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A POSSIBILISTIC VIEW ON FUZZY CONTROL

1 Introduction

Possibility theory can be briefly described as the formalism that allows us to mathematically represent and manipulate *linguistic information*. This is the information contained in and conveyed by affirmative and conditional sentences in natural language.

Affirmative sentences are of the type '(subject) IS (predicate)', where the predicate involved can be either crisp (clear) or fuzzy (vague). To give an example, let us look at the affirmative sentence 'boiler pressure is high'. The imprecision and vagueness of the predicate 'high' does not allow us to exactly determine the pressure in the boiler, and yet such a sentence gives us useful information, that can be mathematically represented by a possibility measure, or equivalently, by a possibility distribution.

Conditional sentences are of the type 'IF (condition) THEN (conclusion)', where both the condition and the conclusion are affirmative sentences of the type described above. Because of the possible imprecision and vagueness of the predicates in these sentences, conditional sentences generally yield conditional, non-deterministic (non-functional) relations between the subjects of their condition and conclusion parts. These can be modeled by conditional possibility measures, or equivalently, by conditional possibility distributions.

Possibility theory then tells us how to manipulate linguistic information, or in other words, how to combine possibility measures and conditional possibility measures, and how to project the combined possibility measures on suitable subspaces, in order to obtain appropriate conclusions.

A *possibilistic system* can in this context be described as a system for which the available information about its behaviour is of a linguistic nature, i.e., is a collection of affirmative and conditional sentences of the type described above. Possibility theory allows us to study the behaviour and the properties of such systems in a mathematical way.

What we call a *generalized fuzzy controller* is made up of three parts: a fuzzifier, a possibilistic system, and a defuzzifier. In teaching fuzzy control, we feel that it is important to stress the possibilistic nature of such fuzzy controllers, to explain the link that exists between these controllers and linguistic information.

With this paper, we mainly intend to serve a didactical aim, in giving a brief outline of how the workings of the interior, possibilistic, part of generalized fuzzy controllers can, in our view, be best explained to students. In section 2, we show how linguistic information can be mathematically represented by possibility measures and/or distributions. In section 3, we show how approximate reasoning can be done by properly manipulating these possibility measures. In section 4 we explain how a number of existing fuzzy controllers fit into this possibilistic picture.

2 Representation of linguistic information

2.1 Informationless v-systems

Let us consider a universe, or nonempty set, U and a variable u that assumes values in this universe. In other words, U is the set of all *a priori* possible values that the variable u may assume. We shall call the couple $\langle U, u \rangle$ a *v-system* (for variable system). If we have no further information about the values that u may take in U, we shall say that $\langle U, u \rangle$ is *informationless*.

As an example, let the variable w be the weight of a particular boy of 6 months old called Wietse, and let W be the set of possible body weights. Saying that $\langle W, w \rangle$ is informationless is now really the same thing as saying that we know nothing at all about Wietse's weight.

We shall now look at ways of *adding information* to v-systems.

2.2 Probabilistic information

First of all, a familiar way in which information can be added to the informationless v-system $\langle U, u \rangle$, is by specifying a probability measure Prob on $(U, \mathcal{S}_U)$, where $\mathcal{S}_U$ is a σ-algebra of subsets of U, with the following interpretation: for any A in $\mathcal{S}_U$, $\text{Prob}(A)$ is the probability that u assumes a value in A. In this case, we can say that we have added stochastic information to $\langle U, u \rangle$, and we can call the corresponding v-system $\langle U, u, \text{Prob} \rangle$ a *stochastic* v-system.

To illustrate this, let us again consider the v-system $\langle W, w \rangle$ representing Wietse's weight. Knowing that the boy is six month's old, we might take a look at the (Gaussian) statistical weight distribution for babies of this age, and add the probabilistic information contained in this curve to the v-system $\langle W, w \rangle$. We would then be able to calculate, say, for any interval $[a, b]$ of body weights, the probability $\text{Prob}([a, b])$ that Wietse's weight lies in that interval.

Until the advent of the new uncertainty paradigm [8], stochastic or probabilistic information was the only kind of information that was recognized, or considered worthy of investigation. In the last few decades, however, it has become recognized that, besides this probabilistic information, there exist other kinds, among which the linguistic information first recognized as such by Zadeh [12].

2.3 Affirmative linguistic information: the crisp case

In fact, the easiest way to add information to an informationless v-system $\langle U, u \rangle$ is the following. Consider an arbitrary subset E of U, i.e., $E \subseteq U$, or $E \in \wp(U)$, where $\wp(U)$ denotes the power set of the universe U. Then, clearly, we add information to $\langle U, u \rangle$ by stating that *u must be an element of E*, because we impose a restriction on the values that u may assume in U.

Let us again consider the example v-system $\langle W, w \rangle$, and an interval $[a, b]$ of body weights. By stating that $w \in [a, b]$, or

Wietse's weight w is between a and b,

we restrict the set of weights that we know are candidates for Wietse's actual weight.

Since the information we add to the v-system takes the form of a statement of the type

(subject) IS (crisp predicate),

and can be expressed in natural language, it is natural to call this type of information '*affirmative linguistic information*'. By adding the information '$u \in E$' to the informationless v-system $\langle U, u \rangle$, we obtain a *(crisp) linguistic v-system* $\langle U, u, E \rangle$.

This linguistic information can be mathematically represented in a number of equivalent ways. The first representation is of course the set E, also called the *certain set*. A second representation is the *characteristic mapping* or indicator function $\chi_E \colon U \to \{0,1\}$, with for any x in U:

$$\chi_E(x) = \begin{cases} 1 & ; \quad x \in E \\ 0 & ; \quad x \notin E. \end{cases}$$

There is also a third equivalent representation, that is closely linked with *possibility theory* [12, 2, 3, 4, 7]. Indeed, consider an arbitrary subset A of U, then there are three possibilities.

- $A \cap E = \emptyset$
  Since $u \in E$, we conclude that in this case it is *impossible* that $u \in A$.
- $A \cap E = \emptyset$, $E \not\subseteq A$
  Since we know that $u \in E$, we find that in this case it is *possible*, but not *necessary* that $u \in A$.
- $E \subseteq A$
  Since $u \in E$, we have in this case that it is *necessary* that $u \in A$.

This observation leads to the introduction of two $\wp(U) - \{0,1\}$-mappings Pos_E and Nec_E, with, for any A in $\wp(U)$:

$$\mathrm{Pos}_E(A) = \begin{cases} 1 & ; \quad A \cap E \neq \emptyset \\ 0 & ; \quad A \cap E = \emptyset \end{cases} \text{ and } \mathrm{Nec}_E(A) = \begin{cases} 1 & ; \quad E \subseteq A \\ 0 & ; \quad E \not\subseteq A. \end{cases}$$

$\mathrm{Pos}_E(A)$ can clearly be interpreted as the *possibility* that u belongs to A, given the information that u belongs to E, and $\mathrm{Nec}_E(A)$ as the *necessity* that u belongs to A, given that u belongs to E. Pos_E is called a *classical possibility measure* and Nec_E a *classical necessity measure*. These set mappings satisfy the following interesting properties, where y is any element of U, A any element of $\wp(U)$ and $(A_j \mid j \in J)$ any family of elements of $\wp(U)$.

$$\mathrm{Nec}_E(A) = 1 - \mathrm{Pos}_E(\mathrm{co}A) \tag{1}$$

$$\mathrm{Pos}_E(A) = \sup_{x \in A} \chi_E(x) \tag{2}$$

$$\mathrm{Nec}_E(A) = \inf_{x \in \mathrm{co}A} \chi_{\mathrm{co}E}(x) \tag{3}$$

$$\chi_E(y) = \mathrm{Pos}_E(\{y\}) = 1 - \mathrm{Nec}_E(\mathrm{co}\{y\}) \tag{4}$$

$$\mathrm{Pos}_E(\bigcup_{j \in J} A_j) = \sup_{j \in J} \mathrm{Pos}_E(A_j) \tag{5}$$

$$\mathrm{Nec}_E(\bigcap_{j \in J} A_j) = \inf_{j \in J} \mathrm{Nec}_E(A_j) \tag{6}$$

If we take into account (1) and (4), we see that E, χ_E, Pos_E and Nec_E are equivalent mathematical representations of the given linguistic information that $u \in E$, because any of them can unequivocally be determined from any other. χ_E is also called the *distribution* of the possibility measure Pos_E, and also the possibility distribution of the variable u. In what follows, we shall mainly use the distribution representation, and we shall also denote the crisp linguistic v-system by $\langle U, u, \chi_E \rangle$.

2.4 Affirmative linguistic information: the fuzzy case

Let us now direct our attention to a more general problem. Let us assume that we have linguistic information about the values that u assumes in U of the following form

$$\text{(subject) IS (predicate)} \tag{7}$$

where the subject is of course u and the predicate can be either clear (crisp) or vague (fuzzy). If the predicate is clear, we have seen in the previous subsection that it can be mathematically represented by a subset E of U. If the predicate is also allowed to be vague, we need a more general and powerful mathematical representation, provided by the notion of a fuzzy set [11]. We shall indeed assume that this predicate can be mathematically represented by a *fuzzy set* h in X, that is, a $U-[0,1]$-mapping with, for any x in U:

$h(x)$ is the degree to which x satisfies the given predicate.

Of course, h is an obvious generalization to the fuzzy case of the characteristic mapping χ_E in the crisp case, and will also be called the *certain fuzzy set*. In very much the same way as E acts as a (hard) restriction on the values that u may assume, in the vague case h acts as an 'elastic' or fuzzy restriction on the values that u may take in U. Thus, by giving the affirmative linguistic information (7) or equivalently 'u IS h, we have added information to our informationless v-system. The *linguistic v-system* thus obtained will be denoted by $\langle U, u, h \rangle$.

As in the crisp case, there are a number of equivalent ways in which the given affirmative linguistic information may be mathematically represented. It was argued by Zadeh [12] and elaborated more recently in [1, 5] that the given information may be represented by the $\wp(U)-[0,1]$-mappings Pos_h and Nec_h, defined by, for any A in $\wp(U)$:

$$\mathrm{Pos}_h(A) = \sup_{x \in A} h(x) \tag{8}$$

$$\mathrm{Nec}_h(A) = \inf_{x \in \mathrm{co}A} (1 - h(x)). \tag{9}$$

$\mathrm{Pos}_h(A)$ will be called the *possibility* that $u \in A$, given the information that u IS h, and $\mathrm{Nec}_h(A)$ the *necessity* that $u \in A$, given the information that u IS h. Pos_h is called the *possibility measure* and Nec_h the *necessity measure* representing the affirmative linguistic information 'u IS h'. They are immediate generalizations of the classical possibility and necessity measures introduced in the previous subsection, and satisfy the following important properties, where y is any element of U, A is any element of $\wp(U)$ and $(A_j \mid j \in J)$ is any family of elements of $\wp(U)$.

$$\mathrm{Nec}_h(A) = 1 - \mathrm{Pos}_h(\mathrm{co}A) \tag{10}$$

$$h(y) = \mathrm{Pos}_h(\{y\}) = 1 - \mathrm{Nec}_h(\mathrm{co}\{y\}) \tag{11}$$

$$\mathrm{Pos}_h(\bigcup_{j \in J} A_j) = \sup_{j \in J} \mathrm{Pos}_h(A_j) \tag{12}$$

$$\mathrm{Nec}_h(\bigcap_{j \in J} A_j) = \inf_{j \in J} \mathrm{Nec}_h(A_j) \tag{13}$$

The above equations tell us that h, Pos_h and Nec_h are equivalent representations of the given linguistic information. h is often called the *distribution* of Pos_h, and also the possibility distribution of the variable u. Remark that, in denoting the linguistic v-system by $\langle U, u, h\rangle$, we are as before using the distribution representation of linguistic information. Also remark that, for any x in U, $h(x)$ is the possibility that $u = x$.

2.5 Linguistic v-systems

These considerations lead to a general definition of a linguistic, or possibilistic, v-system $\langle U, u, h\rangle$ as a v-system $\langle U, u\rangle$ that is provided with a possibility distribution h, i.e., a $U - [0, 1]$-mapping with the following interpretation: for any x in U

$h(x)$ is the possibility that $u = x$.

h is to be regarded as a mathematical representation of some affirmative linguistic information in the form (7).

Needless to say, this definition includes the multidimensional case, where we have a number, say n, of universes $U_1, \ldots, U_n$ and a number of variables $u_1, \ldots, u_n$ assuming values in these respective universes. All we have to do is put $u = (u_1, \ldots, u_n)$ and $U = U_1 \times \cdots \times U_n$. h will then also be called the *joint possibility distribution* of the variables $u_1, \ldots, u_n$.

2.6 Conditional linguistic information

Now that we know how to mathematically represent affirmative linguistic information, we are ready to take a second important conceptual step: the representation of *conditional* linguistic information. In general, this type of information consists of statements of the form

$$\text{IF (condition) THEN (conclusion).} \tag{14}$$

where both (condition) and (conclusion) are affirmative statements of type (7).

Let us first look at the case where the predicates in both (condition) and (conclusion) are crisp. In general, we may start with an informationless v-system $\langle U \times V, (u, v)\rangle$ and assume that (14) takes the form

$$\text{IF } u \in A \text{ THEN } v \in B, \tag{15}$$

where A is a subset of U and B a subset of V. Statement (15) imposes a restriction on the values that u and v may assume, that is equivalent to specifying the *certain set*

$$E = A \times B \cup \text{co}A \times V.$$

We thus find that (15) leads to a linguistic v-system $\langle U \times V, (u, v), \chi_E\rangle$, where the corresponding (conditional) possibility distribution of the variable (u, v) is given by, for any x in U and y in V:

$$\chi_E(x, y) = \chi_A(x) \to \chi_B(y), \tag{16}$$

where $\to$ is the binary implication operator on the Boolean chain $(\{0, 1\}, \leq)$, defined by the following table.

$\to$	0	1
0	1	1
1	0	1

Let us now look at the fuzzy case, where (14) takes the form

$$\text{IF } u \text{ IS } h \text{ THEN } v \text{ IS } g, \tag{17}$$

where h is a fuzzy set in U and g a fuzzy set in V. On the basis of (16), we find that (17) leads to a linguistic v-system $\langle U \times V, (u, v), r\rangle$, where the corresponding *certain fuzzy set*, or (conditional) possibility distribution r of the variable (u, v), is given by

$$r(x, y) = I(h(x), g(x)), \tag{18}$$

where I is a properly chosen implication operator on $[0, 1]$, that is, a properly chosen extension of the binary implication operator $\rightarrow$ to the real unit interval $[0, 1]$.

We are thus led to the important conclusion that in general, both affirmative and conditional linguistic information can be represented by *certain (fuzzy) sets*, or equivalently, by possibility distributions.

3 Manipulation of linguistic information

In this section, we address the question of how linguistic information can be manipulated, i.e., how we can combine, extend and propagate such information.

3.1 Combination of linguistic information

Consider an informationless v-system $\langle U, u\rangle$, and assume that we have linguistic information about the values that u assumes in U from two different sources. The question we now want to address is how this information form both sources can be combined.

We shall first consider the crisp case. It is clear that we now have two linguistic v-systems $\langle U, u, \chi_{E_1}\rangle$ and $\langle U, u, \chi_{E_2}\rangle$, where E_1 and E_2 are elements of $\wp(U)$ that are a mathematical representation of the crisp linguistic information from the respective sources: $u \in E_1$ and $u \in E_2$. χ_{E_1} and χ_{E_2} can also be interpreted as possibility distributions of the variable u. We can combine the given information by observing that

$$(u \in E_1 \text{ and } u \in E_2) \Leftrightarrow u \in E_1 \cap E_2,$$

so that we arrive at a new linguistic v-system $\langle U, u, \chi_{E_1 \cap E_2}\rangle$. From the distribution point of view, we find for the combined possibility distribution $\chi_{E_1 \cap E_2}$ of u that for any x in U:

$$\chi_{E_1 \cap E_2}(x) = \chi_{E_1}(x) \wedge \chi_{E_2}(x), \tag{19}$$

where $\wedge$ is the meet of the Boolean chain $(\{0, 1\}, \leq)$.

In the fuzzy case, we have two linguistic information v-systems $\langle U, u, h_1\rangle$ and $\langle U, u, h_2\rangle$, where h_1 and h_2 are fuzzy sets in U, i.e., $U - [0, 1]$-mappings, that are mathematical representations of the fuzzy linguistic information coming from the respective sources: u IS h_1 and u IS h_2. They can also be interpreted as two possibility distributions for the variable u. Drawing inspiration from the crisp case, we see that the combined information can be obtained by taking an appropriate intersection $h = h_1 \sqcap h_2$ of the fuzzy sets h_1 and h_2, and we get a new linguistic v-system $\langle U, u, h_1 \sqcap h_2\rangle$. For the combined possibility distribution $h_1 \sqcap h_2$ of u we find that, for any x in U:

$$(h_1 \sqcap h_2)(x) = T(h_1(x), h_2(x)), \tag{20}$$

where T is a *triangular norm* [6, 9, 10], an obvious extension of $\wedge$ from $\{0, 1\}$ to $[0, 1]$.

3.2 Extension of linguistic information

Assume now that we have two informationless v-systems $\langle U_1, u_1\rangle$ and $\langle U_2, u_2\rangle$, or equivalently, an informationless v-system $\langle U_1 \times U_2, (u_1, u_2)\rangle$. Suppose that we add linguistic information to one of the v-systems, say, $\langle U_1, u_1\rangle$. What is the linguistic information that is thus effectively added to the v-system $\langle U_1 \times U_2, (u_1, u_2)\rangle$?

Again, we shall first consider the crisp case, in which the information added to $\langle U_1, u_1\rangle$ is can be mathematically represented by a subset E_1 of U_1, i.e., $u_1 \in E_1$, and yields a crisp linguistic v-system $\langle U_1, u_1, \chi_{E_1}\rangle$. χ_{E_1} is the possibility distribution of the variable u_1. For the informationless v-system $\langle U_2, u_2\rangle$, we of course know that $u_2 \in U_2$, that is, we may identify $\langle U_2, u_2\rangle$ with $\langle U_2, u_2, \chi_{U_2}\rangle$. Remark that this is in complete accordance with (19). We may now combine the given information by observing that

$$(u_1 \in E_1 \text{ and } u_2 \in U_2) \Leftrightarrow (u_1, u_2) \in E_1 \times U_2$$

so that we arrive at a new linguistic v-system $\langle U_1 \times U_2, (u_1, u_2), \chi_{E_1 \times U_2}\rangle$. From the distribution point of view, we find for the possibility distribution $\chi_{E_1 \times U_2}$ of (u_1, u_2) that for any (x_1, x_2) in $U_1 \times U_2$:

$$\chi_{E_1 \times U_2}(x_1, x_2) = \chi_{E_1}(x_1) = \overline{\chi_{E_1}}(x_1, x_2) \tag{21}$$

where $\overline{\chi_{E_1}}$ is the well-known cylindrical extension of χ_{E_1} to $U_1 \times U_2$.

In the fuzzy case, the information added to $\langle U_1, u_1\rangle$ can be mathematically represented by a fuzzy set h_1 in U_1, i.e., u_1 IS h_1, and yields a fuzzy linguistic v-system $\langle U_1, u_1, h_1\rangle$. h_1 is the possibility distribution of the variable u_1. For the informationless v-system $\langle U_2, u_2\rangle$, we of course know that $u_2 \in U_2$, that is, we may identify $\langle U_2, u_2\rangle$ with $\langle U_2, u_2, \chi_{U_2}\rangle$. Remark that this is in complete accordance with (20). We may now combine the given information by generalizing (21). From the distribution point of view, we find for the possibility distribution $h_1 \times \chi_{U_2}$ of (u_1, u_2) that for any (x_1, x_2) in $U_1 \times U_2$:

$$(h_1 \times \chi_{U_2})(x_1, x_2) = h_1(x_1) = \overline{h_1}(x_1, x_2) \tag{22}$$

where $\overline{h_1}$ is the cylindrical extension of h_1 to $U_1 \times U_2$. We then find a new linguistic v-system $\langle U_1 \times U_2, (u_1, u_2), \overline{h_1}\rangle$.

We are now ready to look at a problem that is a little more complicated. Let us at once consider the fuzzy case, the crisp variant can always be derived from it as a special case by identifying sets and their characteristic functions. Suppose that we have two linguistic v-systems $\langle U_1, u_1, h_1\rangle$ and $\langle U_2, u_2, h_2\rangle$, where h_k is a fuzzy set in U_k, $k = 1, 2$. How can the information given be represented in terms of the *product v-system* $\langle U_1 \times U_2, (u_1, u_2)\rangle$?

Using (22), we find two linguistic v-systems, $\langle U_1 \times U_2, (u_1, u_2), \overline{h_1}\rangle$ and $\langle U_1 \times U_2, (u_1, u_2), \overline{h_2}\rangle$. using (20), we may combine these into the linguistic v-system $\langle U_1 \times U_2, (u_1, u_2), \overline{h_1} \sqcap \overline{h_2}\rangle$, where $\overline{h_1} \sqcap \overline{h_2}$ is the combined possibility distribution of the variable (u_1, u_2), given by, for any (x_1, x_2) in $U_1 \times U_2$

$$(\overline{h_1} \sqcap \overline{h_2})(x_1, x_2) = (h_1 \times h_2)(x_1, x_2) = T(\overline{h_1}(x_1, x_2), \overline{h_2}(x_1, x_2)) = T(h_1(x_1), h_2(x_2)). \tag{23}$$

3.3 Propagation of linguistic information

Finally, let us address the question of how linguistic or possibilistic information can be transported from one universe to another using a mapping. Let us consider an informationless v-system $\langle U_1, u_1 \rangle$ and a universe U_2, together with a mapping f from U_1 to U_2. Since u_1 is a variable in U_1, we may interpret $f(u_1)$ as a variable assuming values in U_2, that is, we have a second informationless v-system $\langle U_2, f(u_1) \rangle$.

Now assume that we add linguistic information to $\langle U_1, u_1 \rangle$, i.e., in some (crisp or fuzzy) way we restrict the values that u_1 may assume in U_1. We thus effectively add linguistic information to $\langle U_2, f(u_1) \rangle$, because the restriction on u_1 entails a corresponding restriction on $f(u_1)$. How can we represent the linguistic information added in this way to $\langle U_2, f(u_1) \rangle$?

As usual, let us first take a look at the crisp case, and assume that we have a linguistic v-system $\langle U_1, u_1, \chi_{E_1} \rangle$, with $E_1 \subseteq U_1$. Since

$$u_1 \in E_1 \Rightarrow f(u_1) \in f(E_1),$$

with $f(E_1) = \{f(x_1) \mid x_1 \in E_1\}$ the direct image of E_1 under f, we find that we may write the second linguistic v-system as $\langle U_2, f(u_1), \chi_{f(E_1)} \rangle$. Furthermore, we find for the possibility distribution $\chi_{f(E_1)}$ of the variable $f(u_1)$, for any x_2 in U_2:

$$\chi_{f(E_1)}(x_2) = \sup_{f(x_1)=x_2} \chi_{E_1}(x_1) \tag{24}$$

This may immediately be extended to the fuzzy case. Assume that we have a linguistic v-system $\langle U_1, u_1, h_1 \rangle$, where h_1 is a fuzzy set in U_1. In generalizing (24), we find that we may write the second linguistic v-system as $\langle U_2, f(u_1), f(h_1) \rangle$, with, for the possibility distribution $f(h_1)$ of the variable $f(u_1)$, for any x_2 in U_2:

$$f(h_1)(x_2) = \sup_{f(x_1)=x_2} h_1(x_1) \tag{25}$$

To conclude this section, let us consider the important special case of *projection* of linguistic information. Let us immediately go to the case of fuzzy linguistic information, because it contains crisp linguistic information as a special subcase. We start with the linguistic v-system $\langle U_1 \times U_2, (u_1, u_2), h \rangle$, where h is a fuzzy set in $U_1 \times U_2$, that can also be considered as the joint possibility distribution of the variables u_1 and u_2. Of course, the restriction imposed on (u_1, u_2) also imposes restrictions on one of the variables, say, u_2. How can this derived restriction be represented? If we consider the *projection operator*

$$\mathrm{proj}_2\colon U_1 \times U_2 \to U_2\colon (x_1, x_2) \mapsto x_2,$$

it is clear that $\mathrm{proj}_2(u_1, u_2) = u_2$, so that we may use the method explained above to find a linguistic v-system $\langle U_2, u_2, \mathrm{proj}_2(h) \rangle$, with, for the possibility distribution $\mathrm{proj}_2(h)$ of u_2 (also called marginal possibility distribution), for any x_2 in U_2:

$$\mathrm{proj}_2(h)(x_2) = \sup_{\mathrm{proj}_2(x_1', x_2') = x_2} h(x_1', x_2') = \sup_{x_1 \in U_1} h(x_1, x_2). \tag{26}$$

4 An application: fuzzy control

To conclude this paper, let us show how this discussion of linguistic v-systems fits into the framework of fuzzy control. It is well known that any fuzzy controller is made up of three parts: the fuzzifier, the central reasoning system (CRS) and the defuzzifier. We focus our attention here on the central part of a fuzzy controller, the reasoning system CRS.

Let us consider a controller with one input variable u taking values in a set U, and one output variable v assuming values in a set V. Our CRS is constructed using a number n of rules ρ_k, $k = 1, \ldots, n$ of the form

ρ_k: IF u IS h_k THEN v IS g_k

Each rule ρ_k conveys conditional linguistic information that can be mathematically represented by a certain fuzzy set r_k in $U \times V$. According to (18), we may write, for any (x, y) in $U \times V$:

$$r_k(x, y) = I(h_k(x), g_k(y)).$$

Furthermore, using (20), the conditional linguistic information conveyed by the rules ρ_k, i.e., coming from different sources, can be combined into conditional linguistic information (*the rule information*) represented by a certain fuzzy set r in $U \times V$, given by $r_1 \sqcap \cdots \sqcap r_n$, or equivalently, for any (x, y) in $U \times V$:

$$r(x, y) = T_{k=1}^{n} I(h_k(x), g_k(y)), \tag{27}$$

where we use the obvious associative extension of the triangular norm T. This means that the information given by the rules leads to a linguistic v-system $\langle U \times V, (u, v), r\rangle$.

On the other hand, the CRS receives from the fuzzifier a fuzzy set i on U, that can be interpreted as a fuzzy restriction (the *input information*) on the values that the controller input u may assume in the input space U. That is, the fuzzifier provides the CRS with a linguistic v-system $\langle U, u, i\rangle$. At the same time, it will send to the defuzzifier a fuzzy set o on V, that can be interpreted as a fuzzy restriction (*the output information*) on the values that the controller output v may assume in the output space V. That is, the CRS feeds the defuzzifier with a linguistic v-system $\langle V, v, o\rangle$.

The question that we now address, is how the CRS transforms the linguistic v-system $\langle U, u, i\rangle$ into the linguistic v-system $\langle V, v, o\rangle$, or, in other words, the input information into the output information, using the rule information. Using (22), we know that $\langle U, u, i\rangle$ may be transformed into $\langle U \times V, (u, v), \bar{i}\rangle$. This may be combined with $\langle U \times V, (u, v), r\rangle$, using (20), to form the v-system $\langle U \times V, (u, v), r \sqcap \bar{i}\rangle$. $r \sqcap \bar{i}$ is the representation of the combination of input and rule information,and must now be projected on the output space V to obtain the output information. Using an appropriate modification of (26), we obtain the v-system $\langle V, v, o\rangle$, where $o = \mathrm{proj}_2(r \sqcap \bar{i})$, or equivalently, for any y in V:

$$o(y) = \sup_{x \in U} T(i(x), T_{k=1}^{n} I(h_k(x), g_k(y)).$$

which is the input-output relation of the CRS that we were looking for.

Thus it turns out that using the notion of linguistic or possibilistic information, we find a very interesting formula for the input-output behaviour of the central reasoning system of a fuzzy controller, that furthermore generalizes a number of input-output behaviours extant in the literature.

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