A New Approach to Possibilistic Independence

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Abstract— **The introduction of a notion of independence in possibility theory is a problem of long-standing interest. The definitions that have up to now been given in the literature face some difficulties as far as interpretation is concerned. Also, there are inconsistencies between the definition of independence of measurable sets and possibilistic variables. After a discussion of these definitions and their shortcomings, we suggest a new definition, that is consistent in this respect. Furthermore, we show that in the special case of classical, two-valued possibility our definition has a straightforward and natural interpretation.**

Keywords— **possibility theory, possibilistic variables, possibilistic independence, logical independence.**

I. Introduction

SINCE the introduction of possibility theory by Zadeh [12] in 1978, various attempts have been made to define what we shall call here *possibilistic independence*, a notion which is intended to be a possibilistic analogon of the probabilistic notion of stochastic independence. In Section II, we give a brief discussion of the definitions already present in the literature, and point out a few of their shortcomings as far as consistency and interpretation is concerned. In Sections III and IV we give a new definition of the possibilistic independence of possibilistic variables and measurable sets, and derive important results. In Section V, we discuss the interpretation of our definitions in the special case of classical, two-valued possibility. Section VI concludes this paper.

We start with some notational conventions and the introduction of a few basic notions, needed for the proper understanding of what follows. In this paper, we shall denote by $(L,\leq)$ a complete lattice, with smallest element ℓ and greatest element u. We furthermore assume that $\ell \neq u$. The meet of $(L,\leq)$ will be denoted by $\frown$, its join by $\smile$. By T we shall denote an arbitrary t-norm on $(L,\leq)$, i.e., a commutative, associative and isotonic binary operator on L, that satisfies the boundary condition $(\forall\lambda \in L)(T(\lambda,u)=\lambda)$. For a detailed account of triangular norms on complete lattices, or, more general, on bounded posets, we refer to [2], [3], [4].

By Ω we shall denote an arbitrary universe, called the *basic space*. By $\mathcal{R}_\Omega$ we shall mean an arbitrary *ample field* on the universe Ω, i.e., a set of subsets of Ω that is closed under arbitrary unions and intersections, and under complementation. Elements of $\mathcal{R}_\Omega$ will be called *$\mathcal{R}_\Omega$-measurable sets*. The *atom* of $\mathcal{R}_\Omega$ containing the element ω of Ω will be denoted by $[\,\omega\,]_{\mathcal{R}_\Omega}$:

$$[\,\omega\,]_{\mathcal{R}_\Omega} \stackrel{\text{def}}{=} \bigcap \{\, A \mid \omega \in A \text{ and } A \in \mathcal{R}_\Omega \,\}.$$

For a more detailed account of ample fields—also called complete fields—we refer to [3], [5], [11].

By Π_Ω, we shall mean an arbitrary $(L,\leq)$-*possibility measure* on $(\Omega,\mathcal{R}_\Omega)$, i.e., a complete join-morphism between the complete lattices $(\mathcal{R}_\Omega,\subseteq)$ and $(L,\leq)$. By definition, this means that Π_Ω satisfies the following requirement: for an arbitrary family $(A_j \mid j \in J)$ of elements of $\mathcal{R}_\Omega$

$$\Pi_\Omega(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi_\Omega(A_j).$$

This immediately implies that $\Pi_\Omega(\emptyset)=\ell$. We will assume in this paper that Π_Ω is *normalized*, i.e., $\Pi_\Omega(\Omega)=u$. The $\Omega - L$ mapping π_Ω, defined by $(\forall\omega \in \Omega)(\pi_\Omega(\omega) \stackrel{\text{def}}{=} \Pi_\Omega([\,\omega\,]_{\mathcal{R}_\Omega}))$ will be called the *distribution* of Π_Ω. Remark that it completely determines Π_Ω, since for arbitrary A in $\mathcal{R}_\Omega$:

$$\Pi_\Omega(A) = \sup_{\omega\in A} \pi_\Omega(\omega).$$

$(L,\leq)$-possibility measures are of course generalizations towards more general domains and image sets of Zadeh's possibility measures [12], Wang's contactabilities [11], and the possibility measures we introduced in [2]. For a more detailed discussion of these notions, we refer to [2], [3], [5].

By X we shall denote an arbitrary universe, henceforth called *sample space*, and by $\mathcal{R}$ an arbitrary ample field on X. A $\Omega - X$ mapping f will be called a *possibilistic variable* in $(X,\mathcal{R})$ iff f is *$\mathcal{R}_\Omega - \mathcal{R}$-measurable*, i.e.,

$$(\forall A \in \mathcal{R})(f^{-1}(A) \in \mathcal{R}_\Omega).$$

Of course, possibilistic variables are possibilistic analoga of the stochastic variables defined in probability theory (see, for instance, [1]). Also, they are also formalizations of the more intuitive *fuzzy variables* introduced by Zadeh [12]. In accordance with Zadeh's nomenclature, the $(L,\leq)$-possibility measure Π_f on $(X,\mathcal{R})$, defined by

$$(\forall A \in \mathcal{R})(\Pi_f(A) \stackrel{\text{def}}{=} \Pi_\Omega(f^{-1}(A))),$$

is called the *possibility distribution* of f. Its distribution π_f is called the *possibility distribution function* of f, and is of course defined by $(\forall x \in X)(\pi_f(x) \stackrel{\text{def}}{=} \Pi_\Omega(f^{-1}([\,x\,]_{\mathcal{R}})))$. For a more detailed discussion of possibilistic variables, and their importance in a measure- an integral-theoretic approach to possibility theory, we refer to [3].

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II. Possibilistic Independence in the Literature

The notion of possibilistic independence was first introduced by Zadeh in relation with his treatment of *conditional possibility* [12]. His work was later refined by Hisdal [7]. Let us briefly discuss their approach[1].

Zadeh starts with two universes X_1 and X_2. He furthermore considers ξ_1 and ξ_2 as variables that take values in respectively X_1 and X_2. (ξ_1, ξ_2) is also a variable, taking values in $X_1 \times X_2$. Information about the values that (ξ_1, ξ_2) takes in $X_1 \times X_2$, is given in the form of a $([0,1], \leq)$-possibility measure $\Pi_{(\xi_1,\xi_2)}$ on $(X_1 \times X_2, \wp(X_1 \times X_2))$, with distribution $\pi_{(\xi_1,\xi_2)}$. $\Pi_{(\xi_1,\xi_2)}$ is called the possibility distribution of the variable (ξ_1, ξ_2). For arbitrary (x_1, x_2) in $X_1 \times X_2$, $\pi_{(\xi_1,\xi_2)}(x_1, x_2)$ is the possibility that (ξ_1, ξ_2) takes the value (x_1, x_2). Zadeh furthermore defines the $([0,1], \leq)$-possibility measure Π_{ξ_1} on $(X_1, \wp(X_1))$ with distribution π_{ξ_1} by

$$(\forall A_1 \in \wp(X_1))(\Pi_{\xi_1}(A_1) \stackrel{\text{def}}{=} \Pi_{(\xi_1,\xi_2)}(A_1 \times X_2))$$

or equivalently

$$(\forall x_1 \in X_1)(\pi_{\xi_1}(x_1) \stackrel{\text{def}}{=} \sup_{x_2 \in X_2} \pi_{(\xi_1,\xi_2)}(x_1, x_2)),$$

and completely analogously for the $([0,1], \leq)$-possibility measure Π_{ξ_2} on $(X_2, \wp(X_2))$ with distribution π_{ξ_2}. Π_{ξ_1} and Π_{ξ_2} are called by Zadeh and Hisdal the *marginal possibility distributions* of ξ_1 and ξ_2 respectively. Furthermore, Zadeh calls the variables ξ_1 and ξ_2 *noninteractive* iff

$$\begin{gathered}(\forall (A_1, A_2) \in \wp(X_1 \times X_2)) \\ (\Pi_{(\xi_1,\xi_2)}(A_1, A_2) = \min(\Pi_{\xi_1}(A_1), \Pi_{\xi_2}(A_2))),\end{gathered} \qquad (1)$$

or equivalently

$$\begin{gathered}(\forall (x_1, x_2) \in X_1 \times X_2) \\ (\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \min(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2))).\end{gathered} \qquad (2)$$

According to Zadeh this non-interactivity can be considered as an analogon of the notion 'stochastic independence' in probability theory.

From this we may conclude that Zadeh and Hisdal concentrate on the possibilistic independence of *variables*, and do not discuss the possibilistic independence of *measurable sets*.

The notion of possibilistic independence has also been studied by Nahmias [9], Rao and Rashed [10], and Wang [11], albeit under different names. Let us briefly discuss their approach. Nahmias considers a universe X and what in this paper is called a $([0,1], \leq)$-possibility measure[2] Π on $(X, \wp(X))$. He calls the elements $A_1, \ldots, A_n$ $(n \in \mathbb{N}^*)$ of $\wp(X)$ *mutually unrelated* iff for arbitrary k in $\mathbb{N}^*$ with $k \leq n$, for arbitrary and different $j_1, \ldots, j_k$ in $\{1, \ldots, n\}$:

$$\Pi(\bigcap_{\ell=1}^{k} A_{j_\ell}) = \min_{\ell=1}^{k} \Pi(A_{j_\ell}). \qquad (3)$$

He draws inspiration from an analogous formula for the stochastic independence of events in probability theory (see, for instance, [1] Section 11-4). Furthermore, he calls two $X - \mathbb{R}$ mappings[3] f_1 and f_2 *unrelated* iff for arbitrary a and b in $\mathbb{R}^2$:

$$\Pi(f_1^{-1}(\{a\}) \cap f_2^{-1}(\{b\})) = \min(\Pi(f_1^{-1}(\{a\})), \Pi(f_2^{-1}(\{b\}))). \qquad (4)$$

Rao and Rashed point out that according to (3) every element A of $\wp(X)$ is mutually unrelated to itself, since

$$\Pi(A) = \Pi(A \cap A) = \min(\Pi(A), \Pi(A)).$$

With reason, they are a bit worried about the interpretation of this result, and point out that in probability theory an analogous result cannot be derived. Therefore, they suggest to use the phrase min-*related* instead of 'mutually unrelated'.

Wang generalizes Nahmias' notion of a fuzzy variable by considering an ample field $\mathcal{R}$ on X, and calling a $X - \mathbb{R}$ mapping a fuzzy variable iff it is $\mathcal{R} - \wp(\mathbb{R})$-measurable. His definition of *independence* for such fuzzy variables is essentially the same as Nahmias' definition of unrelatedness.

In our opinion, the approach of Nahmias et al. faces a number of difficulties. On the one hand, there is the interpretational difficulty already laid bare by Rao and Rashed. That every event is mutually unrelated to itself—indeed, to each of its subsets—is, to say the least, a little strange. Giving the notion another name in order to evade this interpretational difficulty seems to us to completely miss the point, because it fails to explain how such a radical difference in interpretation can emerge between stochastic independence and this new notion.

On the other hand, we know that events can always be associated—through their characteristic mappings—with special fuzzy (or, in our terminology, possibilistic) variables. None of the above-mentioned authors answers the question whether there exists a relation between the mutual unrelatedness of events and the unrelatedness (or independence) of their characteristic mappings. The existence of an analogous relation is a central idea in probability theory. Moreover, we shall show furtheron that such a relationship starting from the formulas (3) and (4) does not in general exist. Interestingly, this difficulty appears to be linked with the interpretational problem discussed above.

In this paper, we shall construct a more general theory of possibilistic independence, and at the same time give a solution for the above-mentioned difficulties. Although our guiding principle in doing this will be the (formal) analogy with probability theory, we shall also show that our definition of possibilistic independence has an interesting interpretation.

[1] We shall concentrate on Zadeh's work. Hisdal seems to distinguish between what she calls 'possibilistic independence' and Zadeh's 'non-interactivity', but as one of us has shown in [3], her definition of possibilistic independence suffers from internal inconsistencies. A more detailed assessment of Hisdals approach to possibilistic independence and, more interestingly, conditional possibility will be published elsewhere.

[2] Nahmias uses the name *scale*.

[3] Nahmias calls these mappings *fuzzy variables*. Of course, they are possibilistic variables in $(\mathbb{R}, \wp(\mathbb{R}))$ in the sense of the previous section.

III. Independence of Possibilistic Variables

We start our treatment of possibilitic independence with a basic definition, of which the other definitions are in fact special cases.

Definition 1: Consider a non-empty family $(\mathcal{O}_{|} \mid | \in \mathcal{J})$ of subsets of $\mathcal{R}_\Omega$. This family is called (Π_Ω, T)-independent iff for arbitrary n in $\mathbb{N}^*$, for arbitrary and different $j_1, \ldots, j_n$ in J, for arbitrary F_k in $\mathcal{O}_{|_{||}}$ and for arbitrary G_k in $\{F_k, \mathrm{coF_k}\}$ $(k \in \{1, \ldots, n\})$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} G_k) = T_{k=1}^{n} \Pi_\Omega(G_k). \tag{5}$$

Whenever we do not want to be explicit about the possibility measure Π_Ω and/or the t-norm T, we shall simply speak of (possibilistic) independence instead of (Π_Ω, T)-independence.

Why we propose exactly this definition becomes clear when we look at the analogous definition of stochastic independence of families of measurable sets (see, for instance, [8], Definition VI.6.1, [1] Section 11-5).

Definition 2: Let $(\Omega, \mathcal{S}_\Omega, \mathrm{Pr}_\Omega)$ be a probability space. Consider a non-empty family $(\mathcal{O}_{|} \mid | \in \mathcal{J})$ of subsets of $\mathcal{S}_\Omega$. This family is called independent (for Pr_Ω) iff for arbitrary n in $\mathbb{N}^*$, for arbitrary and different $j_1, \ldots, j_n$ in J, for arbitrary F_k in $\mathcal{O}_{|_{||}}$ $(k \in \{1, \ldots, n\})$:

$$\mathrm{Pr}_\Omega(\bigcap_{k=1}^{n} F_k) = \prod_{k=1}^{n} \mathrm{Pr}_\Omega(F_k). \tag{6}$$

We point out that Definition 1 is very similar to this definition, with the exception of one important detail: in the probabilistic definition the phrase 'for arbitrary G_k in $\{F_k, \mathrm{coF_k}\}$' does not appear, and (5) is modified accordingly. Let us briefly indicate the reason for our including this phrase in Definition 1.

Let A and B be arbitrary elements of $\mathcal{S}_\Omega$, and assume that

$$\mathrm{Pr}_\Omega(A \cap B) = \mathrm{Pr}_\Omega(A)\mathrm{Pr}_\Omega(B).$$

Using the additivity and complementation laws for probability measures, we immediately deduce that

$$\mathrm{Pr}_\Omega(A \cap \mathrm{coB}) = \mathrm{Pr}_\Omega(\mathrm{A})\mathrm{Pr}_\Omega(\mathrm{coB}).$$

From this we conclude that the special properties of probability measures render the above-mentioned phrase redundant: the *independence formula* (6) is *formally invariant* under the complementation of an arbitrary number of subsets in the family $(F_k \mid k \in \{1, \ldots, n\})$.

On the other hand, it is shown in the following example that an analogous line of reasoning is not necessarily valid in the possibilistic case. In order to render the definition of possibilistic independence invariant under complementation, we must explicitly add the phrase 'for arbitrary G_k in $\{F_k, \mathrm{coF_k}\}$' to the definition of independence, and correspondingly turn the independence formula (6) into the independence formula (5).

Example 1: We make the following choices: $\Omega = \{1, 2, 3\}$, $\mathcal{R}_\Omega = \wp(\{1, 2, 3\})$, $(L, \leq) = ([0, 1], \leq)$, $T = \min$. Let the $([0, 1], \leq)$-possibility measure Π_Ω be completely determined by $\pi_\Omega(1) \stackrel{\mathrm{def}}{=} 1$, $\pi_\Omega(2) \stackrel{\mathrm{def}}{=} 1$, $\pi_\Omega(3) \stackrel{\mathrm{def}}{=} \frac{1}{2}$. Finally, let $A = \{1\}$ and $B = \{1, 2\}$. Using these assumptions, it is easily verified that

$$\Pi_\Omega(A \cap B) = \min(\Pi_\Omega(A), \Pi_\Omega(B)),$$

whereas

$$\Pi_\Omega(A \cap \mathrm{coB}) \neq \min(\Pi_\Omega(\mathrm{A}), \Pi_\Omega(\mathrm{coB})).$$

From Definition 1 we derive the following definition for the possibilistic independence of possibilistic variables (analogon for probability: [8] Definition VI.6.5). In Theorem 1 a few criteria for this new form of possibilistic independence are deduced.

Definition 3: Consider a non-empty family $(X_j \mid j \in J)$ of universes (sample spaces). For every j in J we consider an ample field $\mathcal{R}_j$ on X_j and a $\mathcal{R}_\Omega - \mathcal{R}_j$-measurable $\Omega - X_j$ mapping f_j, i.e., f_j is a possibilistic variable in $(X_j, \mathcal{R}_j)$. We call the family $(f_j \mid j \in J)$ of possibilistic variables (Π_Ω, T)-independent iff the family $(f_j^{-1}(\mathcal{R}_j) \mid j \in J)$ of subsets of $\mathcal{R}_\Omega$ is (Π_Ω, T)-independent.

Theorem 1: The following propositions are equivalent:
1. the family $(f_j \mid j \in J)$ of possibilistic variables is (Π_Ω, T)-independent;
2. for arbitrary n in $\mathbb{N}^*$, for arbitrary and different $j_1, \ldots, j_n$ in J, for arbitrary A_k in $\mathcal{R}_{j_k}$ $(k \in \{1, \ldots, n\})$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}(A_k)) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}(A_k));$$

3. for arbitrary n in $\mathbb{N}^*$, for arbitrary and different $j_1, \ldots, j_n$ in J, for arbitrary x_k in X_{j_k} $(k \in \{1, \ldots, n\})$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}([x_k]_{\mathcal{R}_{j_k}})) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}([x_k]_{\mathcal{R}_{j_k}})).$$

In Theorem 2, we consider the special case of two possibilistic variables. The analogy with the probabilistic case is striking. Point 3 of this theorem shows that our definition of the independence of possibilistic variables is a generalization of the above-mentioned definitions of Nahmias et al. (see formula (4)). Theorem 2 can also be considered as a formalization and generalization of the case considered by Zadeh and Hisdal, which we also briefly discussed in the previous section (see formulas (1) and (2)). In order to understand this theorem, a few remarks are due. Let us consider two arbitrary possibilistic variables f_1 and f_2 in $(X_1, \mathcal{R}_1)$ and $(X_2, \mathcal{R}_2)$ respectively. Consider the ample field $\mathcal{R}_1 \times \mathcal{R}_2$ on $X_1 \times X_2$, i.e., the smallest ample field on $X_1 \times X_2$ containing the subset

$$\{\, A_1 \times A_2 \mid A_1 \in \mathcal{R}_1 \text{ and } A_2 \in \mathcal{R}_2 \,\}$$

of $\wp(X_1 \times X_2)$, and called the *product ample field* of $\mathcal{R}_1$ and $\mathcal{R}_2$ [5], [11]. Then it is easily proven [3] that the $\Omega - X_1 \times X_2$ mapping (f_1, f_2) is a possibilistic variable in $(X_1 \times X_2, \mathcal{R}_1 \times \mathcal{R}_2)$. Let the possibility distributions of f_1, f_2 and (f_1, f_2)

be denoted by Π_{f_1}, Π_{f_2} and $\Pi_{(f_1,f_2)}$ respectively, and their possibility distribution functions by π_{f_1}, π_{f_2} and $\pi_{(f_1,f_2)}$ respectively.

Theorem 2: The following propositions are equivalent:
1. f_1 and f_2 are (Π_Ω, T)-independent;
2. for arbitrary A_1 in $\mathcal{R}_1$ and arbitrary A_2 in $\mathcal{R}_2$:

$$\Pi_{(f_1,f_2)}(A_1 \times A_2) = T(\Pi_{f_1}(A_1), \Pi_{f_2}(A_2));$$

3. for arbitrary x_1 in X_1 and arbitrary x_2 in X_2:

$$\pi_{(f_1,f_2)}(x_1, x_2) = T(\pi_{f_1}(x_1), \pi_{f_2}(x_2)).$$

IV. Independence of Measurable Sets

Consider an arbitrary subset A of Ω. With this set we can associate the *characteristic mapping* χ_A, i.e., the $\Omega - \{0,1\}$-mapping defined by

$$\chi_A(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & ; \quad \omega \in A \\ 0 & ; \quad \omega \notin A, \end{cases}$$

for arbitrary ω in Ω. We can interpret $\{0,1\}$ as a sample space and consider the power class $\wp(\{0,1\})$ as an ample field of measurable sets on $\{0,1\}$. It is obvious that A is $\mathcal{R}_\Omega$-measurable if and only if its characteristic mapping χ_A is $\mathcal{R}_\Omega - \wp(\{0,1\})$-measurable. This means that an arbitrary element A of $\mathcal{R}_\Omega$ can be identified with a possibilistic variable χ_A in $(\{0,1\}, \wp(\{0,1\}))$. We use this identification in Definition 4 to introduce the notion of possibilistic independence of measurable sets. In Theorem 3 we derive a criterion for the possibilistic independence of measurable sets.

Definition 4: Consider a family $(A_j \mid j \in J)$ of elements of $\mathcal{R}_\Omega$. We call this family (Π_Ω, T)-independent iff the family $(\chi_{A_j} \mid j \in J)$ of possibilistic variables is (Π_Ω, T)-independent.

Theorem 3: A family $(A_j \mid j \in J)$ of elements of $\mathcal{R}_\Omega$ is (Π_Ω, T)-independent if and only if for arbitrary n in $\mathbb{N}^*$, for arbitrary and different $j_1, \ldots, j_n$ in J, for arbitrary F_k in $\{A_{j_k}, \mathrm{coA_{j_k}}\}$ $(k \in \{1, \ldots, n)$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} F_k) = T_{k=1}^{n} \Pi_\Omega(F_k).$$

In Proposition 1 we derive a criterion for the possibilistic independence of two measurable sets. We point out that this criterion involves four conditions instead of only one in the probabilistic case. This is necessary to make sure that the criterion be invariant under the complementation of the measurable sets involved. With this in mind, we also call the reader's attention to Proposition 2.

Proposition 1: Let O_1 and O_2 be two $\mathcal{R}_\Omega$-measurable sets. They are (Π_Ω, T)-independent if and only if

$$\begin{cases} \Pi_\Omega(O_1 \cap O_2) & = & T(\Pi_\Omega(O_1), \Pi_\Omega(O_2)) \\ \Pi_\Omega(O_1 \cap \mathrm{coO_2}) & = & T(\Pi_\Omega(O_1), \Pi_\Omega(\mathrm{coO_2})) \\ \Pi_\Omega(\mathrm{coO_1} \cap \mathrm{O_2}) & = & T(\Pi_\Omega(\mathrm{coO_1}), \Pi_\Omega(\mathrm{O_2})) \\ \Pi_\Omega(\mathrm{coO_1} \cap \mathrm{coO_2}) & = & T(\Pi_\Omega(\mathrm{coO_1}), \Pi_\Omega(\mathrm{coO_2})). \end{cases}$$

Furthermore, for arbitrary O in $\mathcal{R}_\Omega$, the events $\emptyset$, O and Ω are (Π_Ω, T)-independent.

Proposition 2: Let $(A_j \mid j \in J)$ be a family of elements of $\mathcal{R}_\Omega$ and let $(A'_j \mid j \in J)$ be a family of elements of $\mathcal{R}_\Omega$, obtained by the substitution for their complements of an arbitrary number of elements of the first family. Then the family $(A_j \mid j \in J)$ is (Π_Ω, T)-independent if and only if the family $(A'_j \mid j \in J)$ is (Π_Ω, T)-independent.

We point out that our definition of possibilistic independence of measurable sets is different form Nahmias' definition of mutual unrelatedness, even allowing for our more general formulation. Indeed, it follows from the proposition above that our definition is invariant under complementation of an arbitrary number of measurable sets, whereas Example 1 shows that this is not the case for Nahmias' definition. We also stress that this invariance property follows from our definition of the independence of possibilistic variables, of which Nahmias' mutual unrelatedness of fuzzy variables is a special case. Furthermore, the approach we have used to derive the independence of measurable sets from the independence of possibilistic variables is the identification of a measurable set with its characteristic mapping.

Finally, consider an arbitrary element A of $\mathcal{R}_\Omega$. Taking into account Proposition 1, the measurable sets A and A are shown to be (Π_Ω, T)-independent if and only if

$$\begin{cases} T(\Pi_\Omega(A), \Pi_\Omega(A)) & = & \Pi_\Omega(A) \\ T(\Pi_\Omega(\mathrm{coA}), \Pi_\Omega(\mathrm{coA})) & = & \Pi_\Omega(\mathrm{coA}) \\ T(\Pi_\Omega(A), \Pi_\Omega(\mathrm{coA})) & = & \ell. \end{cases}$$

For $(L, \leq) = ([0,1], \leq)$ and $T = \min$ this may be written as $\min(\Pi_\Omega(A), \Pi_\Omega(\mathrm{coA})) = 0$, or equivalently,

$$\Pi_\Omega(A) = 0 \text{ or } \Pi_\Omega(\mathrm{coA}) = 0,$$

a result that is very similar to the probabilistic case.

We conclude this section with a theorem that indicates that there exists yet another relation between the independence of possibilistic variables and measurable sets.

Theorem 4: Let f_1 be a possibilistic variable in $(X_1, \mathcal{R}_1)$ and let f_2 be a possibilistic variable in $(X_2, \mathcal{R}_2)$. Then f_1 and f_2 are (Π_Ω, T)-independent if and only if for every A_1 in $\mathcal{R}_1$ and A_2 in $\mathcal{R}_2$ the elements $A_1 \times X_2$ and $X_1 \times A_2$ of $\mathcal{R}_1 \times \mathcal{R}_2$ are $(\Pi_{(f_1,f_2)}, T)$-independent.

V. Classical Possibility

In the previous section, we have shown that our definition of possibilistic independence is an improvement of the existing definitions from the formal mathematical point of view: not only there exists a close relationship between the independence of possibilistic variables and the independence of measurable sets, but also in the well-known special case studied by Nahmias a measurable set is not as a rule possibilistically independent of itself.

In this section, we illustrate that our definition is also meaningful from the interpretational point of view. In order to do this, we shall study the meaning of our definition in the special and well-known case of *classical, two-valued possibility.* We therefore choose $(L, \leq) = (\{0,1\}, \leq)$,

$T = \frown$ and $\Pi_\Omega = \Pi_A$, A being an arbitrary but fixed element of $\mathcal{R}_\Omega \setminus \{\emptyset\}$. More explicitly, for arbitrary B in $\mathcal{R}_\Omega$:

$$\Pi_A(B) \stackrel{\text{def}}{=} \begin{cases} 1 & ; \quad A \cap B \neq \emptyset \\ 0 & ; \quad A \cap B = \emptyset \end{cases}$$

In order to facilitate the interpretation of our results, we briefly discuss the interpretation of the classical possibility measure Π_A. Let us consider an experiment E for which there exists uncertainty about the value that the outcome o takes in the universe Ω of the possible outcomes. $\mathcal{R}_\Omega$ is the set of the measurable sets (events) associated with this experiment. Let us assume that we have the following information about the outcome of the experiment: *o takes a value in the set A.* This information can be represented by the $(\{0,1\}, \leq)$-possibility measure Π_A. Indeed, for an arbitrary B in $\mathcal{R}$, it is—taking into account the given information—*possible* that o takes a value in B if and only if $\Pi_A(B) = 1$, and *impossible* if and only if $\Pi_A(B) = 0$. Let us therefore use the following nomenclature. We shall say that the event B is *impossible* iff $\Pi_A(B) = 0$, *necessary* iff $\Pi_A(\mathrm{coB}) = 0$, and *strictly possible* whenever it is not necessary nor impossible, i.e., whenever $\Pi_A(B) = 1$ and $\Pi_A(\mathrm{coB}) = 1$. In this last case and only then, there is *uncertainty* about the occurrence of the event B.

Let us first look for an interpretation of the $(\Pi_A, \frown)$-independence of two arbitrary events B and C. According to Proposition 1, a necessary and sufficient condition for the independence of B and C is

$$\begin{cases} \Pi_A(B \cap C) & = & \Pi_A(B) \frown \Pi_A(C) \\ \Pi_A(B \cap \mathrm{coC}) & = & \Pi_A(B) \frown \Pi_A(\mathrm{coC}) \\ \Pi_A(\mathrm{coB} \cap \mathrm{C}) & = & \Pi_A(\mathrm{coB}) \frown \Pi_A(\mathrm{C}) \\ \Pi_A(\mathrm{coB} \cap \mathrm{coC}) & = & \Pi_A(\mathrm{coB}) \frown \Pi_A(\mathrm{coC}). \end{cases} \tag{7}$$

We shall try and turn (7) into an equivalent form, that is more easily interpreted. For arbitrary D and E in $\mathcal{R}_\Omega$ it is easily proven that '$\Pi_A(D \cap E) = \Pi_A(D) \frown \Pi_A(E)$' is equivalent to '$A \cap D \cap E \neq \emptyset$ or $A \cap D = \emptyset$ or $A \cap \mathrm{coD} = \emptyset$ or $A \cap E = \emptyset$ or $A \cap \mathrm{coE} = \emptyset$'. This implies (after a few elementary algebraic manipulations) that B and C are $(\Pi_A, \frown)$-independent if and only if

$$\left.\begin{array}{r} \Pi_A(B) = 1 \\ \Pi_A(\mathrm{coB}) = 1 \\ \Pi_A(C) = 1 \\ \Pi_A(\mathrm{coC}) = 1 \end{array}\right\} \Rightarrow \begin{cases} A \cap B \cap C \neq \emptyset \\ A \cap \mathrm{coB} \cap \mathrm{C} \neq \emptyset \\ A \cap B \cap \mathrm{coC} \neq \emptyset \\ A \cap \mathrm{coB} \cap \mathrm{coC} \neq \emptyset. \end{cases}$$

This means that on the one hand the events B and C are possibilistically independent as soon as one of them is *impossible* or *necessary.* On the other hand, whenever both events are *strictly possible*, they are possibilistically independent if and only if

$$\begin{cases} A \cap C_1 \neq \emptyset \\ A \cap C_2 \neq \emptyset \\ A \cap C_3 \neq \emptyset \\ A \cap C_4 \neq \emptyset, \end{cases} \tag{8}$$

where the events $C_1 \stackrel{\text{def}}{=} B \cap C$, $C_2 \stackrel{\text{def}}{=} \mathrm{coB} \cap \mathrm{C}$, $C_3 \stackrel{\text{def}}{=} B \cap \mathrm{coC}$ and $C_4 \stackrel{\text{def}}{=} \mathrm{coB} \cap \mathrm{coC}$ are called the *constituents* of B and C (see, for instance, [6]).

Let us now assume that the occurrence of B and C is possible but not necessary—and therefore uncertain—, and look for an interpretation of (8). From our assumption, we easily deduce that

$$\begin{cases} A \not\subseteq C_1 \\ A \not\subseteq C_2 \\ A \not\subseteq C_3 \\ A \not\subseteq C_4. \end{cases} \tag{9}$$

Indeed, assume that for instance $A \subseteq C_3$, or equivalently,

$$\emptyset = A \cap \mathrm{co}(\mathrm{B} \cap \mathrm{coC}) = (\mathrm{A} \cap \mathrm{coB}) \cup (\mathrm{A} \cap \mathrm{C}).$$

This implies that $A \cap \mathrm{coB} = \emptyset$ and $A \cap C = \emptyset$, which contradicts our assumption that B and C are strictly possible.

We may therefore conclude that in this case (8) holds if and only if the constituents C_1, C_2, C_3 and C_4 are *strictly possible*, i.e., are uncertain events. In this case, the uncertain events B and C are in the literature called *logically independent* (see, for instance, [6] Section 2.7). *This logical independence means that additional knowledge about the occurrence of either event B or C can on no account change the existing uncertainty about the occurrence of the other event.*

To illustrate this, let us assume that (8) holds and that we know that the event B occurs, in other words $o \in A \cap B$. We now ask ourselves if this additional information can remove the uncertainty about the occurrence of C. This question must be answered in the negative: taking into account (8), we have that $A \cap B \cap C \neq \emptyset$ and $A \cap B \cap \mathrm{coC} \neq \emptyset$, which means that the outcome o can *a priori* belong to C as well as to coC. The uncertainty about the occurrence of C is preserved due to (8).

We may therefore conclude that two arbitrary events B and C are $(\Pi_A, \frown)$-independent if and only if at least one of them is either necessary or impossible, or, whenever they are both strictly possible, if and only if they are logically independent.

Let us now turn to the $(\Pi_A, \frown)$-independence of two *possibilistic variables*, and find out whether this also has a simple interpretation. We shall use the assumptions and notations introduced in Theorem 2, and consider the possibilistic variables f_1 and f_2 in $(X_1, \mathcal{R}_1)$ and $(X_2, \mathcal{R}_2)$ respectively. We shall however in what follows assume that $\mathcal{R}_1 = \wp(X_1)$ and $\mathcal{R}_2 = \wp(X_2)$, which implies that $\mathcal{R}_1 \times \mathcal{R}_2 = \wp(X_1 \times X_2)$. This simplification does not appreciably alter the conclusions we are about to reach, and does make the following discussion far less technical, because considerations of measurability can be omitted altogether. For a more complete treatment of this case, that takes into account aspects of measurability, we refer to the doctoral dissertation of one of us [3].

Let us turn the criterion for the $(\Pi_A, \frown)$-independence of f_1 and f_2 into an equivalent form, that is more readily interpreted. It is easily verified that $\pi_{f_1} = \chi_{f_1(A)}$, $\pi_{f_2} = \chi_{f_2(A)}$ and $\pi_{(f_1,f_2)} = \chi_{(f_1,f_2)(A)}$, where for instance, $\chi_{f_1(A)}$ is the characteristic mapping of the direct image of A under f_1. The interpretation of these formulas is immediate: since

o must belong to A, we know for sure that $f_1(o)$ belongs to $f_1(A) = \{ f_1(\omega) \mid \omega \in A \}$. For f_2 and (f_1, f_2) we have analogous conclusions. Using Theorem 2 we find that the $(\Pi_A, \frown)$-independence of f_1 and f_2 is equivalent to

$$(f_1, f_2)(A) = f_1(A) \times f_2(A). \tag{10}$$

Using a few straightforward set-theoretical manipulations, it is easily proven that (10) implies the proposition:

$$(\forall A_1 \in \mathcal{R}_1) \\ (A_1 \cap f_1(A) \neq \emptyset \Rightarrow f_2(A) \subseteq f_2(f_1^{-1}(f_1(A) \cap A_1))). \tag{11}$$

We are now able to give an interpretation to the possibilistic independence of f_1 and f_2. Indeed, assume that (11) holds. Furthermore, assume that we have additional information about the value that $f_1(o)$ takes in X_1, e.g., $f_1(o)$ belongs to a set A_1 in $\mathcal{R}_1$. Is it possible to deduce from this fact information about the value that $f_2(o)$ takes in X_2 that is more specific than the information we had from the outset, namely that $f_2(o)$ belongs to $f_2(A)$? We shall show that this is not the case. We know that the outcome o satisfies

$$f_1(o) \in f_1(A) \cap A_1.$$

This implies that we know the following about $f_2(o)$:

$$f_2(o) \in f_2(f_1^{-1}(f_1(A) \cap A_1)). \tag{12}$$

But, looking at (11) we find that the additional information about the value $f_1(o)$ takes in X_1 leads to additional information (12) about the value $f_2(o)$ takes in X_2 that can never be more restrictive than the information already present in π_{f_2}. *In other words, additional information about the values of either one of the possibilistic variables* f_1 *and* f_2 cannot change *the uncertainty about the values the other possibilistic variable takes.* In the literature it is said in this case that these variables are *logically independent* (see, for instance, [6] Subsection 2.7.5).

We conclude from this discussion that in the special case of classical, two-valued possibility our possibilistic independence is very closely related to the notion of 'logical independence'.

VI. Conclusion

In this paper, we have discussed the definitions of possibilistic independence of events and variables extant in the literature, and in a few cases pointed out some of their shortcomings.

Starting with a very general definition of possibilistic independence, and a discussion of its relation with the probabilistic definition, we have deduced as special cases a definition for the possibilistic independence of *variables*, which generalizes and formalizes the definition given by Zadeh [12]; and a definition of the possibilistic independence of *events*, which is different from the one found in the literature [9], [10]. It is furthermore shown that our definition of the independence of events does not have the interpretational difficulties that the other definitions in the literature are faced with; and that in the case of classical, two-valued possibility, our definition of independence leads to a notion that is very closely related with 'logical independence' [6].

Of course, a treatment of possibilistic independence cannot be complete without a discussion of the notion of conditional possibility, and of the relation between both notions. This is however beyond the scope of this paper. A thorough treatment of this subject can be found in the doctoral dissertation of one of us [3] and will be published in detail elsewhere.

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