Possibility and necessity integrals

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Abstract: In this paper, we introduce seminormed and semiconormed fuzzy integrals associated with confidence measures. These confidence measures have a field of sets as their domain, and a complete lattice as their codomain. In introducing these integrals, the analogy with the classical introduction of Legesgue integrals is explored and exploited. It is amongst other things shown that our integrals are the most general integrals that satisfy a number of natural basic properties. In this way, our dual classes of fuzzy integrals constitute a significant generalization of Sugeno's fuzzy integrals.

A large number of important general properties of these integrals is studied. Furthermore, and most importantly, the combination of seminormed fuzzy integrals and possibility measures on the one hand, and semiconormed fuzzy integrals and necessity measures on the other hand, is extensively studied. It is shown that these combinations are very natural, and have properties which are analogous to the combination of Lebesgue integrals and classical measures. Using these results, the very basis is laid for a unifying measure- and integral-theoretic account of possibility and necessity theory, in very much the same way as the theory of Lebesgue integration provides a proper framework for a unifying and formal account of probability theory.

Keywords: Operators; measure theory; triangular (semi)norms and (semi)conorms; confidence measures; seminormed and semiconormed fuzzy integrals; possibility and necessity theory.

1 Introduction

In his doctoral dissertation, Sugeno [20] introduced fuzzy measures and fuzzy integrals associated with them. Although Sugeno's definition of a fuzzy integral is clearly a fuzzification of the Choquet integral [3], his integral also bears a formal resemblance to the Lebesgue integral, that plays a very important role in classical measure and probability theory. This was shown in 1980 by Ralescu and Adams [15] in an important paper which explores the relation between Sugeno's fuzzy integral and the Lebesgue integral. Weber has thoroughly studied integrals associated with special types of decomposable fuzzy measures [24, 25, 26]. He has also introduced a number of new integrals associated with these fuzzy measures. Interesting extensions of the fuzzy integral in the same spirit as Weber's work have been introduced by Sugeno and Murofushi [21]. Murofushi and Sugeno [13] have also proposed the Choquet integral as an alternative for the fuzzy integral. In a recent paper [10], Grabisch, Murofushi and Sugeno have introduced a new type of integral that is a generalization of the above-mentioned types. On the other hand, Sugeno's fuzzy integral has been generalized by Suárez García and Gil Álvarez towards seminormed and semiconormed fuzzy integrals [19]. These generalizations do not fit into the general class of integrals introduced by Grabish et. al.

In the theory of the above-mentioned integrals, the chain $([0,1],\leq)$ plays the important part of the codomain of the fuzzy measures they are associated with. In [4], we have introduced possibility integrals associated with $(L,\leq)$-possibility measures. In his doctoral dissertation [5], one of us has extended these notions towards seminormed and semiconormed $(L,\leq)$-fuzzy integrals associated with a very general class of measures, namely $(L,\leq)$-confidence measures having a complete lattice $(L,\leq)$ as their codomain and a field of sets as their domain. He has also convincingly shown that, on the one hand, seminormed $(L, \leq)$-fuzzy integrals and $(L, \leq)$-possibility measures, and, on the other hand, semiconormed $(L, \leq)$-fuzzy integrals and $(L, \leq)$-necessity measures, are a perfect match. This paper reports on these results.

Our seminormed and semiconormed $(L, \leq)$-fuzzy integrals are a generalization of existing integrals, and many properties of Sugeno's fuzzy integrals, and of Suárez and Gil's seminormed and semiconormed fuzzy integrals remain valid for our integrals. This generalization, and especially the transition towards more general codomains for the associated measures, involves a number of complications. In many cases the proofs of the above-mentioned properties must be changed or completely reformulated. We shall therefore in the sequel as often as reasonable explicitly state these proofs.

Our integrals are, in a sense, the most general types of integrals that can be associated with $(L, \leq)$-confidence measures and that satisfy at the same time a number of natural properties. Their importance lies in the fact that they can be harmoniously associated with $(L, \leq)$-possibility and $(L, \leq)$-necessity measures, and help provide a unifying measure- and integral-theoretic foundation to possibility and necessity theory, in very much the same way as the Lebesgue integral provides the basis for a systematic treatment of probability theory.

In section 2, we give a brief summary of the preliminary definitions and notational conventions, necessary for a proper understanding of the main material of the paper. In section 3, we introduce our generalized seminormed and semiconormed fuzzy integrals in a way that is formally analogous to that in which traditionally Lebesgue integrals are introduced (see, for instance, [2, 11, 22]). It is shown here that the definition of these classes of integrals—which are, in a sense, each other's duals—is a general as possible: integrals satisfying a certain number of natural properties must belong to one of these classes. We also deduce a number of important formulas that will facilitate the subsequent use of these integrals. In section 4, general properties of our integrals are proven. In section 5 we take a conceptually very important step: we deduce interesting results and properties for our generalized types of seminormed and semiconormed fuzzy integrals, when they are associated with our generalized possibility respectively necessity measures. Section 6 concludes this paper with a brief discussion of the most important results and their relevance to fuzzy set, possibility and necessity theory. Indeed, using the results discussed in this paper, a general measure- and integral-theoretic treatment can be given for Zadeh's possibility theory [5, 28], which includes a consistent account of product possibility measures and integrals, and of conditional possibility and possibilistic independence.

2 Preliminary definitions

Let us begin with a few preliminary definitions and notational conventions. We shall denote by X an arbitrary universe, that contains at least two different elements, so that there exist proper subsets of X.

By $(L, \leq)$ we shall mean a complete lattice that is arbitrary but *fixed throughout the whole text*. The smallest element of $(L, \leq)$ will be denoted by ℓ and the greatest element by u. We shall also assume that $\ell \neq u$. The meet of $(L, \leq)$ will be denoted by $\frown$, the join of $(L, \leq)$ by $\smile$.

With an arbitrary subset A of a universe X, we can associate its *characteristic* $X - L$ *mapping* χ_A, defined by

$$\chi_A(x) \stackrel{\text{def}}{=} \begin{cases} u & ; \quad x \in A \\ \ell & ; \quad x \in \text{co}A. \end{cases}$$

An arbitrary $X - L$ mapping will be called a $(L, \leq)$-*fuzzy set on* X, in accordance with the terminology introduced by Goguen [9]. The set of the $(L, \leq)$-fuzzy sets on X will be denoted by $\mathcal{F}_{(L,\leq)}(X)$. We shall also need the partial order relation $\sqsubseteq$ on $\mathcal{F}_{(L,\leq)}(X)$, defined as follows: for arbitrary h_1 and h_2 in $\mathcal{F}_{(L,\leq)}(X)$,

$$h_1 \sqsubseteq h_2 \Leftrightarrow (\forall x \in X)(h_1(x) \leq h_2(x)).$$

Of course, the structure $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is a complete lattice. The supremum in this complete lattice is the *pointwise* supremum, and the infimum the *pointwise* infimum. The complete lattice $(\mathcal{P}(X), \subseteq)$ can be embedded in this complete lattice by the injection that maps a subset of X to its characteristic $X - L$ mapping.

For arbitrary λ in L, $\underline{\lambda}$ will denote the constant $X - \{\lambda\}$ mapping. Furthermore, let h be an arbitrary $(L, \leq)$-fuzzy set on X, and let λ be an arbitrary element of L. We shall use the following notations:

$$\begin{aligned} S_\lambda^h &\stackrel{\text{def}}{=} h^{-1}([\lambda, u]) = \{\, x \mid x \in X \text{ and } h(x) \geq \lambda \,\} \\ D_\lambda^h &\stackrel{\text{def}}{=} h^{-1}([\ell, \lambda]) = \{\, x \mid x \in X \text{ and } h(x) \leq \lambda \,\} \\ N_\lambda^h &\stackrel{\text{def}}{=} h^{-1}(\{\lambda\}) = \{\, x \mid x \in X \text{ and } h(x) = \lambda \,\}. \end{aligned}$$

S_λ^h will be called the λ-*cut* of h and is a generalization of the level sets of standard fuzzy set theory. D_λ^h is called the *dual* λ-*cut* of h and is related to the strict level sets of standard fuzzy set theory. Indeed, if $(L, \leq)$=$([0, 1], \leq)$, the complement of D_λ^h is a strict level set of h [27]. N_λ^h will be called the λ-*level* of h.

We shall denote by $\mathcal{V}$ an arbitrary proper field of subsets of a universe X, i.e., a set of subsets of X that contains $\emptyset$, is closed under complementation and finite unions, and also contains at least one proper subset of X. We shall say that a subset A of X is $\mathcal{V}$-*measurable* iff $A \in \mathcal{V}$. Furthermore, an arbitrary $X - L$ mapping h is called $\mathcal{V}$-*cut measurable* iff $(\forall \lambda \in L)(S_\lambda^h \in \mathcal{V})$, *dually* $\mathcal{V}$-*cut measurable* iff $(\forall \lambda \in L)(D_\lambda^h \in \mathcal{V})$ and $\mathcal{V}$-*level measurable* iff $(\forall \lambda \in L)(N_\lambda^h \in \mathcal{V})$.

An isotonic $(\mathcal{V}, \subseteq) - (L, \leq)$ mapping v will be called a $(L, \leq)$-*confidence measure on* $(X, \mathcal{V})$ [5], i.e.,

$$(\forall (A, B) \in \mathcal{V}^2)(A \subseteq B \Rightarrow v(A) \leq v(B)).$$

The triple $(X, \mathcal{V}, v)$ is called a $(L, \leq)$-*confidence space*. If $v(\emptyset) = \ell$, then v is called *normalized from below*; and v is called *normalized from above* if $v(X) = u$. Finally, v is called *normalized* if it is normalized from above and from below. Whenever we want to omit reference to the structure $(L, \leq)$, we shall also in general speak about *confidence measures* and *confidence spaces*. Evidently, confidence measures are a generalization of Sugeno's fuzzy measures [20] and the *mesures de confiance* of Dubois and Prade [8]. For a more detailed account of our theory of confidence measures, we refer to [5].

3 Seminormed and semiconormed fuzzy integrals

In Lebesgue integration theory, simple mappings—there defined as measurable real mappings with a finite range—play an important role. We want to give an extended definition of simple mappings in the same vein as the above-mentioned classical definition, that is however better adapted to the more general framework studied here. We shall define a simple mapping as a mapping with a finite range that furthermore satisfies certain conditions of measurability.

Definition 1 *Let X be an arbitrary universe and let $\mathcal{V}$ be an arbitrary proper field on X. A $X - L$ mapping s is called $\mathcal{V}$-simple iff it has a finite range $s(X) = \{a_1, \ldots, a_n\}$ ($n \in \mathbb{N}^*$) and $(\forall k \in \{1, \ldots, n\})(D_k \stackrel{\text{def}}{=} s^{-1}(\{a_k\}) \in \mathcal{V})$.*

In other words, a $X - L$ mapping with finite range is simple if and only if it is $\mathcal{V}$-level measurable. Moreover, it can be proven [5] that for $X - L$ mappings with finite range the notions $\mathcal{V}$-cut measurability, dual $\mathcal{V}$-cut measurability and $\mathcal{V}$-level measurability *coincide*.

In order to define an integral in classical integration theory, certain types of functionals on the set of simple mappings are first defined. The definition of these functionals draws its inspiration from the form of possible decompositions of the simple mappings. More explicitly, when $(\Omega, \mathcal{S})$ is a measurable space, m is a classical measure on $(\Omega, \mathcal{S})$ and s is a non-negative (classical) simple mapping, we may write with obvious notations (see, for instance, [2])

$$(\forall \omega \in \Omega)(s(\omega) = \sum_{k=1}^{n} a_k \chi_{D_k}(\omega)), \tag{1}$$

with $n \in \mathbb{N}^*$, $s(\Omega) = \{a_1, \ldots, a_n\}$, $a_k \in \mathbb{R}$ and χ_{D_k} the characteristic $\Omega - \{0,1\}$ mapping of $D_k \stackrel{\text{def}}{=} s^{-1}(\{a_k\}) \in \mathcal{S}$ $(k \in \{1, \ldots, n\})$. Let furthermore E be an arbitrary element of $\mathcal{S}$. The value of the functional I_E in s is, by definition, given by

$$I_E(s) \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k m(E \cap D_k). \tag{2}$$

The formal analogy between (1) and (2) is striking. Let us also remark that decompositions of the simple mapping s other than (1) are possible. Indeed, if we furthermore assume that the a_k are ordered in such a way that

$$(\forall (k,l) \in \{1, \ldots, n\}^2)(k \leq l \Rightarrow a_k \leq a_l),$$

we have that

$$(\forall \omega \in \Omega)(s(\omega) = \sum_{l=1}^{n} (a_l - a_{l-1}) \chi_{F_l}(\omega)), \tag{3}$$

with

$$F_l \stackrel{\text{def}}{=} \bigcup_{k=l}^{n} D_k = s^{-1}([a_l, 1]), \qquad l \in \{1, \ldots, n\}$$

and $a_o \stackrel{\text{def}}{=} 0$. Using this decomposition, the functional C_E can be introduced as:

$$C_E(s) \stackrel{\text{def}}{=} \sum_{l=1}^{n} (a_l - a_{l-1}) m(E \cap F_l).$$

It is interesting to note that, while the decomposition (1), and therefore also the functional I_E, are connected with the definition of the Lebesque integral, decomposition (3), and therefore also the functional C_E, essentially lead to the definition of the Choquet integral[1] (see, for instance, [3, 13]).

This observation is the starting point of our attempt to generalize the seminormed and semiconormed fuzzy integrals of Suárez García and Gil Álvarez. Let us first investigate which decompositions of the form (1)—or of a related form—are possible in this more general context.

Theorem 1 *Let X be an arbitrary universe. Let ϕ and ξ be binary operators on L that are commutative and associative. Let ψ and ζ be binary operators on L.*

(i) *The following proposition holds for all $X - L$ mappings s with finite range:*

$$(\forall x \in X)(s(x) = \phi_{k=1}^{n} \psi(a_k, \chi_{D_k}(x))), \tag{4}$$

if and only if ϕ and ψ satisfy

$$\begin{cases} (\forall \lambda \in L)(\psi(\lambda, u) = \lambda) \\ (\forall (\lambda, \mu) \in L^2)(\phi(\lambda, \psi(\mu, \ell)) = \lambda). \end{cases} \tag{5}$$

(ii) *The following proposition holds for all $X - L$ mappings s with finite range:*

$$(\forall x \in X)(s(x) = \xi_{k=1}^{n} \zeta(a_k, \chi_{\text{co}D_k}(x))), \tag{6}$$

if and only if ξ and ζ satisfy

$$\begin{cases} (\forall \lambda \in L)(\zeta(\lambda, \ell) = \lambda) \\ (\forall (\lambda, \mu) \in L^2)(\xi(\lambda, \zeta(\mu, u)) = \lambda). \end{cases} \tag{7}$$

[1] For classical measures, the σ-additivity property holds, which implies that there is no difference between the Lebesgue and Choquet integrals associated with these measures. Differences between these integrals only arise for non-classical measures that are not additive.

Proof. We shall give the proof of (ii). The proof of (i) is analogous. First, let us assume that (6) holds for arbitrary $X-L$ mappings with finite range. Let λ and μ be arbitrary elements of L. For the $X-L$ mapping $\underline{\lambda}$, we deduce from (6) in particular that $\lambda = \zeta(\lambda, \ell)$. Also consider the $X-L$ mapping s, defined by

$$s(x) \stackrel{\text{def}}{=} \begin{cases} \lambda & ; \quad x \in A \\ \mu & ; \quad x \in \text{co}A, \end{cases}$$

where x is an arbitrary element of X and A an arbitrary proper subset of X. Since X is assumed to contain at least two elements, this is always possible. Using (6) and putting $x \in A$, we deduce that

$$\lambda = \xi(\zeta(\lambda, \ell), \zeta(\mu, u)) = \xi(\lambda, \zeta(\mu, u)).$$

Conversely, let us assume that (7) holds. Consider an arbitrary $X-L$ mapping s with finite range. First, if s is a constant mapping, we immediately see that (6) holds. Let us therefore assume that s is not a constant mapping, i.e., with the notations of definition 1, $s(X) = \{a_1, \ldots, a_n\}$ with $n \in \mathbb{N}^*$ and $n > 1$. Consider an arbitrary x in X and assume that $x \in D_k$, where k is an arbitrary element of $\{1, \ldots, n\}$. Then, using (7) and repeatedly taking into account the commutativity and associativity of ξ:

$$\begin{aligned} \xi_{m=1}^{n} \zeta(a_m, \chi_{\text{co}D_m}(x)) &= \xi(\zeta(a_k, \chi_{\text{co}D_k}(x)), \xi_{m \neq k} \zeta(a_m, \chi_{\text{co}D_m}(x))) \\ &= \xi(\zeta(a_k, \ell), \xi_{m \neq k} \zeta(a_m, u)) \\ &= \xi(a_k, \xi_{m \neq k} \zeta(a_m, u)) = a_k = s(x). \end{aligned}$$

We conclude that (6) holds. □

In the rest of this section, we shall denote by ϕ a commutative and associative binary operator on L and by ψ a binary operator on L such that (5) holds for ϕ and ψ. Also, we shall denote by ξ a commutative and associative binary operator on L and by ζ a binary operator on L such that (7) holds for ξ and ζ. This implies that for an arbitrary $\mathcal{V}$-simple $X-L$ mapping the decompositions (4) and (6) are valid. It should in this context also be noted that there always exist ϕ and ψ satisfying (5), e.g., $\phi = \smile$ and $\psi = \frown$. Analogously, there always exist ξ and ζ satisfying (7), e.g., $\xi = \frown$ and $\zeta = \smile$. It is also very important to note that the conditions (5) and (7) are dual in a order-theoretic sense. Our introductory discussion makes the following definition rather obvious.

Definition 2 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let s be a $\mathcal{V}$-simple $X-L$ mapping and let A be an arbitrary element of $\mathcal{V}$. Then the following definitions (with the notations of definition 1) make sense.*

(i) $\alpha^v_{\phi\psi}(A\,; s) \stackrel{\text{def}}{=} \phi_{k=1}^{n} \psi(a_k, v(A \cap D_k))$.

(ii) $\beta^v_{\xi\zeta}(A\,; s) \stackrel{\text{def}}{=} \xi_{k=1}^{n} \zeta(a_k, v(A \cap \text{co}D_k))$.

In the following propositions, we investigate the requirements that the operators ϕ and ψ must have in order that the functional $\alpha^v_{\phi\psi}(\cdot\,;\cdot)$ satisfy some *natural* properties. We say that $\alpha^v_{\phi\psi}(\cdot\,;\cdot)$ is *isotonic in both arguments* iff for arbitrary A_1 and A_2 in $\mathcal{V}$ and for arbitrary $\mathcal{V}$-simple $X-L$ mappings s_1 and s_2:

$$(A_1 \subseteq A_2 \text{ and } s_1 \sqsubseteq s_2) \Rightarrow \alpha^v_{\phi\psi}(A_1\,; s_1) \leq \alpha^v_{\phi\psi}(A_2\,; s_2).$$

Proposition 1 *In order that for an arbitrary $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ the functional $\alpha^v_{\phi\psi}(\cdot\,;\cdot)$ be isotonic in both arguments, it is necessary that ϕ and ψ be isotonic.*

Proof. Let us assume that $\alpha^v_{\phi\psi}(\cdot\,;\cdot)$ is isotonic in both arguments for arbitrary $(L, \leq)$-confidence spaces. Choose arbitrary λ, μ and ν in L and assume that $\lambda \leq \mu$. First, it is always possible to choose a $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ such that there exists an element A of $\mathcal{V}$ for which $v(A) = \nu$. Since $\lambda \leq \mu$ and therefore also $\underline{\lambda} \sqsubseteq \underline{\mu}$, it follows from the assumptions that

$$\psi(\lambda, \nu) = \alpha^v_{\phi\psi}(A\,; \underline{\lambda}) \leq \alpha^v_{\phi\psi}(A\,; \underline{\mu}) = \psi(\mu, \nu).$$

We conclude that ψ is isotonic in its first argument.

Next, it is always possible to choose a $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ such that there exist elements A_1 and A_2 in $\mathcal{V}$ for which $v(A_1) = \lambda$ and $v(A_2) = \mu$ and $A_1 \subseteq A_2$. It now follows from the assumptions that

$$\psi(\nu, \lambda) = \alpha^v_{\phi\psi}(A_1\,; \underline{\nu}) \leq \alpha^v_{\phi\psi}(A_2\,; \underline{\nu}) = \psi(\nu, \mu).$$

We conclude that ψ is also isotonic in its second argument. Hence, ψ is isotonic in both arguments, and therefore also isotonic.

Finally, let λ_1, λ_2, μ_1 and μ_2 be arbitrary elements of L. Assume that $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$. It is always possible to choose a $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ such that $(\forall A \in \mathcal{V} \setminus \{\emptyset\})(v(A) = u)$ and such that $\mathcal{V}$ contains at least one element B other than $\emptyset$ and X. Furthermore, let s_1 and s_2 be two $\mathcal{V}$-simple $X - L$ mappings, defined by

$$s_k(x) \stackrel{\text{def}}{=} \begin{cases} \lambda_k & ; \quad x \in B \\ \mu_k & ; \quad x \in \text{co}B \end{cases}$$

for arbitrary x in X and k in $\{1, 2\}$. Of course, we have that $s_1 \sqsubseteq s_2$, and it therefore follows from the assumptions that

$$\begin{aligned} \phi(\lambda_1, \mu_1) &= \phi(\psi(\lambda_1, u), \psi(\mu_1, u)) \\ &= \alpha^v_{\phi\psi}(X\,; s_1) \\ &\leq \alpha^v_{\phi\psi}(X\,; s_2) \\ &= \phi(\psi(\lambda_2, u), \psi(\mu_2, u)) = \phi(\lambda_2, \mu_2). \end{aligned}$$

We conclude that ϕ is isotonic as well. $\square$

Proposition 2 *In order that for an arbitrary $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ the following proposition hold:*

$$(\forall A \in \mathcal{V})(\alpha^v_{\phi\psi}(A\,; \underline{u}) = v(A)),$$

it is necessary that

$$(\forall \lambda \in L)(\psi(u, \lambda) = \lambda). \tag{8}$$

Proof. Let λ be an arbitrary element of L. It is always possible to choose a $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ and an element A of $\mathcal{V}$, such that $v(A) = \lambda$. It now follows from the assumption that

$$\lambda = \alpha^v_{\phi\psi}(A\,; \underline{u}) = \psi(u, v(A)) = \psi(u, \lambda). \quad \square$$

These propositions lead us to consider ϕ that are isotonic, and ψ that are isotonic and satisfy (8). If we summarize the requirements for ψ, we find that it must be an isotonic binary operator on L, that satisfies

$$(\forall \lambda \in L)(\psi(\lambda, u) = \psi(u, \lambda) = \lambda).$$

Such an operator has been studied before (see [4, 6]). It is called a *t-seminorm on* $(L, \leq)$. It is a generalization of the t-seminorms on $([0, 1], \leq)$, that were introduced by Suárez García and Gil Álvarez [19]. It is easily shown that such a ψ also satisfies

$$(\forall \lambda \in L)(\psi(\lambda, \ell) = \psi(\ell, \lambda) = \ell).$$

If we substitute this last expression in the second expression of (5), we find that ϕ satisfies

$$(\forall \lambda \in L)(\phi(\lambda, \ell) = \lambda). \tag{9}$$

Summarizing the requirements for ϕ, we find that it must be a commutative, associative and isotonic binary operator on L that satisfies (9). Again, such an operator has been studied before (see [4, 6]). It is called a *t-conorm on* $(L, \leq)$. It is a generalization of the t-conorms on $([0,1], \leq)$, that were introduced by Schweizer and Sklar [17, 18]. In the sequel, we shall denote by P an arbitrary t-seminorm on $(L, \leq)$, by S an arbitrary t-conorm on $(L, \leq)$, and only concern ourselves with functionals of the type $\alpha^v_{SP}(\cdot\,;\cdot)$.

In a fairly similar way, we could investigate the properties that the operators ξ and ζ must have in order that the functional $\beta^v_{\xi\zeta}(\cdot\,;\cdot)$ satisfy similar natural properties, and thus arrive at similar, dual results. Due to limitations of space, we shall in this paper omit this investigation and simply assume from the outset that the restrictions imposed on ξ and ζ are in a sense dual to those imposed on ϕ and ψ respectively. This means that from now on, we shall assume that ξ is a *t-norm* on $(L, \leq)$ and that ζ is a *t-semiconorm* on $(L, \leq)$. This means that ζ is an isotonic binary operator on L, that satisfies

$$(\forall \lambda \in L)(\zeta(\lambda, \ell) = \zeta(\ell, \lambda) = \lambda),$$

and that ξ is an isotonic, commutative and associative binary operator on L satisfying

$$(\forall \lambda \in L)(\xi(\lambda, u) = \xi(u, \lambda) = \lambda).$$

In the sequel, we shall denote by Q an arbitrary t-semiconorm on $(L, \leq)$, by T an arbitrary t-norm on $(L, \leq)$, and only make use of functionals of the type $\beta^v_{TQ}(\cdot\,;\cdot)$. We shall not explicitly concern ourselves with the discussion of the properties of the t-(semi)norms and t-(semi)conorms that in general can be defined on bounded partially ordered sets. For a detailed discussion of these dual classes of operators, we again refer to [4, 6]. It should however be explicitly mentioned here that $\frown$ is a t-(semi)norm on $(L, \leq)$ and that $\smile$ is a t-(semi)conorm on $(L, \leq)$.

Let us now use the functionals $\alpha^v_{SP}(\cdot\,;\cdot)$ and $\beta^v_{TQ}(\cdot\,;\cdot)$ to try and define functionals (integrals) on the set $\mathcal{F}_{(L,\leq)}(X)$ and not just on the set of the $\mathcal{V}$-simple $X - L$ mappings. In definition 3 (i) an integral is introduced in very much the same way as the Lebesgue integral is traditionally defined. The integral in definition 3 (ii) is in a sense dual to the first one.

Definition 3 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let h be a $X - L$ mapping and let A be a $\mathcal{V}$-measurable set.*

(i) $(SP)\displaystyle\int_A h\mathrm{d}v \overset{\text{def}}{=} \sup\{\, \alpha^v_{SP}(A\,;s) \mid s$ *is $\mathcal{V}$-simple and* $s \sqsubseteq h\,\}$ *is called the $(L, \leq)$-fuzzy SP-integral of h on A (associated with v).*

(ii) $(TQ)\displaystyle\int_A h\mathrm{d}v \overset{\text{def}}{=} \inf\{\, \beta^v_{TQ}(A\,;s) \mid s$ *is $\mathcal{V}$-simple and* $h \sqsubseteq s\,\}$ *is called the $(L, \leq)$-fuzzy TQ-integral of h on A (associated with v).*

In general, an integral of a mapping over a set can be considered as a 'weighted aggregation' of the values the mapping takes over the set. The 'weights' in this aggregation are expressed in terms of the 'measure' the integral is associated with. It is in the first place very important to study the behaviour of an integral when it acts on constant mappings. In the following theorem, we investigate this behaviour for $(L, \leq)$-fuzzy SP- and TQ-integrals.

Theorem 2 (i) *In order that for an arbitrary $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ with v normalized, for an arbitrary t-seminorm P on $(L, \leq)$ and for arbitrary μ in L the following hold:*

$$(SP)\int_X \underline{\mu}\mathrm{d}v = \mu, \tag{10}$$

it is necessary and sufficient that $S = \smile$.

(ii) *In order that for an arbitrary $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$ with v normalized, for an arbitrary t-semiconorm Q on $(L, \leq)$ and for arbitrary μ in L the following hold:*

$$(TQ)\int_X \underline{\mu}\mathrm{d}v = \mu, \tag{11}$$

it is necessary and sufficient that $T = \frown$.

Proof. As an example, we shall prove (ii). The proof of (i) is analogous. We shall prove in corollary 1 furtheron that for arbitrary μ in L

$$(\frown Q)\int_X \underline{\mu}\mathrm{d}v = \mu.$$

Let us therefore now concentrate on the reverse implication. If L contains only two elements, there exists only one t-norm on $(L,\leq)$, that is precisely the meet $\frown$ of $(L,\leq)$ [6], and therefore the proof is trivial. Let us now assume that L contains more than two elements, which implies that there exists more than one t-norm on $(L,\leq)$ [6]. Consider an arbitrary t-norm T on $(L,\leq)$ that differs from $\frown$. This means that there exist λ and μ in L such that $T(\lambda,\mu) < \lambda \frown \mu$ [4, 6] and therefore also $T(\lambda,\mu) < \lambda$ and $T(\lambda,\mu) < \mu$. Remark that this implies that $\ell < \lambda$ and $\ell < \mu$. Choose $Q = \smile$, $X = \{x_1, x_2\}$, $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, $\mathcal{V} = \{\emptyset, A_1, A_2, X\}$ and the $(L,\leq)$-confidence measure v on $(X,\mathcal{V})$ such that $v(A_1) < \lambda$ and $v(A_2) < \mu$, which is always possible. Define the $\mathcal{V}$-simple $X - L$ mapping s_o by

$$s_o(x) \stackrel{\text{def}}{=} \begin{cases} \lambda & ; \quad x \in A_2 \\ \mu & ; \quad x \in A_1. \end{cases}$$

For these choices we have that $\underline{\lambda \frown \mu} \sqsubseteq s_o$, whence

$$\begin{aligned}(T\smile)\int_X \underline{\lambda \frown \mu}\mathrm{d}v &= \inf_{\underline{\lambda\frown\mu}\sqsubseteq s} \beta^v_{T\smile}(X\,;s) \\ &\leq \beta^v_{T\smile}(X\,;s_o) \\ &= T(\lambda \smile v(\mathrm{co}A_2), \mu \smile v(\mathrm{co}A_1)) \\ &= T(\lambda \smile v(A_1), \mu \smile v(A_2)) \\ &= T(\lambda,\mu) < \lambda \frown \mu. \quad \Box\end{aligned}$$

Whenever the 'weight' $v(X)$ of the universe X equals u, it seems very natural to demand that the 'weighted aggregation' of a constant mapping on this universe be equal to the constant value of that mapping. We shall therefore in the sequel only concern ourselves with those classes of integrals for which formulas (10) and (11) hold. This leads to the following basic definition.

Definition 4 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. Let h be a $X - L$ mapping and let A be a $\mathcal{V}$-measurable set.*

(i) $(P)\int_A h\mathrm{d}v \stackrel{\text{def}}{=} (\smile P)\int_A h\mathrm{d}v$ *is called the $(L,\leq)$-fuzzy P-integral of h on A (associated with v).*

(ii) $(Q)\int_A h\mathrm{d}v \stackrel{\text{def}}{=} (\frown Q)\int_A h\mathrm{d}v$ *is called the $(L,\leq)$-fuzzy Q-integral of h on A (associated with v).*

In general, we shall also call these integrals seminormed respectively semiconormed $(L,\leq)$-fuzzy integrals. Both types of integral will be given the collective name '$(L,\leq)$-fuzzy integral'. Whenever we do not want to be explicit about the complete lattice $(L,\leq)$ we shall simply speak of seminormed fuzzy integrals, semiconormed fuzzy integrals and fuzzy integrals.

This definition marks the endpoint of our search for a fairly general definition of seminormed and semiconormed fuzzy integrals. It can be verified that Sugeno's fuzzy integrals are special seminormed fuzzy integrals, indeed special instances of $([0,1],\leq)$-fuzzy min-integrals. Furthermore, the semi(co)normed fuzzy integrals of Suárez García and Gil Álvarez are also generalized by our definition.

To conclude this section, we shall deduce a few formulas that facilitate the calculation and subsequent treatment of fuzzy integrals. First of all, we want to stress that the $(L,\leq)$-fuzzy P- and Q-integrals of an arbitrary $X - L$ mapping on an arbitrary $\mathcal{V}$-measurable subset of X always exist, since we have assumed from the outset that $(L,\leq)$ is a complete lattice.

Theorem 3 gives us formulas for the calculation of $(L,\leq)$-fuzzy integrals of arbitrary $X - L$ mappings. In theorem 4 we show that these formulas can be further simplified whenever the mappings considered satisfy certain measurability requirements. Let us point out that these measurability conditions are different—indeed, dual—for the two types of fuzzy integrals.

Theorem 3 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let h be a $X - L$ mapping and let A be a $\mathcal{V}$-measurable set. Then*

(i) $(P)\int_A h\mathrm{d}v = \sup_{B\in\mathcal{V}} P(\inf_{x\in B} h(x), v(A \cap B))$;

(ii) $(Q)\int_A h\mathrm{d}v = \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B))$.

Proof. As an example, we shall prove (ii). The proof of (i) is similar. On the one hand, consider an arbitrary B in $\mathcal{V}$ and let $\lambda_B \stackrel{\text{def}}{=} \sup_{x\in B} h(x)$. Also consider the $\mathcal{V}$-simple $X - L$ mapping s_B, defined by

$$s_B(x) \stackrel{\text{def}}{=} \begin{cases} \lambda_B & ; \quad x \in B \\ u & ; \quad x \in \mathrm{co}B, \end{cases}$$

for arbitrary x in X. Then $h \sqsubseteq s_B$, and therefore also by definition

$$\begin{aligned} (Q)\int_A h\mathrm{d}v &= \inf_{h\sqsubseteq s} \beta^v_{\frown Q}(A\,;s) \\ &\leq \beta^v_{\frown Q}(A\,;s_B) \\ &= Q(\lambda_B, v(A \cap \mathrm{co}B)) \frown Q(u, v(A \cap B)) \\ &= Q(\lambda_B, v(A \cap \mathrm{co}B)) \\ &= Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B)). \end{aligned}$$

Taking into account the definition of infimum, this implies that

$$(Q)\int_A h\mathrm{d}v \leq \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B)).$$

On the other hand, consider an arbitrary $\mathcal{V}$-simple $X - L$ mapping s for which $h \sqsubseteq s$. Then, with the notations of definition 1, for $k \in \{1, \ldots, n\}$, $(\forall x \in D_k)(a_k \geq h(x))$, and therefore also, taking into account the definition of supremum

$$(\forall k \in \{1, \ldots, n\})(a_k \geq \sup_{x\in D_k} h(x)),$$

whence, taking into account the isotonicity of Q and the $\mathcal{V}$-measurability of all the sets taken into consideration,

$$(\forall k \in \{1, \ldots, n\})(Q(a_k, v(A \cap \mathrm{co}D_k)) \geq Q(\sup_{x\in D_k} h(x), v(A \cap \mathrm{co}D_k))).$$

Taking into account the isotonicity of infimum, this leads to

$$\begin{aligned} \beta^v_{\frown Q}(A\,;s) &= \inf_{k=1}^{n} Q(a_k, v(A \cap \mathrm{co}D_k)) \\ &\geq \inf_{k=1}^{n} Q(\sup_{x\in D_k} h(x), v(A \cap \mathrm{co}D_k)) \\ &\geq \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B)), \end{aligned}$$

which implies that $(Q)\int_A h\mathrm{d}v \geq \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B))$. $\square$

Theorem 4 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let h be a $X - L$ mapping and let A be a $\mathcal{V}$-measurable set.*

(i) *If h is $\mathcal{V}$-cut measurable, then* $(P)\int_A h\mathrm{d}v = \sup_{\lambda\in L} P(\lambda, v(A \cap S^h_\lambda))$.

(ii) *If h is dually $\mathcal{V}$-cut measurable, then* $(Q)\int_A h\mathrm{d}v = \inf_{\lambda\in L} Q(\lambda, v(A \cap \mathrm{co}D^h_\lambda))$.

Proof. Let us prove (ii). The proof of (i) is completely analogous. On the one hand, consider an arbitrary element B of $\mathcal{V}$ and let $\lambda_B \overset{\text{def}}{=} \sup_{x\in B} h(x)$. Then $(\forall x \in B)(h(x) \leq \lambda_B)$ and therefore also $\mathrm{co}D^h_{\lambda_B} \subseteq \mathrm{co}B$. This implies that $A\cap\mathrm{co}D^h_{\lambda_B} \subseteq A\cap\mathrm{co}B$, and since all the sets considered are by assumption $\mathcal{V}$-measurable, also that $v(A \cap \mathrm{co}D^h_{\lambda_B}) \leq v(A \cap \mathrm{co}B)$. Taking into account the isotonicity of Q, this leads to

$$(\forall B \in \mathcal{V})(Q(\lambda_B, v(A \cap \mathrm{co}D^h_{\lambda_B})) \leq Q(\lambda_B, v(A \cap \mathrm{co}B))),$$

whence, by definition of λ_B and by definition of infimum,

$$(\forall B \in \mathcal{V})(\inf_{\lambda\in L} Q(\lambda, v(A \cap \mathrm{co}D^h_\lambda)) \leq Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B))),$$

and therefore also, again taking into account the definition of infimum,

$$\inf_{\lambda\in L} Q(\lambda, v(A \cap \mathrm{co}D^h_\lambda) \leq \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B))).$$

On the other hand, we have for arbitrary λ in L that $(\forall x \in D^h_\lambda)(h(x) \leq \lambda)$, and therefore also

$$\sup_{x\in D^h_\lambda} h(x) \leq \lambda.$$

Since all the sets considered are $\mathcal{V}$-measurable, it follows from the isotonicity of Q that

$$(\forall \lambda \in L)(Q(\lambda, v(A \cap \mathrm{co}D^h_\lambda)) \geq Q(\sup_{x\in D^h_\lambda} h(x), v(A \cap \mathrm{co}D^h_\lambda))).$$

Taking into account the isotonicity of infimum this implies that

$$\begin{aligned} \inf_{\lambda\in L} Q(\lambda, v(A \cap \mathrm{co}D^h_\lambda)) &\geq \inf_{\lambda\in L} Q(\sup_{x\in D^h_\lambda} h(x), v(A \cap \mathrm{co}D^h_\lambda)) \\ &\geq \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A \cap \mathrm{co}B)). \quad \square \end{aligned}$$

Let us briefly discuss the originality of the definitions and results of this section. The items (i) of definitions 2, 3 and 4 and of theorems 3 and 4 are generalizations—towards more general codomains for the mappings and towards more general types of measures—of definitions and theorems by Suárez García and Gil Álvarez [19], which in turn are based upon the work of Ralescu and Adams [15] and Sugeno [20]. Our introduction of semiconormed fuzzy integrals—generalizations of the semiconormed fuzzy integrals of Suárez García and Gil Álvarez—is on the other hand completely new and exposes the dual analogy between seminormed and semiconormed fuzzy integrals. Furthermore, since the items (ii) of the above-mentioned theorems, and definitions, for that matter, are entirely original, we have only given the proofs of these items. The proofs of the items (i) can be found by dual analogy, or by the not always trivial extension of the proofs found in the literature [15, 19, 20]. Theorem 2 deserves some extra attention. It is a modified and at the same time generalized version of theorem 3.5 in [19]. As is shown in [5], the latter theorem is not valid, and can only be proven in the modified form given above.

4 General properties

In this section, we have gathered a few important general properties of our fuzzy integrals. We stress that in many cases these properties are analogous to properties of Lebesgue integrals. Proposition 3 and its corollaries 1 and 2 describe the behaviour of fuzzy integrals acting on constant mappings. We point out that corollary 1 completes the proof of theorem 2.

Proposition 3 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let μ be an element of L and let A be a $\mathcal{V}$-measurable set. Then*

(i) $(P)\displaystyle\int_A \underline{\mu}\mathrm{d}v = P(\mu, v(A)) \smile v(\emptyset)$;

(ii) $(Q)\displaystyle\int_A \underline{\mu}\mathrm{d}v = Q(\mu, v(\emptyset)) \frown v(A)$.

Proof. We shall give the proof of (i). The proof of (ii) is fairly analogous. For arbitrary λ in L we have by definition that

$$S_\lambda^{\underline{\mu}} = \{\, x \mid x \in X \text{ and } \lambda \leq \mu \,\} = \begin{cases} X & ; \quad \lambda \leq \mu \\ \emptyset & ; \quad \lambda \not\leq \mu. \end{cases}$$

which implies that $\underline{\mu}$ is $\mathcal{V}$-cut measurable. Taking into account theorem 4 (i) we may write that

$$\begin{aligned} (P)\int_A \underline{\mu}\mathrm{d}v &= \sup_{\lambda \in L} P(\lambda, v(A \cap S_\lambda^{\underline{\mu}})) \\ &= \sup(\sup_{\lambda \leq \mu} P(\lambda, v(A \cap X)), \sup_{\lambda \not\leq \mu} P(\lambda, v(A \cap \emptyset))) \\ &= \sup(\sup_{\lambda \leq \mu} P(\lambda, v(A)), \sup_{\lambda \not\leq \mu} P(\lambda, v(\emptyset))). \end{aligned}$$

Taking into account the properties of t-seminorms [4, 6], it is now easily verified that on the one hand

$$\sup_{\lambda \leq \mu} P(\lambda, v(A)) = P(\mu, v(A)),$$

and on the other hand

$$\sup_{\lambda \not\leq \mu} P(\lambda, v(\emptyset)) = \begin{cases} v(\emptyset) & ; \quad \mu < u \\ \ell & ; \quad \mu = u. \end{cases}$$

This implies that

$$(P)\int_A \underline{\mu}\mathrm{d}v = \begin{cases} P(\mu, v(A)) \smile v(\emptyset) & ; \quad \mu < u \\ v(A) & ; \quad \mu = u \end{cases} = P(\mu, v(A)) \smile v(\emptyset),$$

since $v(\emptyset) \leq v(A)$. □

Corollary 1 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space, with v normalized. For arbitrary μ in L:*

(i) $(P)\displaystyle\int_X \underline{\mu}\mathrm{d}v = \mu$;

(ii) $(Q)\displaystyle\int_X \underline{\mu}\mathrm{d}v = \mu$.

Proof. At once from the proposition above, taking into account $v(\emptyset) = \ell$, $v(X) = u$ and the boundary conditions for t-seminorms and t-semiconorms on $(L, \leq)$ [4, 6]. □

Corollary 2 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. For arbitrary A in $\mathcal{V}$:*

(i) $(P)\!\!\!\!\!\!\!\int_A \mathrm{d}v \stackrel{\mathrm{def}}{=} (P)\!\!\!\!\!\!\!\int_A \underline{u}\mathrm{d}v = v(A)$;

(ii) $(Q)\!\!\!\!\!\!\!\int_A \mathrm{d}v \stackrel{\mathrm{def}}{=} (Q)\!\!\!\!\!\!\!\int_A \underline{u}\mathrm{d}v = v(A)$.

Proof. At once from the proposition above, taking into account $v(\emptyset) \leq v(A)$, and the boundary conditions for t-seminorms and t-semiconorms on $(L,\leq)$ [4, 6]. $\square$

The following proposition shows that the value of a fuzzy integral of an arbitrary mapping on a set with confidence measure ℓ is always equal to ℓ.

Proposition 4 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. Let h be a $X-L$ mapping and let A be a $\mathcal{V}$-measurable set. Then*

(i) $v(A) = \ell \Rightarrow (P)\!\!\!\!\!\!\!\int_A h\mathrm{d}v = \ell$;

(ii) $v(A) = \ell \Rightarrow (Q)\!\!\!\!\!\!\!\int_A h\mathrm{d}v = \ell$.

Proof. Assume that $v(A) = \ell$. Taking into account the isonicity of v, we have that $(\forall B \in \mathcal{V})(v(A\cap B) = \ell)$. Taking into account theorem 3 (i), this implies that

$$(P)\!\!\!\!\!\!\!\int_A h\mathrm{d}v = \sup_{B\in\mathcal{V}} P(\inf_{x\in B} h(x), v(A\cap B)) = \sup_{B\in\mathcal{V}} P(\inf_{x\in B} h(x), \ell) = \ell,$$

and analogously, taking into account theorem 3 (ii), we have that

$$\begin{aligned}(Q)\!\!\!\!\!\!\!\int_A h\mathrm{d}v &= \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), v(A\cap \mathrm{co}B)) \\ &= \inf_{B\in\mathcal{V}} Q(\sup_{x\in B} h(x), \ell) \\ &= \inf_{B\in\mathcal{V}} \sup_{x\in B} h(x) = \ell,\end{aligned}$$

since $\sup\emptyset = \ell$ holds in the complete lattice $(L,\leq)$, and furthermore $\emptyset \in \mathcal{V}$. $\square$

Propositions 5, 6, and 7, and their corollaries 3 and 4 express the isotonicity of our fuzzy integrals.

Proposition 5 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. Let h_1 and h_2 be $X-L$ mappings with $h_1 \sqsubseteq h_2$. Then for arbitrary A in $\mathcal{V}$:*

(i) $(P)\!\!\!\!\!\!\!\int_A h_1\mathrm{d}v \leq (P)\!\!\!\!\!\!\!\int_A h_2\mathrm{d}v$;

(ii) $(Q)\!\!\!\!\!\!\!\int_A h_1\mathrm{d}v \leq (Q)\!\!\!\!\!\!\!\int_A h_2\mathrm{d}v$.

Proof. As an example, we shall prove (i). The proof of (ii) is analogous. For arbitrary B in $\mathcal{V}$, we deduce from the assumptions and the isotonicity of infimum, that

$$\inf_{x\in B} h_1(x) \leq \inf_{x\in B} h_2(x)$$

whence, since P is isotonic,

$$P(\inf_{x\in B} h_1(x), v(A\cap B)) \leq P(\inf_{x\in B} h_2(x), v(A\cap B)).$$

Taking into account the isotonicity of supremum it follows that

$$\sup_{B\in\mathcal{V}} P(\inf_{x\in B} h_1(x), v(A\cap B)) \leq \sup_{B\in\mathcal{V}} P(\inf_{x\in B} h_2(x), v(A\cap B)),$$

which, taking into account theorem 3 (i), proves (i). $\square$

Proposition 6 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. Let h_1 and h_2 be $X-L$ mappings. Let A be an element of $\mathcal{V}$, and assume that $(\forall x \in A)(h_1(x) \leq h_2(x))$. Then the following propositions hold.*

(i) *If h_1 and h_2 are $\mathcal{V}$-cut measurable, then $(P)\int_A h_1 \mathrm{d}v \leq (P)\int_A h_2 \mathrm{d}v$.*

(ii) *If h_1 and h_2 are dually $\mathcal{V}$-cut measurable, then $(Q)\int_A h_1 \mathrm{d}v \leq (Q)\int_A h_2 \mathrm{d}v$.*

Proof. As an example, we shall prove (ii). The proof of (i) is analogous. Assume that h_1 and h_2 are dually $\mathcal{V}$-cut measurable. Consider an arbitrary λ in L and an arbitrary x in $A \cap \mathrm{co}D_\lambda^{h_1}$. By definition, we have that $h_1(x) \not\leq \lambda$. From the assumptions it now also follows that $h_2(x) \not\leq \lambda$. Indeed, should $h_2(x) \leq \lambda$, then the validity of $h_1(x) \leq h_2(x)$ and the transitivity of $\leq$ would imply that $h_1(x) \leq \lambda$, a contradiction. This implies that $x \in A \cap \mathrm{co}D_\lambda^{h_2}$, whence $A\cap \mathrm{co}D_\lambda^{h_1} \subseteq A \cap \mathrm{co}D_\lambda^{h_2}$. Taking into account the isotonicity of v and Q, and the fact that all the sets considered are $\mathcal{V}$-measurable, we obtain

$$Q(\lambda, v(A\cap \mathrm{co}D_\lambda^{h_1})) \leq Q(\lambda, v(A\cap \mathrm{co}D_\lambda^{h_2})).$$

The isotonicity of infimum now implies that

$$\inf_{\lambda\in L} Q(\lambda, v(A\cap \mathrm{co}D_\lambda^{h_1})) \leq \inf_{\lambda\in L} Q(\lambda, v(A\cap \mathrm{co}D_\lambda^{h_2})),$$

which, taking into account theorem 4 (ii), proves (ii). $\square$

Corollary 3 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. Let $(h_j \mid j\in J)$ be a family of $X-L$ mappings. Then for arbitrary A in $\mathcal{V}$:*

(i) $(P)\int_A \sup_{j\in J} h_j \mathrm{d}v \geq \sup_{j\in J} (P)\int_A h_j \mathrm{d}v$;

(ii) $(P)\int_A \inf_{j\in J} h_j \mathrm{d}v \leq \inf_{j\in J} (P)\int_A h_j \mathrm{d}v$;

(iii) $(Q)\int_A \sup_{j\in J} h_j \mathrm{d}v \geq \sup_{j\in J} (Q)\int_A h_j \mathrm{d}v$;

(iv) $(Q)\int_A \inf_{j\in J} h_j \mathrm{d}v \leq \inf_{j\in J} (Q)\int_A h_j \mathrm{d}v$.

Proposition 7 *Let $(X,\mathcal{V},v)$ be an arbitrary $(L,\leq)$-confidence space. Let A and B be $\mathcal{V}$-measurable sets, with $A\subseteq B$. Then for an arbitrary $X-L$ mapping h:*

(i) $(P)\int_A h\mathrm{d}v \leq (P)\int_B h\mathrm{d}v$;

(ii) $(Q)\int_A h\mathrm{d}v \leq (Q)\int_B h\mathrm{d}v$.

Proof. As an example, we shall prove (ii). The proof of (i) is analogous. Let C be an arbitrary element of $\mathcal{V}$, then $A \cap \text{co}C \subseteq B \cap \text{co}C$, whence, since v and Q are isotonic,

$$Q(\sup_{x\in C} h(x), v(A \cap \text{co}C)) \leq Q(\sup_{x\in C} h(x), v(B \cap \text{co}C)).$$

Since infimum is isotonic, this implies that

$$\inf_{C\in\mathcal{V}} Q(\sup_{x\in C} h(x), v(A \cap \text{co}C)) \leq \inf_{C\in\mathcal{V}} Q(\sup_{x\in C} h(x), v(B \cap \text{co}C)),$$

which, taking into account theorem 3 (ii), proves (ii). □

Corollary 4 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let $(A_j \mid j \in J)$ be a family of $\mathcal{V}$-measurable sets. Then for an arbitrary $X - L$ mapping h:*

(i) $(P)\displaystyle\fint_{\bigcup_{j\in J} A_j} h\mathrm{d}v \geq \sup_{j\in J} (P)\fint_{A_j} h\mathrm{d}v$;

(ii) $(P)\displaystyle\fint_{\bigcap_{j\in J} A_j} h\mathrm{d}v \leq \inf_{j\in J} (P)\fint_{A_j} h\mathrm{d}v$;

(iii) $(Q)\displaystyle\fint_{\bigcup_{j\in J} A_j} h\mathrm{d}v \geq \sup_{j\in J} (Q)\fint_{A_j} h\mathrm{d}v$;

(iv) $(Q)\displaystyle\fint_{\bigcap_{j\in J} A_j} h\mathrm{d}v \leq \inf_{j\in J} (Q)\fint_{A_j} h\mathrm{d}v$.

Propositions 8 and 9 describe the behaviour of fuzzy integrals when characteristic mappings appear in the integrand.

Proposition 8 *Let $(X, \mathcal{V}, v)$ be an arbitrary $(L, \leq)$-confidence space. Let A be an element of $\mathcal{V}$ with characteristic $X - L$ mapping χ_A. Then*

(i) $v(A) = (P)\displaystyle\fint_X \chi_A \mathrm{d}v$;

(ii) $v(A) = (Q)\displaystyle\fint_X \chi_A \mathrm{d}v$.

Proof. As an example, we shall prove (ii). The proof of (i) is analogous. Since $A \in \mathcal{V}$ it can easily be shown that χ_A is dually $\mathcal{V}$-cut measurable. Taking into account theorem 4 (ii), we may write that

$$\begin{aligned}
(Q)\fint_X \chi_A \mathrm{d}v &= \inf_{\lambda\in L} Q(\lambda, v(\text{co}D_\lambda^{\chi_A})) \\
&= \inf(\inf_{\lambda<u} Q(\lambda, v(A)), Q(u, v(\emptyset))) \\
&= \inf(\inf_{\lambda<u} Q(\lambda, v(A)), u) \\
&= \inf_{\lambda<u} Q(\lambda, v(A)) \\
&\leq Q(\ell, v(A)) = v(A).
\end{aligned}$$

On the other hand, we have that $(\forall\lambda \in L)(Q(\lambda, v(A)) \geq v(A))$, whence, taking into account the definition of infimum, $\inf_{\lambda<u} Q(\lambda, v(A)) \geq v(A)$. □

Proposition 9 *Let* $(X, \mathcal{V}, v)$ *be an arbitrary* $(L, \leq)$*-confidence space. Let* A *be an element of* $\mathcal{V}$ *with characteristic* $X - L$ *mapping* χ_A*, and let* h *be a* $X - L$ *mapping.*

(i) *If* h *is* $\mathcal{V}$*-cut measurable, then* $(P)\int_A h\mathrm{d}v = (P)\int_X (\chi_A \frown h)\mathrm{d}v.$

(ii) *If* h *is dually* $\mathcal{V}$*-cut measurable, then* $(Q)\int_A h\mathrm{d}v = (Q)\int_X (\chi_A \frown h)\mathrm{d}v.$

Of course, $\chi_A \frown h$ *is the* $X - L$ *mapping defined pointwise by*

$$(\forall x \in X)((\chi_A \frown h)(x) \stackrel{\text{def}}{=} \chi_A(x) \frown h(x)).$$

Proof. As an example, we give the proof of (ii). The proof of (i) is analogous. For arbitrary λ in L we have that

$$D_\lambda^{\chi_A \frown h} = \{\, x \mid x \in X \text{ and } \chi_A(x) \frown h(x) \leq \lambda \,\} = \mathrm{co}A \cup D_\lambda^h,$$

and therefore $\chi_A \frown h$ is dually $\mathcal{V}$-cut measurable as well. Theorem 4 (ii) therefore implies that

$$(Q)\int_X (\chi_A \frown h)\mathrm{d}v = \inf_{\lambda \in L} Q(\lambda, v(\mathrm{co}D_\lambda^{\chi_A \frown h})) = \inf_{\lambda \in L} Q(\lambda, v(A \cap \mathrm{co}D_\lambda^h)) = (Q)\int_A h\mathrm{d}v. \quad \square$$

In the next theorem, we explicitly show that our two types of fuzzy integrals are in a sense dual. In order to be able to formulate this theorem, we introduce a few preliminary notions. Let n be a dual order-automorphism on $(L, \leq)$, i.e., an order-isomorphism between the structures $(L, \leq)$ and $(L, \geq)$ (see, for instance, [1]). In order that such a dual order-isomorphism exist, $(L, \leq)$ must of course be self-dual. Let us assume in the rest of this section that this is indeed the case. Consider an arbitrary $(L, \leq)$-confidence space $(X, \mathcal{V}, v)$. Then the $\mathcal{V} - L$ mapping v_n, defined by

$$(\forall A \in \mathcal{V})(v_n(A) \stackrel{\text{def}}{=} n^{-1}(v(\mathrm{co}A))),$$

is a $(L, \leq)$-confidence measure on $(X, \mathcal{V})$, and is called the *dual* $(L, \leq)$*-confidence measure of* v *w.r.t.* n [5]. Also, consider an arbitrary t-seminorm P on $(L, \leq)$. The $L^2 - L$ mapping P_n, defined by

$$(\forall(\lambda, \mu) \in L^2)(P_n(\lambda, \mu) \stackrel{\text{def}}{=} n^{-1}(P(n(\lambda), n(\mu)))),$$

is a t-semiconorm on $(L, \leq)$, called the *dual* t*-semiconorm of* P *w.r.t.* n [6]. Dually, consider an arbitrary t-semiconorm Q on $(L, \leq)$. Then the $L^2 - L$ mapping Q_n, defined by

$$(\forall(\lambda, \mu) \in L^2)(Q_n(\lambda, \mu) \stackrel{\text{def}}{=} n^{-1}(Q(n(\lambda), n(\mu)))),$$

is a t-seminorm on $(L, \leq)$, called the *dual* t*-seminorm of* Q *w.r.t.* n [6].

Theorem 5 *Let* $(X, \mathcal{V}, v)$ *be an arbitrary* $(L, \leq)$*-confidence space. Also, let* n *be a dual order-automorphism of* $(L, \leq)$ *and let* h *be a* $X - L$ *mapping. Then*

(i) $(P)\int_X h\mathrm{d}v = n((P_n)\int_X (n^{-1} \circ h)\mathrm{d}v_n)$*, or equivalently,* $(P_n)\int_X h\mathrm{d}v_n = n^{-1}((P)\int_X (n \circ h)\mathrm{d}v)$*;*

(ii) $(Q)\int_X h\mathrm{d}v = n((Q_n)\int_X (n^{-1} \circ h)\mathrm{d}v_n)$*, or equivalently,* $(Q_n)\int_X h\mathrm{d}v_n = n^{-1}((Q)\int_X (n \circ h)\mathrm{d}v)$*.*

Proof. As an example, we give the proof of (i). The proof of (ii) is analogous. We deduce from theorem 3 that

$$\begin{aligned}
n((P_n)\int_X (n^{-1}\circ h)\mathrm{d}v_n) &= n(\inf_{B\in\mathcal{V}} P_n(\sup_{x\in B}(n^{-1}\circ h)(x), v_n(\mathrm{co}B))) \\
&= \sup_{B\in\mathcal{V}} n(P_n(\sup_{x\in B}(n^{-1}\circ h)(x), v_n(\mathrm{co}B))) \\
&= \sup_{B\in\mathcal{V}} n(n^{-1}(P(n(\sup_{x\in B}(n^{-1}\circ h)(x)), n(v_n(\mathrm{co}B))))) \\
&= \sup_{B\in\mathcal{V}} P(n(\sup_{x\in B}(n^{-1}\circ h)(x)), n(v_n(\mathrm{co}B))) \\
&= \sup_{B\in\mathcal{V}} P(n(\sup_{x\in B}(n^{-1}\circ h)(x)), n(n^{-1}(v(\mathrm{co}(\mathrm{co}B))))) \\
&= \sup_{B\in\mathcal{V}} P(\inf_{x\in B} h(x), v(B)) \\
&= (P)\int_X h\mathrm{d}v.
\end{aligned}$$

This immediately implies that $(P_n)\int_X h\mathrm{d}v_n = n^{-1}((P)\int_X (n\circ h)\mathrm{d}v)$. □

5 Possibility and necessity integrals

In this section, we want to combine seminormed fuzzy integrals and possibility measures on the one hand, and semiconormed fuzzy integrals and necessity measures on the other hand. We shall show that these combinations are indeed very fruitful.

We start by introducing a few preliminary notions, and by discussing the assumptions made in this section. From now on, whenever we speak about a t-seminorm P on $(L,\leq)$, we shall assume that P is *completely distributive* w.r.t. supremum, i.e., for arbitrary λ in L and an arbitrary family $(\mu_j \mid j\in J)$ of elements of L:

$$\begin{cases} P(\lambda, \sup_{j\in J}\mu_j) = \sup_{j\in J} P(\lambda,\mu_j) \\ P(\sup_{j\in J}\mu_j, \lambda) = \sup_{j\in J} P(\mu_j,\lambda). \end{cases}$$

Analogously, whenever we speak about a t-semiconorm Q on $(L,\leq)$, we shall assume that Q is *completely distributive* w.r.t. infimum. In short, we assume that the structure $(L,\leq,P)$ is a complete lattice with t-seminorm and that the structure $(L,\geq,Q)$ is a complete lattice with t-semiconorm [6].

By $\mathcal{R}$, we shall mean an arbitrary *ample field* on the universe X, i.e., a set of subsets of X that is closed under *arbitrary* unions and intersections, and under complementation. We shall furthermore assume that $\mathcal{R}$ is a *proper* ample field, i.e., $\{\emptyset, X\}\subset\mathcal{R}$. The atom of $\mathcal{R}$ containing the element x of X will be denoted by $[\,x\,]_{\mathcal{R}}$:

$$[\,x\,]_{\mathcal{R}} \overset{\text{def}}{=} \bigcap\{\,A \mid x\in A \text{ and } A\in\mathcal{R}\,\}.$$

For a more detailed account of ample fields, we refer to [7, 23].

By Π, we shall mean an arbitrary $(L,\leq)$-*possibility measure* on $(X,\mathcal{R})$, i.e., a complete join-morphism between the complete lattices $(\mathcal{R},\subseteq)$ and $(L,\leq)$. This means by definition that Π satisfies the following requirement: for an arbitrary family $(A_j \mid j\in J)$ of elements of $\mathcal{R}$

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J}\Pi(A_j).$$

This definition immediately implies that $\Pi(\emptyset)=\ell$. Also, $(X,\mathcal{R},\Pi)$ is a special kind of $(L,\leq)$-confidence space, that will henceforth be called a $(L,\leq)$-*possibility space*. For obvious reasons, Π will be called normalized iff $\Pi(X)=u$. The $X-L$ mapping π, defined by

$$(\forall x\in X)(\pi(x) \overset{\text{def}}{=} \Pi([\,x\,]_{\mathcal{R}}))$$

will be called the *distribution* of Π. Remark that it completely determines Π, since for arbitrary A in $\mathcal{R}$

$$\Pi(A) = \sup_{x \in A} \pi(x). \tag{12}$$

By N, we shall mean an arbitrary $(L, \leq)$-*necessity measure* on $(X, \mathcal{R})$, i.e., a complete meet-morphism between the complete lattices $(\mathcal{R}, \subseteq)$ and $(L, \leq)$. This means by definition that N satisfies the following requirement: for an arbitrary family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$

$$\mathrm{N}(\bigcap_{j \in J} A_j) = \inf_{j \in J} \mathrm{N}(A_j).$$

This definition immediately implies that $\mathrm{N}(X) = u$. Also, $(X, \mathcal{R}, \mathrm{N})$ is a special kind of $(L, \leq)$-confidence space, that will henceforth be called a $(L, \leq)$-*necessity space*. For obvious reasons, N will be called normalized iff $\mathrm{N}(\emptyset) = \ell$. The $X - L$ mapping ν, defined by

$$(\forall x \in X)(\nu(x) \stackrel{\text{def}}{=} \mathrm{N}(\mathrm{co}[\, x \,]_{\mathcal{R}}))$$

will be called the *distribution* of N. Remark that it completely determines N, since for arbitrary A in $\mathcal{R}$

$$\mathrm{N}(A) = \inf_{x \in \mathrm{co} A} \nu(x). \tag{13}$$

$(L, \leq)$-possibility and $(L, \leq)$-necessity measures are generalizations towards more general domains and codomains of Zadeh's possibility measures [28], Dubois and Prade's necessity measures [8], Wang's fuzzy contactabilities [23], and the possibility and necessity measures we introduced in [4]. For a more detailed discussion of these generalizations, we refer to [4, 5, 7].

A $X - L$ mapping h will be called $\mathcal{R}$-*measurable* iff it is constant on the atoms of $\mathcal{R}$. It can be proven that the notions $\mathcal{R}$-measurability, $\mathcal{R}$-cut measurability, dual $\mathcal{R}$-cut measurability and $\mathcal{R}$-level measurability in fact *coincide* [7].

Definition 5 *A $X - L$ mapping h—a $(L, \leq)$-fuzzy set in X—is called a $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$ iff h is $\mathcal{R}$-measurable. The set of the $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$ is denoted by $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. Whenever we want to omit reference to the structures $(L, \leq)$ and $(X, \mathcal{R})$, we shall simply speak of fuzzy variables.*

Our fuzzy variables are generalizations towards more general codomains and measurability conditions of the fuzzy variables introduced by Nahmias [14], and further refined by Wang [23]. They are to a certain extent also in spirit related to Ralescu's fuzzy variables [16]. They are primarily meant to serve as a possibilistic equivalent of the real stochastic variables in probability theory. On the other hand, $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$ can also be considered as obvious extensions—or indeed "*fuzzifications*"—of $\mathcal{R}$-measurable[2] sets: the characteristic $X - L$ mapping χ_A of a subset A of X is a $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$ if and only if A is $\mathcal{R}$-measurable. Moreover, the fuzzification $(\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X), \sqsubseteq)$ of $(\mathcal{R}, \subseteq)$ is a *complete sublattice* of the fuzzification $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ of $(\mathcal{P}(X), \subseteq)$, in a similar way as $(\mathcal{R}, \subseteq)$ is a complete sublattice of $(\mathcal{P}(X), \subseteq)$ [5, 7].

We shall now derive a few important formulas for seminormed and semiconormed fuzzy integrals, when the confidence measure they are associated with is a possibility, respectively a necessity measure. These formulas will be used furtheron to derive results that constitute the foundation of a general measure- and integral-theoretic approach to possibility and necessity theory [5].

[2]We note that the fuzzy variables introduced by Zadeh [28] are more general than the fuzzy variables discussed here. The former are intended as possibilistic analoga of general, not necessarily real-valued, stochastic variables. Furthermore, they cannot in general be interpreted as measurable fuzzy sets. We therefore suggest that the more appropriate name '*possibilistic variable*' be used for Zadeh's notion, and that the name 'fuzzy variables' be reserved for measurable fuzzy sets. For a detailed discussion of possibilistic variables, conditional possibility and possibilistic independence of possibilistic variables, we refer to the doctoral dissertation of one of us [5].

Definition 6 *Let h be a $X - L$ mapping and A a $\mathcal{R}$-measurable set.*

(i) $(P)\int_A h\mathrm{d}\Pi$ *is called the $(L, \leq, P)$-possibility integral of h on A (associated with Π).*

(ii) $(Q)\int_A h\mathrm{d}\mathrm{N}$ *is called the $(L, \geq, Q)$-necessity integral of h on A (associated with N).*

Whenever we want to omit reference to the complete lattice $(L, \leq)$, the t-seminorm P and/or the t-semiconorm Q, we shall simple speak of possibility and necessity integrals.

Theorem 6 *Let A be a $\mathcal{R}$-measurable set and let h be a $X - L$ mapping. Then*

(i) $(P)\int_A h\mathrm{d}\Pi = \sup_{x\in A} P(\inf_{y\in[\,x\,]_{\mathcal{R}}} h(y), \pi(x))$;

(ii) $(Q)\int_A h\mathrm{d}\mathrm{N} = \mathrm{N}(A) \frown \inf_{x\in X} Q(\sup_{y\in[\,x\,]_{\mathcal{R}}} h(y), \nu(x))$.

Proof. We shall prove (ii). The proof of (i) is fairly analogous. We already know, taking into account theorem 3 (ii) and the fact that Q is completely distributive w.r.t. infimum, that

$$\begin{aligned}(Q)\int_A h\mathrm{d}\mathrm{N} &= \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \mathrm{N}(A\cap \mathrm{co}B))\\ &= \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \mathrm{N}(A) \frown \mathrm{N}(\mathrm{co}B))\\ &= \inf_{B\in\mathcal{R}} (Q(\sup_{y\in B} h(y), \mathrm{N}(A)) \frown Q(\sup_{y\in B} h(y), \mathrm{N}(\mathrm{co}B)))\\ &= \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \mathrm{N}(A)) \frown \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \mathrm{N}(\mathrm{co}B))\\ &= Q(\inf_{B\in\mathcal{R}} \sup_{y\in B} h(y), \mathrm{N}(A)) \frown \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \mathrm{N}(\mathrm{co}B)).\end{aligned}$$

Since $\emptyset \in \mathcal{R}$ and $\sup_{y\in\emptyset} h(y) = \sup\emptyset = \ell$ holds in the complete lattice $(L, \leq)$, it follows that

$$\inf_{B\in\mathcal{R}} \sup_{y\in B} h(y) = \ell.$$

Therefore, we find that, taking into account the complete distributivity of Q w.r.t. infimum, the associativity of infimum and the boundary conditions of t-semiconorms [4, 6],

$$\begin{aligned}(Q)\int_A h\mathrm{d}\mathrm{N} &= \mathrm{N}(A) \frown \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \mathrm{N}(\mathrm{co}B))\\ &= \mathrm{N}(A) \frown \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \inf_{x\in X} Q(\chi_{\mathrm{co}B}(x), \nu(x)))\\ &= \mathrm{N}(A) \frown \inf_{x\in X} \inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), Q(\chi_{\mathrm{co}B}(x), \nu(x)))\\ &= \mathrm{N}(A) \frown \inf_{x\in X} \inf_{B\in\mathcal{R}} Q(Q(\sup_{y\in B} h(y), \chi_{\mathrm{co}B}(x)), \nu(x))\\ &= \mathrm{N}(A) \frown \inf_{x\in X} Q(\inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \chi_{\mathrm{co}B}(x)), \nu(x)).\end{aligned}$$

The proof of (ii) is complete if we can show that for arbitrary x in X:

$$\inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \chi_{\mathrm{co}B}(x)) = \sup_{y\in[\,x\,]_{\mathcal{R}}} h(y).$$

To this end, consider an arbitrary x in X. We have on the one hand that

$$\inf_{B\in\mathcal{R}} Q(\sup_{y\in B} h(y), \chi_{\mathrm{co}B}(x)) = \inf\{\sup_{y\in B} h(y) \mid B\in\mathcal{R} \text{ and } x\in B\} \le \sup_{y\in[\,x\,]_{\mathcal{R}}} h(y),$$

taking into account the definition of infimum and the fact that $x \in [\,x\,]_{\mathcal{R}}$ and $[\,x\,]_{\mathcal{R}} \in \mathcal{R}$ [7]. On the other hand, since by definition of $[\,x\,]_{\mathcal{R}}$, for every B in $\mathcal{R}$ for which $x \in B$, $[\,x\,]_{\mathcal{R}} \subseteq B$, and therefore also

$$\sup_{y\in[\,x\,]_{\mathcal{R}}} h(y) \le \sup_{y\in B} h(y),$$

it follows from the definition of infimum that

$$\inf\{\sup_{y\in B} h(y) \mid B\in\mathcal{R} \text{ and } x\in B\} \ge \sup_{y\in[\,x\,]_{\mathcal{R}}} h(y). \quad \square$$

Corollary 5 *Let A be a $\mathcal{R}$-measurable set and let h be a $X - L$ mapping. Then*

(i) $(Q)\int_X h\mathrm{dN} = \inf_{x\in X} Q(\sup_{y\in[\,x\,]_{\mathcal{R}}} h(y), \nu(x))$;

(ii) $(Q)\int_A h\mathrm{dN} = \mathrm{N}(A) \frown (Q)\int_X h\mathrm{dN}$.

Proof. This is an immediate consequence of the previous theorem, since $\mathrm{N}(X) = u$. $\square$

Corollary 6 *Let A be an element of $\mathcal{R}$ and let h be a $X - L$ mapping that is $\mathcal{R}$-measurable. Then*

(i) $(P)\int_A h\mathrm{d}\Pi = \sup_{x\in A} P(h(x), \pi(x))$;

(ii) $(Q)\int_A h\mathrm{dN} = \mathrm{N}(A) \frown \inf_{x\in X} Q(h(x), \nu(x))$.

Proof. This is an immediate consequence of theorem 6 and the fact that h is by assumption constant on the atoms of $\mathcal{R}$. $\square$

Using these basic results, we are now able to investigate the meaning of the notions ‘possibility integral’ and ‘necessity integral’. We shall do this by further exploring the analogy between fuzzy variables and measurable sets, already mentioned at the beginning of this section.

In corollary 6 we have shown that the possibility and necessity integral of a fuzzy variable assume a very simple form. Theorems 7, 8 and 9 shed some light on other, related aspects of the combination of fuzzy variables and possibility and necessity integrals. They give us an impression of the ‘empathy’ between seminormed fuzzy integrals and possibility measures, and semiconormed fuzzy integrals and necessity measures. In theorem 7 we investigate the commutativity of the supremum operator and the possibility integral[3], and the infimum operator and the necessity integral. In theorem 8 we give a related result. A comparison between these theorems and corollaries 3 and 4 of the previous section, should give the reader a first indication that the combinations introduced in definition 6 are indeed rather special.

Theorem 7 *Let $(h_j \mid j \in J)$ be an arbitrary family of $(L, \le)$-fuzzy variables in $(X, \mathcal{R})$, and let A be an arbitrary $\mathcal{R}$-measurable set.*

[3]Weber [26] has proven an analogous result for finite suprema for the combination of Sugeno’s fuzzy integral and what he calls ‘*σ-$\smile$-decomposable measures*’. Furthermore, Sugeno [20] has derived an analogous result for his fuzzy integral and what he calls ‘*F-additive fuzzy measures*.’

(i) $(P)\int_A \sup_{j\in J} h_j \mathrm{d}\Pi = \sup_{j\in J} (P)\int_A h_j \mathrm{d}\Pi.$

(ii) $(Q)\int_A \inf_{j\in J} h_j \mathrm{dN} = \inf_{j\in J} (Q)\int_A h_j \mathrm{dN}.$

Proof. We shall give the proof of (i). The proof of (ii) is fairly analogous. Remark that $\sup_{j\in J} h_j$ is a $(L,\leq)$-fuzzy variable in $(X,\mathcal{R})$. We may therefore write, taking into account corollary 6 (i) and the complete distributivity of P w.r.t. supremum,

$$\begin{aligned}(P)\int_A \sup_{j\in J} h_j \mathrm{d}\Pi &= \sup_{x\in A} P((\sup_{j\in J} h_j)(x), \pi(x))\\ &= \sup_{x\in A} P(\sup_{j\in J} h_j(x), \pi(x))\\ &= \sup_{x\in A}\sup_{j\in J} P(h_j(x), \pi(x))\\ &= \sup_{j\in J}\sup_{x\in A} P(h_j(x), \pi(x)) = \sup_{j\in J} (P)\int_A h_j \mathrm{d}\Pi. \quad \square\end{aligned}$$

Theorem 8 *Let $(A_j \mid j\in J)$ be an arbitrary familiy of elements of $\mathcal{R}$, and let h be an arbitrary $(L,\leq)$-fuzzy variable in $(X,\mathcal{R})$.*

(i) $(P)\int_{\bigcup_{j\in J} A_j} h \mathrm{d}\Pi = \sup_{j\in J} (P)\int_{A_j} h \mathrm{d}\Pi.$

(ii) $(Q)\int_{\bigcap_{j\in J} A_j} h \mathrm{dN} = \inf_{j\in J} (Q)\int_{A_j} h \mathrm{dN}.$

Proof. We shall give the proof of (i). The proof of (ii) is immediate, taking into account corollary 5 (ii) and the definition of necessity measures. Taking into account corollary 6 (i) and the associativity of supremum, we may write

$$(P)\int_{\bigcup_{j\in J} A_j} h \mathrm{d}\Pi = \sup_{x\in \bigcup_{j\in J} A_j} P(h(x), \pi(x)) = \sup_{j\in J}\sup_{x\in A_j} P(h(x), \pi(x)) = \sup_{j\in J} (P)\int_{A_j} h \mathrm{d}\Pi. \quad \square$$

Theorem 9 (i) expresses that for $\mathcal{R}$-simple $X-L$ mappings, which are of course also special $(L,\leq)$-fuzzy variables in $(X,\mathcal{R})$, the $(L,\leq)$-possibility integral coincides with the functional used to define it. This result is an analogon of a well-known theorem[4] in the theory of classical measures and integrals (see, for instance, [2]). This is a very interesting point, since Ralescu and Adams [15] have shown that an analogous theorem is *not necessarily* valid in general[5] for fuzzy integrals associated with arbitrary confidence measures.

Theorem 9 *Let A be a $\mathcal{R}$-measurable set and let s be a $\mathcal{R}$-simple $X-L$ mapping.*

[4]Weber [26] has proven an analogous result for the combination of Sugeno's fuzzy integral and what he calls σ-$\smile$-*decomposable measures*.

[5]As a matter of fact, Ralescu and Adams have proven that an analogous result is not valid for the special case of $([0,1],\leq)$-fuzzy min-integrals associated with Sugeno's fuzzy measures, by giving a counterexample. Of course, this counterexample remains valid for the more general result discussed here.

(i) $(P)\int_A s\mathrm{d}\Pi = \alpha^{\Pi}_{\smile P}(A\,;s)$.

(ii) $(Q)\int_A s\mathrm{dN} = \mathrm{N}(A) \frown \beta^{\mathrm{N}}_{\frown Q}(X\,;s)$.

Proof. We shall give the proof of (ii). The proof of (i) is fairly analogous. Since the $\mathcal{R}$-simple mapping s is by definition $\mathcal{R}$-measurable, we may write, taking into account corollary 6 (ii), using the notations of definition 1, the associativity of infimum and the complete distributivity of Q w.r.t. infimum, that

$$\begin{aligned}(Q)\int_A s\mathrm{dN} &= \mathrm{N}(A) \frown \inf_{x\in X} Q(s(x),\nu(x))\\ &= \mathrm{N}(A) \frown \inf_{k=1}^{n} \inf_{x\in D_k} Q(s_k,\nu(x))\\ &= \mathrm{N}(A) \frown \inf_{k=1}^{n} Q(s_k, \inf_{x\in D_k} \nu(x))\\ &= \mathrm{N}(A) \frown \inf_{k=1}^{n} P(s_k,\mathrm{N}(\mathrm{co}D_k))\\ &= \mathrm{N}(A) \frown \beta^{\mathrm{N}}_{\frown Q}(X\,;s). \quad \square\end{aligned}$$

Item (ii) of theorem 9 is not completely analogous to item (i), but is on the other hand a necessary consequence of proposition 9, the fact that the infimum operator and the necessity integral operator commute and the following equality

$$(Q)\int_X s\mathrm{dN} = \beta^{\mathrm{N}}_{\frown Q}(X\,;s).$$

We now have sufficient knowledge about the combination of fuzzy variables and possibility and necessity integrals, to be able to make the next step in our investigation of the analogy between the structures $(\mathcal{R},\subseteq)$ and $(\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X),\sqsubseteq)$: the extension of the notions 'possibility measure' and 'necessity measure'. We shall introduce a generalized kind of possibility and necessity measures, the arguments of which are no longer measurable sets, but measurable *fuzzy* sets, i.e., fuzzy variables.

Definition 7 *A $(L,\leq)$-possibility measure Π' on $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$ is a $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) - L$ mapping satisfying the following requirement: for an arbitrary family $(h_j \mid j\in J)$ of $(L,\leq)$-fuzzy variables in $(X,\mathcal{R})$*

$$\Pi'(\sup_{j\in J} h_j) = \sup_{j\in J} \Pi'(h_j).$$

A $(L,\leq)$-necessity measure N' on $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$ is a $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) - L$ mapping satisfying the following requirement: for an arbitrary family $(h_j \mid j\in J)$ of $(L,\leq)$-fuzzy variables in $(X,\mathcal{R})$

$$\mathrm{N}'(\inf_{j\in J} h_j) = \inf_{j\in J} \mathrm{N}'(h_j).$$

This means that the mapping Π' respectively N' is a complete join-morphism respectively complete meet-morphism between the complete lattices $(\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X),\sqsubseteq)$ and $(L,\leq)$. We shall call Π' normalized iff $\Pi'(\chi_X) = u$, and N' normalized iff $\mathrm{N}'(\chi_\emptyset) = \ell$. Whenever we do not want to be specific about the domain or the codomain of these morphisms, we shall simply speak of extended possibility and necessity measures.

Corollary 7 $\Pi'(\chi_\emptyset) = \ell$ *and* $\mathrm{N}'(\chi_X) = u$.

Proposition 10 indicates in precisely what way the extended possibility and necessity measures are extensions of the ordinary possibility and necessity measures. Its proof is immediate, and is therefore omitted.

Proposition 10 *Let* Π' *be an arbitrary* $(L,\leq)$*-possibility measure on* $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$*. Consider the* $\mathcal{R}-L$ *mapping* Π'' *defined by*

$$(\forall A\in\mathcal{R})(\Pi''(A)\stackrel{\text{def}}{=}\Pi'(\chi_A)),$$

where χ_A *is the characteristic* $X-L$ *mapping of the set* A*. Then* Π'' *is a* $(L,\leq)$*-possibility measure on* $(X,\mathcal{R})$*. Similarly, let* N' *be an arbitrary* $(L,\leq)$*-necessity measure on* $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$*. Consider the* $\mathcal{R}-L$ *mapping* N'' *defined by*

$$(\forall A\in\mathcal{R})(\mathrm{N}''(A)\stackrel{\text{def}}{=}\mathrm{N}'(\chi_A)).$$

Then N'' *is a* $(L,\leq)$*-necessity measure on* $(X,\mathcal{R})$*.*

From proposition 10 we deduce that, starting from an arbitrary extended possibility (necessity) measure, we can easily construct an ordinary possibility (necessity) measure by 'restriction of the domain'. This of course raises the following question: can we construct an extended possibility (necessity) measure starting from an ordinary possibility (necessity) measure? In proposition 11 we show that this is indeed the case, whenever it is possible to define on the complete lattice $(L,\leq)$ a t-seminorm P (t-semiconorm Q), that is completely distributive w.r.t. supremum (infimum).

Proposition 11 *Consider the* $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)-L$ *mapping* Π_P*, defined as*

$$(\forall h\in\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))(\Pi_P(h)\stackrel{\text{def}}{=}(P)\int_X h\mathrm{d}\Pi).$$

Then Π_P *is a* $(L,\leq)$*-possibility measure on* $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$*, and* $\Pi_P(\chi_A)=\Pi(A)$*, for arbitrary* A *in* $\mathcal{R}$*. Similarly, consider the* $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)-L$ *mapping* N_Q*, defined as*

$$(\forall h\in\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))(\mathrm{N}_Q(h)\stackrel{\text{def}}{=}(Q)\int_X h\mathrm{d}\mathrm{N}).$$

Then N_P *is a* $(L,\leq)$*-necessity measure on* $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$*, and* $\mathrm{N}_Q(\chi_A)=\mathrm{N}(A)$*, for arbitrary* A *in* $\mathcal{R}$*.*

Π_P can be considered as an extension of the $(L,\leq)$-possibility measure Π, and N_Q an extension of the $(L,\leq)$-necessity measure N. This leads to the following definition.

Definition 8 *The* $(L,\leq)$*-possibility measure* Π_P *on the structure* $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$ *will be called the* P*-extension of the* $(L,\leq)$*-possibility measure* Π *on* $(X,\mathcal{R})$*. The* $(L,\leq)$*-necessity measure* N_Q *on the structure* $(X,\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))$ *will be called the* Q*-extension of the* $(L,\leq)$*-necessity measure* N *on* $(X,\mathcal{R})$*. Also, for an arbitrary* $(L,\leq)$*-fuzzy variable* h *in* $(X,\mathcal{R})$*,* $\Pi_P(h)$ *is called the* $(L,\leq,P)$*-possibility of* h*, and* $\mathrm{N}_Q(h)$ *the* $(L,\geq,Q)$*-necessity of* h*.*

6 Conclusion

From the results of the previous section we may conclude that possibility integrals enable us to extend the domain of possibility measures from collections of measurable sets towards collections of measurable fuzzy sets. In a similar way, necessity integrals allow the extension of the domain of necessity measures. We are thus led to consider such notions as the *possibility* and the *necessity* of *measurable fuzzy sets*. Moreover, consider an arbitrary dual order-automorphism n on $(L,\leq)$. Also consider a t-seminorm P on $(L,\leq)$, and its dual t-semiconorm $Q\stackrel{\text{def}}{=}P_n$ w.r.t. n. Finally, consider a $(L,\leq)$-possibility measure Π on $(X,\mathcal{R})$, with distribution π. It is easily proven that the dual confidence measure $\mathrm{N}\stackrel{\text{def}}{=}\Pi_n$, defined as

$$(\forall A\in\mathcal{R})(\mathrm{N}(A)\stackrel{\text{def}}{=}n^{-1}(\Pi(\mathrm{co}A))), \tag{14}$$

is a $(L, \leq)$-necessity measure on $(X, \mathcal{R})$, called the *dual* $(L, \leq)$*-necessity measure of* Π *w.r.t.* n [5]. The distribution ν of N satisfies $\nu = n^{-1} \circ \pi$. Furthermore, $P = Q_{n^{-1}}$ and $\Pi = \mathrm{N}_{n^{-1}}$. Then, taking into account theorem 5, we find that for arbitrary h in $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$, with the notations of proposition 11,

$$\mathrm{N}_Q(h) = (Q)\!\!\int_X h \mathrm{dN} = (P_n)\!\!\int_X h \mathrm{d}\Pi_n = n^{-1}((P)\!\!\int_X (n \circ h) \mathrm{d}\Pi) = n^{-1}(\Pi_P(n \circ h)),$$

whence, putting $\mathrm{co}_n h \stackrel{\text{def}}{=} n \circ h$,

$$\mathrm{N}_Q(h) = n^{-1}(\Pi_P(\mathrm{co}_n h)). \tag{15}$$

In this expression, co_n is a truth-functional *complement operator* for $(L, \leq)$-fuzzy sets in X, an obvious extension of the well-known complement operators for $([0,1], \leq)$-fuzzy sets (see, for instance, [5, 12]). Noticing the strong formal analogy between the formulas (14) and (15), we may say that $\mathrm{N}_Q = (\Pi_n)_{P_n}$ can be considered as the dual 'extended' possibility measure of the 'extended' necessity measure Π_P w.r.t. n.

Let us now look at a well-known special case in order to reveal the significance of the results given above. Indeed, choose $(L, \leq) = ([0,1], \leq)$, $\mathcal{R} = \mathcal{P}(X)$—the power class of X—, $P = \min$, $Q = \max$ and let n be the involutive dual order-automorphism on $([0,1], \leq)$ defined by

$$(\forall a \in [0,1])(n(a) \stackrel{\text{def}}{=} 1 - a).$$

It is fairly well known that the t-norm min on the complete chain $([0,1], \leq)$ is completely distributive w.r.t. supremum, and that the t-conorm max on $([0,1], \leq)$ is completely distributive w.r.t. infimum. Also, $\max_n = \min$ and $\min_n = \max$. Let furthermore Π be an arbitrary $([0,1], \leq)$-possibility measure on $(X, \mathcal{P}(X))$—and therefore a possibility measure according to Zadeh [28]—with distribution π, i.e.

$$(\forall x \in X)(\pi(x) \stackrel{\text{def}}{=} \Pi(\{x\})).$$

Finally, let N be the dual $([0,1], \leq)$-necessity measure—and therefore a necessity measure according to Dubois and Prade [8]—of Π w.r.t. n, i.e, $\mathrm{N} \stackrel{\text{def}}{=} \Pi_n$, or since n is involutive

$$(\forall A \in \mathcal{P}(X))(\mathrm{N}(A) = n(\Pi(\mathrm{co}A)) = 1 - \Pi(\mathrm{co}A)).$$

The distribution ν of N satisfies, for arbitrary x in X,

$$\nu(x) = \mathrm{N}(\mathrm{co}\{x\}) = n(\Pi(\{x\})) = 1 - \pi(x).$$

For an arbitrary $([0,1], \leq)$-fuzzy variable h in $(X, \mathcal{P}(X))$—the membership function of a fuzzy set according to Zadeh [28]—we have, with the obvious notations

$$\Pi_{\min}(h) \stackrel{\text{def}}{=} (\min)\!\!\int_X h \mathrm{d}\Pi \text{ and } \mathrm{N}_{\max}(h) \stackrel{\text{def}}{=} (\max)\!\!\int_X h \mathrm{dN} \tag{16}$$

that

$$\begin{cases} \Pi_{\min}(h) = \sup_{x \in X} \min(h(x), \pi(x)) \\ \mathrm{N}_{\max}(h) = \inf_{x \in X} \max(h(x), \nu(x)) = \inf_{x \in X} \max(h(x), 1 - \pi(x)). \end{cases} \tag{17}$$

Formulas (17) are precisely the ones used by Zadeh [28] to define possibility for his type of fuzzy sets, and by by Dubois and Prade to define the notion of necessity for Zadeh's fuzzy sets (see [8] section 1.7, formula (1.65)). In order to arrive at these definitions, Zadeh, Dubois and Prade start with the well-known expressions[6]

$$\begin{cases} \Pi(A) &= \sup_{x \in A} \pi(x) \\ \mathrm{N}(A) &= \inf_{x \in \mathrm{co}A} (1 - \pi(x)) \end{cases} \tag{18}$$

[6] We want to stress that the notion of distribution for necessity measures is not used by Zadeh nor by Dubois and Prade, who only use distributions of possibility measures.

for arbitrary A in $\mathcal{P}(X)$. They remark that these expressions can alo be written as

$$\begin{cases} \Pi(A) &= \sup_{x\in X} \min(\chi_A(x), \pi(x)) \\ \mathrm{N}(A) &= \inf_{x\in X} \max(\chi_A(x), 1-\pi(x)), \end{cases} \tag{19}$$

where, of course, χ_A is the characteristic $X - [0,1]$ mapping of the set A. The step towards the formulas (17) is obvious.

Although our approach and the one followed by Zadeh, Dubois and Prade lead to the same result, we want to stress here that their respective backgrounds are completely different. In the approach of Zadeh et. al. the crucial step is the transition from (19) to (17). In our approach, possibility and necessity integrals play a very important part. Moreover, it combines three important notions in a very natural way: fuzzy sets, possibility-necessity measures and fuzzy integrals.

To conclude this paper, let us briefly discuss the analogy probability-possibility. Although our treatment of fuzzy variables was formally inspired by the theory of the real stochastic variables, there are obvious differences between both notions, especially from the point of view of their interpretation. On the one hand, fuzzy variables have in this paper been interpreted as generalizations, or fuzzifications, of *measurable* sets—they could equally well have been called measurable fuzzy sets. On the other hand, real stochastic variables can be considered as a formal mathematical concretization of the abstract notion 'variable that takes real values'. In probability theory, Lebesgue integrals are used to define the notion of *expectation* or *mean* of real stochastic variables. In the discussion above, possibility (necessity) integrals were used to extend the domain of possibility (necessity) measures from measurable sets towards measurable fuzzy sets.

It is nevertheless possible to introduce the notion of a possibilistic variable, which is a possibilistic instance of the abstract notion of a variable, and is therefore a perfect possibilistic analogon of the stochastic variables, known in probability theory. Using these possibilistic variables and the possibility and necessity integrals discussed here, a general measure- and integral-theoretic treatment of possibility and necessity theory can be given, which unifies a diversity of results in fuzzy set and possibility theory. This treatment includes amongst many other things a consistent discussion of product possibility measures and integrals, and of conditional possibility of possibilistic variables and possibilistic independence, until now an outstanding problem in this field. A detailed account of this theory can be found in the doctoral dissertation of one of us [5], and will be published in detail elsewhere. The groundwork for this unifying treatment of possibility and necessity theory has however been laid in this paper.

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Footnotes

[1] For classical measures, the σ-additivity property holds, which implies that there is no difference between the Lebesgue and Choquet integrals associated with these measures. Differences between these integrals only arise for non-classical measures that are not additive.

[2] We note that the fuzzy variables introduced by Zadeh [28] are more general than the fuzzy variables discussed here. The former are intended as possibilistic analoga of general, not necessarily real-valued, stochastic variables. Furthermore, they cannot in general be interpreted as measurable fuzzy sets. We therefore suggest that the more appropriate name '*possibilistic variable*' be used for Zadeh's notion, and that the name 'fuzzy variables' be reserved for measurable fuzzy sets. For a detailed discussion of possibilistic variables, conditional possibility and possibilistic independence of possibilistic variables, we refer to the doctoral dissertation of one of us [5].

[3] Weber [26] has proven an analogous result for finite suprema for the combination of Sugeno's fuzzy integral and what he calls 'σ-$\smile$-*decomposable measures*'. Furthermore, Sugeno [20] has derived an analogous result for his fuzzy integral and what he calls '*F-additive fuzzy measures*.'

[4] Weber [26] has proven an analogous result for the combination of Sugeno's fuzzy integral and what he calls σ-$\smile$-*decomposable measures*.

[5] As a matter of fact, Ralescu and Adams have proven that an analogous result is not valid for the special case of $([0,1],\leq)$-fuzzy min-integrals associated with Sugeno's fuzzy measures, by giving a counterexample. Of course, this counterexample remains valid for the more general result discussed here.

[6] We want to stress that the notion of distribution for necessity measures is not used by Zadeh nor by Dubois and Prade, who only use distributions of possibility measures.