A POSSIBILISTIC DANIELL-KOLMOGOROV THEOREM

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Abstract. We define a possibilistic process as a special family of possibilistic variables, and show how its possibility distribution functions can be constructed. We introduce and study the notions of inner and outer regularity for possibility measures. Using these notions, we prove an analogon for possibilistic processes (and possibility measures) of the well-known probabilistic Daniell-Kolmogorov theorem, in the important special case that the variables assume values in compact spaces, and that the possibility measures involved are regular.
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1 Introduction

A process can be informally defined as a variable which changes in time. A process is *uncertain* if there is not enough information available in order to specify its value for all times.

A very general formal model for an uncertain process consists of a non-empty set Ω, called *basic space*, a non-empty set X, called *sample space*, and a family of $\Omega - X$-mappings $(f_t \mid t \in T)$, indexed by a non-empty time set T. Information about the process is represented by an imprecise probability model on the basic space [12].

The paper provides a closer look at uncertain processes of a special type, namely for which the given imprecise probability model consists of a possibility measure. For such *possibilistic processes*, we show that it is possible to prove a counterpart to a fundamental result in the theory of *stochastic* processes, namely the Daniell-Kolmogorov theorem.

2 Preliminary results and definitions

In this section, we have collected the preliminary material, requisite for understanding the formal developments in the rest of this paper.

2.1 Plump fields, ample fields and measurability

Throughout we shall denote by X a non-empty set. A non-empty subset $\mathcal{D}$ of the power set $\wp(X)$ of X is called a *plump field* on X iff it is closed under arbitrary unions and intersections [14]. The *atom* $[x]_\mathcal{D}$ of $\mathcal{D}$ containing the element x of X is defined as $[x]_\mathcal{D} = \bigcap\{A \mid A \in \mathcal{D} \text{ and } x \in A\}$. The set of the atoms of $\mathcal{D}$ is denoted by $X_\mathcal{D}$. Note that $X_\mathcal{D} \subseteq \mathcal{D}$ and that $(\forall x \in X)(x \in [x]_\mathcal{D})$. Also, for any subset A of X, $A \in \mathcal{D} \Leftrightarrow A = \bigcup_{x\in A}[x]_\mathcal{D}$.

A plump field is in particular a topology. Let $\mathcal{S}$ be a non-empty collection of subsets of X, then $\mathfrak{T}_\mathcal{S}$ denotes the topology on X with subbase $\mathcal{S}$, and $\mathcal{D}_\mathcal{S}$ denotes the *plump field generated by* $\mathcal{S}$ on X, i.e., the smallest plump field on X including $\mathcal{S}$.

An *ample field* $\mathcal{R}$ on X is a plump field on X that is closed under complementation [7, 14]. The couple $(X, \mathcal{R})$ is called *ample space*. The set of the atoms $X_\mathcal{R}$ constitutes a partition of X. For a subset $\mathcal{S}$ of the power set $\wp(X)$, $\tau_X(\mathcal{S})$ denotes the smallest ample field on X which includes $\mathcal{S}$ and is called *the ample field generated by* $\mathcal{S}$ (on X).

An ample field is a special topology, for which all open sets are closed, and vice versa. An element of this ample field is compact iff it is a finite union of atoms.

We continue with a number of measurability definitions. If $\mathcal{S} \subseteq \wp(X)$, and $A \subseteq X$, then we call A *$\mathcal{S}$-measurable* iff $A \in \mathcal{S}$. If $\mathcal{S}_1 \subseteq \wp(X_1)$ and $\mathcal{S}_2 \subseteq \wp(X_2)$, where X_1 and X_2 are non-empty sets, then a $X_1 - X_2$-mapping f is called $\mathcal{S}_1 - \mathcal{S}_2$-*measurable* iff $(\forall B \in \mathcal{S}_2)(f^{-1}(B) \in \mathcal{S}_1)$.

2.2 Products of sets, ample fields and mappings

In this subsection, we have collected a number of basic definitions and results concerning products of sets, mappings and ample fields. A more detailed discussion of this topic can be found in [10].

Let $(X_t \mid t \in T)$ be a non-empty family of non-empty sets. The *Cartesian product* $\times_{t\in T} X_t$ of $(X_t \mid t \in T)$ is the set of all $T - \bigcup_{t\in T} X_t$-mappings x such that $x(t) \in X_t$, $t \in T$. For simplicity of notation, we shall denote this Cartesian product by X_T. If all the sets X_t, $t \in T$, are equal to a given set X, the Cartesian product is the set of all the $T - X$-mappings, which is also given the standard notation X^T.

We need the following projection operators. For any $t \in T$, $\mathbf{pr}_{T,t}$ is the *t-th projection mapping* from X_T onto X_t, defined by $\mathbf{pr}_{T,t}(x) = x(t)$, $x \in X_T$. If S is a non-empty subset of T, then $\mathbf{pr}_{T,S}$ is the $X_T - X_S$-mapping such that for any $x \in X_T$, $\mathbf{pr}_{T,S}(x) = x|_S$ is the restriction of the mapping x to S.

This brings us to the notion of product mapping. Let f_t be a mapping from a set X to X_t, $t \in T$. The *product mapping* $\times_{t\in T} f_t$ of the family of mappings $(f_t \mid t \in T)$ is the unique $X - X_T$-mapping f such that $f_t = \mathbf{pr}_{T,t} \circ f$, $t \in T$. For notational simplicity, it will also be denoted by f_T.

Product mappings allow us to define product ample fields. Let $\mathcal{R}_t$ be an ample field on X_t, $t \in T$. The *product* $\prod_{t\in T} \mathcal{R}_t$ of the family $(\mathcal{R}_t \mid t \in T)$ of ample fields is the smallest ample field $\mathcal{H}$ on X_T such that $\mathbf{pr}_{T,t}$ is a $\mathcal{H} - \mathcal{R}_t$-measurable mapping, $t \in T$. For notational simplicity, it is also denoted by $\mathcal{R}_T$. $(X_T, \mathcal{R}_T)$ is called the *product ample space* of the family $((X_t, \mathcal{R}_t) \mid t \in T)$ of ample spaces.

Proposition 1 *Let S be a non-empty subset of T, and let $x \in X_S$. Then the following statements hold:*

1. $\mathbf{pr}_{T,S} = \times_{t\in S} \mathbf{pr}_{T,t}$;
2. $\mathbf{pr}_{T,S}$ *is a $\mathcal{R}_T - \mathcal{R}_S$-measurable mapping;*
3. $[x]_{\mathcal{R}_S} = \bigcap_{t\in S} \mathbf{pr}_{S,t}^{-1}([x(t)]_{\mathcal{R}_t}) = \times_{t\in S}[x(t)]_{\mathcal{R}_t}$.

Moreover, let $(X, \mathcal{R})$ be an ample space and consider the mappings $f_t \colon X \to X_t$, $t \in S$. Then the product mapping f_S is $\mathcal{R} - \mathcal{R}_S$-measurable iff f_t is $\mathcal{R} - \mathcal{R}_t$-measurable, $t \in S$.

We also introduce measurable cylinders. Consider a *finite* subset S of T – we denote this as $S \Subset T$ – which is furthermore non-empty. We define $\mathcal{C}_{T,S} = \{\mathbf{pr}_{T,S}^{-1}(E) \mid E \in \mathcal{R}_S\}$ and $\mathcal{C}_T = \bigcup_{\emptyset \neq S \Subset T} \mathcal{C}_{T,S}$. Note that $C_T \subseteq \mathcal{R}_T$. An element of $\mathcal{C}_T$ is called a *measurable cylinder* of $(X_T, \mathcal{R}_T)$. It will also be useful to consider special subsets of the measurable cylinders: $\mathcal{A}_{T,S} = \{\mathbf{pr}_{T,S}^{-1}(A) \mid A \in (X_S)_{\mathcal{R}_S}\}$ and $\mathcal{A}_T = \bigcup_{\emptyset \neq S \Subset T} \mathcal{A}_{T,S}$. Any element of $\mathcal{A}_T$ is called an *atomic* measurable cylinder of $(X_T, \mathcal{R}_T)$.

Proposition 2 *Let S, S_1 and S_2 be non-empty, finite subsets of T. Then the following statements hold:*

1. *$\mathcal{C}_{T,S}$ is an ample field on X_T, with set of atoms $\mathcal{A}_{T,S}$;*
2. *if $S_1 \subseteq S_2$, then $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$.*

Moreover, $\mathcal{C}_T$ is a field on X_T.

We may consider the *smallest* topology on X_T for which the mappings $\mathbf{pr}_{T,t}$, $t \in T$ are continuous, given the topologies $\mathcal{R}_t$ on X_t, $t \in T$. By definition this is the *product topology* of the topologies $\mathcal{R}_t$ on X_t, $t \in T$, and is denoted by $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$. Note that $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) \subseteq \mathcal{R}_T$.

Proposition 3 *$\mathcal{C}_T$ is a base for the product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T. Moreover, $\tau_{X_T}(\mathcal{C}_T) = \tau_{X_T}(\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)) = \mathcal{R}_T$.*

In the following proposition, it is made clear under what conditions the field of measurable cylinders $\mathcal{C}_T$ is also an ample field on X_T.

Proposition 4 *The following statements are equivalent:*

1. *$\mathcal{C}_T$ is an ample field on X_T;*
2. *$\mathcal{C}_T = \mathcal{R}_T$;*
3. *there exists a non-empty, finite subset S of T such that $\mathcal{C}_T = \mathcal{C}_{T,S}$;*
4. *$\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ is an ample field on X_T;*
5. *$\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) = \mathcal{R}_T$.*

Moreover, if T is a finite set, then $\mathcal{C}_T$ is an ample field on X_T.

2.3 *Product lattices*

Throughout, $(L, \leq)$ denotes a complete lattice with top 1_L and bottom 0_L. Furthermore, if $\big((L_j, \leq_j) \mid j \in J\big)$ is a non-empty family of lattices, then we can equip the Cartesian product $\times_{j\in J} L_j$ with the usual product order $\times_{j\in J} \leq_j$, i.e., if λ and μ are elements of $\times_{j\in J} L_j$, then $\lambda \times_{j\in J} \leq_j \mu$ iff $\lambda(j) \leq_j \mu(j)$, $j \in J$. The lattice $(\times_{j\in J} L_j, \times_{j\in J} \leq_j)$ is called the *product* of the lattices $\big((L_j, \leq_j) \mid j \in J\big)$ [1]. If $(L_j, \leq_j)$ is a complete lattice, $j \in J$, then it is well known that $(\times_{j\in J} L_j, \times_{j\in J} \leq_j)$ is also a complete lattice. Let $\mathbf{pr}_k$ denote the k-th projection mapping from $\times_{j\in J} L_j$ onto L_k, $k \in J$. Then, for any subset A of $\times_{j\in J} L_j$ and $k \in J$, it follows that $\mathbf{pr}_k(\sup A) = \sup \mathbf{pr}_k(A)$ and $\mathbf{pr}_k(\inf A) = \inf \mathbf{pr}_k(A)$. In the special case that all the lattices $(L_j, \leq_j)$ coincide with the complete chain $([0,1], \leq)$, we denote the product complete lattice $(\times_{j\in J} L_j, \times_{j\in J} \leq_j)$ by $([0,1]^J, \leq^J)$.

2.4 *Possibility measures*

In the context of this paper, we need a definition of a possibility measure that slightly generalises Zadeh's original notion [15]. A set function Π defined on an ample field $\mathcal{R}$ and taking values in a complete lattice $(L, \leq)$ is called a $(L, \leq)$-*possibility measure* on $(X, \mathcal{R})$ iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$ [7]. The triple $(X, \mathcal{R}, \Pi)$ is then called a $(L, \leq)$-*possibility space*. A *distribution* for Π is an $X - L$-mapping π that is $\mathcal{R} - \wp(L)$-measurable, i.e., constant on the atoms of $\mathcal{R}$, and satisfies $\Pi(A) = \sup_{x\in A} \pi(x)$, $A \in \mathcal{R}$. Clearly, such a distribution is unique and completely determined by $\pi(x) = \Pi([x]_{\mathcal{R}})$, $x \in X$.

3 Formal definition of a possibilistic process

As a first step towards the formal introduction of a possibilistic process, we introduce the notion of a possibilistic variable. Informally, this is a variable for which the available information about the values it may assume, takes the form of a possibility measure. In a formal approach, we consider a *basic space* Ω, provided with an ample field $\mathcal{R}_\Omega$, and a *sample space* X, provided with an ample field $\mathcal{R}$. The available information is represented by a $(L, \leq)$-possibility measure Π_Ω on the basic space $(\Omega, \mathcal{R}_\Omega)$. A $\Omega - X$-mapping f that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, is called a *possibilistic variable* in $(X, \mathcal{R})$. The $(L, \leq)$-possibility measure Π_f on $(X, \mathcal{R})$, defined by $\Pi_f(B) = \Pi_\Omega(f^{-1}(B))$, $B \in \mathcal{R}$, represents the available information about the values that the variable f may assume in X. Π_f is completely characterised by its distribution π_f, given by

$$\pi_f(x) = \Pi_\Omega(f^{-1}([x]_\mathcal{R})) = \sup_{\omega \in f^{-1}([x]_\mathcal{R})} \pi_\Omega(\omega),$$

which is called the *possibility distribution function* of f. More information about possibilistic variables and their possibility distribution functions can be found in [3, 4, 5, 6]. Unless stated to the contrary, it will be implicitly assumed that $(\Omega, \mathcal{R}_\Omega)$ is the basic space for any possibilistic variable we consider in this paper. Knowledge of Π_Ω then enables us to determine the possibility distribution functions of any possibilistic variable considered.

A possibilistic process will now be formally defined as a family of possibilistic variables having the same sample space.

Definition 5 *Let T be a non-empty set and $(X, \mathcal{R})$ an ample space. A family $(f_t \mid t \in T)$ such that f_t is a possibilistic variable in $(X, \mathcal{R})$, $t \in T$, is called a possibilistic process in $(X, \mathcal{R})$ with index set T. A possibilistic process is called discrete iff its index set is countable, and continuous iff its index set is a non-degenerate real interval.*

T is also called time set. Now, consider an arbitrary family of possibilistic variables $(f_t \mid t \in T)$, where f_t is a possibilistic variable in the sample space $(X_t, \mathcal{R}_t)$, $t \in T$. If S is any subset of T, we may consider the product mapping f_S of the family $(f_t \mid t \in S)$. It follows from Proposition 1 that f_S is a possibilistic variable in $(X_S, \mathcal{R}_S)$, and we may therefore consider its possibility distribution function $\pi_{f_S} : X_S \to L$, given for any x in X_S by

$$\pi_{f_S}(x) = \sup_{(\forall t \in S)(f_t(\omega) \in [x(t)]_{\mathcal{R}_t})} \pi_\Omega(\omega).$$

π_{f_S} is also called the *joint* possibility distribution function of the possibilistic variables f_t, $t \in S$, and completely characterises the values that these variables may assume jointly, that is as a product, in the set X_S.

So, when a basic space $(\Omega, \mathcal{R}_\Omega)$ and the possibilistic information Π_Ω are present, we are able to calculate the joint possibility distribution function of any family of possibilistic variables. In particular, we are able to calculate the joint possibility distribution functions of the values of a possibilistic process at any collection of times.

In this paper, we investigate an important and practical special case of the general converse problem: given a collection of mappings $\pi_S : X_S \to L$, where S belongs to a collection Λ of subsets of T, does there exist a basic space with possibility measure, and a possibilistic process with this basic space, which has the given functions as the corresponding joint possibility distribution functions? The special case we consider here is when Λ is the collection of all finite subsets of the time set T. We call the corresponding problem the *possibilistic Daniell-Kolmogorov problem*, as it is the possibilistic counterpart of a problem solved in probability theory by Daniell and Kolmogorov [2].

Intuitively, it is easily understood from the foregoing discussion that specifying a collection of mappings $\{\pi_S \mid S \in \Lambda\}$ and interpreting them as joint possibility distribution functions amounts to specifying the values of a set function on a collection of subsets of a basic space. A crucial question will therefore be whether this set function can be extended to a possibility measure on the basic space. In the next section, we present a brief overview of the existing general results about extending set functions to possibility measures. In Section 5, we devote some attention to the notion of regularity for possibility measures. We use these results in Section 6 to solve the possibilistic Daniell-Kolmogorov problem in a number of important special cases, namely when the sets X_t are compact, $t \in T$, and the collection of mappings $\{\pi_S \mid S \in \Lambda\}$ leads to regular possibility measures.

4 P-consistency of set mappings

Consider a non-empty collection $\mathcal{S}$ of subsets of the non-empty set X, and a mapping μ defined on $\mathcal{S}$ and taking values in a complete lattice $(L, \leq)$. We can ask ourselves if the set function μ is *extendable to a $(L, \leq)$-possibility measure*, that is, if there exists an ample field $\mathcal{R}$ on X and a $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$, such that Π coincides with μ on $\mathcal{S}$: $\Pi(A) = \mu(A)$, $A \in \mathcal{S}$. It is clear that we must at least have that $\mathcal{S} \subseteq \mathcal{R}$.

This so-called *possibilistic extension problem* was solved by Wang [13, 14] for $[0, 1]$-valued set functions. Quite recently, Boyen et al. [11] considered and partially solved the more general problem for $(L, \leq)$-valued set functions.

Boyen et al. have generalised Wang's definition of *P-consistency* for set mappings as follows.

Definition 6 *Let X be a non-empty set and let $\mathcal{S}$ be a non-empty collection of subsets of X. A $\mathcal{S} - L$-mapping μ is called P-consistent iff for any family*

$(A_j \mid j \in J)$ *of elements of* $\mathcal{S}$ *and any element* A *of* $\mathcal{S}$*:*

$$A \subseteq \bigcup_{j \in J} A_j \Rightarrow \mu(A) \leq \sup_{j \in J} \mu(A_j).$$

They have shown that P-consistency is a necessary condition for extendability to a possibility measure. Moreover, they have proven the following result, which tells us that it is also a sufficient condition in a number of special cases.

Theorem 7 *Let* X *be a non-empty set and let* $\mathcal{S}$ *be a non-empty collection of subsets of* X*. Then, for a P-consistent* $\mathcal{S} - L$*-mapping* μ*, any of the following conditions is sufficient for the extendability of* μ *to a* $(L, \leq)$*-possibility measure.*

(E_1) $(L, \leq)$ *is a complete chain.*

(E_2) $\mathcal{S}$ *is a plump field.*

(E_3) $(L, \leq) = (\mathcal{B}, \supseteq)$*, where* $\mathcal{B}$ *is a plump field on some set* Y*.*

In the following sections, we shall often associate a special possibility measure Π^g_μ with a $\mathcal{S} - L$-mapping μ. Π^g_μ is the possibility measure on $(X, \wp(X))$ with distribution π^g_μ, defined by

$$\pi^g_\mu(x) = \inf_{A \in \mathcal{S}, x \in A} \mu(A), \quad x \in X.$$

It is straightforward to show that Π^g_μ is the greatest possibility measure that is dominated on $\mathcal{S}$ by μ. Obviously μ is extendable to a $(L, \leq)$-possibility measure iff Π^g_μ is the greatest possibility measure that coincides with μ on $\mathcal{S}$. In case that $\mathcal{S}$ is an ample field on X and μ is a $(L, \leq)$-possibility measure on $(X, \mathcal{S})$ with distribution π, then $\pi^g_\mu = \pi$. Note also that these remarks remain valid if we substitute for $\wp(X)$ any ample field $\mathcal{R}$ on X satisfying $\tau_X(\mathcal{S}) \subseteq \mathcal{R} \subseteq \wp(X)$.

We may add a fourth case in which P-consistency is sufficient for extendability to a possibility measure. Indeed, assume that $(L, \leq)$ is the product complete lattice $(\times_{j \in J} L_j, \times_{j \in J} \leq_j)$ of the non-empty family of complete lattices $\big((L_j, \leq_j) \mid j \in J\big)$. The following results are immediate and their proof is therefore omitted.

Proposition 8 *Let* X *be a non-empty set, let* $\mathcal{S}$ *be a non-empty collection of subsets of* X*, and let* μ *be a* $\mathcal{S} - \times_{j \in J} L_j$*-mapping. Then* μ *is P-consistent iff* $\mathbf{pr}_j \circ \mu$ *is P-consistent for any* $j \in J$*.* μ *is extendable to a* $(\times_{j \in J} L_j, \times_{j \in J} \leq_j)$*-possibility measure iff* $\mathbf{pr}_j \circ \mu$ *is extendable to a* $(L_j, \leq_j)$*-possibility measure,* $j \in J$*.*

Therefore, using Theorem 7, we obtain that a P-consistent $\mathcal{S} - L$-mapping μ is extendable to a $(L, \leq)$-possibility measure if

(E_4) $(L, \leq) = (\times_{j \in J} L_j, \times_{j \in J} \leq_j)$*, where for all* $j \in J$*,* $(L_j, \leq_j)$ *satisfies* (E_1) *or* (E_3)*.*

So, for $([0,1], \leq)$-possibility measures, condition (E_1) is satisfied, and this means that P-consistency is a necessary and sufficient condition for possibilistic extendability. In general, for set functions valued on a complete lattice, P-consistency is only necessary and not sufficient for extendability to a possibility measure. However, Boyen et al. also proved the following result, which provides an interesting way to circumvent this problem.

Theorem 9 *Let* X *be a non-empty set and let* $\mathcal{S}$ *be a non-empty collection of subsets of* X*. The complete lattice* $(L, \leq)$ *can always be embedded using a supremum preserving mapping* ϕ *in a second complete lattice* $(L', \leq')$*, in such a way that for any P-consistent* $\mathcal{S} - L$*-mapping* μ*,* $\phi \circ \Pi$ *is a P-consistent* $\mathcal{S} - L'$*-mapping, which is furthermore extendable to a* $(L', \leq')$*-possibility measure.*

5 Regularity

In this section, we introduce and study inner and outer regularity of possibility measures. Unless explicitly stated otherwise, X will be a non-empty set, $\mathfrak{T}$ is a topology on X and $\mathcal{R}$ is an ample field on X.

Let us first extend the well-known notions of inner and outer regularity [9] towards set mappings which have a complete lattice $(L, \leq)$ as their codomain.

Definition 10 *Let* $\mathcal{S}$ *be a non-empty collection of subsets of* X *and let* μ *be a* $\mathcal{S} - L$*-mapping. Let* A *be an element of* $\mathcal{S}$*.*

1. μ *is called inner regular w.r.t.* $\mathfrak{T}$ *in* A *iff* $\mu(A) = \sup\{\mu(C) \mid A \supseteq C \in \mathcal{S}$ *and* C *is compact in* $(X, \mathfrak{T})\}$*, and inner regular w.r.t.* $\mathfrak{T}$ *iff this equality holds for any* A *in* $\mathcal{S}$*.*

2. μ *is called outer regular w.r.t.* $\mathfrak{T}$ *in* A *iff* $\mu(A) = \inf\{\mu(O) \mid A \subseteq O \in \mathcal{S}$ *and* O *is open in* $(X, \mathfrak{T})\}$*, and outer regular w.r.t.* $\mathfrak{T}$ *iff this equality holds for any* A *in* $\mathcal{S}$*.*

3. μ *is called regular w.r.t.* $\mathfrak{T}$ *in* A *iff* μ *is both inner and outer regular w.r.t.* $\mathfrak{T}$ *in* A*.*

4. μ *is called regular w.r.t.* $\mathfrak{T}$ *iff* μ *is both inner and outer regular w.r.t.* $\mathfrak{T}$*.*

If the codomain $(L, \leq)$ of the set function μ is a product complete lattice, then it is obvious that the inner and outer regularity of μ depend on the inner and outer regularity of the components of μ.

Proposition 11 *Let* $(L, \leq)$ *be the product complete lattice of the non-empty family of complete lattices* $\big((L_j, \leq_j) \mid j \in J\big)$*. Let* $\mathcal{S}$ *be a non-empty collection of subsets of* X *and let* μ *be a* $\mathcal{S} - L$*-mapping. Consider an arbitrary* A *in* $\mathcal{S}$*.*

1. *μ is inner regular w.r.t. $\mathfrak{T}$ in A iff $\mathbf{pr}_j \circ \mu$ is inner regular w.r.t. $\mathfrak{T}$ in A for any $j \in J$.*

2. *μ is outer regular w.r.t. $\mathfrak{T}$ in A iff $\mathbf{pr}_j \circ \mu$ is outer regular w.r.t. $\mathfrak{T}$ in A for any $j \in J$.*

3. *μ is regular w.r.t. $\mathfrak{T}$ in A iff $\mathbf{pr}_j \circ \mu$ is regular w.r.t. $\mathfrak{T}$ in A for any $j \in J$.*

The following proposition gives an immediate, but rather weak sufficient condition for regularity.

Proposition 12 *A $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ is (inner and outer) regular w.r.t. $\mathcal{R}$.*

Possibility measures which are inner regular with respect to a given topology can be easily characterised.

Proposition 13 *Let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π. Let $\mathfrak{T}$ be a topology on X. Then Π is inner regular w.r.t. $\mathfrak{T}$ iff*

$$(\forall x \in X)(\pi(x) > 0_L \Rightarrow [x]_{\mathcal{R}} \text{ is compact in } (X, \mathfrak{T})).$$

Corollary 14 *Let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$. Let $\mathfrak{T}$ be a topology on X. If $\mathcal{R} = \wp(X)$ or more generally $\mathfrak{T} \subseteq \mathcal{R}$, then Π is inner regular w.r.t. $\mathfrak{T}$. In particular, any possibilistic extension of Π to $\wp(X)$ is always inner regular w.r.t. $\mathfrak{T}$.*

The last statement in this corollary tells us that inner regularity is a very natural property for possibility measures: any possibility measure can be made inner regular by extending it to a possibility measure on $\wp(X)$, which is always possible.

On the other hand, possibility measures are generally not outer regular, even if we make their domains as large as possible, as the following counterexample tells us.

Example 15 *Consider $x_0 \in \mathbb{R}$ and let Π be the $([0,1], \leq)$-possibility measure on $(\mathbb{R}, \wp(\mathbb{R}))$ with distribution π, such that $\pi(x) = 1$ for all $x \in \mathbb{R} \setminus \{x_0\}$ and $\pi(x_0) = 0$. Then Π is not outer regular w.r.t. the Euclidean topology on $\mathbb{R}$ in $\{x_0\}$.*

However, we now show that a special class of possibility measures is always outer regular (w.r.t. a specific topology) in the atoms of the ample fields on which they are defined.

Proposition 16 *Let $\mathcal{S}$ be a non-empty collection of subsets of X, and let μ be a $\mathcal{S}-L$-mapping. Then Π^g_μ is outer regular w.r.t. $\mathfrak{T}_\mathcal{S}$ in any $E \in \wp(X)$ for which there exists an $x \in X$ such that $x \in E \subseteq [x]_{\mathcal{D}_\mathcal{S}}$. In particular, the restriction $\Pi^g_\mu|_{\tau_X(\mathcal{S})}$ of Π^g_μ to $\tau_X(\mathcal{S})$ is outer regular w.r.t. $\mathfrak{T}_\mathcal{S}$ in any atom of $\tau_X(\mathcal{S})$ and in any atom of $\mathcal{D}_\mathcal{S}$.*

Proposition 17 *Let $\mathcal{S}$ be a non-empty collection of subsets of X, let μ be a $\mathcal{S}-L$-mapping, and let $\mathfrak{T}$ be a topology on X. Then Π^g_μ is inner regular w.r.t. $\mathfrak{T}$. Assume furthermore that μ is extendable to a $(L, \leq)$-possibility measure, i.e., that Π^g_μ coincides with μ on $\mathcal{S}$. Let $A \in \mathcal{S}$. If μ is outer regular w.r.t. $\mathfrak{T}$ in A then so is Π^g_μ.*

In the rest of this section, we investigate outer regularity in the important special case of possibility measures which assume values in the unit interval. In particular, we explore the link between the outer regularity of a $([0,1], \leq)$-possibility measure and the continuity of its distribution.

Let $\mathfrak{T}_{[0,1]} = \mathfrak{T}_\mathbb{R} \cap [0,1]$ be the relativisation of the Euclidean topology $\mathfrak{T}_\mathbb{R}$ on $\mathbb{R}$ to $[0,1]$. Then the following proposition tells us that continuity of the distribution is a sufficient condition for the outer regularity of the corresponding possibility measure in the atoms of the ample field on which it is defined.

Proposition 18 *Let Π be a $([0,1], \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π, and let $x \in X$. Assume that π is continuous in x w.r.t. the topologies $\mathfrak{T}$ on X and $\mathfrak{T}_{[0,1]}$ on $[0,1]$. If $\mathfrak{T} \subseteq \mathcal{R}$, then Π is outer regular w.r.t. $\mathfrak{T}$ in $[x]_\mathcal{R}$.*

Let $\mathfrak{T}^n_{[0,1]} = \mathfrak{T}^n_\mathbb{R} \cap [0,1]^n$ be the relativisation of the Euclidean topology $\mathfrak{T}^n_\mathbb{R}$ on $\mathbb{R}^n$ to $[0,1]^n$, where $n \in \mathbb{N} \setminus \{0\}$. By Proposition 11, we obtain the following result.

Corollary 19 *Let Π be a $([0,1]^n, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π. Let $x \in X$. Assume π is continuous in x w.r.t. the topologies $\mathfrak{T}$ on X and $\mathfrak{T}^n_{[0,1]}$ on $[0,1]^n$. If $\mathfrak{T} \subseteq \mathcal{R}$, then Π is outer regular w.r.t. $\mathfrak{T}$ in $[x]_\mathcal{R}$.*

Continuity of the distribution π in $x \in X$ is however not necessary for outer regularity w.r.t. $\mathfrak{T}$ of Π in $[x]_\mathcal{R}$. To see this, let $X = \mathbb{R}$, and let $\mathfrak{T} = \mathfrak{T}_\mathbb{R} \subseteq \mathcal{R}$. This implies that $\mathcal{R} = \wp(\mathbb{R})$. If π is an isotone, right-continuous mapping, or an antitone, left-continuous mapping, or if π is everywhere zero except on a finite subset of $\mathbb{R}$, then Π is outer regular w.r.t. the Euclidean topology on $\mathbb{R}$ in the singletons of $\mathbb{R}$.

6 A possibilistic Daniell-Kolmogorov theorem

In this section, we derive a possibilistic analogon for the probabilistic Daniell-Kolmogorov theorem [8]. For a given family of $(L, \leq)$-valued mappings on finite Cartesian powers of a sample space that satisfies a natural consistency condition, we want to prove that there always exist a basic space with possibility measure and a family of possibilistic variables that have the given $(L, \leq)$-valued mappings as their possibility distribution functions.

Throughout this section, T denotes a non-empty set and $((X_t, \mathcal{R}_t) \mid t \in T)$ a family of ample spaces.

Furthermore, $(\pi_S \mid \emptyset \neq S \Subset T)$ is a family of mappings, such that $\pi_S\colon X_S \to L$ is $\mathcal{R}_S - \wp(L)$-measurable, $S \Subset T$. π_S is interpreted as the distribution of a $(L, \leq)$-possibility measure Π_S on the ample space $(X_S, \mathcal{R}_S)$. For such families of distributions, we introduce the following consistency condition.

Definition 20 *$(\pi_S \mid \emptyset \neq S \Subset T)$ is called consistent iff for any two sets S_1 and S_2 such that $\emptyset \neq S_1 \subseteq S_2 \Subset T$, and for any $x \in X_{S_1}$:*

$$\pi_{S_1}(x) = \sup_{\mathbf{pr}_{S_2,S_1}(y)=x} \pi_{S_2}(y).$$

If the family of mappings $(\pi_S \mid \emptyset \neq S \Subset T)$ is consistent, we can define in a consistent way the following $(L, \leq)$-valued set function $\mathfrak{M}$ on the field $\mathcal{C}_T$ of measurable cylinders of the product ample space $(X_T, \mathcal{R}_T)$.

Definition 21 *Let the family $(\pi_S \mid \emptyset \neq S \Subset T)$ be consistent. Then $\mathfrak{M}$ is the $\mathcal{C}_T - L$-mapping, given for an element B of $\mathcal{C}_T$ by $\mathfrak{M}(B) = \Pi_S(A)$, where S is a non-empty, finite subset of T, and A is an element of $\mathcal{R}_S$ such that $B = \mathbf{pr}_{T,S}^{-1}(A)$.*

We now investigate if the set function $\mathfrak{M}$ on $\mathcal{C}_T$ can be extended to a possibility measure. Recall that $\tau_{X_T}(\mathcal{C}_T) = \mathcal{R}_T$ is the smallest ample field on which a possibilistic extension of $\mathfrak{M}$, if any, may be defined. If we find a possibilistic extension of $\mathfrak{M}$ to $\mathcal{R}_T$, then possibilistic extension to any ample field including $\mathcal{C}_T$ – and therefore also $\wp(X_T)$ – is straightforward. For example, the greatest possibilistic extension of $\mathfrak{M}$ on any ample field including $\mathcal{C}_T$ will then have distribution $\pi^g_{\mathfrak{M}}$. In Section 4, we saw that a necessary condition for possibilistic extendability is that $\mathfrak{M}$ should be P-consistent on $\mathcal{C}_T$. This is investigated in the next proposition.

Proposition 22 *Let the family $(\pi_S \mid \emptyset \neq S \Subset T)$ be consistent, so that the mapping $\mathfrak{M}$ is well defined. It then has the following properties.*

1. *$\mathfrak{M}$ preserves finite suprema, and is therefore an isotone mapping from $(\mathcal{C}_T, \subseteq)$ to $(L, \leq)$.*
2. *If S is a non-empty, finite subset of T, then the restriction $\mathfrak{M}|_{\mathcal{C}_{T,S}}$ of $\mathfrak{M}$ to $\mathcal{C}_{T,S}$ is a $(L, \leq)$-possibility measure on the ample space $(X_T, \mathcal{C}_{T,S})$.*
3. *If $\mathcal{C}_T$ is an ample field on X_T, then $\mathfrak{M}$ is a $(L, \leq)$-possibility measure on $(X_T, \mathcal{C}_T)$.*
4. *If $\mathfrak{M}$ is P-consistent on $\mathcal{A}_T$, then $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.*
5. *If $(X_t, \mathcal{R}_t)$ is a compact topological space for any $t \in T$, then $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.*

So, we know that if the ample spaces $(X_t, \mathcal{R}_t)$, $t \in T$, are compact topological spaces, our set function $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$. Going from P-consistency to extendability is now but a small step.

Theorem 23 *Let $(X_t, \mathcal{R}_t)$ be compact for any $t \in T$, and assume that the family $(\pi_S \mid \emptyset \neq S \Subset T)$ is consistent. In general, there exist a complete lattice $(L', \leq')$, a supremum preserving order-embedding ϕ from $(L, \leq)$ to $(L', \leq')$ and a $(L', \leq')$-possibility measure Π' on $(X_T, \mathcal{R}_T)$, such that $\Pi'|_{\mathcal{C}_T} = \phi \circ \mathfrak{M}$.*

If the mapping $\mathfrak{M}$ satisfies at least one of the conditions (E_1), (E_2), (E_3) or (E_4), then there exists a $(L, \leq)$-possibility measure Π on $(X_T, \mathcal{R}_T)$ – and therefore also on $(X_T, \wp(X_T))$ –, such that $\Pi|_{\mathcal{C}_T} = \mathfrak{M}$. The greatest such possibility measure $\Pi^g_{\mathfrak{M}}$ has distribution $\pi^g_{\mathfrak{M}}$, determined by

$$\pi^g_{\mathfrak{M}}(x) = \inf_{\emptyset \neq S \Subset T} \pi_S(\mathbf{pr}_{T,S}(x)), \quad x \in X_T.$$

The compactness of the topological spaces $(X_t, \mathcal{R}_t)$ is a very strong requirement. Indeed, $(X_t, \mathcal{R}_t)$ is compact iff X_t has a finite number of atoms. The above theorem therefore states that there is possibilistic extendability for $\mathfrak{M}$ if all the sets X_t, $t \in T$ are essentially finite. This result is not strong enough for many practical purposes. We prove a stronger result for the case that the $(X_t, \mathcal{R}_t)$, $t \in T$, are not necessarily compact, by imposing a number of additional conditions on the distributions $(\pi_S \mid \emptyset \neq S \Subset T)$. This is precisely where regularity comes in. Because compactness is essential when we have to prove that $\mathfrak{M}$ is P-consistent, we shall still require that the X_t should be compact, but with respect to a topology $\mathfrak{T}_t$ that is not necessarily the topology $\mathcal{R}_t$, $t \in T$. Both classes of topologies will then be implicitly linked by imposing regularity conditions on the possibility measures $(\Pi_S \mid \emptyset \neq S \Subset T)$.

Proposition 24 *Let $\mathfrak{T}_t$ be a topology on X_t such that the topological space $(X_t, \mathfrak{T}_t)$ is compact, $t \in T$. Let $(L, \leq)$ be the product complete lattice $([0,1]^J, \leq^J)$, where J is a non-empty set, or let $(L, \leq) = ([0,1], \leq)$. Let the family of distributions $(\pi_S \mid \emptyset \neq S \Subset T)$ be consistent, and assume that for any finite, non-empty subset S of T the corresponding $(L, \leq)$-possibility measure Π_S is regular with respect to the product topology $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S)$ in the atoms of $\mathcal{R}_S$. Then $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.*

Consider a non-empty, finite subset S of T. Since Π_S is a possibility measure on $(X_S, \mathcal{R}_S)$, it is always extendable to a possibility measure on $(X_S, \wp(X_S))$. The greatest such possibilistic extension $\Pi^g_{\Pi_S}$ has distribution $\pi^g_{\Pi_S} = \pi_S$. This leads to the following even stronger result.

Theorem 25 *Let $\mathfrak{T}_t$ be a topology on X_t such that the topological space $(X_t, \mathfrak{T}_t)$ is compact, $t \in T$. Let*

$(L,\leq)$ be the product complete lattice $([0,1]^J,\leq^J)$, where J is a non-empty set, or let $(L,\leq)=([0,1],\leq)$. Let the family of distributions $(\pi_S\mid\emptyset\neq S\Subset T)$ be consistent. Assume that the greatest possibilistic extension $\Pi^g_{\Pi_S}$ of Π_S to $\wp(X_S)$ is outer regular w.r.t. $\mathcal{W}((X_t,\mathfrak{T}_t)\mid t\in S)$ in the singletons of X_S, $\emptyset\neq S\Subset T$. Then $\mathfrak{M}$ is extendable to a $(L,\leq)$-possibility measure on $(X_T,\mathcal{R}_T)$ – and therefore also on $(X_T,\wp(X_T))$. The greatest possibilistic extension $\Pi^g_{\mathfrak{M}}$ of $\mathfrak{M}$ has distribution $\pi^g_{\mathfrak{M}}$, given by

$$\pi^g_{\mathfrak{M}}(x)=\inf_{\emptyset\subset S\Subset T}\pi_S(\mathbf{pr}_{T,S}(x)),\quad x\in X_T.$$

By Proposition 3, $\mathcal{C}_T$ is a base for the product topology $\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T)$ on X_T. Using Proposition 16 we have that $\Pi^g_{\mathfrak{M}}$ is outer regular w.r.t. $\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T)$ in the atoms of $\mathcal{R}_T$.

We are now ready to prove the main result of this paper.

Theorem 26 *Let the family $(\pi_S\mid\emptyset\neq S\Subset T)$ be consistent. Consider the following conditions.*

(C_1) *$(X_t,\mathcal{R}_t)$ is compact for any $t\in T$.*

(C_2) *$\mathcal{C}_T$ is an ample field on X_T.*

(C_3) *$(L,\leq)=([0,1]^J,\leq^J)$ where J is a non-empty set, or $(L,\leq)=([0,1],\leq)$. For any $t\in T$, $\mathfrak{T}_t$ is a topology on X_t such that X_t is compact, and for any $\emptyset\neq S\Subset T$, the greatest possibilistic extension $\Pi^g_{\Pi_S}$ is outer regular w.r.t. $\mathcal{W}((X_t,\mathfrak{T}_t)\mid t\in S)$ in the singletons of X_S.*

If condition (C_1) holds, then there exist a $(L',\leq')$-possibility space $(\Omega,\mathcal{R}_\Omega,\Pi_\Omega)$, where $(L',\leq')$ is a complete lattice in which $(L,\leq)$ is embedded using a supremum preserving mapping ϕ, and a family $(f_t\mid t\in T)$ of possibilistic variables in $((X_t,\mathcal{R}_t)\mid t\in T)$, with basic space $(\Omega,\mathcal{R}_\Omega,\Pi_\Omega)$, such that for any non-empty, finite subset S of T, $\pi_{f_S}=\phi\circ\pi_S$. If the mapping $\mathfrak{M}$ moreover satisfies at least one of the sufficient conditions for extendability (E_1), (E_2), (E_3) or (E_4), then one can take $(L,\leq)$ for $(L',\leq')$ and the identical transformation $\mathbf{1}_L$ of L for ϕ.

If (C_2) or (C_3) holds, then there exist a $(L,\leq)$-possibility space $(\Omega,\mathcal{R}_\Omega,\Pi_\Omega)$ and a family $(f_t\mid t\in T)$ of $(L,\leq)$-possibilistic variables in $((X_t,\mathcal{R}_t)\mid t\in T)$ with basic space $(\Omega,\mathcal{R}_\Omega,\Pi_\Omega)$, such that for any non-empty, finite subset S of T, $\pi_{f_S}=\pi_S$.

From the previous theorem we can immediately derive a possibilistic Daniell-Kolmogorov theorem [8], by assuming that for any $t\in T$, $(X_t,\mathcal{R}_t)$ coincides with a given ample space $(X,\mathcal{R})$. It should also be noted that condition (C_2) holds if the index set T is finite.

References

1. B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order.* Cambridge University Press, Cambridge, 1990.
2. C. W. Burrill. *Measure, Integration and Probability.* McGraw-Hill, New York, 1972.
3. G. de Cooman. “Possibility Theory I: The Measure- and Integral-Theoretic Groundwork”. *International Journal of General Systems.* **8**, 291–323, 1997.
4. G. de Cooman. “Possibility Theory II: Conditional Possibility”. *International Journal of General Systems.* **25**, 325–351, 1997.
5. G. de Cooman. “Possibility Theory III: Possibilistic Independence”. *International Journal of General Systems.* **25**, 353–371, 1997.
6. G. de Cooman. *Evaluatieverzamelingen en -afbeeldingen - Een ordetheoretische benadering van vaagheid en onzekerheid [Evaluation Sets and Mappings - An Order-Theoretic Approach to Vagueness and Uncertainty].* PhD thesis, Universiteit Gent, Ghent, 1993.
7. G. de Cooman and E. E. Kerre. “Ample Fields”. *Simon Stevin.* **67**, 235–244, 1993.
8. J. L. Doob. *Stochastic Processes.* John Wiley & Sons, New York, 1967.
9. P. R. Halmos. *Measure Theory.* Springer Verlag, New York, 1974.
10. H.J. Janssen, G. de Cooman and E.E. Kerre. “First Results for a Mathematical Theory of Possibilistic Processes”. Proc. EMCSR ’96, Vienna, 1996, pp. 341–346.
11. L. Boyen, G. de Cooman and E.E. Kerre. “On the extension of P-consistent mappings”. Proc. FAPT ’95, Gent, 1995, pp. 88–98.
12. P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman & Hall, London, 1991.
13. Z. Wang. “Extension of possibility measures defined on an arbitrary nonempty class of sets”. Proc. IFSA ’85, Palma de Mallorca, 1985.
14. Z. Wang and G. J. Klir. *Fuzzy Measure Theory.* Plenum Press, New York, 1992.
15. L. A. Zadeh. “Fuzzy sets as a basis for a theory of possibility”. *Fuzzy Sets and Systems.* **1**, 3–28, 1978.