leftrightarrow \Xi(a) \preceq \Xi(b)$ and that $Z(\inf A) = \inf_{a\in A} Z(a)$ and $\Xi(\sup A) = \sup_{a\in A} \Xi(a)$ for any a and b in A and $A \subseteq \mathbf{L}$. So Z is an infimum-preserving order-embedding and Ξ a supremum-preserving order-embedding of the complete lattice $\mathbf{L}$ into the complete lattice $L^{\mathbf{L}}$.

Let T denote a triangular norm on L that is completely distributive with respect to supremum. In other words, T is a binary operation on L satisfying,

1. $T(\alpha, \beta) = T(\beta, \alpha)$ [commutativity]
2. $T(T(\alpha, \beta), \gamma) = T(\alpha, T(\beta, \gamma))$ [associativity]
3. $\alpha \leq \beta$ and $\gamma \leq \delta$ imply $T(\alpha, \gamma) \leq T(\beta, \delta)$ [isotonicity]
4. $T(\alpha, 1_L) = \alpha$ [neutral element]
5. $\sup_{j\in J} T(\alpha_j, \beta) = T(\sup_{j\in J} \alpha_j, \beta)$ [complete distributivity]

where α, β, γ, δ and α_j, $j \in J$, are elements of L. It follows from these assumptions that $T(\alpha, 0_L) = T(0_L, \alpha) = 0_L$.

For any $a \in \mathbf{L}$ and $A \subseteq \mathbf{L}$ we also introduce the following notations:

$$a \mathbin{\bar\curlywedge} A = \sup\{x \in A \mid x \leq a\} \quad \text{and} \quad a \curlywedge A = \sup\{x \in A \mid x \not\geq a\}.$$

Moreover, for any $f\colon \mathbf{L} \to L$ and $\alpha \in L$, we write $f_\alpha = \{x \in \mathbf{L} \mid f(x) \geq \alpha\}$. If A is any subset of $\mathbf{L}$, $\chi_A\colon \mathbf{L} \to L$ is defined by $\chi_A(x) = 1_L$ if $x \in A$ and $\chi_A(x) = 0_L$ if $x \notin A$.

Definition 10. For a in $\mathbf{L}$ and $f\colon \mathbf{L} \to L$, we define two T-possibility integrals of f on a with respect to π:

$$(Z)\int_a f \mathrm{d}\pi = \sup_{\alpha\in L} T(\alpha, \sup_{x\in\mathbf{L}} T(\zeta_a(x), \chi_{f_\alpha}(x), \pi(x)))$$
$$(\Xi)\int_a f \mathrm{d}\pi = \sup_{\alpha\in L} T(\alpha, \sup_{x\in\mathbf{L}} T(\xi_a(x), \chi_{f_\alpha}(x), \pi(x))).$$

Note that the above procedure could be used to associate an integral with every order-embedding Γ of $\mathbf{L}$ into $L^{\mathbf{L}}$. Define, for all $a \in \mathbf{L}$ and $f \in L^{\mathbf{L}}$, the corresponding integral as follows:

$$(\Gamma)\int_a f \mathrm{d}\pi = \sup_{\alpha\in L} T(\alpha, \sup_{x\in\mathbf{L}} T(\gamma_a(x), \chi_{f_\alpha}(x), \pi(x))),$$

where $\gamma_a = \Gamma(a) \in L^{\mathbf{L}}$. We shall however concentrate on the integrals associated with the embeddings Z and Ξ, as they seem the easiest to work with. In what follows, we derive alternative expressions for these integrals, and prove a number of important properties.

Theorem 11. *Let $a \in \mathbf{L}$ and let f be mapping from $\mathbf{L}$ to L. Then*

$$(Z)\int_a f \mathrm{d}\pi = \sup_{\alpha\in L} T(\alpha, \pi(a \mathbin{\bar\curlywedge} f_\alpha)) = \sup_{E\subseteq\mathbf{L}} T(\inf_{x\in E} f(x), \pi(a \mathbin{\bar\curlywedge} E))$$
$$(\Xi)\int_a f \mathrm{d}\pi = \sup_{\alpha\in L} T(\alpha, \pi(a \curlywedge f_\alpha)) = \sup_{E\subseteq\mathbf{L}} T(\inf_{x\in E} f(x), \pi(a \curlywedge E)).$$

Proof. We give a proof for the equalities involving $(Z)\int_a f \mathrm{d}\pi$. The other equalities are proven similarly. The first equality follows immediately from the definitions of

ζ_a and $\bar{\curlywedge}$. We therefore concentrate on the second equality. For every $\alpha \in L$, we have $\inf_{x\in f_\alpha} f(x) \geq \alpha$ and therefore

$$T(\alpha, \pi(a \,\bar{\curlywedge}\, f_\alpha)) \leq T(\inf_{x\in f_\alpha} f(x), \pi(a \,\bar{\curlywedge}\, f_\alpha)) \leq \sup_{E\subseteq \mathbf{L}} T(\inf_{x\in E} f(x), \pi(a \,\bar{\curlywedge}\, E)).$$

Consequently, $(Z)\int_a f\mathrm{d}\pi \leq \sup_{E\subseteq\mathbf{L}} T(\inf_{x\in E} f(x), \pi(a \,\bar{\curlywedge}\, E))$. To prove the converse inequality, consider any $E \subseteq \mathbf{L}$ and let $\alpha_E = \inf_{x\in E} f(x)$. Then $E \subseteq f_{\alpha_E}$. Consequently, $\pi(a \,\bar{\curlywedge}\, E) \leq \pi(a \,\bar{\curlywedge}\, f_{\alpha_E})$, and therefore

$$T(\inf_{x\in E} f(x), \pi(a \,\bar{\curlywedge}\, E)) \leq T(\alpha_E, \pi(a \,\bar{\curlywedge}\, f_{\alpha_E})) \leq \sup_{\alpha\in L} T(\alpha, \pi(a \,\bar{\curlywedge}\, f_\alpha)) = (Z)\int_a f\mathrm{d}\pi.$$

This implies that $\sup_{E\subseteq\mathbf{L}} T(\inf_{x\in E} f(x), \pi(a \,\bar{\curlywedge}\, E)) \leq (Z)\int_a f\mathrm{d}\pi$. □

In the context of this paper, a mapping $s\colon \mathbf{L} \to L$ will be called *simple* iff it has a finite range, i.e. $s(\mathbf{L}) = \{s_1, s_2, \ldots, s_n\}$, where n is a strictly positive natural number. It will be convenient to introduce the following generic notation for simple mappings: $D_k = s^{-1}(\{s_k\}) = \{x \in \mathbf{L} \mid s(x) = s_k\}$, $k = 1, \ldots, n$. Note that the D_k's constitute a partition of $\mathbf{L}$. Consequently, for every $x \in \mathbf{L}$:

$$s(x) = \sup_{k=1}^{n} T(s_k, \chi_{D_k}(x)).$$

For every $a \in \mathbf{L}$, we shall associate with such a simple mapping s two elements $Q_T^a(s)$ and $R_T^a(s)$ of L, defined as follows:

$$Q_T^a(s) = \sup_{k=1}^{n} T(s_k, \pi(a \,\bar{\curlywedge}\, D_k)) \quad \text{and} \quad R_T^a(s) = \sup_{k=1}^{n} T(s_k, \pi(a \curlywedge D_k)).$$

Theorem 12. *Let $a \in \mathbf{L}$ and $f\colon \mathbf{L} \to L$. Then*

$$(Z)\int_a f\mathrm{d}\pi = \sup_{s\in S(f)} Q_T^a(s) \quad \textit{and} \quad (\Xi)\int_a f\mathrm{d}\pi = \sup_{s\in S(f)} R_T^a(s),$$

where $S(f) = \{s \mid s\colon \mathbf{L} \to L$ is simple and $s \preceq f\}$.

Proof. We prove the second equality. For every $E \subseteq \mathbf{L}$, let $\alpha_E = \inf_{x\in E} f(x)$, and define the simple function $s_E\colon \mathbf{L} \to L$ as follows: $s_E(x) = T(\alpha_E, \chi_E(x))$, $x \in \mathbf{L}$. Note that $s_E \preceq f$, so $s_E \in S(f)$. Hence

$$\sup_{s\in S(f)} R_T^a(s) \geq R_T^a(s_E) = T(\alpha_E, \pi(a \curlywedge E)) = T(\inf_{x\in E} f(x), \pi(a \curlywedge E)).$$

Consequently, $\sup_{s\in S(f)} R_T^a(s) \geq (\Xi)\int_a f\mathrm{d}\pi$.

Conversely, for every simple function $s = \sup_{k=1}^n T(s_k, \chi_{D_k})$ in $S(f)$, we have, for $1 \leq k \leq n$ and $x \in D_k$, that $s_k = s(x) \leq f(x)$, so $s_k \leq \inf_{x\in D_k} f(x)$. Hence

$$T(s_k, \pi(a \curlywedge D_k)) \leq T(\inf_{x\in D_k} f(x), \pi(a \curlywedge D_k)) \leq \sup_{E\subseteq\mathbf{L}} T(\inf_{x\in E} f(x), \pi(a \curlywedge E)).$$

Consequently, for every $s \in S(f)$, $R_T^a(s) \leq (\Xi)\int_a f\mathrm{d}\pi$, which leads to the desired inequality $\sup_{s\in S(f)} R_T^a(s) \leq (\Xi)\int_a f\mathrm{d}\pi$. □

Theorem 13. *Let $a \in \mathbf{L}$ and $f\colon \mathbf{L} \to L$. Then*

$$(Z)\int_a f\mathrm{d}\pi = \sup_{x\leq a} T(f(x), \pi(x)) \quad \textit{and} \quad (\Xi)\int_a f\mathrm{d}\pi = \sup_{x\not\geq a} T(f(x), \pi(x)).$$

Proof. We prove the first equality. First of all, it is easy to prove that for every $x \in \mathbf{L}$, $f(x) = \sup_{\alpha \in L} T(\alpha, \chi_{f_\alpha}(x))$. Consequently, using the complete distributivity of T with respect to sup and the fact that T is associative:

$$\begin{aligned}
(Z)\int_a f\mathrm{d}\pi &= \sup_{\alpha\in L} T(\alpha, \sup_{x\in\mathbf{L}} T(\chi_{f_\alpha}(x), \zeta_a(x), \pi(x))) \\
&= \sup_{\alpha\in L}\sup_{x\in\mathbf{L}} T(\alpha, T(\chi_{f_\alpha}(x), \zeta_a(x), \pi(x))) \\
&= \sup_{x\in\mathbf{L}}\sup_{\alpha\in L} T(T(\alpha, \chi_{f_\alpha}(x)), \zeta_a(x), \pi(x)) \\
&= \sup_{x\in\mathbf{L}} T(\sup_{\alpha\in L} T(\alpha, \chi_{f_\alpha}(x)), \zeta_a(x), \pi(x)) \\
&= \sup_{x\in\mathbf{L}} T(f(x), \zeta_a(x), \pi(x)) \\
&= \sup_{x\le a} T(f(x), \pi(x)). \quad \square
\end{aligned}$$

Corollary 14. *Let a and b be elements of $\mathbf{L}$ with $a \le b$. Then $(Z)\int_b \zeta_a \mathrm{d}\pi = \pi(a)$. Moreover, if π is strict then $(\Xi)\int_b \xi_a \mathrm{d}\pi = \pi(a)$.*

Corollary 15. *For every $a \in \mathbf{L}$, $(Z)\int_a s\mathrm{d}\pi = Q_T^a(s)$ and $(\Xi)\int_a s\mathrm{d}\pi = R_T^a(s)$.*

Corollary 16. *Let $a \in \mathbf{L}$ and $\lambda \in L$. Then $(Z)\int_a \lambda\mathrm{d}\pi = T(\lambda, \pi(a))$. Moreover, if π is strict then $(\Xi)\int_a \lambda\mathrm{d}\pi = T(\lambda, \pi(a))$.*

Note that in this last corollary, we have denoted the constant $\mathbf{L} - \{\lambda\}$-mapping by λ.

Proposition 17. *Let $a \in \mathbf{L}$ and let $f_1 \colon \mathbf{L} \to L$ and $f_2 \colon \mathbf{L} \to L$. Then $f_1 \preceq f_2$ implies that $(Z)\int_a f_1\mathrm{d}\pi \le (Z)\int_a f_2\mathrm{d}\pi$ and $(\Xi)\int_a f_1\mathrm{d}\pi \le (\Xi)\int_a f_2\mathrm{d}\pi$.*

Proof. We give the proof for the second inequality. It follows from Theorem 13 that

$$(\Xi)\int_a f_1\mathrm{d}\pi = \sup_{x\not\geq a} T(f_1(x), \pi(x)) \le \sup_{x\not\geq a} T(f_2(x), \pi(x)) = (\Xi)\int_a f_2\mathrm{d}\pi. \quad \square$$

Corollary 18. *Let $a \in \mathbf{L}$ and let $f_1 \colon \mathbf{L} \to L$ and $f_2 \colon \mathbf{L} \to L$. Then we have both $(Z)\int_a \inf\{f_1, f_2\}\mathrm{d}\pi \le \inf\{(Z)\int_a f_1\mathrm{d}\pi, (Z)\int_a f_2\mathrm{d}\pi\}$ and $(\Xi)\int_a \inf\{f_1, f_2\}\mathrm{d}\pi \le \inf\{(\Xi)\int_a f_1\mathrm{d}\pi, (\Xi)\int_a f_2\mathrm{d}\pi\}$.*

Proposition 19. *Let $f \colon \mathbf{L} \to L$ and let a and b be elements of $\mathbf{L}$. If $a \le b$ then $(Z)\int_a f\mathrm{d}\pi \le (Z)\int_b f\mathrm{d}\pi$ and $(\Xi)\int_a f\mathrm{d}\pi \le (\Xi)\int_b f\mathrm{d}\pi$.*

Proof. This follows at once from Theorem 13 and the fact that given $a \le b$, $x \le a$ implies $x \le b$ and $x \not\geq a$ implies $x \not\geq b$. $\square$

Proposition 20. *Let $A \subseteq \mathbf{L}$ and $f \colon \mathbf{L} \to L$. Then*

$$(Z)\int_{\inf A} f\mathrm{d}\pi \le \inf_{a\in A}(Z)\int_a f\mathrm{d}\pi \quad \textit{and} \quad (\Xi)\int_{\inf A} f\mathrm{d}\pi \le \inf_{a\in A}(\Xi)\int_a f\mathrm{d}\pi.$$

Moreover,

$$(\Xi)\int_{\sup A} f\mathrm{d}\pi = \sup_{a\in A}(\Xi)\int_a f\mathrm{d}\pi.$$

Proof. The proof of the first two inequalities is trivial. We prove the equality. Using Theorem 13, we find that

$$\begin{aligned}(\Xi)\int_{\sup A} f\mathrm{d}\pi &= \sup_{x\not\geq \sup A} T(f(x),\pi(x))\\ &= \sup_{(\exists a\in A)(x\not\geq a)} T(f(x),\pi(x))\\ &= \sup_{a\in A}\sup_{x\not\geq a} T(f(x),\pi(x))\\ &= \sup_{a\in A}(\Xi)\int_a f\mathrm{d}\pi. \quad\square\end{aligned}$$

Corollary 21. *Let $f\colon \mathbf{L}\to L$. Define the mapping $\mu_f\colon \mathbf{L}\to L$ by*

$$\mu_f(a) = (\Xi)\int_a f\mathrm{d}\pi, \quad a\in\mathbf{L}.$$

Then μ_f is an L-possibility measure on $\mathbf{L}$.

Proposition 22. *Let $\{f_i \mid i\in I\}$ be an arbitrary family of $\mathbf{L}-L$-mappings and $\{\alpha_i \mid i\in I\}$ a corresponding family of elements of L. For every $a\in\mathbf{L}$,*

$$\begin{aligned}(Z)\int_a \sup_{i\in I} T(\alpha_i, f_i)\mathrm{d}\pi &= \sup_{i\in I} T(\alpha_i, (Z)\int_a f_i\mathrm{d}\pi)\\ (\Xi)\int_a \sup_{i\in I} T(\alpha_i, f_i)\mathrm{d}\pi &= \sup_{i\in I} T(\alpha_i, (\Xi)\int_a f_i\mathrm{d}\pi).\end{aligned}$$

Proof. We prove the first equality. By Theorem 13 and the complete distributivity of T with respect to sup,

$$\begin{aligned}(Z)\int_a \sup_{i\in I} T(\alpha_i, f_i)\mathrm{d}\pi &= \sup_{x\leq a} T((\sup_{i\in I} T(\alpha_i, f_i))(x), \pi(x))\\ &= \sup_{x\leq a} T(\sup_{i\in I} T(\alpha_i, f_i(x)), \pi(x))\\ &= \sup_{x\leq a}\sup_{i\in I} T(T(\alpha_i, f_i(x)), \pi(x))\\ &= \sup_{i\in I}\sup_{x\leq a} T(\alpha_i, T(f_i(x), \pi(x)))\\ &= \sup_{i\in I} T(\alpha_i, \sup_{x\leq a} T(f_i(x), \pi(x)))\\ &= \sup_{i\in I} T(\alpha_i, (Z)\int_a f_i\mathrm{d}\pi). \quad\square\end{aligned}$$

Corollary 23. *Let $a\in\mathbf{L}$, $\alpha\in L$ and let $f\colon\mathbf{L}\to L$. Then*

$$(Z)\int_a T(\alpha, f)\mathrm{d}\pi = T(\alpha, (Z)\int_a f\mathrm{d}\pi) \quad \textit{and} \quad (\Xi)\int_a T(\alpha, f)\mathrm{d}\pi = T(\alpha, (\Xi)\int_a f\mathrm{d}\pi).$$

Corollary 24. *Let $a\in\mathbf{L}$ and let $\{f_i \mid i\in I\}$ be an arbitrary family of $\mathbf{L}-L$-mappings. Then*

$$(Z)\int_a \sup_{i\in I} f_i\mathrm{d}\pi = \sup_{i\in I}(Z)\int_a f_i\mathrm{d}\pi \quad \textit{and} \quad (\Xi)\int_a \sup_{i\in I} f_i\mathrm{d}\pi = \sup_{i\in I}(\Xi)\int_a f_i\mathrm{d}\pi.$$

This last result tells us that we may use the integrals to extend a possibility measure π on $\mathbf{L}$ to a possibility measure on $L^{\mathbf{L}}$. To conclude this paper, let us take a closer look at the integral involving Ξ and a strict possibility measure π on $\mathbf{L}$. Completely similar remarks can be made about the integral involving Z and a possibility measure π (not necessarily strict in that case). Define the function σ which maps elements f of $L^{\mathbf{L}}$ to elements $\sigma(f)$ of L as follows:

$$\sigma(f) = (\Xi) \int_{1_{\mathbf{L}}} f \mathrm{d}\pi.$$

Then, by Corollary 24, σ is an L-possibility measure on $L^{\mathbf{L}}$. Since, by Corollary 14, $\sigma(\Xi(a)) = \pi(a)$ for any a in $\mathbf{L}$, we see that σ extends π from $\mathbf{L}$ to $L^{\mathbf{L}}$, if we identify elements a of $\mathbf{L}$ with elements $\Xi(a) = \xi_a$ of $L^{\mathbf{L}}$.

It is instructive to relate this to the results of Section 2. Identifying elements a of $\mathbf{L}$ with elements $\Xi(a) = \xi_a$ of $L^{\mathbf{L}}$ implicitly amounts to specifiying a set function μ on the subset $\Xi(\mathbf{L}) = \{\xi_a \mid a \in \mathbf{L}\}$ of $L^{\mathbf{L}}$ as follows: $\mu(\xi_a) = \pi(a)$. We already know that μ is extendable to a possibility measure on $L^{\mathbf{L}}$, as we have already constructed such an extension σ through the integration process. Proposition 4 implies that μ must be P-consistent, and this can indeed be verified explicitly using the definition. If we assume that L is a complete chain, we may use Theorems 7 and 8 to identify the greatest possibilistic extension π_μ^g of μ to $L^{\mathbf{L}}$: for any f in $L^{\mathbf{L}}$,

$$\pi_\mu^g(f) = \inf_{a \in \mathbf{L}, \uparrow a \subseteq f^o} \pi(a),$$

where $\uparrow a = \{x \in \mathbf{L} \mid x \geq a\}$ and $f^o = \{x \in \mathbf{L} \mid f(x) = 0_L\}$. It is easy to verify explicitly that π_μ^g dominates σ on $L^{\mathbf{L}}$. Note also that the only information needed to construct $\pi_\mu^g(f)$ is where f assumes the value 0_L. Theorems 11 and 13 tell us that σ is much richer in this respect: we need information about where f assumes other values than 0_L as well.

4. Acknowledgements

Research for this paper is supported by the International Projects of the Flemish Community – Co-operation with China No 9706B. Guangquan Zhang is on leave from the Mathematics department of Hebei University (China) and is visiting Curtin University of Technology (Australia) as a Research Fellow. He would like to thank the National Natural Science Foundation of the People's Republic of China for partially funding this research. Gert de Cooman is a Postdoctoral Fellow of the Fund for Scientific Research – Flanders (Belgium).

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