SOME REMARKS ON STATIONARY POSSIBILISTIC PROCESSES

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We investigate the following extendability problem for systems, for which the available information is given by a monotone set mapping $\mathfrak{M}$ on the field $\mathcal{C}_T$ of measurable cylinders of a product ample space $(X^T, \mathcal{R}^T)$: given that $\mathfrak{M}$ is invariant under a $\mathcal{R}^T - \mathcal{R}^T$- measurable transformation H of X^T, i.e. $\mathfrak{M}(H^{-1}(B)) = \mathfrak{M}(B)$ for all $B \in \mathcal{C}_T$, is it possible to find H-invariant monotone extensions of $\mathfrak{M}$ to the powerclass of X^T? We first show that the outer and inner measures of $\mathfrak{M}$ always have the desired invariance property. If the system that we are dealing with is possibilistic, a number of sufficient conditions are derived to ensure the H-invariance of the greatest possibilistic extension $\Pi^g_{\mathfrak{M}}$ of $\mathfrak{M}$. Consequently stationary possibilistic processes can be represented by a shift-invariant possibility measure on their basic space. As an illustration for our results, we show that possibilistic Markov processes with stationary transition possibilities and stationary initial possibilities are stationary processes.

1 Motivation

In previous papers [1,2] we derived a Daniell-Kolmogorov theorem in order to set up a measure-theoretic theory of *possibilistic processes*, that is, processes for which the available information about their behaviour is given by possibility measures. For the case where the observations of a given system lead to a collection of possibility measures defined on the finite Cartesian powers of some set X and satisfying a consistency condition, this theorem allows us to represent this information by a possibilistic process $(f_t \mid t \in T)$, i.e. a collection of possibilistic variables sharing a common basic and a common sample space, indexed by some set T, acting as the 'time set' for the system. Since possibility measures are determined by their unique distribution this information can alternatively be given by a collection of distributions satisfying a similar consistency condition, which – as expected – coincide with the finite joint distribution functions that can be constructed for the variables in $(f_t \mid t \in T)$.

In many practical situations it may be reasonable to judge that the available information satisfies additional structural properties. For instance, we may judge that the distributions that were derived from our observations do not change in time. In that case the following question naturally arises: does there exist a shift-invariant (or time-invariant) possibility measure on the basic space representing the given information? We shall discuss a slightly more general problem in Section 3. A first case in which the problem can be solved is when the time 'shift-operator' is bijective. When this is not the case – for instance when we take as time set the natural numbers $\mathbb{N}$, provided with the usual linear ordering, we can still construct a suitable shift-invariant possibility measure when the distributions have an extra decomposability property or satisfy an extra continuity condition.

In Section 4 we introduce *'strictly stationary' possibilistic processes*. As an illustration we shall indicate that a possibilistic Markov process [3] is a strictly stationary possibilistic process when its transition possibilities and initial possibilities are stationary.

For the sake of clarity we recall in the next section a number of definitions and results that are needed for a clear understanding of the paper. In a forthcoming paper we will give a more detailed treatment and explicit proofs of the results mentioned here.

2 Preliminaries

2.1 Ample fields and topologies

Throughout X is a nonempty set. If A is a finite subset of X, we denote this as $A \Subset X$. A subset $\mathcal{R}$ of the power set $\wp(X)$ of X is called an *ample field* on X iff it is closed under arbitrary unions and under complementation. The couple $(X, \mathcal{R})$ is called an *ample space*. The atom $[\,x\,]_{\mathcal{R}}$ of $\mathcal{R}$ containing the element x of X is defined as $[\,x\,]_{\mathcal{R}} = \bigcap\{A \mid A \in \mathcal{R} \text{ and } x \in A\}$. Note that for any $A \in \wp(X)$, $A \in \mathcal{R}$ iff $A = \bigcup_{x \in A} [\,x\,]_{\mathcal{R}}$.

We continue with a number of measurability definitions. If $\mathcal{S} \subseteq \wp(X)$ then we call a subset A of X *$\mathcal{S}$-measurable* iff $A \in \mathcal{S}$. If $\mathcal{S}_1 \subseteq \wp(X_1)$ and $\mathcal{S}_2 \subseteq \wp(X_2)$, where X_1 and X_2 are nonempty sets, then a $X_1 - X_2$-mapping f is called *$\mathcal{S}_1 - \mathcal{S}_2$-measurable* iff $(\forall B \in \mathcal{S}_2)(f^{-1}(B) \in \mathcal{S}_1)$.

Assume that $T \neq \emptyset$. Then the Cartesian product X^T is the set of all the $T - X$-mappings. For any $t \in T$, $\mathbf{pr}_{T,t}$ is the *projection mapping* from X^T onto X, defined by $\mathbf{pr}_{T,t}(x) = x(t)$, $\forall x \in X^T$. If S is a nonempty subset of T, then $\mathbf{pr}_{T,S}$ is the $X^T - X^S$-mapping such that for any $x \in X^T$, $\mathbf{pr}_{T,S}(x) = x|_S$

is the restriction of the mapping x to S. If $\mathfrak{T}$ is a topology on X, then the product topology $\mathfrak{T}^T$ is the smallest topology on X^T for which all projection operators $\mathbf{pr}_{T,t}$, $t \in T$ are continuous, when their codomains are provided with the topology $\mathfrak{T}$. Furthermore, $\mathcal{R}^T$ denotes the product ample field on X^T, whose atoms are given by $[\,x\,]_{\mathcal{R}^T} = \bigcap_{t\in T} \mathbf{pr}_{T,t}^{-1}([\,x(t)\,]_{\mathcal{R}})$, $\forall x \in X^T$. The subset $\mathcal{C}_T = \{\mathbf{pr}_{T,S}^{-1}(E) \mid \emptyset \subset S \Subset T \text{ and } E \in \mathcal{R}^S\}$ of $\mathcal{R}^T$ is the *field of all measurable cylinders* of $(X^T, \mathcal{R}^T)$.

Let A be any set and consider a nonempty family $(f_t \mid t \in T)$ of mappings from A into X. The unique mapping $f\colon A \to X^T$, such that $f_t = \mathbf{pr}_{T,t} \circ f$, $t \in T$, is denoted by f_T and is called the *product mapping* of $(f_t \mid t \in T)$.

2.2 Possibility measures

A set mapping Π is a possibility measure [4,5] on an ample space $(X, \mathcal{R})$ iff Π is a $\mathcal{R} - [0,1]$-mapping such that $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$ for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$. A *distribution* for Π is a $\mathcal{R} - \wp([0,1])$-measurable $X - [0,1]$-mapping π such that $\Pi(A) = \sup_{x\in A} \pi(x)$, $\forall A \in \mathcal{R}$. Obviously, such a distribution is unique and completely determined by $\pi(x) = \Pi([\,x\,]_{\mathcal{R}})$, $\forall x \in X$. Π is called *normal* iff $\Pi(X) = 1$. The triple $(X, \mathcal{R}, \Pi)$ is called a *possibility space*. Π is called *outer regular* with respect to a topology $\mathfrak{T}$ on X iff $\Pi(E) = \inf\{\Pi(O) \mid E \subseteq O \in \mathfrak{T}\}$, $\forall E \in \mathcal{R}$, iff π is upper semi-continuous with respect to $\mathfrak{T}$ on X.

Let $\mathcal{S}$ be a nonempty collection of subsets of X and let μ be a $\mathcal{S} - [0,1]$-mapping. If g is a $\mathcal{S} - \mathcal{S}$-measurable transformation of X then μ is said to be *g-invariant* iff $\mu(g^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{S}$.

A possibilistic variable [6] is a variable for which the available information about the values it may assume, takes the form of a possibility measure. Formally, we have a *basic space* Ω, provided with an ample field $\mathcal{R}_\Omega$, and a *sample space* X, provided with an ample field $\mathcal{R}$. The available information is represented by a possibility measure Π_Ω on $(\Omega, \mathcal{R}_\Omega)$. A $\Omega - X$-mapping f that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, is called a *possibilistic variable* in $(X, \mathcal{R})$. The $X - [0,1]$-mapping π_f, given for any $x \in X$ by $\pi_f(x) = \Pi_\Omega(f^{-1}([\,x\,]_{\mathcal{R}}))$, is called the *possibility distribution function* of f. A *possibilistic process* in an ample space $(X, \mathcal{R})$ is defined as a nonempty family of possibilistic variables $(f_t \mid t \in T)$ having $(X, \mathcal{R})$ as sample space. When all variables in $(f_t \mid t \in T)$ have the possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as their basic space, then the possibility distribution function of the product f_S of $(f_s \mid s \in S)$ (where $\emptyset \subset S \Subset T$) is given for any $x \in X^S$ by $\pi_{f_S}(x) = \Pi_\Omega(f_S^{-1}([\,x\,]_{\mathcal{R}^S})) = \Pi_\Omega(\bigcap_{s\in S} f_s^{-1}([\,x(s)\,]_{\mathcal{R}}))$, and is called the *joint possibility distribution function* of the variables $(f_s \mid s \in S)$.

3 Invariance of set mappings under measurable transformations

3.1 Preliminary definitions and results

Throughout $(X, \mathcal{R})$ is an ample space. We assume that information about the behaviour of a system is given by a consistent collection of monotone set mappings $(\Lambda_S \mid \emptyset \subset S \Subset T)$ on the finite products of $(X, \mathcal{R})$, where T is a nonempty set. So, for all $\emptyset \subset S \Subset T$, Λ_S is a $([0,1], \leq)$-valued monotone set mapping on $(X^S, \mathcal{R}^S)$, and $(\Lambda_S \mid \emptyset \subset S \Subset T)$ is assumed to be consistent, i.e. for any two sets S_1 and S_2 such that $\emptyset \subset S_1 \subseteq S_2 \Subset T$:

$$\Lambda_{S_1} = \Lambda_{S_2} \circ \mathbf{pr}_{S_2,S_1}^{-1}, \qquad (C)$$

i.e. Λ_{S_1} is the *marginal* of Λ_{S_2} on $(X^{S_1}, \mathcal{R}^{S_1})$. This allows us to define consistently a $([0,1], \leq)$-valued set mapping $\mathfrak{M}_T$ on the field $\mathcal{C}_T$ of measurable cylinders of $(X^T, \mathcal{R}^T)$ as follows:

$$\mathfrak{M}(B) = \Lambda_S(A)$$

for any $B \in \mathcal{C}_T$, where $\emptyset \subset S \Subset T$ and $A \in \mathcal{R}^S$ such that $B = \mathbf{pr}_{T,S}^{-1}(A)$.

Consider an injective transformation h of the index set T. Then the transformation H of X^T given by $H(x) = x \circ h, \forall x \in X^T$, is an $\mathcal{R}^T - \mathcal{R}^T$-measurable mapping. In particular, H is also $\mathcal{C}_T - \mathcal{C}_T$-measurable, i.e. the inverse image $H^{-1}(B)$ of a measurable cylinder B of $(X^T, \mathcal{R}^T)$ is also a measurable cylinder of $(X^T, \mathcal{R}^T)$.

Example 1 Let $T = \mathbb{N}$ and let h be the transformation of $\mathbb{N}$ that maps an element of T on its successor, i.e. $h(n) = n+1$ for all $n \in \mathbb{N}$. The transformation H of $X^{\mathbb{N}}$ corresponding with h is the (left) shift-operator. Note that, for all $x \in X^{\mathbb{N}}$, $H(x)(n) = x(n+1)$ for all $n \in \mathbb{N}$, and $H^{-1}(\{x\}) = \times_{n \in \mathbb{N}} \Delta_n$ where

$$\Delta_n = \begin{cases} X & \text{if } n = 0; \\ \{x(n-1)\} & \text{if } n \neq 0. \end{cases}$$

The following mappings will be useful in the sequel. For any $\emptyset \subset S \Subset T$, H_S is $X^{h(S)} - X^S$-mapping given by $H_S(x) = x \circ h|_S, \forall x \in X^{h(S)}$. Furthermore, H_S is a $\mathcal{R}^{h(S)} - \mathcal{R}^S$-measurable bijection, such that $H_S([\, y \,]_{\mathcal{R}^{h(S)}}) = [\, H_S(y) \,]_{\mathcal{R}^S}$ for all $y \in X^{h(S)}$.

$\mathfrak{M}$ is H-invariant (see Section 2.2) iff

$$\mathfrak{M}(H^{-1}(B)) = \mathfrak{M}(B), \forall B \in \mathcal{C}_T, \qquad (I)$$

Obviously this is equivalent to the requirement that the information given by $(\Lambda_S \mid \emptyset \subset S \Subset T)$ should satisfy:

$$\Lambda_{h(S)} = \Lambda_S \circ H_S, \forall\, \emptyset \subset S \Subset T. \qquad (I')$$

This brings us to the main problem of the paper: given an H-invariant set mapping $\mathfrak{M}$, or, equivalently, given a system with information $(\Lambda_S \mid \emptyset \subset S \Subset T)$ satisfying (I'), are there H-invariant monotone extensions of $\mathfrak{M}$ to $\wp(X^T)$? Naturally, a monotone extension of $\mathfrak{M}$ to $\wp(X^T)$ dominates the inner measure $\mathfrak{M}_*$ and is dominated by the outer measure $\mathfrak{M}^*$. Moreover, they both have the desired invariance property.

Theorem 2 *$\mathfrak{M}^*$ is H-invariant, i.e. $\mathfrak{M}^*(H^{-1}(E)) = \mathfrak{M}^*(E)$, $\forall E \in \wp(X^T)$. $\mathfrak{M}_*$ is H-invariant, i.e. $\mathfrak{M}_*(H^{-1}(E)) = \mathfrak{M}_*(E)$, $\forall E \in \wp(X^T)$.*

If the elements of $(\Lambda_S \mid \emptyset \subset S \Subset T)$ have additional properties, it is sometimes possible to construct a monotone extension of $\mathfrak{M}$ which is smaller than $\mathfrak{M}^*$. This is for instance the case when the information about the system is possibilistic, i.e. each set mapping Λ_S, $\emptyset \subset S \Subset T$ is a possibility measure on $(X^S, \mathcal{R}^S)$. Since any possibility measure Λ_S, $\emptyset \subset S \Subset T$ is completely determined by its distribution λ_S, we can reformulate the consistency condition (C) as follows: for any two sets S_1 and S_2 such that $\emptyset \subset S_1 \subseteq S_2 \Subset T$, and for any $x \in X^{S_1}$:

$$\lambda_{S_1}(x) = \sup_{\mathbf{pr}_{S_2,S_1}(y)=x} \lambda_{S_2}(y).$$

For any $t \in T$, we shall denote by Λ_t the possibility measure on $(X, \mathcal{R})$ with distribution $\lambda_t = \lambda_{\{t\}} \circ \mathbf{pr}_{\{t\},t}^{-1}$.

If the given possibility measures are normal, then $\mathfrak{M}$ is a 2-alternating set mapping. By a theorem of Peter Walley [7] the natural extension of $\mathfrak{M}$ to $\wp(X^T)$, i.e. the greatest coherent extension of $\mathfrak{M}$ to $\wp(X^T)$, is precisely the outer measure $\mathfrak{M}^*$. Theorem 2 ensures that this extension is H-invariant. In fact, this result still holds for the more general case where the set mappings λ_S, $\emptyset \subset S \Subset T$ are coherent upper probabilities.

Since we are dealing with possibilistic information it is natural to ask whether or not $\mathfrak{M}$ can be extended to a possibility measure on $(X^T, \wp(X^T))$. The next theorem, which can be regarded as a possibilistic counterpart of the Daniell-Kolmogorov theorem, summarises a number of conditions that are sufficient.

Theorem 3 *Assume that $(\lambda_S \mid \emptyset \subset S \Subset T)$ is consistent and that at least one of the following conditions holds.*

1. *$\mathfrak{T}$ is a compact topology on X such that Λ_S is outer regular with respect to the product topology $\mathfrak{T}^S$ for all $\emptyset \subset S \Subset T$.*

2. *T is countable.*

3. *The collection $(\lambda_S \mid \emptyset \subset S \Subset T)$ consists of finite* min*-products, i.e. $\lambda_S(x) = \min_{t\in S} \lambda_t(x(t))$, $\forall x \in X^S$ where $\emptyset \subset S \Subset T$.*

Then $\mathfrak{M}$ is extendable to a possibility measure on $(X^T, \mathcal{R}^T)$ - and therefore also on $(X^T, \wp(X^T))$. The greatest such possibility measure, and therefore the greatest possibilistic extension $\Pi^g_{\mathfrak{M}}$ of $\mathfrak{M}$ has distribution $\pi^g_{\mathfrak{M}}$, given by

$$\pi^g_{\mathfrak{M}}(x) = \inf_{\emptyset\subset S\Subset T} \lambda_S(\mathbf{pr}_{T,S}(x)), \forall x \in X^T.$$

In particular, when condition 1 holds, then $\Pi^g_{\mathfrak{M}}$ is outer regular with respect to $\mathfrak{T}^T$. Moreover, there exists a possibilistic process $(f_t \mid t \in T)$ in $(X, \mathcal{R})$ with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ such that, for all $\emptyset \subset S \Subset T$, the possibility distribution function of f_S is given by $\pi_{f_S} = \pi_S$.
From the result above we may conclude that, if at least one of the sufficient conditions holds, we have that $\Pi^g_{\mathfrak{M}}$ is the greatest possibilistic extension of $\mathfrak{M}$. In particular, if the given possibility measures are normal, we have that $\Pi^g_{\mathfrak{M}}$ is a coherent extension of $\mathfrak{M}$.

3.2 Sufficient conditions for the H-invariance of $\Pi^g_{\mathfrak{M}}$

Let us now derive a number of conditions that are sufficient for $\Pi^g_{\mathfrak{M}}$ to be an H-invariant set mapping on $(X^T, \wp(X^T))$, when $\mathfrak{M}$ is constructed from a consistent collection of possibility measures $(\Lambda_S \mid \emptyset \subset S \Subset T)$ having invariance property (I'). As already explained in the foregoing subsection this means that $\mathfrak{M}$ is H-invariant. In terms of the corresponding distributions $(\lambda_S \mid \emptyset \subset S \Subset T)$ this is also equivalent to the condition:

$$\lambda_{h(S)} = \lambda_S \circ H_S, \forall\ \emptyset \subset S \Subset T. \qquad (I'')$$

Since possibility measures are supremum preserving set mappings we can simplify the problem by taking into account the following consideration: *a possibility measure Π on an ample space $(X, \mathcal{R})$ is g-invariant where g is a $\mathcal{R}-\mathcal{R}$-measurable transformation of X iff the invariance property holds for all atoms of its domain $\mathcal{R}$,i.e. $\Pi(g^{-1}([\,x\,]_{\mathcal{R}})) = \Pi([\,x\,]_{\mathcal{R}}), \forall x \in X$.* We therefore conclude that $\Pi^g_{\mathfrak{M}}$ is H-invariant iff $\Pi^g_{\mathfrak{M}}(H^{-1}(\{x\})) = \Pi^g_{\mathfrak{M}}(\{x\}), \forall x \in X^T$.

Consider an element x of X^T, then

$$\begin{aligned}
\Pi^g_{\mathfrak{M}}(H^{-1}(\{x\})) &= \sup_{z\in H^{-1}(\{x\})} \pi^g_{\mathfrak{M}}(z) \\
&= \sup_{z\in H^{-1}(\{x\})} \inf_{\emptyset\subset S\Subset T} \lambda_S(\mathbf{pr}_{T,S}(z)) \\
&\leq \sup_{z\in H^{-1}(\{x\})} \inf_{\emptyset\subset S\Subset T} \lambda_{h(S)}(\mathbf{pr}_{T,h(S)}(z)) \qquad (1)\\
&= \sup_{z\in H^{-1}(\{x\})} \inf_{\emptyset\subset S\Subset T} \lambda_S((z\circ h)|_S) \\
&= \sup_{z\in H^{-1}(\{x\})} \inf_{\emptyset\subset S\Subset T} \lambda_S(\mathbf{pr}_{T,S}(x)) \\
&= \pi^g_{\mathfrak{M}}(x),
\end{aligned}$$

In particular, if h is bijective, then the equality holds in (1), giving us a first condition that is sufficient for the H-invariance of $\Pi^g_{\mathfrak{M}}$.

Theorem 4 *$\Pi^g_{\mathfrak{M}}(H^{-1}(\{x\})) \leq \Pi^g_{\mathfrak{M}}(x)$, $\forall x \in X^T$. Moreover, if h is bijective, then $\Pi^g_{\mathfrak{M}}$ is H-invariant.*

A second sufficient condition can be derived as follows. Consider an element x of X. Since $\mathbf{pr}_{T,t}\circ H = \mathbf{pr}_{T,h(t)}, \forall t \in T$, it follows that $H^{-1}(\{x\})$ is a Cartesian product, namely $H^{-1}(\{x\}) = \bigcap_{t\in T} \mathbf{pr}^{-1}_{T,h(t)}(\{x(t)\}) = \times_{t\in T}\Delta_t$ where

$$\Delta_t = \begin{cases} X & \text{if } t \in T\setminus h(T); \\ \{x(h^{-1}(t))\} & \text{if } t\in h(T). \end{cases}$$

It can be verified that $\Pi^g_{\mathfrak{M}}$ and $\mathfrak{M}^*$ coincide on the subsets of X^T that are Cartesian products, if we additionally assume that $(\lambda_S \mid \emptyset \subset S \Subset T)$ consists of finite min-products. Using Theorem 2 we find:

$$\Pi^g_{\mathfrak{M}}(H^{-1}(\{x\})) = \mathfrak{M}^*(H^{-1}(\{x\})) = \mathfrak{M}^*(\{x\}) = \Pi^g_{\mathfrak{M}}(\{x\}),$$

leading us to the following result.

Theorem 5 *Assume that $(\lambda_S \mid \emptyset\subset S\Subset T)$ is consists of finite* min-*products. Then $\Pi^g_{\mathfrak{M}}$ is an H-invariant extension of $\mathfrak{M}$.*

When none of the foregoing extra conditions are satisfied, we can still impose regularity conditions on the given possibility measures. The next theorem ensures that, if we do this, we indeed find that $\Pi^g_{\mathfrak{M}}$ is an H-invariant possibility measure.

Theorem 6 *Suppose that $\mathfrak{T}$ is a compact topology on X. Assume that, for all $\emptyset\subset S\Subset T$, Λ_S is outer regular with respect to $\mathfrak{T}^S$. If $(\lambda_S \mid \emptyset\subset S\Subset T)$ satisfies (I''), then $\Pi^g_{\mathfrak{M}}$ is an H-invariant extension of $\mathfrak{M}$.*

4 Stationary possibilistic processes

We can define strictly stationary processes in a similar way as in probability theory.

Definition 7 *Let $(f_t \mid t \in T)$ be a possibilistic process in $(X, \mathcal{R})$, indexed by $T = \mathbb{N}$ or $T = \mathbb{Z}$. Then $(f_t \mid t \in T)$ is called a strictly stationary possibilistic process in $(X, \mathcal{R})$ iff for any nonempty finite subset $S = \{t_1, \ldots, t_n\} \subseteq T$ (where $n \in \mathbb{N}\setminus\{0\}$) and for any $s \in T$ such that $t_i + s \in T$ for all $i \in \{1, \ldots, n\}$ it follows that the variables $(f_{t_i} \mid i : 1 \ldots n)$ and $(f_{t_i+s} \mid i : 1 \ldots n)$ have the same joint possibility distribution function.*

Using the notations and results of the previous section we can reformulate this as follows: $(f_t \mid t \in T)$ is a strictly stationary process in $(X, \mathcal{R})$ iff $(\pi_{f_S} \mid \emptyset \subset S \Subset T)$ satisfies (I'') (where $h : T \to T$ such that $h(t) = t + 1$, $\forall t \in T$). Note that the transformation H corresponding with h is precisely the (left) shift-operator (see also Example 1).

Using the results in Section 3, sufficient conditions can be derived in order to represent the information $(\pi_{f_S} \mid \emptyset \subset S \Subset T)$ by a shift-invariant possibility measure.

To give an example, assume that for a system, assuming its states in a nonempty set X, the following information was obtained after observation at times that are numbered by the natural numbers $\mathbb{N}$:

- *a possibilistic matrix* $\mathbb{P}$, i.e. a $X^2 - [0,1]$-mapping $\mathbb{P}$ such that, for any couple $(x, y) \in X^2$, $\mathbb{P}(x, y)$ denotes the *one-step transition possibility* from state x at time $n \in \mathbb{N}$ to state y at time $n + 1$, and that satisfies the following condition: $\sup_{y \in X} \mathbb{P}(x, y) = 1$ for all $x \in X$;

- *initial possibilities* $\overline{q}$, i.e. a $X - [0,1]$-mapping $\overline{q}$ such that $\overline{q}(x)$ expresses the possibility that the system is in state $x \in X$ at time 0.

Using the information above we may define for any $x \in X^{\{0,\cdots,n\}}$ where $n \in \mathbb{N}$:

$$\pi_{\{0,\ldots,n\}}(x) = \begin{cases} \overline{q}(x(0)) & \text{if } n = 0; \\ \overline{q}(x(0)) \prod_{i=0}^{n-1} \mathbb{P}(x(i), x(i+1)) & \text{otherwise;} \end{cases}$$

where the last expression involves the n-ary algebraic product. For any $x \in X_S$ where $\emptyset \subset S \Subset T$ let

$$\pi_S(x) = \sup_{\mathbf{pr}_{\{0,\ldots,\max S\},S}(y) = x} \pi_{\{0,\ldots,\max S\}}(y),$$

i.e. the possibility that the system assumes the states $x(s)$, $s \in S$ at the corresponding times in S.

By Theorem 3 there exists a possibilistic process $(f_n \mid n \in \mathbb{N})$ in $(X, \wp(X))$ with $(X^{\mathbb{N}}, \wp(X^{\mathbb{N}}), \Pi_{\mathbb{N},\overline{q}})$ as basic space, where $\Pi_{\mathbb{N},\overline{q}}$ has distribution $\pi_{\mathbb{N},\overline{q}}$ given for any $x \in X^{\mathbb{N}}$ by

$$\pi_{\mathbb{N},\overline{q}}(x) = \inf_{n \in \mathbb{N}} \overline{q}(x(0)) \prod_{i=0}^{n-1} \mathbb{P}(x(i), x(i+1)), = \overline{q}(x(0)) \prod_{i=0}^{+\infty} \mathbb{P}(x(i), x(i+1)),$$

and such that $\pi_{f_S} = \pi_S$, $\forall \emptyset \subset S \Subset T$.

$(f_n \mid n \in \mathbb{N})$ can be interpreted as a possibilistic Markov process satisfying a possibilistic counterpart of the Chapman-Kolmogorov equation. Furthermore, $(f_n \mid n \in \mathbb{N})$ is a strictly stationary possibilistic process iff $\overline{q}$ is an eigenvector of $\mathbb{P}$ in the $(\times, \max)$-algebra, i.e. $\overline{q}(y) = \sup_{x \in X} \overline{q}(x)\mathbb{P}(x, y), \forall y \in X$. In that case it is easily verified that $\Pi_{\mathbb{N},\overline{q}}$ is shift-invariant.

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