Implicator and coimplicator integrals

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Abstract

We introduce and study implicator and coimplicator integrals, and investigate their possible application in defining the possibility and necessity of fuzzy sets. First, the definition and properties of implicators and coimplicators on bounded posets are discussed. Then, in analogy with the theory of seminormed and semiconormed fuzzy integrals, implicator and coimplicator integrals are defined. Next, we study the properties of these dual types of integrals. We uncover an interesting relationship between implicator and coimplicator integrals, and seminormed and semiconormed fuzzy integrals, which could also be called conjunctor and disjunctor integrals. Finally, we show that coimplicator and implicator integrals can be used to extend the domain of possibility measures and necessity measures from sets to fuzzy sets.

1 INTRODUCTION

Seminormed and semiconormed fuzzy integrals were introduced in 1986 by Suárez García and Gil Álvarez [10] as generalizations of Sugeno's fuzzy integral [11]. They were generalized by De Cooman and Kerre [7], who also showed that they are, in a sense, the most general types of integrals satisfying a number of desired properties, and that they are very well suited for combination with possibility respectively necessity measures. As their name suggests, these integrals are respectively associated with triangular seminorms and triangular semiconorms. In this paper, we introduce two other types of fuzzy integrals, associated with border implicators and border coimplicators, and show that they too are well suited for combination with possibility respectively necessity measures.

2 EXTENDED LOGICAL OPERATORS

In this section, we discuss the four basic extended logical operators on a bounded poset $(P,\leq)$, with top 1_P and bottom 0_P. More extensive results concerning conjunctors and disjunctors can be found in [3, 6], concerning implicators in [2, 3], and concerning coimplicators in [1].

A decreasing unary operator $\mathcal{N}$ on $(P,\leq)$ is called a *negator* iff $\mathcal{N}(0_P) = 1_P$ and $\mathcal{N}(1_P) = 0_P$. A *strong negator* $\mathcal{N}$ on $(P,\leq)$ is a permutation of P that satisfies $\alpha \leq \beta \Leftrightarrow \mathcal{N}(\alpha) \geq \mathcal{N}(\beta)$, α and β in P. Note that an involutive, decreasing transformation is a strong negator.

An increasing binary operator $\mathcal{O}$ on $(P,\leq)$ is called *a conjunctor* iff $\mathcal{O}(0_P,1_P) = \mathcal{O}(1_P,0_P) = 0_P$ and $\mathcal{O}(1_P,1_P) = 1_P$, and a *disjunctor* iff $\mathcal{O}(0_P,1_P) = \mathcal{O}(1_P,0_P) = 1_P$ and $\mathcal{O}(0_P,0_P) = 0_P$.

Any conjunctor $\mathcal{C}$ satisfies $\mathcal{C}(0_P,\alpha) = \mathcal{C}(\alpha,0_P) = 0_P$, while any disjunctor $\mathcal{D}$ satisfies $\mathcal{D}(1_P,\alpha) = \mathcal{D}(\alpha,1_P) = 1_P$. This means that by definition the values of a conjunctor are fixed on two of the four 'borders'; similarly for a disjunctor, but on the 'opposite borders'. Fixing the values on the remaining 'borders' leads to t-seminorms and t-semiconorms.

Definition 1 *An increasing binary operator $\mathcal{O}$ on $(P,\leq)$ is called a t-seminorm iff for any α in P, $\mathcal{O}(1_P,\alpha) = \mathcal{O}(\alpha,1_P) = \alpha$; and a t-semiconorm iff for any α in P, $\mathcal{O}(0_P,\alpha) = \mathcal{O}(\alpha,0_P) = \alpha$.*

Clearly, a t-seminorm is a conjunctor and a t-semiconorm a disjunctor.

A hybrid monotonous binary operator $\mathcal{O}$ on $(P,\leq)$ (with decreasing first and increasing second partial mappings) is called an *implicator* iff $\mathcal{O}(0_P,0_P) = \mathcal{O}(1_P,1_P) = 1_P$ and $\mathcal{O}(1_P,0_P) = 0_P$, and a *coimplicator* iff $\mathcal{O}(0_P,0_P) = \mathcal{O}(1_P,1_P) = 0_P$ and $\mathcal{O}(0_P,1_P) = 1_P$.

Implicators are extensions of the Boolean implication $\Rightarrow$ ($p \Rightarrow q$ meaning that p is sufficient for q), while coimplicators are extensions of the Boolean coimplication $\not\Leftarrow$ ($p \not\Leftarrow q$ meaning that p is not necessary for q).

Any implicator $\mathcal{I}$ satisfies $\mathcal{I}(0_P, \alpha) = \mathcal{I}(\alpha, 1_P) = 1_P$, while any coimplicator $\mathcal{J}$ satisfies $\mathcal{J}(1_P, \alpha) = \mathcal{J}(\alpha, 0_P) = 0_P$. Again, we can say that the values of an implicator or coimplicator are fixed on two of the four 'borders'. The behaviour on one of the remaining borders is of particular interest. The partial mapping $\mathcal{I}(\cdot, 0_P)$ of an implicator $\mathcal{I}$ clearly is a negator, and is denoted by $\mathcal{N}_{\mathcal{I}}$, i.e. $\mathcal{N}_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha, 0_P)$, and called the *negator induced by* $\mathcal{I}$. Similarly, the partial mapping $\mathcal{J}(\cdot, 1_P)$ of a coimplicator $\mathcal{J}$ is a negator, and is denoted by $\mathcal{N}_{\mathcal{J}}$, i.e. $\mathcal{N}_{\mathcal{J}}(\alpha) = \mathcal{J}(\alpha, 1_P)$, and called the *negator induced by* $\mathcal{J}$. Fixing the values on the remaining border leads to border implicators and border coimplicators.

Definition 2 *An implicator $\mathcal{I}$ on $(P, \leq)$ that satisfies $\mathcal{I}(1_P, \beta) = \beta$, $\beta \in P$, is called a border implicator. A coimplicator $\mathcal{J}$ on $(P, \leq)$ that satisfies $\mathcal{J}(0_P, \beta) = \beta$, $\beta \in P$, is called a border coimplicator.*

Consider a strong negator $\mathcal{N}$ and a binary operator $\mathcal{O}$ on $(P, \leq)$. The dual operator of $\mathcal{O}$ w.r.t. $\mathcal{N}$ is the binary operator $D_{\mathcal{N}}[\mathcal{O}]$ on $(P, \leq)$ defined by, for any α and β in P:

$$D_{\mathcal{N}}[\mathcal{O}](\alpha, \beta) = \mathcal{N}^{-1}(\mathcal{O}(\mathcal{N}(\alpha), \mathcal{N}(\beta))).$$

With $\mathcal{O}$ and $\mathcal{N}$ we also associate two binary operators $\Lambda_{\mathcal{N}}[\mathcal{O}]$ and $\Omega_{\mathcal{N}}[\mathcal{O}]$ on $(P, \leq)$ defined by, for any α and β in P:

$$\Lambda_{\mathcal{N}}[\mathcal{O}](\alpha, \beta) = \mathcal{O}(\beta, \mathcal{N}(\alpha))$$
$$\Omega_{\mathcal{N}}[\mathcal{O}](\alpha, \beta) = \mathcal{O}(\mathcal{N}(\beta), \alpha).$$

These notions lead to interesting relationships between our four basic types of extended logical operators.

Proposition 3

(i) *Consider a t-seminorm $\mathcal{T}$, then $D_{\mathcal{N}}[\mathcal{T}]$ is a t-semiconorm and $\Lambda_{\mathcal{N}}[\mathcal{T}]$ is a border coimplicator.*

(ii) *Consider a t-semiconorm $\mathcal{S}$, then $D_{\mathcal{N}}[\mathcal{S}]$ is a t-seminorm and $\Lambda_{\mathcal{N}}[\mathcal{S}]$ is a border implicator.*

(iii) *Consider a border implicator $\mathcal{I}$, then $D_{\mathcal{N}}[\mathcal{I}]$ is a border coimplicator. Moreover, $\Omega_{\mathcal{N}}[\mathcal{I}]$ is a t-semiconorm iff $\mathcal{N}_{\mathcal{I}} = \mathcal{N}^{-1}$.*

(iv) *Consider a border coimplicator $\mathcal{J}$, then $D_{\mathcal{N}}[\mathcal{J}]$ is a border implicator. Moreover, $\Omega_{\mathcal{N}}[\mathcal{J}]$ is a t-seminorm iff $\mathcal{N}_{\mathcal{J}} = \mathcal{N}^{-1}$.*

A mapping f between complete lattices $(L, \leq)$ and $(M, \leq')$ is called a *complete meet-morphism* iff for any nonempty subset A of L, $f(\inf A) = \inf f(A)$; a *complete join-morphism* iff for any nonempty subset A of L, $f(\sup A) = \sup f(A)$; a *complete dual meet-morphism* iff for any nonempty subset A of L, $f(\inf A) = \sup f(A)$; and a *complete dual join-morphism* iff for any nonempty subset A of L, $f(\sup A) = \inf f(A)$. Note that a strong negator on a complete lattice always is a complete dual meet-morphism and a complete dual join-morphism.

3 IMPLICATOR AND COIMPLICATOR INTEGRALS

In the rest of this paper, Ω will denote an arbitrary universe of discourse, and $\mathcal{V}$ a field of subsets of Ω, with $\{\emptyset, \Omega\} \subset \mathcal{V}$. By $(L, \leq)$ we denote a complete lattice, with top 1_L and bottom 0_L. The meet of $(L, \leq)$ will be denoted by $\frown$, its join by $\smile$. Moreover, $\mathcal{T}$ is a t-seminorm, $\mathcal{S}$ a t-semiconorm, $\mathcal{I}$ a border implicator and $\mathcal{J}$ a border coimplicator on $(L, \leq)$.

By v we denote a $(L, \leq)$-*confidence measure* on $(\Omega, \mathcal{V})$, i.e., a $\mathcal{V} - L$-mapping that is increasing: $A \subseteq B \Rightarrow v(A) \leq v(B)$, for any A and B in $\mathcal{V}$. If $\mathcal{N}$ is a strong negator, then the $\mathcal{V} - L$-mapping $v_{\mathcal{N}}$, defined by $v_{\mathcal{N}}(A) = \mathcal{N}^{-1}(v(\text{co}A))$, $A \in \mathcal{V}$, is a $(L, \leq)$-confidence measure on $(\Omega, \mathcal{V})$, called the *dual confidence measure of v w.r.t. $\mathcal{N}$.*

A mapping from Ω to L is also called a $(L, \leq)$-*fuzzy set* in Ω. The set of all these mappings is denoted by L^{Ω}. The partial order $\leq$ on L allows us to define a partial order $\sqsubseteq$ on L^{Ω} as follows: for any h_1 and h_2 in L^{Ω}, $h_1 \sqsubseteq h_2 \Leftrightarrow (\forall \omega \in \Omega)(h_1(\omega) \leq h_2(\omega))$.

For any element μ of L, we denote the constant $\Omega - \{\mu\}$-mapping by $\underline{\mu}$. For any subset A of Ω, its *characteristic $\Omega - L$-mapping* is denoted by χ_A and defined by $\chi_A(\omega) = 1_L$, $\omega \in A$ and $\chi_A(\omega) = 0_L$, $\omega \in \text{co}A$.

Let h be a $\Omega - L$-mapping and let λ be an element of L. We define the *cut set* $S^h_{\lambda} = \{\omega \mid h(\omega) \geq \lambda\}$ and the *dual cut set* $D^h_{\lambda} = \{\omega \mid h(\omega) \leq \lambda\}$ of h at level λ. h is called $\mathcal{V}$-*cut-measurable* iff $(\forall \lambda \in L)(S^h_{\lambda} \in \mathcal{V})$ and *dually $\mathcal{V}$-cut-measurable* iff $(\forall \lambda \in L)(D^h_{\lambda} \in \mathcal{V})$. Note that for any $A \subseteq \Omega$, the following statements are equivalent: χ_A is $\mathcal{V}$-cut-measurable; χ_A is dually $\mathcal{V}$-cut-measurable; and $A \in \mathcal{V}$.

We call a $\Omega - L$-mapping s $\mathcal{V}$-*simple* iff it has a finite range $s(\Omega) = \{s_1, \ldots, s_n\}$ and is $\mathcal{V}$-measurable, i.e., $D_k = s^{-1}(\{s_k\}) \in \mathcal{V}$, $k = 1, \ldots, n$. It is easily verified that for any ω in Ω:

$$s(\omega) = \inf_{k=1}^{n} \mathcal{I}\big(\chi_{D_k}(\omega), s_k\big) = \sup_{k=1}^{n} \mathcal{J}\big(\chi_{\text{co}D_k}(\omega), s_k\big).$$

Inspired by these expressions, we associate two functionals $\gamma_{\mathcal{I}}^{v}(\cdot;\cdot)$ and $\delta_{\mathcal{J}}^{v}(\cdot;\cdot)$ with the $(L,\leq)$-confidence measure v in the following way. If $A \in \mathcal{V}$ and s is a $\mathcal{V}$-simple $\Omega - L$-mapping, then, with obvious notations,

$$\gamma_{\mathcal{I}}^{v}(A;s) = \inf_{k=1}^{n} \mathcal{I}(v(A \cap D_k), s_k)$$

and

$$\delta_{\mathcal{J}}^{v}(A;s) = \sup_{k=1}^{n} \mathcal{J}(v(A \cap \mathrm{co}D_k), s_k).$$

In the standard way (see also [7]), we use these functionals to construct two new functionals which are defined for arbitrary $\Omega - L$-mappings, and not just for $\mathcal{V}$-simple mappings.

Definition 4 *Let h be a $\Omega - L$-mapping and let A be an element of $\mathcal{V}$. Then*

$$(\mathcal{I}) \fint_A h \mathrm{d}v = \inf\{\gamma_{\mathcal{I}}^{v}(A;s) \mid h \sqsubseteq s\}$$

is called the $\mathcal{I}$-integral (or implicator integral) of h on A; and

$$(\mathcal{J}) \fint_A h \mathrm{d}v = \sup\{\delta_{\mathcal{J}}^{v}(A;s) \mid s \sqsubseteq h\}$$

is called the $\mathcal{J}$-integral (or coimplicator integral) of h on A.

The following theorems give explicit formulas for the calculation of implicator and coimplicator integrals.

Theorem 5 *Let h be a $\Omega - L$-mapping and let A be an element of $\mathcal{V}$. Then*

$$(\mathcal{I}) \fint_A h \mathrm{d}v = \inf_{B \in \mathcal{V}} \mathcal{I}\Big(v(A \cap B), \sup_{\omega \in B} h(\omega)\Big) \qquad (1)$$

and

$$(\mathcal{J}) \fint_A h \mathrm{d}v = \sup_{B \in \mathcal{V}} \mathcal{J}\Big(v(A \cap \mathrm{co}B), \inf_{\omega \in B} h(\omega)\Big). \qquad (2)$$

Proof. We give the proof of (2). The proof of (1) is completely analogous. Consider an arbitrary B in $\mathcal{V}$, let $\lambda_B = \inf_{\omega \in B} h(\omega)$ and consider the $\mathcal{V}$-simple mapping s_B assuming the value λ_B on B and 0_L elsewhere. Clearly $s_B \sqsubseteq h$, whence $(\mathcal{J})\fint_A h\mathrm{d}v \geq \delta_{\mathcal{J}}^{v}(A;s_B) = \mathcal{J}\big(v(A \cap \mathrm{co}B), \inf_{\omega \in B} h(\omega)\big)$, and therefore

$$(\mathcal{J}) \fint_A h \mathrm{d}v \geq \sup_{B \in \mathcal{V}} \mathcal{J}\Big(v(A \cap \mathrm{co}B), \inf_{\omega \in B} h(\omega)\Big).$$

Conversely, consider a $\mathcal{V}$-simple mapping s with $s \sqsubseteq h$, then clearly, with obvious notations, $s_k \leq \inf_{\omega \in D_k} h(\omega)$, and therefore also $\mathcal{J}(v(A \cap \mathrm{co}D_k), s_k) \leq \mathcal{J}\big(v(A \cap \mathrm{co}D_k), \inf_{\omega \in D_k} h(\omega)\big)$, $k = 1, \ldots, n$. Therefore $\delta_{\mathcal{J}}^{v}(A;s) \leq \sup_{k=1}^{n} \mathcal{J}\big(v(A \cap \mathrm{co}D_k), \inf_{\omega \in D_k} h(\omega)\big) \leq \sup_{B \in \mathcal{V}} \mathcal{J}\big(v(A \cap \mathrm{co}B), \inf_{\omega \in B} h(\omega)\big)$, whence

$$(\mathcal{J}) \fint_A h \mathrm{d}v \leq \sup_{B \in \mathcal{V}} \mathcal{J}\Big(v(A \cap \mathrm{co}B), \inf_{\omega \in B} h(\omega)\Big).$$

This completes the proof. □

Theorem 6 *Let h be a $\Omega - L$-mapping and let A be an element of $\mathcal{V}$. If h is dually $\mathcal{V}$-cut-measurable, then*

$$(\mathcal{I}) \fint_A h \mathrm{d}v = \inf_{\lambda \in L} \mathcal{I}(v(A \cap D_\lambda^h), \lambda). \qquad (3)$$

If h is $\mathcal{V}$-cut-measurable, then

$$(\mathcal{J}) \fint_A h \mathrm{d}v = \sup_{\lambda \in L} \mathcal{J}(v(A \cap \mathrm{co}S_\lambda^h), \lambda). \qquad (4)$$

Proof. We give the proof of (3). The proof of (4) is analogous. Assume that h is dually $\mathcal{V}$-cut-measurable. Consider an arbitrary B in $\mathcal{V}$ and let $\lambda_B = \sup_{\omega \in B} h(\omega)$. Clearly $B \subseteq D_{\lambda_B}^h$, whence $\mathcal{I}(v(A \cap D_{\lambda_B}^h), \lambda_B) \leq \mathcal{I}(v(A \cap B), \sup_{\omega \in B} h(\omega))$. Therefore,

$$\inf_{\lambda \in L} \mathcal{I}(v(A \cap D_\lambda^h), \lambda) \leq \inf_{B \in \mathcal{V}} \mathcal{I}\Big(v(A \cap B), \sup_{\omega \in B} h(\omega)\Big).$$

Conversely, for λ in L, $\sup_{\omega \in D_\lambda^h} h(\omega) \leq \lambda$, whence $\mathcal{I}\big(v(A \cap D_\lambda^h), \sup_{\omega \in D_\lambda^h} h(\omega)\big) \leq \mathcal{I}(v(A \cap D_\lambda^h), \lambda)$. Therefore,

$$\inf_{\lambda \in L} \mathcal{I}(v(A \cap D_\lambda^h), \lambda) \geq \inf_{B \in \mathcal{V}} \mathcal{I}\Big(v(A \cap B), \sup_{\omega \in B} h(\omega)\Big).$$

This completes the proof. □

We continue with a number of interesting properties of implicator and coimplicator integrals.

Proposition 7 *Let A be an element of $\mathcal{V}$. Then*

$$(\mathcal{I}) \fint_\Omega \chi_A \mathrm{d}v = \mathcal{N}_{\mathcal{I}}(v(\mathrm{co}A))$$

and

$$(\mathcal{J}) \fint_\Omega \chi_A \mathrm{d}v = \mathcal{N}_{\mathcal{J}}(v(\mathrm{co}A)).$$

Proof. We give the proof of the first equality. The other equality is proven analogously. Since χ_A is dually $\mathcal{V}$-cut-measurable, we find that, using (3),

$$\begin{aligned}
&(\mathcal{I}) \fint_\Omega \chi_A \mathrm{d}v \\
&= \inf_{\lambda \in L} \mathcal{I}(v(D_\lambda^{\chi_A}), \lambda) \\
&= \mathcal{I}(v(\Omega), 1_L) \frown \inf_{\lambda < 1_L} \mathcal{I}(v(\mathrm{co}A), \lambda) \\
&= \mathcal{I}(v(\mathrm{co}A), 0_L) = \mathcal{N}_{\mathcal{I}}(v(\mathrm{co}A)).
\end{aligned}$$

This completes the proof. □

Proposition 8 *Let A be an element of $\mathcal{V}$ and let μ be an element of L. Then*

$$(\mathcal{I})\int_A \underline{\mu}\mathrm{d}v = \mathcal{N}_{\mathcal{I}}(v(\emptyset)) \frown \mathcal{I}(v(A),\mu)$$

and

$$(\mathcal{J})\int_A \underline{\mu}\mathrm{d}v = \mathcal{N}_{\mathcal{J}}(v(A)) \smile \mathcal{J}(v(\emptyset),\mu).$$

Proof. We give the proof of the second equality. The proof of the first equality is similar. Consider any λ in L. Since, for $\lambda \leq \mu$, $S_{\lambda}^{\underline{\mu}} = \Omega$ and for $\lambda \not\leq \mu$, $S_{\lambda}^{\underline{\mu}} = \emptyset$, $\underline{\mu}$ is $\mathcal{V}$-cut-measurable, and therefore, using (4),

$$\begin{aligned}&(\mathcal{J})\int_A \underline{\mu}\mathrm{d}v\\ &= \sup_{\lambda\in L} \mathcal{J}(v(A\cap \mathrm{co}S_{\lambda}^{\underline{\mu}}),\lambda)\\ &= \sup_{\lambda\leq\mu} \mathcal{J}(v(\emptyset),\lambda) \smile \sup_{\lambda\not\leq\mu} \mathcal{J}(v(A),\lambda)\\ &= \begin{cases} \mathcal{J}(v(\emptyset),1_L) & ;\quad \mu = 1_L\\ \mathcal{J}(v(\emptyset),\mu) \smile \mathcal{J}(v(A),1_L) & ;\quad \mu < 1_L\end{cases}\\ &= \mathcal{N}_{\mathcal{J}}(v(A)) \smile \mathcal{J}(v(\emptyset),\mu).\end{aligned}$$

This completes the proof. □

As a corollary of this, we have for any A in $\mathcal{V}$ that $(\mathcal{I})\int_A \underline{0_L}\mathrm{d}v = \mathcal{N}_{\mathcal{I}}(v(A))$ and $(\mathcal{I})\int_A \underline{1_L}\mathrm{d}v = \mathcal{N}_{\mathcal{I}}(v(\emptyset))$. Also, $(\mathcal{J})\int_A \underline{0_L}\mathrm{d}v = \mathcal{N}_{\mathcal{J}}(v(A))$ and $(\mathcal{J})\int_A \underline{1_L}\mathrm{d}v = \mathcal{N}_{\mathcal{J}}(v(\emptyset))$. Moreover, if μ is any element of L, we find that if $v(\emptyset) = 0_L$, then $(\mathcal{I})\int_A \underline{\mu}\mathrm{d}v = \mathcal{I}(v(A),\mu)$ and $(\mathcal{J})\int_A \underline{\mu}\mathrm{d}v = \mathcal{N}_{\mathcal{J}}(v(A)) \smile \mu$. If $v(A) = 1_L$, then $(\mathcal{I})\int_A \underline{\mu}\mathrm{d}v = \mathcal{N}_{\mathcal{I}}(v(\emptyset)) \frown \mu$ and $(\mathcal{J})\int_A \underline{\mu}\mathrm{d}v = \mathcal{J}(v(\emptyset),\mu)$. Finally, if both $v(\emptyset) = 0_L$ and $v(A) = 1_L$, then $(\mathcal{I})\int_A \underline{\mu}\mathrm{d}v = (\mathcal{J})\int_A \underline{\mu}\mathrm{d}v = \mu$.

The following proposition is an immediate consequence of Theorem 5. It shows that implicator and coimplicator integrals are increasing in their integrand but decreasing in their domain of integration, which is a somewhat undesirable property. Indeed, in [7] it was shown that the only integrals which have a selection of desired properties, among which isotonicity in the integration domain, are the seminormed and semiconormed fuzzy integrals. It is therefore not surprising that these new integrals have a number of less desirable properties. We shall nevertheless see further on that, like seminormed and semiconormed fuzzy integrals, they have a part to play in possibility theory.

Proposition 9 *Let A, A_1 and A_2 be elements of $\mathcal{V}$. Let h, h_1 and h_2 be $\Omega - L$-mappings. Assume that $A_1 \subseteq A_2$ and $h_1 \sqsubseteq h_2$. Then $(\mathcal{I})\int_A h_1\mathrm{d}v \leq (\mathcal{I})\int_A h_2\mathrm{d}v$ and $(\mathcal{J})\int_A h_1\mathrm{d}v \leq (\mathcal{J})\int_A h_2\mathrm{d}v$. Also, $(\mathcal{I})\int_{A_1} h\mathrm{d}v \geq (\mathcal{I})\int_{A_2} h\mathrm{d}v$ and $(\mathcal{J})\int_{A_1} h\mathrm{d}v \geq (\mathcal{J})\int_{A_2} h\mathrm{d}v$.*

To end this section, we give a number of propositions relating implicator and coimplicator integrals both to each other and to seminormed and semiconormed fuzzy integrals. For a detailed account of the latter, we refer to [7]. In the context of this paper, we merely point out that for the $(L,\leq)$-*fuzzy $\mathcal{T}$-integral* of a $\Omega - L$-mapping h on a set $A \in \mathcal{V}$:

$$(\mathcal{T})\int_A h\mathrm{d}v = \sup_{B\in\mathcal{V}} \mathcal{T}(\inf_{\omega\in B} h(\omega), v(A\cap B)),$$

and for the $(L,\leq)$-*fuzzy $\mathcal{S}$-integral* of h on A:

$$(\mathcal{S})\int_A h\mathrm{d}v = \inf_{B\in\mathcal{V}} \mathcal{S}(\sup_{\omega\in B} h(\omega), v(A\cap \mathrm{co}B)).$$

Proposition 10 *Let $\mathcal{N}$ be a strong negator on $(L,\leq)$ and let h be a $\Omega - L$-mapping. Then*

$$(D_{\mathcal{N}}[\mathcal{I}])\int_\Omega h\mathrm{d}v_{\mathcal{N}} = \mathcal{N}^{-1}\Big((\mathcal{I})\int_\Omega (\mathcal{N}\circ h)\mathrm{d}v\Big)$$

and

$$(D_{\mathcal{N}}[\mathcal{J}])\int_\Omega h\mathrm{d}v_{\mathcal{N}} = \mathcal{N}^{-1}\Big((\mathcal{J})\int_\Omega (\mathcal{N}\circ h)\mathrm{d}v\Big).$$

Proof. We prove the first equality. The proof of the second is similar. Since $D_{\mathcal{N}}[\mathcal{I}]$ is a border coimplicator on $(L,\leq)$, Theorem 5 tells us that

$$\begin{aligned}&(D_{\mathcal{N}}[\mathcal{I}])\int_\Omega h\mathrm{d}v_{\mathcal{N}}\\ &= \sup_{B\in\mathcal{V}} D_{\mathcal{N}}[\mathcal{I}]\Big(v_{\mathcal{N}}(\mathrm{co}B), \inf_{\omega\in B} h(\omega)\Big)\\ &= \sup_{B\in\mathcal{V}} \mathcal{N}^{-1}\Big(\mathcal{I}(\mathcal{N}(v_{\mathcal{N}}(\mathrm{co}B)), \mathcal{N}(\inf_{\omega\in B} h(\omega)))\Big)\\ &= \mathcal{N}^{-1}\Big(\inf_{B\in\mathcal{V}} \mathcal{I}\big(v(B), \sup_{\omega\in B}(\mathcal{N}\circ h)(\omega)\big)\Big),\end{aligned}$$

which completes the proof. □

The proof of the next proposition is similar and therefore omitted.

Proposition 11 *Let $\mathcal{N}$ be a strong negator on $(L,\leq)$ and let h be a $\Omega - L$-mapping. Then*

$$(\Lambda_{\mathcal{N}}[\mathcal{T}])\int_\Omega h\mathrm{d}v_{\mathcal{N}} = (\mathcal{T})\int_\Omega h\mathrm{d}v$$

and

$$(\Lambda_{\mathcal{N}}[\mathcal{S}])\int_\Omega h\mathrm{d}v_{\mathcal{N}} = (\mathcal{S})\int_\Omega h\mathrm{d}v.$$

Moreover, if $\mathcal{N}_{\mathcal{I}} = \mathcal{N}^{-1}$, then

$$(\Omega_{\mathcal{N}}[\mathcal{I}])\int_\Omega h\mathrm{d}v_{\mathcal{N}} = (\mathcal{I})\int_\Omega h\mathrm{d}v,$$

and if $\mathcal{N}_{\mathcal{J}} = \mathcal{N}^{-1}$, then

$$(\Omega_{\mathcal{N}}[\mathcal{J}])\int_\Omega h\mathrm{d}v_{\mathcal{N}} = (\mathcal{J})\int_\Omega h\mathrm{d}v,$$

4 POSSIBILITY AND NECESSITY OF FUZZY SETS

Let us now consider an *ample field* $\mathcal{R}$ [5, 12] of subsets of Ω, i.e., a class of subsets of Ω that is closed under arbitrary unions and under complementation. The *atom* $[\omega]_{\mathcal{R}}$ of $\mathcal{R}$ containing ω is the element of $\mathcal{R}$ defined by $[\omega]_{\mathcal{R}} = \bigcap\{A \mid A \in \mathcal{R} \text{ and } \omega \in A\}$. Note that for any $A \subseteq \Omega$, $A \in \mathcal{R} \Leftrightarrow A = \bigcup_{\omega\in A}[\omega]_{\mathcal{R}}$.

For any $\Omega - L$-mapping h, the following statements are readily shown to be equivalent: h is $\mathcal{R}$-cut-measurable; h is dually $\mathcal{R}$-cut-measurable; and h is constant on the atoms of $\mathcal{R}$. Whenever h is constant on the atoms of $\mathcal{R}$, we say that h is $\mathcal{R}$*-measurable.*

A mapping Π from $\mathcal{R}$ to L is called a $(L, \leq)$*-possibility measure* (or simply possibility measure) on $(\Omega, \mathcal{R})$ iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$. Note that $\Pi(\emptyset) = 0_L$. Π is called *normal* iff $\Pi(\Omega) = 1_L$. A $\Omega - L$-mapping π is called a *distribution* of Π iff it is constant on the atoms of $\mathcal{R}$ ($\mathcal{R}$-measurable) and if for any A in $\mathcal{R}$, $\Pi(A) = \sup_{\omega\in A} \pi(\omega)$. Such a distribution is unique and given by $\pi(\omega) = \Pi([\omega]_{\mathcal{R}})$, $\omega \in \Omega$.

A mapping N from $\mathcal{R}$ to L is called a $(L, \leq)$*-necessity measure* (or simply necessity measure) on $(\Omega, \mathcal{R})$ iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\mathrm{N}(\bigcap_{j\in J} A_j) = \inf_{j\in J} \mathrm{N}(A_j)$. Note that $\mathrm{N}(\Omega) = 1_L$. N is called *normal* iff $\mathrm{N}(\emptyset) = 0_L$. A $\Omega - L$-mapping ν is called a *distribution* of N iff it is constant on the atoms of $\mathcal{R}$ and if for any A in $\mathcal{R}$, $\mathrm{N}(A) = \inf_{\omega\in \mathrm{co}A} \nu(\omega)$. Again, such a distribution is unique. It is given by $\nu(\omega) = \mathrm{N}(\mathrm{co}[\omega]_{\mathcal{R}})$, $\omega \in \Omega$. For more information about possibility and necessity measures, we refer to [4, 8, 9, 13]

In this section, Π is a $(L, \leq)$-possibility measure on $(\Omega, \mathcal{R})$, with distribution π; and N is a $(L, \leq)$-necessity measure on $(\Omega, \mathcal{R})$ with distribution ν. Furthermore, for the border implicator $\mathcal{I}$, we assume that its first partial mappings are complete dual join-morphisms and that its second partial mappings are complete meet-morphisms. For the border coimplicator $\mathcal{J}$, it will dually be assumed that its first partial mappings are complete dual meet-morphisms and that its second partial mappings are complete join-morphisms. Note that this implies that $\mathcal{N}_{\mathcal{I}}$ is a complete dual join-morphism and $\mathcal{N}_{\mathcal{J}}$ a complete dual meet-morphism.

The following results tell us what happens if we associate an implicator integral with Π, and a coimplicator integral with N.

Theorem 12 *Let A be an element of $\mathcal{V}$ and let h be a $\Omega - L$-mapping. Then*

$$(\mathcal{I})\int_A h\mathrm{d}\Pi = \inf_{\omega\in A} \mathcal{I}\Big(\pi(\omega), \sup_{x\in[\omega]_{\mathcal{R}}} h(x)\Big)$$

and

$$(\mathcal{J})\int_A h\mathrm{dN} = \mathcal{N}_{\mathcal{J}}(\mathrm{N}(A)) \smile \sup_{\omega\in\Omega} \mathcal{J}\Big(\nu(\omega), \inf_{x\in[\omega]_{\mathcal{R}}} h(x)\Big).$$

Proof. We give the proof of the second equality. The proof of the first equality is analogous, but somewhat less complicated. Since the first partial mappings of $\mathcal{J}$ are complete dual meet-morphisms, we find, taking into account Theorem 5,

$$\begin{aligned}(\mathcal{J})&\int_A h\mathrm{dN}\\ &= \sup_{B\in\mathcal{R}} \mathcal{J}\Big(N(A\cap \mathrm{co}B), \inf_{x\in B} h(x)\Big)\\ &= \sup_{B\in\mathcal{R}} \mathcal{J}\Big(N(A) \frown \inf_{\omega\in B} \nu(\omega)), \inf_{x\in B} h(x)\Big)\\ &= \mathcal{N}_{\mathcal{J}}(\mathrm{N}(A)) \smile \sup_{B\in\mathcal{R}} \mathcal{J}\Big(\inf_{\omega\in B} \nu(\omega), \inf_{x\in B} h(x)\Big)\end{aligned}$$

Since moreover it is easily verified that for $\mu \in L$, $\mathcal{J}(\chi_{\mathrm{co}B}(\omega) \smile \nu(\omega), \mu) = \mathcal{J}(\nu(\omega), \chi_B(\omega) \frown \mu)$, and since the second partial mappings of $\mathcal{J}$ are complete join-morphisms, we find that

$$\begin{aligned}&\sup_{B\in\mathcal{R}} \mathcal{J}\Big(\inf_{\omega\in B} \nu(\omega), \inf_{x\in B} h(x)\Big)\\ &\quad= \sup_{\omega\in\Omega}\sup_{B\in\mathcal{R}} \mathcal{J}\Big(\chi_{\mathrm{co}B}(\omega) \smile \nu(\omega), \inf_{x\in B} h(x)\Big)\\ &\quad= \sup_{\omega\in\Omega} \mathcal{J}\Big(\nu(\omega), \sup_{B\in\mathcal{R}} \chi_B(\omega) \frown \inf_{x\in B} h(x)\Big)\\ &\quad= \sup_{\omega\in\Omega} \mathcal{J}\Big(\nu(\omega), \inf_{x\in[\omega]_{\mathcal{R}}} h(x)\Big),\end{aligned}$$

This completes the proof. □

The following propositions give interesting expressions for implicator and coimplicator integrals associated with possibility respectively necessity measures, when the integrand is $\mathcal{R}$-measurable. They immediately follow from the theorem above.

Proposition 13 *Let A be an element of $\mathcal{V}$ and let h be an $\mathcal{R}$-measurable $\Omega - L$-mapping. Then*

$$(\mathcal{I})\int_A h\mathrm{d}\Pi = \inf_{\omega\in A} \mathcal{I}(\pi(\omega), h(\omega))$$

and

$$(\mathcal{J})\int_A h\mathrm{dN} = \mathcal{N}_{\mathcal{J}}(\mathrm{N}(A)) \smile \sup_{\omega\in\Omega} \mathcal{J}(\nu(\omega), h(\omega)).$$

Proposition 14 *Let h be an $\mathcal{R}$-measurable $\Omega - L$-mapping. Then*

$$(\mathcal{I})\int_\Omega h\mathrm{d}\Pi = \inf_{\omega\in\Omega} \mathcal{I}(\pi(\omega), h(\omega))$$

and

$$(\mathcal{J})\int_{\Omega} h\mathrm{dN} = \sup_{\omega\in\Omega} \mathcal{J}(\nu(\omega), h(\omega)).$$

Moreover, for any A in $\mathcal{R}$,

$$(\mathcal{J})\int_{A} h\mathrm{dN} = \mathcal{N}_{\mathcal{J}}(\mathrm{N}(A)) \smile (\mathcal{J})\int_{\Omega} h\mathrm{dN}.$$

As a corollary of this proposition, we have for any $\mathcal{R}$-simple $\Omega - L$-mapping s that $(\mathcal{I})\int_A s\mathrm{d}\Pi = \gamma_{\mathcal{I}}^{\Pi}(A; s)$ and $(\mathcal{J})\int_A s\mathrm{dN} = \mathcal{N}_{\mathcal{J}}(\mathrm{N}(A)) \smile \delta_{\mathcal{J}}^{\mathrm{N}}(\Omega; s)$.

Proposition 15 *Let A be an element of $\mathcal{R}$ and let $(h_j \mid j \in J)$ be a family of $\mathcal{R}$-measurable $\Omega - L$-mappings. Then $\inf_{j\in J} h_j$ and $\sup_{j\in J} h_j$ are $\Omega - L$-measurable. Moreover,*

$$(\mathcal{I})\int_A \inf_{j\in J} h_j\mathrm{d}\Pi = \inf_{j\in J} (\mathcal{I})\int_A h_j\mathrm{d}\Pi$$

and

$$(\mathcal{J})\int_A \sup_{j\in J} h_j\mathrm{dN} = \sup_{j\in J} (\mathcal{J})\int_A h_j\mathrm{dN}.$$

Proposition 16 *Let $(A_j \mid j \in J)$ be a family of elements of $\mathcal{R}$ and let h be a $\mathcal{R}$-measurable $\Omega - L$-mapping. Then*

$$(\mathcal{I})\int_{\bigcup_{j\in J} A_j} h\mathrm{d}\Pi = \inf_{j\in J} (\mathcal{I})\int_{A_j} h_j\mathrm{d}\Pi$$

and

$$(\mathcal{J})\int_{\bigcap_{j\in J} A_j} h\mathrm{dN} = \sup_{j\in J} (\mathcal{J})\int_{A_j} h\mathrm{dN}.$$

In [7] it was shown that seminormed and semiconormed fuzzy integrals can be used to extend the domain of possibility respectively necessity measures from measurable sets to measurable fuzzy sets. Propositions 7 and 15 tell us that something similar can be done with implicator and coimplicator integrals. For any $\mathcal{R}$-measurable $\Omega - L$ mapping h, define $\Pi_{\mathcal{I}}(h) = (\mathcal{I})\int_\Omega h\mathrm{d}\Pi$ and $\mathrm{N}_{\mathcal{J}}(h) = (\mathcal{J})\int_\Omega h\mathrm{dN}$. Then clearly $\Pi_{\mathcal{I}}$ is infimum preserving and $\mathrm{N}_{\mathcal{J}}$ supremum preserving. Moreover, for any A in $\mathcal{R}$, $\Pi_{\mathcal{I}}(\chi_A) = \mathcal{N}_{\mathcal{I}}(\Pi(\mathrm{co}A))$ and $\mathrm{N}_{\mathcal{J}}(\chi_A) = \mathcal{N}_{\mathcal{J}}(\mathrm{N}(\mathrm{co}A))$. An implicator integral therefore allows us to turn a possibility measure Π into an extended necessity measure $\Pi_{\mathcal{I}}$, and dually, a coimplicator integral allows us to turn a necessity measure N into a possibility measure $\mathrm{N}_{\mathcal{J}}$. If $\mathcal{N}_{\mathcal{I}}$ and $\mathcal{N}_{\mathcal{J}}$ are strong negators, then $\Pi_{\mathcal{I}}$ is the extension to fuzzy sets of the dual necessity measure $\Pi_{\mathcal{N}_{\mathcal{I}}^{-1}}$ of Π w.r.t. $\mathcal{N}_{\mathcal{I}}^{-1}$ and dually, $\mathrm{N}_{\mathcal{J}}$ is the extension to fuzzy sets of the dual possibility measure $\mathrm{N}_{\mathcal{N}_{\mathcal{J}}^{-1}}$ of N w.r.t. $\mathcal{N}_{\mathcal{J}}^{-1}$.

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