A POSSIBILISTIC UNCERTAINTY MODEL IN CLASSICAL RELIABILITY THEORY

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ABSTRACT

In this paper, it is argued that a possibilistic uncertainty model can be used to represent linguistic uncertainty about the states of a system and of its components. Furthermore, the basic properties of the application of this model to classical reliability theory are studied. The notion of the possibilistic reliability of a system or a component is defined. Based upon the concept of a binary structure function, the important notion of a possibilistic structure function is introduced. It allows us to calculate the possibilistic reliability of a system in terms of the possibilistic reliabilities of its components.

1. Introduction

The classical approach to reliability theory has two important characteristics. On the one hand, *binary structure functions* are used to model the logical structure of systems. Let us consider a system made up of n components. The classical state space of the system and each of its components is the set $\mathcal{S} = \{\mathit{fail}, \mathit{work}\}$. The state x_s of the system can be expressed as a function ϕ of the states of its components x_k, where $\phi: \mathcal{S}^n \to \mathcal{S}$, is the binary structure function of the system under consideration.

On the other hand, probability theory is used to represent the available information about the states of the components. This means that for each component the probability r_k is given that the component does not fail during a given time interval. This probability is called the *reliability* of the component. Using standard probability-theoretic methods, the reliability of the system r_s can be calculated from the reliabilities of the components. Under a few assumptions, amongst which the stochastic independence of the states of the components, there exists a function $\varphi: [0,1]^n \to [0,1]$, such that $r_s = \varphi(r_1, \ldots, r_n)$. Furthermore, there is a consistent way in which φ can be determined from ϕ. For a more detailed account of how to calculate φ if ϕ is known, we refer to the work of Barlow and Proschan[2].

In this paper, we show that analogous results are obtained when the information about the states of the components is not probabilistic in nature, but rather a linguistic assessment, that can be translated into a possibilistic description. This means that for each component we start from a *possibilistic reliability* π_{x_k}. It is shown using our measure- and integral-theoretic account of possibility theory[4] that, under a few assumptions, there exists a function ψ, such that the possibilistic reliability π_{x_s} of the system is given by $\pi_{x_s} = \psi(\pi_{x_1}, \ldots, \pi_{x_n})$. Furthermore, a consistent method is given for determining ψ from ϕ.

2. Systems, Components, States and Binary Structure Functions

Let us consider* a system S with n ($n \in \mathbb{N}^*$) components C_k ($k \in \{1, \ldots, n\}$). We shall assume that this system and each of its components can only be in either one of the two possible *states work* and *fail*. The *state set* of the system and of each of its components is therefore $\mathcal{S} \overset{\text{def}}{=} \{\mathit{fail}, \mathit{work}\}$. We can define a total order relation $\preceq$ on $\mathcal{S}$ by $\mathit{fail} \prec \mathit{work}$. Of course, $(\mathcal{S}, \preceq)$ is a Boolean chain. The meet of this chain will be denoted by $\wedge$, the join by $\vee$.

We shall also assume that there exists a set Par_S of *parameter combinations* that completely determine (or describe) the workings and therefore also the states of the system and of its components. In particular, this implies that there exists a *system state mapping* $x_s \colon \mathrm{Par}_S \to \mathcal{S}$, which maps any parameter combination p_s to the corresponding state $x_s(p_s)$ of the system. Similarly, there exist n *component state mappings* $x_k \colon \mathrm{Par}_S \to \mathcal{S}$, which map any parameter combination p_s into the corresponding state $x_k(p_s)$ of component C_k ($k \in \{1, \ldots, n\}$). Equivalently, we could say that there exists an *assembly state mapping* $(x_1, \ldots, x_n) \colon \mathrm{Par}_S \to \mathcal{S}^n$, which maps any parameter combination p_s to the corresponding state $(x_1(p_s), \ldots, x_n(p_s))$ of the *assembly*† $(C_1, \ldots, C_n)$.

Finally, we shall make an assumption which is central in classical, two-valued reliability theory, namely, that there exists a mapping $\phi \colon \mathcal{S}^n \to \mathcal{S}$ which maps the states of the components to the corresponding state of the system:

$$x_s = \phi \circ (x_1, \ldots, x_n). \tag{1}$$

ϕ is called the *binary structure function* of the system S. We want to stress at this point that Eq. 1 is an equality of mappings. It is furthermore assumed that ϕ is isotonic – if the components work better, the system as a whole cannot do worse –, that $\phi(\mathit{fail}, \ldots, \mathit{fail}) = \mathit{fail}$ – if all the components fail then the system fails – and that $\phi(\mathit{work}, \ldots, \mathit{work}) = \mathit{work}$ – if all the components work then the system works. It should be noted that the binary structure function ϕ provides us with a logical-structural rather than a physical model of the system S. This structural model can

*For a good treatment of the formal definitions of systems, components and states, we refer to the doctoral dissertation of one of us[3].

†We use the new term 'assembly' here, because the components of the system are considered one by one, without reference to their final place in the system, as it were before the system is assembled.

at least in principle be derived from the physical model, of which the parameter set Par_S is one of the aspects.

A minimal set of components, such that if these components fail, the system S fails, is called a *minimal cut (set)* of the system S. Dually, a minimal set of components, such that if these components work, the system S works, is called a *minimal path (set)* of the system S. With obvious notations, a minimal cut will be denoted by C_l, ($l \in \{1, \ldots, n_c\}$), where n_c is the number of minimal cuts of the system. Similarly, a minimal path will be denoted by P_r, ($r \in \{1, \ldots, n_p\}$), where n_p is the number of minimal paths of the system. Birnbaum, Esary and Saunders[1] have proven the following decompositions of ϕ in minimal paths respectively cuts:

$$\phi(\nu_1, \ldots, \nu_n) = \bigvee_{1 \leq r \leq n_p} \bigwedge_{i \in P_r} \nu_i = \bigwedge_{1 \leq l \leq n_c} \bigvee_{i \in C_l} \nu_i. \tag{2}$$

3. Basic Notions from Possibility Theory

In this section, we very briefly introduce the basic notions necessary for the proper understanding of the rest of this paper. For a more detailed introduction to the measure- and integral-theoretic treatment of possibility theory, we refer to the doctoral dissertation of one of us[4].

In what follows, we shall denote by $(L, \leq)$ a *complete lattice*. The greatest element $\inf \emptyset$ of $(L, \leq)$ will be denoted by u, its smallest element $\sup \emptyset$ by ℓ.

A binary operator T on L that is commutative, associative, isotonic and which satisfies the boundary condition: $(\forall \lambda \in L)(T(\lambda, u) = \lambda)$, is called a *triangular norm*[4,5] (or t-norm) on $(L, \leq)$. If the triangular norm T on $(L, \leq)$ is *completely distributive* w.r.t. supremum, i.e., if for an arbitrary λ in L and for an arbitrary family $(\mu_j \mid j \in J)$ of elements of L:

$$\sup_{j \in J} T(\lambda, \mu_j) = T(\lambda, \sup_{j \in J} \mu_j),$$

we shall call the structure $(L, \leq, T)$ a *complete lattice with t-norm*[4,5].

A mapping h from a universe X into the set L will be called a $(L, \leq)$-*fuzzy set* on X, or simply a fuzzy set on X. In this paper, we shall call such a fuzzy set h *normalized* iff $\sup_{x \in X} h(x) = u$.

An ample field[4,6] $\mathcal{R}$ on an universe X is a collection of subsets of X, that is closed under complementation and under arbitrary unions. Of course, it contains $\emptyset$ and X, and is also closed under arbitrary intersections. A $X - L$-mapping is called $\mathcal{R}$-*measurable* iff it is constant on the atoms of $\mathcal{R}$. Let us furthermore consider two universes X and Y, provided with the respective ample fields $\mathcal{R}_X$ and $\mathcal{R}_Y$. Then a $X - Y$-mapping f is called $\mathcal{R}_X - \mathcal{R}_Y$-*measurable* iff $(\forall B \in \mathcal{R}_Y)(f^{-1}(B) \in \mathcal{R}_X)$.

Let us consider a universe X and an ample field $\mathcal{R}$ of subsets of X. A $(L, \leq)$-*possibility measure* Π *on* $(X, \mathcal{R})$ is a $\mathcal{R} - L$-mapping such that for an arbitrary family

$(A_j \mid j \in J)$ of elements of $\mathcal{R}$:

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j). \tag{3}$$

For every $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$, there exists a unique $\mathcal{R}$-measurable $X - L$-mapping π such that

$$\Pi(A) = \sup_{x\in A} \pi(x). \tag{4}$$

This mapping is called the *distribution* of Π. A $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$ always satisfies $\Pi(\emptyset) = \ell$. It is called *normalized* iff $\Pi(X) = u$. In that case, its distribution is a normalized $(L, \leq)$-fuzzy set in X.

It is also possible to define *possibilistic variables*, which are possibilistic equivalents of the stochastic variables in probability theory. In order to do this, we shall also consider a universe Ω and an ample field $\mathcal{R}_\Omega$ on Ω. This universe will be called *basic space*. X will be called *sample space*. A $\Omega - X$-mapping that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, will be called a *possibilistic variable in* $(X, \mathcal{R})$. If we also consider a $(L, \leq)$-possibility measure Π_Ω on $(\Omega, \mathcal{R}_\Omega)$, we can use the possibilistic variable f to transform Π_Ω to a $(L, \leq)$-possibility measure Π_f on $(X, \mathcal{R})$, defined by $\Pi_f \stackrel{\text{def}}{=} \Pi_\Omega \circ f^{-1}$. In this expression f^{-1} is the inverse image of f. Π_f is called the *possibility distribution* of the possibilistic variable f. The distribution π_f of Π_f will be called the *possibility distribution function* of f, and satisfies

$$\pi_f(x) = \sup_{f(\omega)=x} \pi_\Omega(\omega), \tag{5}$$

where x is an element of X and π_Ω is the distribution of Π_Ω.

4. A Possibilistic Uncertainty Model

Let us now return to the system S in order to discuss the possibilistic reliability model. First of all, we must deal with some aspects of measurability. With the set Par_S of the parameters associated with the system S, we associate an ample field $\mathcal{R}_{\text{Par}_S}$, assumed to contain the measurable subsets of Par_S. On the state space $\mathcal{S}$ of both the components and the system, we consider the power class $\mathcal{P}(\mathcal{S}) = \{\emptyset, \{\mathit{fail}\}, \{\mathit{work}\}, \{\mathit{fail}, \mathit{work}\}\}$ as the ample field of measurable subsets of $\mathcal{S}$, simply because we want to be able to distinguish between the states *work* and *fail*.

Now, consider an arbitrary component C_k of the system S. In what follows, we shall assume that the component state mapping x_k is $\mathcal{R}_{\text{Par}_S} - \mathcal{P}(\mathcal{S})$-measurable, which is equivalent to $x_k^{-1}(\{\mathit{work}\}) \in \mathcal{R}_{\text{Par}_S}$ and $x_k^{-1}(\{\mathit{fail}\}) \in \mathcal{R}_{\text{Par}_S}$. This means that in the ample field $\mathcal{R}_{\text{Par}_S}$, we must be able to distinguish between those values of the parameters which make the component C_k work, and the ones which make it fail. It also means that x_k is a possibilistic variable in the sample space $(\mathcal{S}, \mathcal{P}(\mathcal{S}))$, where the role of the basic space is played by the structure $(\text{Par}_S, \mathcal{R}_{\text{Par}_S})$. Instead of looking at the component state mappings one by one, we can consider the assembly state mapping $(x_1, \ldots, x_n)$. It is easily verified that this mapping is $\mathcal{R}_{\mathrm{Par}_S} - \mathcal{P}(\mathcal{S}^n)$-measurable, since each of the component state mappings x_k is assumed to be $\mathcal{R}_{\mathrm{Par}_S} - \mathcal{P}(\mathcal{S})$-measurable. Indeed, we have the following proposition.

Proposition 1 *The component state mappings x_k $(k = 1, \ldots, n)$ are $\mathcal{R}_{\mathrm{Par}_S} - \mathcal{P}(\mathcal{S})$-measurable if and only if the assembly state mapping is $\mathcal{R}_{\mathrm{Par}_S} - \mathcal{P}(\mathcal{S}^n)$-measurable.*

In a similar way, it can be proven that the system state mapping x_s is $\mathcal{R}_{\mathrm{Par}_S} - \mathcal{P}(\mathcal{S})$-measurable, and is therefore a possibilistic variable in $(\mathcal{S}, \mathcal{P}(\mathcal{S}))$.

The next step consists in assuming that we have linguistic information[4] about the values that the parameter combination p_s, associated with the system S, takes in Par_S, and that this information is represented by the *normalized* $(L, \leq)$-possibility measure Π_{Par_S} on $(\mathrm{Par}_S, \mathcal{R}_{\mathrm{Par}_S})$. The distribution of Π_{Par_S} will be denoted by π_{Par_S}. Let us now use this information to derive information about the states of the system S and its components.

Proposition 2 *For arbitrary k in $\{1, \ldots, n\}$, the $\mathcal{P}(\mathcal{S}) - L$-mapping Π_{x_k}, defined by $\Pi_{x_k} \overset{\mathrm{def}}{=} \Pi_{\mathrm{Par}_S} \circ x_k^{-1}$, is a normalized $(L, \leq)$-possibility measure on $(\mathcal{S}, \mathcal{P}(\mathcal{S}))$. The distribution π_{x_k} of Π_{x_k} is the normalized $\mathcal{S} - L$-mapping, satisfying*

$$\pi_{x_k}(\nu) = \sup_{x_k(p_s) = \nu} \pi_{\mathrm{Par}_S}(p_s). \tag{6}$$

We shall call π_{x_k} the possibilistic reliability of component C_k.

The $\mathcal{P}(\mathcal{S}) - L$-mapping Π_{x_s}, defined by $\Pi_{x_s} \overset{\mathrm{def}}{=} \Pi_{\mathrm{Par}_S} \circ x_s^{-1}$, is a normalized $(L, \leq)$-possibility measure on $(\mathcal{S}, \mathcal{P}(\mathcal{S}))$. The distribution π_{x_s} of Π_{x_s} is the normalized $\mathcal{S} - L$-mapping, satisfying

$$\pi_{x_s}(\nu) = \sup_{x_s(p_s) = \nu} \pi_{\mathrm{Par}_S}(p_s). \tag{7}$$

We shall call π_{x_s} the possibilistic reliability of the system S.

The $\mathcal{P}(\mathcal{S}^n) - L$-mapping $\Pi_{(x_1,\ldots,x_n)}$, defined by $\Pi_{(x_1,\ldots,x_n)} \overset{\mathrm{def}}{=} \Pi_{\mathrm{Par}_S} \circ (x_1, \ldots, x_n)^{-1}$, is a normalized $(L, \leq)$-possibility measure on $(\mathcal{S}^n, \mathcal{P}(\mathcal{S}^n))$. The distribution $\pi_{(x_1,\ldots,x_n)}$ of $\Pi_{(x_1,\ldots,x_n)}$ is the normalized $\mathcal{S}^n - L$-mapping, satisfying

$$\pi_{(x_1,\ldots,x_n)}(\nu_1, \ldots, \nu_n) = \sup_{(x_1,\ldots,x_n)(p_s) = (\nu_1,\ldots,\nu_n)} \pi_{\mathrm{Par}_S}(p_s). \tag{8}$$

We shall call $\pi_{(x_1,\ldots,x_n)}$ the possibilistic reliability of the assembly $(C_1, \ldots, C_n)$.

The distributions π_{x_k}, π_{x_s} and $\pi_{(x_1,\ldots,x_n)}$ are of course the *possibility distributions* of the respective possibilistic variables x_k, x_s and $(x_1, \ldots, x_n)$. For a component C_k, $\pi_{x_k}(work)$ is the $(L, \leq)$-possibility that this component works, and $\pi_{x_k}(fail)$ the $(L, \leq)$-possibility that it fails. For the system S, analogous conclusions can be drawn regarding $\pi_{x_s}(work)$ and $\pi_{x_s}(fail)$. We want to prove in this paper that possibilistic reliabilities play an analogous part in the possibilistic reliability theory as the well-known reliabilities do in the probabilistic approach[‡]

[‡]The reader will notice that the probabilistic reliability r_k of, say, a component k is a single real

Definition 1 *We shall denote by $\tilde{\mathcal{S}}$ the set of the normalized $\mathcal{S} - L$-mappings.*

We are led to the following result, relating the possibilistic reliability of the system S to that of the assembly $(C_1, \ldots, C_n)$.

Proposition 3 *For arbitrary ν in $\mathcal{S}$:*

$$\pi_{x_s}(\nu) = \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} \pi_{(x_1,\ldots,x_n)}(\nu_1, \ldots, \nu_n). \tag{9}$$

Now, what we really would like to find, is a relation between π_{x_s} and the possibilistic reliabilities π_{x_k} of the components C_k. In order to derive such a result, we must make a small digression, and consider the the following *state projection operators*.

Definition 2 *Let k be an element of $\{1, \ldots, n\}$. The $\mathcal{S}^n - \mathcal{S}$-mapping proj_k, defined by $(\forall(\nu_1, \ldots, \nu_n) \in \mathcal{S}^n)(\mathrm{proj}_k(\nu_1, \ldots, \nu_n) \stackrel{\text{def}}{=} \nu_k)$, is called the k-th state projection operator.*

Proposition 4 *For arbitrary ν_k in $\mathcal{S}$:*

$$\pi_{x_k}(\nu_k) = \sup_{\mathrm{proj}_k(\nu)=\nu_k} \pi_{(x_1,\ldots,x_n)}(\nu). \tag{10}$$

In order to achieve our goal, however, we must be able to derive the possibilistic reliability of the configuration from the possibilistic reliabilities of the components. This will in general not be possible unless the possibilistic variables x_k are in some way *possibilistically independent*[4].

Definition 3 *Let T be a triangular norm on $(L, \leq)$ such that $(L, \leq, T)$ is a complete lattice with t-norm. The possibilistic variables x_k in $(\mathcal{S}, \mathcal{P}(\mathcal{S}))$ are $(\Pi_{\mathrm{Par}_S}, T)$-independent (or, in general, possibilistically independent) iff*

$$(\forall(\nu_1, \ldots, \nu_n) \in \mathcal{S}^n)(\pi_{(x_1,\ldots,x_n)}(\nu_1, \ldots, \nu_n) = T_{k=1}^n \pi_{x_k}(\nu_k)).$$

This leads to the result we were, in a sense, looking for: an important relation between the possibilistic reliabilities of the system and its components.

Theorem 1 *Let T be a triangular norm on $(L, \leq)$ such that $(L, \leq, T)$ is a complete lattice with t-norm. If the possibilistic variables x_k are $(\Pi_{\mathrm{Par}_S}, T)$-independent, we have, for arbitrary ν in $\mathcal{S}$:*

$$\pi_{x_s}(\nu) = \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} T_{k=1}^n \pi_{x_k}(\nu_k). \tag{11}$$

Definition 4 *The $\tilde{\mathcal{S}}^n - \tilde{\mathcal{S}}$-mapping $\tilde{\phi}_T$, defined by*

$$\tilde{\phi}_T(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} T_{k=1}^n t_k(\nu_k), \tag{12}$$

for arbitrary $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$, is called the $(L, \leq, T)$-possibilistic reliability mapping associated with the binary structure function ϕ.

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number in $[0, 1]$. Its possibilistic counterpart π_{x_k} can be represented by a *couple* of elements $(\pi_{x_k}(\mathit{fail}), \pi_{x_k}(\mathit{work}))$ of the complete lattice $(L, \leq)$. The reason for this difference is that from r_k, we can immediately infer the probability $1 - r_k$ that the component fails, whereas in general in the possibilistic approach we cannot determine $\pi_{x_k}(\mathit{fail})$ from $\pi_{x_k}(\mathit{work})$ or *vice versa*.

Of course, from definition 4 and theorem 1, it immediately follows that if the possibilistic variables x_k are (Π_{Par_S}, T)-independent ($k \in \{1, \ldots, n\}$), we have that

$$\pi_{x_s} = \tilde{\phi}_T(\pi_{x_1}, \ldots, \pi_{x_n}). \tag{13}$$

The following important theorem tells us that there is an easy way to calculate $\tilde{\phi}_T$ from ϕ, by simply extending the decompositions of ϕ in minimal paths and cuts.

Theorem 2 *Let T be a triangular norm on $(L, \leq)$ such that $(L, \leq, T)$ is a complete lattice with t-norm. Assume that the possibilistic variables x_k are (Π_{Par_S}, T)-independent ($k \in \{1, \ldots, n\}$). Then:*

$$\pi_{x_s}(\mathit{work}) = \sup_{1 \leq r \leq n_p} T_{i \in P_r}\, \pi_{x_i}(\mathit{work}) \quad \text{and} \quad \pi_{x_s}(\mathit{fail}) = \sup_{1 \leq l \leq n_c} T_{i \in C_l}\, \pi_{x_i}(\mathit{fail}). \tag{14}$$

5. Conclusion

The results outlined in this paper give us a practical way of treating the possibilistic aspects of the reliability of a system, stress the formal analogy with the probabilistic approach and show that a possibilistic treatment of reliability need not be more complicated than a classical, probabilistic one.

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7. References

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