EVALUATION SETS AND MAPPINGS: THE ORDER-THEORETIC ASPECT OF THE MEANING OF PROPERTIES

Gert DE COOMAN
Research Assistant N.F.S.R.
Vakgroep Elektrische Energietechniek
Universiteit Gent, Belgium

Abstract : In this contribution, we study some aspects of the mathematical representation of the solutions of problems encountered in everyday life, called evaluation problems. These problems consist in having to check whether objects in a given universe satisfy one or more properties.
First, evaluation problems under a single property are considered. It is argued that (partial-pre)order relations play a central role in this study. Three equivalent ways of representing the order-theoretic aspect of these solutions are presented, the last of which bears a close resemblance to the representations extant in the literature concerning fuzzy set theory. This resemblance is investigated in several examples.
After this basic work, the more complicated study of evaluation problems under more than one property is undertaken. It is argued that this approach is necessary in order to investigate the relations between properties and of course the combinations of properties that lead to such logical operations as *not*, *and*, *or*, ... The approach followed for a single property is generalized and the same order-theoretic methods are used to represent the solutions of problems of this kind. Using this approach, the link with fuzzy sets and their set theoretical operations is made through the study of property combinators, truth-functionality and combination functions. This link is studied in several examples.

1 Introduction

Let us consider the following example problems.

- In selecting suitable candidates for a job in a certain company, the personnel department must look for those applicants who meet the requirements for the job.
- A teacher of French gives his students an extract from the book *Un amour de Swann* by M. Proust and tells them to underline all the verb forms in the *subjonctif imparfait*.
- A pattern recognition problem consists in selecting from a basic set those patterns which closely resemble a given pattern.
- Worn parts of a machine are subject to replacement.
- A manager of a multinational seriously considers building factories in those European countries that still provide cheap labour.
- A mathematics student is given a list of complex functions and asked to answer the following questions. Which of these functions are holomorphic, and where? Which of these functions are sufficiently smooth in the neighbourhood of the point $i\pi$?

These, and also most of the problems people are faced with every day, are of the same type, that can be characterized as follows.

Problem
Let X be the (non-empty) set of all the objects x we want to consider. We shall henceforth call X a universe. The evaluation problem in X under the property P consists in having to check whether the objects x in X satisfy the property P.

In this contribution, we want to investigate how the solutions of problems of this type can be represented mathematically.

1.1 Classical Sets

Starting from a universe X, classical mathematics divides the possible properties in two classes. For the members of the first class of properties it is objectively possible for any object x in X to decide whether or not they are satisfied by x. We shall say that these properties *have a classical representation*[1] in X, or equivalently that they are *crisp* in X. Properties with a classical representation generate what we shall call *well-posed* evaluation problems.
The properties in the second class have no classical representation in X. This means that for members of this class it is not objectively possible for any object in X to tell whether or not they are satisfied by x; they lead to *ill-posed* evaluation problems. Classical mathematics takes no notice of these and only enables us to give representations of the solutions of well-posed problems. Let us give a brief outline of how this is done.
Let P be a property having a classical representation in the universe X. The solution of the evaluation problem under this property can be represented by the subset A_P of X, defined as

$$A_P \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x \text{ satisfies } P \,\}.$$

Equivalently, this solution can be represented using the *characteristic mapping* χ_{A_P} of the *set of solutions* A_P,

$$\chi_{A_P}\colon X \to \{0,1\}\colon x \mapsto \begin{cases} 1 & ; \quad x \text{ satisfies } P \\ 0 & ; \quad x \text{ does not satisfy } P. \end{cases}$$

A_P is also called the *extension* of P. For any x in X, $\chi_{A_P}(x)$ is called the *degree of membership* of x in A_P, or the degree to which x satisfies P.

Figure 1: The universe X and the set of solutions A_P of the evaluation problem in X under a property P that has a classical representation in X.

Example 1 Let X be the set $\mathbb{R}^+$ of the strictly positive real numbers. The property $P_1 \stackrel{\text{def}}{=}$ "is greater than or equal to 5" generates the well-posed problem "Find all the strictly positive real numbers greater than or equal to 5", the solution of which can be represented by the set of solutions A_{P_1}

$$A_{P_1} \stackrel{\text{def}}{=} \{\, x \mid x \in \mathbb{R}^+ \text{ and } x \geq 5 \,\}$$

or equivalently by the characteristic mapping $\chi_{A_{P_1}}$ of A_{P_1}

$$\chi_{A_{P_1}}\colon \mathbb{R}^+ \to \{0,1\}\colon x \mapsto \begin{cases} 1 & ; \quad x \geq 5 \\ 0 & ; \quad x < 5. \end{cases}$$

On the other hand the property $P_2 \stackrel{\text{def}}{=}$ "is a large number" does not generate a well-posed problem, since it is impossible to give an objectively unique and sharp definition to the extension of the property "large number", without violating its meaning.

It follows from these considerations that well-posed evaluation problems are very closely connected with the basis of classical set theory and hence with classical two-valued logic. Since on the one hand we start from a universe X such that for any imaginable object we can *decide* whether or not it belongs to X, and on the other hand the problems considered are well-posed, the paradoxes of naïve set theory are avoided.

The model of classical set theory is manifestly of considerable value in numerous applications, almost all in the province of the exact sciences (especially mathematics), because these almost by definition only concern themselves with well-posed problems. It is characteristic of the paradigmatic way of thinking in exact science that very often the other ill-posed problems are discarded as unimportant and even nonsensical. Nevertheless, many of the ill-posed problems play a central role in the so-called soft sciences and even more so in everyday life. Finding mathematical representations for the solutions of these problems can only extend the usefulness and applicability of mathematics in science.

Let us therefore start from a universe X and also consider what we have up to now called ill-posed problems. More specifically, we shall concentrate on evaluation problems generated by properties P that can be described as *fuzzy* in X. This means that no sharp boundary exists between satisfying and not satisfying P, i.e., the extension of a fuzzy property P in X cannot be sharply delineated in X.

Definition 1 *We shall call a property P fuzzy in a universe X when every object x in X satisfies one of the following conditions:*

(i) *x completely satisfies P;*
(ii) *x completely does not satisfy P;*
(iii) *x neither completely satisfies nor completely does not satisfy P, but lies in-between these extremes;*

and there exists at least one element in X satisfying condition (iii).

Until recently, fuzziness was considered an "ill"-ness, the treatment of which consisted in imposing sharp definitions on fuzzy properties. For instance, the property "is a large number" in example 1 could have been redefined as "is greater than or equal to 10000". It goes without saying that this "treatment" is artificial and arbitrary, and affects the meaning of the property under consideration. Many properties people handle in everyday life are intrinsically fuzzy. By imposing sharp boundaries on the extensions of these properties valuable information is lost. In the first instance[2], the model provided by classical mathematics is therefore inadequate.

1.2 Flou Sets

A second model was proposed by Gentilhomme (1968), but the ideas behind it go back to Aristotle and more recently the work of Kleene (1952). It can be described as follows.

Let X be a universe. With a property P that is fuzzy or crisp in X we associate a threefold partition of X, consisting of the *certain region* (or region of minimal extension) Z_P, containing the objects that completely satisfy the property P; the *region of maximal extension* E_P, the complement of which (the excluded region) contains the objects that completely do not satisfy P; and the *flou region* $E_P \backslash Z_P$, containing the objects that neither completely satisfy nor completely do not satisfy P. This means that the solution of the evaluation problem generated by P is represented by a *flou set* W_P, generally defined as a couple

$$W_P \stackrel{\text{def}}{=} (Z_P, E_P) \quad ; \quad W_P \in \mathcal{P}(X)^2 \text{ and } Z_P \subseteq E_P.$$

An equivalent representation is the $X - \{0, 1/2, 1\}$ mapping γ_{W_P}, with, for any x in X,

$$\gamma_{W_P}(x) \stackrel{\text{def}}{=} \begin{cases} 1 & ; \quad x \in Z_P \\ 1/2 & ; \quad x \in E_P \backslash Z_P \\ 0 & ; \quad x \in \text{co}E_P. \end{cases}$$

For any object x in X, $\gamma_{W_P}(x)$ is called the *degree of membership* of x in the flou set W_P or the degree to which x satisfies P.

Figure 2: The universe X, the certain region Z_P and the region of maximal extension E_P in the Gentilhomme model of the representation of the solution of an evaluation problem in X generated by property P that is fuzzy in X.

Gentilhomme also introduced set-theoretic operations such as union, intersection and complementation for flou sets. We do not want to delve into this matter too deeply, but nevertheless remark that just like classical set theory, flou set theory can be linked to a particular brand of logic, in this case the so-called strong ternary Kleene logic (Kleene, 1952; Rescher, 1969). We also remark that for crisp properties, $E_P \backslash Z_P = \emptyset$, which implies $Z_P = E_P$, or $W_P = (Z_P, Z_P)$. We can identify (Z_P, Z_P) and Z_P, which amounts to defining a bijection between the diagonal of $\mathcal{P}(X)^2$ and $\mathcal{P}(X)$. Furthermore, this identification links the set-theoretic operations for diagonal flou sets in $\mathcal{P}(X)^2$ to the classical set-theoretic operations on $\mathcal{P}(X)$. In this sense, the model proposed by Gentilhomme is a natural extension of the model of classical set theory.

The Gentilhomme model however, although reasonable and clearly a step in the right direction, is not a panacea. Two problems emerge when we use it to try and represent the solution of the ill-posed problem in example 1. First of all, we must admit that in this case there exists a certain arbitrariness in the definition of the certain and the excluded region : when is a number called large, when is it called not large ? Furthermore, we would want to be able to distinguish between the numbers in the flou region, because they do not all satisfy the property "large" to the same extent.

1.3 Fuzzy Sets

Zadeh proposes a solution to both of these problems by considering a gradual transition from not satisfying a property to satisfying a property. We give a brief description of this method. For a detailed account of the theory of fuzzy sets we refer to the first part of this volume.

Let X be a universe. The solution of the evaluation problem generated by a property P that is a crisp or fuzzy property in X is represented by a *fuzzy set* h_P in X, i.e., a $X - [0, 1]$ mapping with the following interpretation: for an arbitrary x in X, $h_P(x)$ is the degree to which x satisfies P and is called the *degree of membership* of x in the fuzzy set h_P. $h_P(x) = 0$ means that x completely does not satisfy P, whereas $h_P(x) = 1$ is taken to mean that x completely satisfies P. Between these extremes lies a region characterized by a gradual transition from not satisfying to satisfying P, which is modelled by an increasing degree of membership. Zadeh's model can be linked with the variant infinite-valued standard logic of Kleene and Dienes (Dienes, 1949; Rescher, 1969). In this sense, the previously described models are special cases of the Zadeh model.

This approach was taken even further by Goguen (1967), who introduced *L-fuzzy sets*, i.e., $X - L$ mappings, where the evaluation set $(L, \leq)$ is at least a *partially ordered set* (or poset). This very interesting generalization has not received the attention it deserves, however, probably because the theory of posets (and more specifically lattices) is still lacking from the toolbox of the "ordinary" scientist. It has nevertheless proven to be of fundamental importance in many a branch of science.

A few important questions remain unanswered. The problem of the sharp boundaries still stands: with every degree of membership there is associated a sharply delineated region of the universe X, namely, the inverse image of the singleton containing this degree of membership under the (L-)fuzzy set considered. Whence this subdivision of the universe X? Why is a particular object assigned a particular degree of membership and not another one? Whereas for classical sets the notion of degree of membership has a precise meaning, the origin and meaning of the degree of membership for (L-)fuzzy sets and fuzzy properties remains obscure. In this contribution, we want to further investigate the notions of degree of membership and of evaluation set, using elementary mathematical tools. We aim at constructing a solid basis for a general theory of (L-)fuzzy sets by linking it to the theory of preference relations.

2 Relations on a Universe X induced by a Property P

2.1 Evaluation Sets and Evaluation Mappings

Let us consider a large set of straight sticks and the property "long". The evaluation problem generated by this property consists in having to investigate which of the sticks are long. Given the intrinsic fuzziness of the property "long", we are faced with serious problems when we want to obtain a *classical* representation of the solution of this evaluation problem: it is impossible to make a sharp distinction between sticks that are long and sticks that are not. On the other hand, it is possible to check whether one stick is longer than another, for instance by juxtaposing them. We are therefore able to associate with the property "long" a relation on the set of the straight sticks, defined as "is at most as long as". This relation represents an important part of the information that can be derived concerning the evaluation problem under consideration, which allows us to say that it is a representation of the solution of the evaluation problem.
This line of reasoning can be generalized. Let us consider a universe X and the evaluation problem in X, generated by a crisp or fuzzy property P. We can always try and compare any two objects in X w.r.t. the property P. In other words, we can define a relation R_P on X as follows.

Definition 2 $R_P \subseteq X^2$, *with for any* x *and* y *in* X,

$$xR_Py \Leftrightarrow x \text{ is at most as } P \text{ as } y.$$

In the sequel, we shall assume that this relation is reflexive and transitive[3], i.e.,

Assumption I

R_P is a quasi-order (or partial-preorder) relation.

The relation R_P is the representation of the information about the solution of the evaluation problem considered, that *can* be derived using a comparative approach. In what follows, we shall frequently disregard how this information is derived or acquired, whether it is subjective, or objective... We are mainly concerned here with the investigation of the representation of this information. This means that in what follows, unless otherwise noted, the relation R_P will be the starting point of our investigation.
We want to stress that it is not necessary for any two elements x and y of X to be comparable w.r.t. P, i.e., (x, y) need not belong to R_P or its inverse relation R_P^{-1}. The relation

$$\begin{aligned} O_P &\stackrel{\text{def}}{=} \text{co}(R_P \cup R_P^{-1}) \\ &= \{\, (x, y) \mid (x, y) \in X^2 \text{ and not } xR_Py \text{ and not } yR_Px \,\} \end{aligned}$$

contains exactly those couples (x, y) for which x and y are not comparable w.r.t. P. It is easily verified that O_P is irreflexive because R_P is reflexive, and symmetrical by construction. We must mention that in the literature O_P is often supposed to be empty, which amounts to saying that R_P must be *strongly complete*. In (Norwich, 1982) for instance, we can find a definition that strongly resembles definition 2, but under the different assumption that the relation mentioned is a weak order, i.e., a transitive and strongly complete relation. In the field of preference logic (Apostel, 1986; Rescher, 1968; Von Wright, 1963), the completeness of similar relations is always required. Nevertheless, this completeness is a fairly stringent property, that cannot always be accounted for. When we think about the beauty of cars or girls, or the goodness of possible solutions to a multicriteria decision problem, the (strong) completeness of the associated relations is not altogether obvious. In order to keep this discussion as general as possible, we shall not require that R_P be strongly complete.

The relation R_P need not be antisymmetric either. This means that when a couple (x, y) in X^2 belongs to both R_P and R_P^{-1}, x and y need not be identical. When we define the new relation

$$\begin{aligned} I_P &\stackrel{\text{def}}{=} R_P \cap R_P^{-1} \\ &= \{\, (x,y) \mid (x,y) \in X^2 \text{ and } xR_Py \text{ and } yR_Px \,\}, \end{aligned}$$

containing those couples of X^2 that satisfy P equally well, this means that I_P is not necessarily equal to the diagonal $\mathbf{1}_X$ of X^2. The relation I_P is reflexive and transitive because R_P is reflexive and transitive, and is symmetrical by construction. Hence, I_P is an *equivalence relation* on X.
It is possible to construct a third relation from R_P. The relation

$$\begin{aligned} S_P &\stackrel{\text{def}}{=} R_P \cap (\text{co}R_P)^{-1} \\ &= \{\, (x,y) \mid (x,y) \in X^2 \text{ and } xR_Py \text{ and not } yR_Px \,\} \end{aligned}$$

contains those couples (x, y) of X^2 for which x satisfies P less well than y does. It is asymmetrical (and therefore irreflexive) by construction and transitive because R_P is transitive.
Using the triple (S_P, I_P, O_P), we can link this discussion with the theory of *preference relations* (Roubens, 1985). This triple is indeed a special case (a quasi-order structure) of what is called there a *preference structure*. S_P is also called the *strict-preference relation*, I_P the *indifference relation* and O_P the *incomparability relation*. Furthermore $R_P = S_P \cup I_P$ is called the *large-preference relation* or *characteristic relation* of this preference structure. From the definitions, it follows that S_P, S_P^{-1}, I_P and O_P are mutually disjoint relations and that $S_P \cup S_P^{-1} \cup I_P \cup O_P = X^2$. One readily verifies that R_P on the one hand and the triple (S_P, I_P, O_P) on the other hand are *equivalent* ways of representing the same information. We can find a third equivalent but admittedly more transparant representation starting from the indifference relation I_P by introducing a few new concepts. For an arbitrary x in X, we call

$$\begin{aligned} x/_{I_P} &\stackrel{\text{def}}{=} \{\, y \mid y \in X \text{ and } xI_Py \,\} \\ &= \text{the set of objects that satisfy } P \text{ equally well as } x \end{aligned}$$

the *equivalence class* of x under I_P. Note that $x/_{I_P}$ is precisely the I_P-afterset of x (see part I). Also, we call

$$X/_{I_P} \stackrel{\text{def}}{=} \{\, x/_{I_P} \mid x \in X \,\}$$

the *quotient set* of X induced by I_P and finally

$$q_P \colon X \to X/_{I_P} \colon x \mapsto x/_{I_P}$$

the *quotient mapping* induced by I_P.
From the transitivity of R_P, it follows that

$$(\forall (x,y) \in X^2)(xR_Py \Leftrightarrow (\forall x' \in x/_{I_P})(\forall y' \in y/_{I_P})(x'R_Py')),$$

which enables us to define the following relation on $X/_{I_P}$.

Definition 3 $\rho_P \subseteq (X/_{I_P})^2$*, with, for any* α *and* β *in* $X/_{I_P}$

$$\begin{aligned} \alpha\rho_P\beta &\Leftrightarrow (\exists x \in \alpha)(\exists y \in \beta)(xR_Py) \\ &\Leftrightarrow (\forall x \in \alpha)(\forall y \in \beta)(xR_Py). \end{aligned} \tag{1}$$

This relation is reflexive and transitive because R_P is reflexive and transitive. In order to prove its antisymmetry, let α and β be arbitrary elements of $X/_{I_P}$ and suppose that $\alpha\rho_P\beta$ and $\beta\rho_p\alpha$. This implies

$$(\forall x \in \alpha)(\forall y \in \beta)(xR_Py \text{ and } yR_Px)$$

and also

$$(\forall x \in \alpha)(\forall y \in \beta)(xI_Py).$$

Hence $\alpha = \beta$. We may therefore conclude that

Corollary 1 *ρ_P is a partial-order relation on $X/_{I_P}$, or equivalently, the structure $(X/_{I_P}, \rho_P)$ is a partially ordered set.*

Now let $(L, \leq)$ be an arbitrary poset, such that an *injective* $X/_{I_P} - L$ mapping ξ_P exists, satisfying

$$(\forall(\alpha,\beta) \in (X/_{I_P})^2)(\alpha\rho_P\beta \Leftrightarrow \xi_P(\alpha) \leq \xi_P(\beta)) \tag{2}$$

and let $\epsilon_P \stackrel{\text{def}}{=} \xi_P \circ q_P$. A mapping ξ_P satisfying these properties is called an *order-embedding* of $(X/_{I_P}, \rho_P)$ in $(L, \leq)$. Immediately[4],

Proposition 1

(i) *$(\forall(x,y) \in X^2)(xR_Py \Leftrightarrow \epsilon_P(x) \leq \epsilon_P(y))$;*
(ii) *$(\forall(x,y) \in X^2)(xS_Py \Leftrightarrow \epsilon_P(x) < \epsilon_P(y))$;*
(iii) *$(\forall(x,y) \in X^2)(xI_Py \Leftrightarrow \epsilon_P(x) = \epsilon_P(y))$;*
(iv) *$(\forall(x,y) \in X^2)(xO_Py \Leftrightarrow \epsilon_P(x) \parallel \epsilon_P(y))$.*

We shall call $(L, \leq)$ an *evaluation set* and ϵ_P the associated *evaluation mapping* of X under P. For arbitrary x in X, $\epsilon_P(x)$ is called the *degree* to which x satisfies P. In the idiom of the first section, we can also call ϵ_P a *L-fuzzy set* representing P and $\epsilon_P(x)$ the *degree of membership* of the object x in the L-fuzzy set ϵ_P. It is obvious from the proposition above that ϵ_P characterizes R_P completely and can be thought of as a third way to represent the solution of the evaluation problem in X generated by P.
Whenever ξ_P is not only injective but also surjective, ξ_P is an order-isomorphism between $(X/_{I_P}, \rho_P)$ and $(L, \leq)$. In this case, ϵ_P will be called a *minimal* evaluation mapping and $(L, \leq)$ a *minimal* evaluation set. Minimal evaluation sets and mappings are therefore determined up to order-isomorphism.

2.2 Approximations

More generally, let $(L', \leq')$ be a poset and ξ'_P an *isotone* or *increasing* $X/_{I_P} - L'$ mapping, i.e.,

$$(\forall(\alpha,\beta) \in (X/_{I_P})^2)(\alpha\rho_P\beta \Rightarrow \xi'_P(\alpha) \leq' \xi'_P(\beta)) \tag{3}$$

and let $\epsilon'_P \stackrel{\text{def}}{=} \xi'_P \circ q_P$. Then of course,

$$(\forall(x,y) \in X^2)(xR_Py \Rightarrow \epsilon'_P(x) \leq' \epsilon'_P(y)). \tag{4}$$

Since the reverse implication in (4) need not hold, ϵ'_P does not necessarily characterize R_P. For reasons that will become clear furtheron, we shall call $(L', \leq')$ an *approximate* evaluation set and ϵ'_P the associated *approximate* evaluation mapping of X under P. We stress that from the knowledge of ϵ'_P we can only infer negative knowledge about R_P. Indeed, by contraposition of the implication in formula (4), we have that

$$(\forall(x,y) \in X^2)(\epsilon'_P(x) \not\leq' \epsilon'_P(y) \Rightarrow x\text{co}R_Py).$$

On the other hand, we can consider the relation R'_P on X, defined by

$$(\forall(x,y) \in X^2)(xR'_Py \Leftrightarrow \epsilon'_P(x) \leq' \epsilon'_P(y)). \tag{5}$$

One readily verifies that R'_P is also a partial-preorder relation. In the same vein as before, we define the indifference relation $I'_P \stackrel{\text{def}}{=} R'_P \cap (R'_P)^{-1}$, the incomparability relation $O'_P \stackrel{\text{def}}{=} (\text{co}R'_P) \cap (\text{co}R'_P)^{-1}$ and the strict-preference relation $S'_P \stackrel{\text{def}}{=} R'_P \cap (\text{co}R'_P)^{-1}$ and construct a new *approximate* preference structure. (4) and (5) yield

$$(\forall(x,y) \in X^2)(xR_Py \Rightarrow xR'_Py),$$

whence

$$\begin{cases} R_P \subseteq R'_P \\ I_P \subseteq I'_P \\ O'_P \subseteq O_P \\ (\forall x \in X)(x/_{I_P} \subseteq x/_{I'_P}). \end{cases} \tag{6}$$

By introducing the approximation R'_P rather than R_P, the set of the indifferent couples does not become smaller and the set of incomparable couples does not become larger. Furthermore, the partition $X/_{I_P}$ is at least as fine as the partition $X/_{I'_P}$. The relation I_P (and therefore also R_P) reveals at least as much texture on X as the relation I'_P (R'_P) does. This justifies the use of the word "approximation". The less texture I'_P reveals on X, the rougher the approximation is.

2.3 A Few Examples

In this subsection, we give a few examples in order to illustrate the ideas developed so far. Amongst other things, these examples show that the models discussed in the previous paragraph—classical sets, flou and (L-)fuzzy sets—are special cases of the model proposed above.

Example 2 (Classical Sets)
Let us again consider a universe X and a property having a classical representation in X. It is possible for any object x in X to objectively verify whether or not x satisfies P. Let A_P be the set of objects that satisfy P. We can objectively derive the characteristic relation R_P as follows. Let x and y be arbitrary elements of X.

- When x does not satisfy P, x always satisfies P at most as well as y:

$$x \in \text{co}A_P \Rightarrow (\forall y \in X)(xR_Py).$$

- When x satisfies P, x only satisfies P at most as well as y if y also satisfies P:

$$x \in A_P \Rightarrow ((\forall y \in A_P)(xR_Py) \text{ and } (\forall y \in \text{co}A_P)(x\text{co}R_Py)).$$

This leads to

$$R_P = (\text{co}A_P \times X) \cup (A_P \times A_P).$$

Furthermore, after a few elementary manipulations

$$\begin{aligned} I_P &= (A_P \times A_P) \cup (\text{co}A_P \times \text{co}A_P) \\ S_P &= \text{co}A_P \times A_P \\ O_P &= \emptyset \\ X/_{I_P} &= \{A_P, \text{co}A_P\} \\ q_P(x) &= \begin{cases} A_P & ; \quad x \in A_P \\ \text{co}A_P & ; \quad x \in \text{co}A_P. \end{cases} \end{aligned}$$

The order relation ρ_P on $X/_{I_P}$ is defined by

$$\rho_P = \{(coA_P, \text{co}A_P), (coA_P, A_P), (A_P, A_P)\}.$$

Therefore $(X/_{I_P}, \rho_P)$ is a Boolean chain of length 2. Any Boolean chain $(B, \leq)$ with two elements 0 and 1 such that $0 < 1$ can be used as a minimal evaluation set and the associated minimal evaluation mapping is given by

$$\epsilon_P : X \to B : x \mapsto \begin{cases} 1 & ; \quad x \in A_P \\ 0 & ; \quad x \in \text{co}A_P. \end{cases}$$

Starting from the relation R_P, the procedure above returns in a fairly trivial way the model of classical set theory.

Example 3 (Flou Sets as an Approximation)
Let us consider an evaluation problem in a universe X under a property P that is fuzzy in X and therefore has no classical representation in X. Let us furthermore assume that the objets in the subset Z_P of X completely satisfy P and that the objects in the subset $\text{co}E_P$ of X completely do not satisfy P. There is a boundary zone $E_P \backslash Z_P$ containing those objects that neither completely satisfy nor completely do not

satisfy P. We shall discuss here the case $Z_P \neq \emptyset$ and co$E_P \neq \emptyset$. The discussion of the other possible cases is completely similar.

Since the elements of the boundary zone $E_P \backslash Z_P$ satisfy P to a lesser extent than the members of the minimal extension Z_P and to a greater extent than the members of the excluded region coE_P, we must have

$$R_P = (Z_P \times Z_P) \cup (E_P \backslash Z_P \times Z_P) \cup (\mathrm{co}E_P \times X) \cup F.$$

In this expression F is a quasi-order relation in $E_P \backslash Z_P$, i.e., F is transitive and satisfies

$$\mathbf{1}_{E_P \backslash Z_P} \subseteq F \subseteq (E_P \backslash Z_P)^2, \tag{7}$$

where

$$\begin{aligned} \mathbf{1}_{E_P \backslash Z_P} &\stackrel{\text{def}}{=} \{ (x, x) \mid x \in E_P \backslash Z_P \} \\ &= \text{the } \textit{diagonal} \text{ of } (E_P \backslash Z_P)^2. \end{aligned}$$

F is the quasi-order relation defined on the flou region $E_P \backslash Z_P$, representing the order-theoretic aspect of the solution of the evaluation problem in $E_P \backslash Z_P$ under the property P.

Taking into account (6) and (7), the crudest possible approximation R'_P corresponds with $F = (E_P \backslash Z_P)^2$:

$$\begin{aligned} R'_P &= (Z_P \times Z_P) \cup (E_P \backslash Z_P \times Z_P) \cup (\mathrm{co}E_P \times X) \cup (E_P \backslash Z_P)^2 \\ &= (Z_P \times Z_P) \cup (E_P \backslash Z_P \times E_P) \cup (\mathrm{co}E_P \times X). \end{aligned}$$

After a few manipulations we find, with obvious notations

$$\begin{aligned} I'_P &= (Z_P \times Z_P) \cup (E_P \backslash Z_P \times E_P \backslash Z_P) \cup (\mathrm{co}E_P \times \mathrm{co}E_P) \\ S'_P &= (E_P \backslash Z_P \times Z_P) \cup (\mathrm{co}E_P \times E_P) \\ O'_P &= \emptyset \\ X/_{I'_P} &= \{Z_P, E_P \backslash Z_P, \mathrm{co}E_P\} \\ q'_P(x) &= \begin{cases} Z_P & ; \quad x \in Z_P \\ E_P \backslash Z_P & ; \quad x \in E_P \backslash Z_P \\ \mathrm{co}E_P & ; \quad x \in \mathrm{co}E_P \end{cases} \end{aligned}$$

and finally also for the corresponding partial-order relation ρ'_P on $X/_{I'_P}$

$$\begin{aligned} \rho'_P = &\{(\mathrm{co}E_P, \mathrm{co}E_P), (\mathrm{co}E_P, E_P \backslash Z_P), (\mathrm{co}E_P, Z_P), \\ &(E_P \backslash Z_P, E_P \backslash Z_P), (E_P \backslash Z_P, Z_P), (Z_P, Z_P)\}. \end{aligned}$$

The Hasse diagram of the bounded chain $(X/_{I'_P}, \rho'_P)$ of length 3 is depicted in figure 3. This shows that the Gentilhomme model gives the crudest possible approximation of the solution of the evaluation problem at hand. This approximation clearly consists in making no further distinction between the elements of the flou region.

Figure 3: Hasse diagram of the bounded chain $(X/_{I'_P}, \rho'_P)$ in example 3.

Example 4 (Fuzzy Sets)
Let us consider the evaluation problem discussed in the previous example and adopt the notations introduced there. Putting $T_P \stackrel{\text{def}}{=} E_P \backslash Z_P$ for notational simplicity, we may write

$$R_P = (Z_P \times Z_P) \cup (T_P \times Z_P) \cup (\text{co}E_P \times X) \cup F,$$

where F is a quasi-order relation on T_P, *in casu* the large-preference relation for the evaluation problem in T_P under P. With obvious notations, we define

$$\begin{aligned} I_F &\stackrel{\text{def}}{=} F \cap F^{-1} \\ S_F &\stackrel{\text{def}}{=} F \cap (T_P^2 \backslash F)^{-1} \\ O_F &\stackrel{\text{def}}{=} (T_P^2 \backslash F) \cap (T_P^2 \backslash F)^{-1}. \end{aligned}$$

The equivalence relation I_F on T_P allows us to define the quotient set

$$T_P/_{I_F} \stackrel{\text{def}}{=} \{\, x/_{I_F} \mid x \in T_P \,\},$$

the quotient mapping

$$q_F \colon T_P \to T_P/_{I_F} \colon x \mapsto x/_{I_F},$$

and the partial-order relation ρ_F on $T_P/_{I_F}$ in the usual way. Using these definitions, we may write

$$\begin{aligned} I_P &= (Z_P \times Z_P) \cup (\text{co}E_P \times \text{co}E_P) \cup I_F \\ S_P &= (T_P \times Z_P) \cup (\text{co}E_P \times E_P) \cup S_F \\ O_P &= O_F \\ X/_{I_P} &= \{Z_P, \text{co}E_P\} \cup T_P/_{I_F} \\ q_P(x) &= \begin{cases} Z_P & ; \quad x \in Z_P \\ q_F(x) & ; \quad x \in T_P \\ \text{co}E_P & ; \quad x \in \text{co}E_P \end{cases} \end{aligned}$$

and also

$$\begin{aligned} \rho_P = \{&(\text{co}E_P, \text{co}E_P), (\text{co}E_P, Z_P), (Z_P, Z_P)\} \cup \\ &\{\, (\text{co}E_P, \eta) \mid \eta \in T_P/_{I_F} \,\} \cup \{\, (\eta, Z_P) \mid \eta \in T_P/_{I_F} \,\} \cup \rho_F. \end{aligned}$$

This means that $(X/_{I_P}, \rho_P)$ is a *bounded poset* with greatest element Z_P and smallest element coE_P. Between these extremes, the ordering is given by the partial-order relation ρ_F.
Let us now investigate whether the model of fuzzy sets as proposed by Zadeh can be used to represent this situation mathematically. Using (Roubens, 1985), section 3.6.3., we have that there exists a $T_P/_{I_F}-]0,1[$ mapping ζ_F, satisfying

$$(\forall(\alpha,\beta)\in(T_P/_{I_F})^2)((\alpha\neq\beta \text{ and } \alpha\rho_F\beta)\Rightarrow\zeta_F(\alpha)<\zeta_F(\beta)).$$

Of course, ζ_F is an injection. If we now define the $X/_{I_P}-[0,1]$ mapping ζ_P by

$$\zeta_P(\alpha)\stackrel{\text{def}}{=}\begin{cases}1 & ;\quad \alpha=Z_P\\ \zeta_F(\alpha) & ;\quad \alpha\in T_P/_{I_F}\\ 0 & ;\quad \alpha=\text{co}E_P,\end{cases}$$

we have

$$(\forall(\alpha,\beta)\in(X/_{I_P})^2)((\alpha\neq\beta \text{ and } \alpha\rho_P\beta)\Rightarrow\zeta_P(\alpha)<\zeta_P(\beta)), \tag{8}$$

whence, for arbitrary x and y in X, putting $\eta_P\stackrel{\text{def}}{=}\zeta_P\circ q_P$,

$$\begin{cases}xS_Py\Rightarrow\eta_P(x)<\eta_P(y)\\ xI_Py\Rightarrow\eta_P(x)=\eta_P(y).\end{cases} \tag{9}$$

We are now left with two possibilities. In the first case, the order relation ρ_F (and also ρ_P) is linear, which is equivalent to $O_P=O_F=\emptyset$. Let α and β be arbitrary elements of $X/_{I_P}$. On the one hand $\alpha\rho_P\beta$ implies $\zeta_P(\alpha)\leq\zeta_P(\beta)$, taking into account (8). Let us on the other hand assume that $\zeta_P(\alpha)\leq\zeta_P(\beta)$ or equivalently $\neg(\zeta_P(\beta)<\zeta_P(\alpha))$. Contraposition of the implication in (8) yields $\neg(\alpha\neq\beta$ and $\beta\rho_P\alpha)$, which is equivalent to $\alpha=\beta$ or $\neg(\beta\rho_P\alpha)$. Since ρ_P is linear, this is equivalent to $\alpha\rho_P\beta$. We may conclude that

$$(\forall(\alpha,\beta)\in(X/_{I_P})^2)(\alpha\rho_P\beta\Leftrightarrow\zeta_P(\alpha)\leq\zeta_P(\beta)),$$

which means that ζ_P is an order-embedding of $(X/_{I_P},\rho_P)$ in $(L,\leq)$. In this case, therefore, the real unit interval $[0,1]$ provided with the natural ordering of real numbers is perfectly suited as an evaluation set and the associated evaluation mapping η_P completely characterizes the relation R_P.
In the second case, ρ_F and ρ_P are not linear, which is equivalent to $O_F=O_P\neq\emptyset$. For arbitrary α and β in $X/_{I_P}$ we may write, taking into account (8),

$$\alpha\rho_P\beta\Rightarrow\zeta_P(\alpha)\leq\zeta_P(\beta).$$

The reverse implication is not necessarily valid, however. Indeed, let $(x_o,y_o)\in O_P$ and call $\alpha_o\stackrel{\text{def}}{=}x_o/_{I_P}$ and $\beta_o\stackrel{\text{def}}{=}y_o/_{I_P}$. Since $([0,1],\leq)$ is a chain, $\zeta_P(\alpha_o)$ and $\zeta_P(\beta_o)$ must be comparable w.r.t. the natural ordering $\leq$ of the reals. If the reverse implication were valid for this choice of $\alpha=\alpha_o$ and $\beta=\beta_o$, we should be able to conclude that α_o and β_o are comparable w.r.t. ρ_P and hence that x_o and y_o are comparable w.r.t. R_P, which is a contradiction. This means that, in this case, the Zadeh approach only yields an approximation in the sense of equations (3)–(6). Zadeh's model is unable to represent incomparability. In order to represent incomparability we must use the more general model of L-fuzzy sets, introduced by Goguen.

Example 5 (The Representation of Nonsensicality)
Let us consider a universe X and a property P that has a classical representation within a subset Y_P of X and is nonsensical within $N_P\stackrel{\text{def}}{=}X\backslash Y_P$. Let furthermore A_P be the subset of Y_P containing those elements of X that satisfy P, and $B_P=Y_P\backslash A_P$ the subset of Y_P containing those elements of X that do not satisfy P. Of course, $\{A_P,B_P,N_P\}$ is a partition of X.

As an example, we might take $X \stackrel{\text{def}}{=} \mathbb{R}$ and P such that an object x satisfies P if and only if

$$\max(\frac{1}{(x-2)^2}, \frac{1}{|x|}) \geq 1.$$

In this case

$$\begin{aligned} A_P &= [-1, 3[\setminus\{0, 2\} \\ B_P &=]-\infty, -1[\cup]3, +\infty[\\ N_P &= \{0, 2\}. \end{aligned}$$

For any object in N_P, it is completely nonsensical to ask whether it satisfies P better than any other object in X. We can only compare it with itself (almost by definition). Therefore, the large-preference relation R_P for the evaluation problem under consideration takes the following form:

$$R_P = (B_P \times Y_P) \cup (A_P \times A_P) \cup \mathbf{1}_{N_P}$$

where, of course,

$$\mathbf{1}_{N_P} \stackrel{\text{def}}{=} \{\, (x, x) \mid x \in N_P \,\}.$$

Although the property P is not fuzzy in the sense of definition 1, we have derived a large preference relation R_P associated with the evaluation problem on X under this property. Furthermore, R_P is clearly a partial-preorder relation on X. We can therefore apply the techniques introduced earlier for properties that are crisp or fuzzy in X. This leads to

$$\begin{aligned} I_P &= (A_P \times A_P) \cup (B_P \times B_P) \cup \mathbf{1}_{N_P} \\ S_P &= B_P \times A_P \\ O_P &= (Y_P \times N_P) \cup (N_P \times Y_P) \cup (N_P^2 \setminus \mathbf{1}_{N_P}) \\ X/_{I_P} &= \{A_P, B_P\} \cup \{\, \{x\} \mid x \in N_P \,\} \\ q_P(x) &= \begin{cases} A_P & ; \quad x \in A_P \\ B_P & ; \quad x \in B_P \\ \{x\} & ; \quad x \in N_P \end{cases} \end{aligned}$$

and also for the partial-order relation ρ_P on $X/_{I_P}$

$$\rho_P = \{(B_P, B_P), (B_P, A_P), (A_P, A_P)\} \cup \{\, (\{x\}, \{x\}) \mid x \in N_P \,\}.$$

This means that when $N_P \neq \emptyset$, $(X/_{I_P}, \rho_P)$ is a partially ordered set that is not bounded and is not a lattice.
An obvious approximation consists in assuming that any two elements of N_P satisfy P equally well. Let $(L', \leq')$ be the partially ordered set of three elements 0, o and 1, such that $0 < 1$, $0 \parallel o$ and $1 \parallel o$. The above-mentioned approximation appears through the introduction of the $X/_{I_P} - L'$ mapping ξ'_P, defined by

$$\begin{cases} \xi'_P(A_P) \stackrel{\text{def}}{=} 1 \\ \xi'_P(B_P) \stackrel{\text{def}}{=} 0 \\ \xi'_P(\{x\}) \stackrel{\text{def}}{=} o \quad ; \quad x \in N_P. \end{cases}$$

Remark that ξ'_P is isotone in the sense of equation (3). The approximate large-preference relation, defined using (5), is given by

$$R'_P = (B_P \times Y_P) \cup (A_P \times A_P) \cup (N_P \times N_P).$$

Hence,

$$
\begin{aligned}
I'_P &= (B_P \times B_P) \cup (A_P \times A_P) \cup (N_P \times N_P) \\
S'_P &= B_P \times A_P \\
O'_P &= (N_P \times Y_P) \cup (Y_P \times N_P) \\
X/_{I'_P} &= \{A_P, B_P, N_P\} \\
q'_P(x) &= \begin{cases} A_P & ; \quad x \in A_P \\ B_P & ; \quad x \in B_P \\ N_P & ; \quad x \in N_P \end{cases}
\end{aligned}
$$

and ρ'_P is given by

$$\rho'_P = \{(B_P, B_P), (B_P, A_P), (A_P, A_P), (N_P, N_P)\}$$

By construction, $(X/_{I'_P}, \rho'_P)$ is order-isomorphic to $(L', \leq')$. This approximate evaluation set is precisely the same (up to order-isomorphism, of course) as the partially ordered set of truth values of the internal three-valued Bochvar logic. Bochvar introduced this logic in order to be able to represent the truth value of nonsensical propositions, alongside of the (classical) truth values of (classical) propositions in classical logic (see for instance (Rescher, 1969)). We may conclude that the model proposed here, although primarily intended for the representation of fuzzy properties, is also able to generate an elementary representation of nonsensicality.

2.4 Conclusion

An important aspect of an evaluation problem in a universe X under a property P is the order relation generated by P on this universe. In a very natural way, this relation leads to evaluation sets and evaluation mappings characterizing it completely. This order relation (or equivalently the evaluation set and mapping associated with this relation) represents the most elementary information that can be obtained w.r.t. the evaluation problem. In this sense, it may be argued that the concept of ordering (order relations) is fundamental to the mathematical representation of the solutions of evaluation problems. Whatever may be the (crisp or fuzzy) property P, ordering is the starting point.

There is, however, another point that calls for our attention. Up to now we have regarded evaluation mappings as mappings that characterize the ordering generated by properties on the universe considered. We have shown that evaluation sets and evaluation mappings are determined up to an order-embedding. In general therefore, there is a class of evaluation sets and mappings that are mutually equivalent characterizations of the ordering generated by a property on a universe. When we want to consider other concepts associated with a certain property besides ordering, we will have to select an appropriate subclass of this class. In other words, the more detail we want to incorporate into the description of properties, the more mathematical structure the evaluation sets and mappings used to that end will have to reveal. In general, evaluation sets and mappings must be made to represent those aspects of the meaning of properties that are considered important for the evaluation problem at hand. In this context, we want to remind the reader of the related discussion about the *ordinal* or *cardinal* aspect of *utility functions* in the (economically flavoured) preference logic (see for instance (Apostel, 1986)).

Choosing an appropriate subclass of the class of evaluation sets and mappings is also necessary when more than one property is considered at the same time. As an example, let us consider a universe X and the properties P and $P' =$ *very* P. It can be argued that the quasi-order relations R_P and $R_{P'}$, separately induced by these properties on the universe X, are one and the same. This does not mean, however, that on the basis of ordering alone no distinction can be made between P and P'. In order to compare and distinguish between P and P', we must use a poset $(L, \leq)$ that is an evaluation set of X under P *as well as* under P'. This means that we must be able to find an order-embedding of both the structures $(X/_{I_P}, \rho_P)$ and $(X/_{I_{P'}}, \rho_{P'})$ in the structure $(L, \leq)$. This clearly imposes restrictions on the poset $(L, \leq)$. Only the members of the common subclass of the classes of the evaluation sets of X under P and P' qualify as evaluation sets of X under P and P' together. However, the study of a number of properties taken together and hence of the relations between these properties is much more complicated than the study undertaken in this section and will be the subject of the following sections.

3 Relations on a Universe X Induced by Several Properties

In the foregoing sections, we have discussed evaluation problems in a universe X under a single property. It is however very important to also consider evaluation problems under more than one property at the same time. The solution of this broader type of problem does not necessarily consist in separately solving the evaluation problems under every single property, since this approach can not do justice to any relations that might exist between the properties under consideration. We shall nevertheless be able to treat this broader type of problem in very much the same way as we treated evaluation problems under a single property.

3.1 Basic Assumptions

Let us make the following assumption, in order to simplify the ensuing discussion.

Assumption II
We assume that it is possible for any two properties P and Q considered furtheron to verify whether they are identical ($P = Q$) or not ($P \neq Q$). Also, when talking about sets of properties, we take for granted that the collections of properties considered are indeed sets and that the necessary precautions have been made to avoid the paradoxes of naïve set theory.

Now, let X be a universe and $\mathcal{E}$ a set of properties that are fuzzy or crisp in X. With every property P in $\mathcal{E}$ we can associate an evaluation problem in X and consequently define a large-preference relation R_P and a preference structure (S_P, I_P, O_P). Using the methods of the previous section, we can then construct the minimal evaluation set $(X/_{I_P}, \rho_P)$ and the evaluation mapping q_P.
Let us now construct a binary relation $R_{\mathcal{E}}^i$ on $X \times \mathcal{E}$ that has the same information content as all the separate relations R_P $(P \in \mathcal{E})$ taken together. Of course, this relation is given by

$$R_{\mathcal{E}}^i \stackrel{\text{def}}{=} \{ ((x,P),(y,P)) \mid P \in \mathcal{E} \text{ and } (x,y) \in R_P \},$$

which means that for arbitrary (x,P) and (y,Q) in $X \times \mathcal{E}$

$$(x,P)R_{\mathcal{E}}^i(y,Q) \Leftrightarrow P = Q \text{ and } xR_Py. \tag{10}$$

Moreover, $R_{\mathcal{E}}^i$ is reflexive and transitive and can be interpreted as a characteristic or large-preference relation, with an associated preference structure $(S_{\mathcal{E}}^i, I_{\mathcal{E}}^i, O_{\mathcal{E}}^i)$, where

$$\begin{aligned} S_{\mathcal{E}}^i &= \{ ((x,P),(y,P)) \mid P \in \mathcal{E} \text{ and } (x,y) \in S_P \} \\ I_{\mathcal{E}}^i &= \{ ((x,P),(y,P)) \mid P \in \mathcal{E} \text{ and } (x,y) \in I_P \} \\ O_{\mathcal{E}}^i &= \{ ((x,P),(y,P)) \mid P \in \mathcal{E} \text{ and } (x,y) \in O_P \} \cup \\ &\quad \{ ((x,P),(y,Q)) \mid (x,y) \in X^2 \text{ and } (P,Q) \in \mathcal{E}^2 \text{ and } P \neq Q \}. \end{aligned} \tag{11}$$

Also, with obvious notations, for arbitrary (x,P) in $X \times \mathcal{E}$,

$$q_{\mathcal{E}}^i(x,P) = (x,P)/_{I_{\mathcal{E}}^i} = x/_{I_P} \times \{P\}$$

which leads to

$$X \times \mathcal{E}/_{I_{\mathcal{E}}^i} = \{ x/_{I_P} \times \{P\} \mid (x,P) \in X \times \mathcal{E} \}.$$

The partial-order relation $\rho_{\mathcal{E}}^i$ is defined as

$$(\forall (x,y) \in X^2)(\forall (P,Q) \in \mathcal{E}^2)$$
$$(((x,P)/_{I_{\mathcal{E}}^i}, (y,Q)/_{I_{\mathcal{E}}^i}) \in \rho_{\mathcal{E}}^i \Leftrightarrow P = Q \text{ and } (x/_{I_P}, y/_{I_P}) \in \rho_P).$$

This implies that we can construct the Hasse diagram of $(X/_{I_{\mathcal{E}}^i}, \rho_{\mathcal{E}}^i)$ by simply juxtaposing the Hasse diagrams of the $(X/_{I_P}, \rho_P)$ $(P \in \mathcal{E})$, without adding any connecting arcs. The large-preference relation $R_{\mathcal{E}}^i$ does not allow the comparison of any two objects in X on the basis of their satisfying any two different properties in $\mathcal{E}$. This is explicitly stated by formula (11).
We should not, however, forgo the possibility that objects in X are indeed comparable w.r.t. two different properties $\mathcal{E}$. Let us therefore define a more general binary relation $R_{\mathcal{E}}$ on $X \times \mathcal{E}$.

Definition 4 *For x and y in X, and P and Q in $\mathcal{E}$:*

$$(x,P)R_{\mathcal{E}}(y,Q) \Leftrightarrow x \textit{ is at most as } P \textit{ as } y \textit{ is } Q.$$

Intuition leads to the following assumption.

Assumption III

$R_{\mathcal{E}}$ is a quasi-order (or partial-preorder) relation.

This relation represents the ordering obtained on the basis of the properties in $\mathcal{E}$. We shall consider $R_{\mathcal{E}}$ as the starting point of the discussion in the following sections.
It is obvious that the relation $R_{\mathcal{E}}$ must not contain any information that is in contradiction with the relations R_P ($P \in \mathcal{E}$). In other words, we must have that

$$(\forall P \in \mathcal{E})(\forall (x,y) \in X^2)((x,P)R_{\mathcal{E}}(y,P) \Leftrightarrow xR_Py).$$

Indeed, comparing objects on the basis of a set of properties should not change the information already obtained by the comparison of these objects on the basis of each separate property. More generally, when we consider two sets $\mathcal{E}_1$ and $\mathcal{E}_2$ of properties that are crisp or fuzzy in X, we should have, with obvious notations, that

$$(\forall (P,Q) \in \mathcal{E}_1 \cap \mathcal{E}_2)(\forall (x,y) \in X^2)((x,P)R_{\mathcal{E}_1}(y,Q) \Leftrightarrow (x,P)R_{\mathcal{E}_2}(y,Q)),$$

because adding properties to a set of properties should not change the ordering already present. To formalize this, we start from the following definition.

Definition 5
Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be any two sets of properties that are crisp or fuzzy in a universe X. Whenever

$$(\forall (P,Q) \in \mathcal{E}_1 \cap \mathcal{E}_2)(\forall (x,y) \in X^2)((x,P)R_{\mathcal{E}_1}(y,Q) \Leftrightarrow (x,P)R_{\mathcal{E}_2}(y,Q)),$$

we shall call $R_{\mathcal{E}_1}$ consistent with $R_{\mathcal{E}_2}$.

Corollary 2 *Let $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ be arbitrary sets of properties that are crisp or fuzzy in X. We then have*

(i) *$R_{\mathcal{E}}$ is consistent with itself;*
(ii) *if $R_{\mathcal{E}_1}$ is consistent with $R_{\mathcal{E}_2}$, then $R_{\mathcal{E}_2}$ is consistent with $R_{\mathcal{E}_1}$;*
(iii) *if $\mathcal{E}_3 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_1$, $R_{\mathcal{E}_1}$ is consistent with $R_{\mathcal{E}_2}$ and $R_{\mathcal{E}_2}$ is consistent with $R_{\mathcal{E}_3}$, then $R_{\mathcal{E}_1}$ is consistent with $R_{\mathcal{E}_3}$.*

Proof. Trivial. □

Definition 6 *Let $\mathcal{E}$ be an arbitrary set of properties that are crisp or fuzzy in X. We shall call $R_{\mathcal{E}}$ consistent iff for any two subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$, $R_{\mathcal{E}_1}$ is consistent with $R_{\mathcal{E}_2}$.*

In what follows, we shall take the consistency[5] for granted, i.e.,

Assumption IV
whenever we consider the large-preference relation $R_{\mathcal{E}}$, associated with a set $\mathcal{E}$ of properties that are crisp or fuzzy in a universe X, we shall assume that $R_{\mathcal{E}}$ is consistent.

Corollary 3 $R_P = \{\, (x,y) \mid ((x,P),(y,P)) \in R_{\mathcal{E}} \,\}$.

Let us also remark that assumptions III and IV imply assumption I. We shall therefore in the sequel omit reference to this first assumption.

3.2 Commensurability

A special case of a large-preference relation on $X \times \mathcal{E}$ is the relation $R^i_{\mathcal{E}}$ considered before. It is the ordering obtained when all the *different* properties in $\mathcal{E}$ are *mutually incommensurable* in the universe X, which explains the notation $R^i_{\mathcal{E}}$ for this relation. Let us clarify this in the next definition.

Definition 7
We shall call any two elements P and Q of $\mathcal{E}$ incommensurable in X iff no objects in X are comparable w.r.t. these properties, i.e.,

$$(\forall(x,y)\in X^2)((x,P)\text{co}R_{\mathcal{E}}(y,Q) \text{ and } (y,Q)\text{co}R_{\mathcal{E}}(x,P))$$

and commensurable in X iff they are not incommensurable, i.e.,

$$(\exists(x,y)\in X^2)((x,P)R_{\mathcal{E}}(y,Q) \text{ or } (y,Q)R_{\mathcal{E}}(x,P)).$$

Also, we shall call P and Q completely commensurable in X iff any two objects in X are comparable w.r.t. the properties P and Q, in other words,

$$(\forall(x,y)\in X^2)((x,P)R_{\mathcal{E}}(y,Q) \text{ or } (y,Q)R_{\mathcal{E}}(x,P)).$$

The partial-preorder relation $R_{\mathcal{E}}$ has an immediate interpretation as a characteristic or large-preference relation with associated preference structure $(S_{\mathcal{E}}, I_{\mathcal{E}}, O_{\mathcal{E}})$. Using these relations, the definition above can be reformulated as follows.

Corollary 4 *Let P and Q be arbitrary properties in $\mathcal{E}$. We then have*

(i) *P and Q are incommensurable in X iff* $(\forall(x,y)\in X^2)((x,P)O_{\mathcal{E}}(y,Q))$;
(ii) *P and Q are commensurable in X iff* $(\exists(x,y)\in X^2)((x,P)\text{co}O_{\mathcal{E}}(y,Q))$;
(iii) *P and Q are completely commensurable in X iff* $(\forall(x,y)\in X^2)((x,P)\text{co}O_{\mathcal{E}}(y,Q))$.

Furthermore, all the pairs of properties in $\mathcal{E}$ are completely commensurable in X iff $O_{\mathcal{E}}=\emptyset$. Also, any property P in $\mathcal{E}$ is always commensurable with itself in X; it is on the other hand completely commensurable with itself in X iff $O_P=\emptyset$.

3.3 Evaluation Sets and Mappings

It is obvious that we can use the methods described in the previous section to construct the quotient class $X\times\mathcal{E}/_{I_{\mathcal{E}}}$ and the $X\times\mathcal{E}-X\times\mathcal{E}/_{I_{\mathcal{E}}}$ quotient mapping $q_{\mathcal{E}}$, starting from the equivalence relation $I_{\mathcal{E}}$ on $X\times\mathcal{E}$. Similarly, we can define the partial-order relation $\rho_{\mathcal{E}}$ on $X\times\mathcal{E}/_{I_{\mathcal{E}}}$. Whenever there exists an order-embedding $\xi_{\mathcal{E}}$ of $(X\times\mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$ in a partially ordered set $(L,\leq)$, we shall call $(L,\leq)$ an evaluation set of X under the properties in $\mathcal{E}$ and $\epsilon_{\mathcal{E}} \overset{\text{def}}{=} \xi_{\mathcal{E}}\circ q_{\mathcal{E}}$ the associated evaluation mapping. The following generalization of proposition 1 is readily proven.

Proposition 2
Let x and y be arbitrary objects in X, and P and Q arbitrary properties in $\mathcal{E}$. Then, with obvious notations,

(i) $(x,P)R_{\mathcal{E}}(y,Q) \Leftrightarrow \epsilon_{\mathcal{E}}(x,P)\leq\epsilon_{\mathcal{E}}(y,Q)$;
(ii) $(x,P)S_{\mathcal{E}}(y,Q) \Leftrightarrow \epsilon_{\mathcal{E}}(x,P)<\epsilon_{\mathcal{E}}(y,Q)$;
(iii) $(x,P)I_{\mathcal{E}}(y,Q) \Leftrightarrow \epsilon_{\mathcal{E}}(x,P)=\epsilon_{\mathcal{E}}(y,Q)$;
(iv) $(x,P)O_{\mathcal{E}}(y,Q) \Leftrightarrow \epsilon_{\mathcal{E}}(x,P)\parallel\epsilon_{\mathcal{E}}(y,Q)$.

The large-preference relation $R_{\mathcal{E}}$ (and therefore also the associated preference structure $(S_{\mathcal{E}}, I_{\mathcal{E}}, O_{\mathcal{E}})$) is completely characterized by $\epsilon_{\mathcal{E}}$.
We now turn to the important question of whether the methods given in the previous section lead to results that are consistent with the results obtained by the present generalization. Let therefore P be an arbitrary property in $\mathcal{E}$, and x and y be arbitrary objects in X. Taking into account proposition 2,

$$(x,P)R_{\mathcal{E}}(y,P) \Leftrightarrow \epsilon_{\mathcal{E}}(x,P)\leq\epsilon_{\mathcal{E}}(y,P).$$

On the other hand, from assumption IV and corollary 3

$$(x,P)R_{\mathcal{E}}(y,P) \Leftrightarrow xR_Py.$$

We may therefore conclude that for arbitrary P in $\mathcal{E}$

$$(\forall(x,y)\in X^2)(xR_Py \Leftrightarrow \epsilon_{\mathcal{E}}(x,P)\leq \epsilon_{\mathcal{E}}(y,P)).$$

This implies that there exists an order-embedding of $(X/_{I_P}, \rho_P)$ in the evaluation set $(L,\leq)$ of X under the properties in $\mathcal{E}$. Hence, $(L,\leq)$ is an evaluation set of X under the property P in the sense of the previous section[6]. Furthermore the partial mapping $\epsilon_{\mathcal{E}}(\,\cdot\,,P)$ is the evaluation mapping of X under P, associated with the evaluation set $(L,\leq)$. This proves that the methods discussed in section 2 are entirely consistent with the approach followed here.
We can use the above-mentioned partial mappings $\epsilon_{\mathcal{E}}(\cdot\,,P)$ $(P\in\mathcal{E})$ to define a notion that will be of use furtheron. We have already stressed that all the discussions in this contribution start from a universe X and are therefore relative to this universe. When we cannot distinguish between two properties in the universe X, we must be able to identify these properties to a certain extent.

Definition 8 *Two properties P and Q in $\mathcal{E}$ coincide in X iff for any object x in X, x satisfies P and Q equally well. We shall use the notation $P =_X Q$ for this coincidence.*

It must be clear from the discussion above that the coincidence in X of two properties P and Q in $\mathcal{E}$ must be reflected in the properties of $R_{\mathcal{E}}$. This remark is further clarified by the following proposition.

Proposition 3 *Let $\mathcal{E}$ be a set of properties that are fuzzy or crisp in X. Let $(L,\leq)$ be an evaluation set of X under the properties in $\mathcal{E}$ and $\epsilon_{\mathcal{E}}$ the associated evaluation mapping. Furthermore, let P and Q be arbitrary elements of $\mathcal{E}$. P and Q coincide in X iff $\epsilon_{\mathcal{E}}(\,\cdot\,,P)=\epsilon_{\mathcal{E}}(\,\cdot\,,Q)$, i.e., $(\forall x\in X)(\epsilon_{\mathcal{E}}(x,P)=\epsilon_{\mathcal{E}}(x,Q))$.*

Proof. Using definition 8 and assumption IV, we have that

$$P =_X Q \Leftrightarrow (\forall x\in X)((x,P)I_{\mathcal{E}}(x,Q)).$$

The proposition then immediately follows from proposition 2. $\square$

Corollary 5 *"$=_X$" is an equivalence relation on $\mathcal{E}$.*

3.4 A Few Examples

Example 6 (Classical Sets)
Let us assume that the properties in $\mathcal{E}$ have a classical representation in X. For every property P in $\mathcal{E}$ we shall use the notations of example 2: A_P is the set of the objects in X that satisfy the property P. Any two properties P and Q that have a classical representation in X are also completely commensurable in this universe. Indeed, for any two objects x and y in X we can objectively deduce the following. If x satisfies P then x can only be at most as P as y is Q if also y satisfies Q. If on the other hand x does not satisfy P, x will always be at most as P as y is Q. This means that the objectively derivable large-preference relation $R_{\mathcal{E}}$ is given by

$$R_{\mathcal{E}}=\bigcup_{(P,Q)\in\mathcal{E}^2}\{\,((x,P),(y,Q))\mid (x,y)\in (A_P\times A_Q)\cup(\mathrm{co}A_P\times X)\,\}.$$

We shall use $R_{\mathcal{E}}$ as the starting point for the discussion in this example. Applying the methods introduced above, we find

$$S_{\mathcal{E}}=\bigcup_{(P,Q)\in\mathcal{E}^2}\{\,((x,P),(y,Q))\mid (x,y)\in \mathrm{co}A_P\times A_Q\,\}$$
$$I_{\mathcal{E}}=\bigcup_{(P,Q)\in\mathcal{E}^2}\{\,((x,P),(y,Q))\mid (x,y)\in (A_P\times A_Q)\cup(\mathrm{co}A_P\times \mathrm{co}A_Q)\,\}.$$

Furthermore $R_{\mathcal{E}}\cup R_{\mathcal{E}}^{-1}=(X\times\mathcal{E})^2$, whence $O_{\mathcal{E}}=\emptyset$, confirming that there are no incomparable couples in $(X\times\mathcal{E})^2$. Also

$$X\times\mathcal{E}/_{I_{\mathcal{E}}}=\{\alpha,\beta\},$$

where

$$\begin{cases} \alpha \stackrel{\text{def}}{=} \bigcup_{P\in\mathcal{E}} \text{co}A_P \times \{P\} \\ \beta \stackrel{\text{def}}{=} \bigcup_{P\in\mathcal{E}} A_P \times \{P\}. \end{cases}$$

The partial-order relation $\rho_{\mathcal{E}}$ is defined as

$$\rho_{\mathcal{E}} = \{(\alpha,\alpha),(\alpha,\beta),(\beta,\beta)\},$$

and the minimal evaluation mapping $q_{\mathcal{E}}$ satisfies

$$q_{\mathcal{E}}(x,P) = \begin{cases} \alpha & ; \quad (x,P)\in\alpha, \text{ or equivalently, } x \in \text{co}A_P \\ \beta & ; \quad (x,P)\in\beta, \text{ or equivalently, } x \in A_P. \end{cases}$$

The minimal evaluation set $(X\times\mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$ is a Boolean chain of length 2 and our methods again lead to results compatible with classical set theory.

Example 7 (Flou Sets and Fuzzy Sets)
Let us now investigate a more general problem by also admitting fuzzy properties in $\mathcal{E}$. This means that for an arbitrary P in $\mathcal{E}$ there exists a set Z_P containing the objects in X that completely satisfy P; a set coE_P containing the objects in X that completely do not satisfy P; and a set $T_P \stackrel{\text{def}}{=} E_P\backslash Z_P$ containing the objects in X that lie between these extremes. We shall assume that neither Z_P nor coE_P is empty. The set T_P may be empty, implying that the property P has a classical representation in X. One easily extends the discussion of examples 3 and 4 to this more general case. This leads to

$$R_{\mathcal{E}} = \bigcup_{(P,Q)\in\mathcal{E}^2} \{\, ((x,P),(y,Q)) \mid (x,y)\in R_{(P,Q)} \,\} \cup G,$$

where

$$R_{(P,Q)} \stackrel{\text{def}}{=} (Z_P\times Z_Q)\cup(T_P\times Z_Q)\cup(\text{co}E_P\times X)$$

and G is a quasi-order relation on $T_G \stackrel{\text{def}}{=} \bigcup_{P\in\mathcal{E}} T_P\times\{P\}$, that can be interpreted as a large-preference relation on T_G. By further exploiting the analogy between this general case and the evaluation problem of examples 3 and 4, we may infer that the crudest approximation $R'_{\mathcal{E}}$ is obtained by substituting

$$\begin{aligned} G &= \Big(\bigcup_{P\in\mathcal{E}} T_P\times\{P\}\Big)^2 \\ &= \bigcup_{(P,Q)\in\mathcal{E}^2} \{\, ((x,P),(y,Q)) \mid (x,y)\in T_P\times T_Q \,\} \end{aligned}$$

in the expression for $R_{\mathcal{E}}$. Hence

$$R'_{\mathcal{E}} = \bigcup_{(P,Q)\in\mathcal{E}^2} \{\, ((x,P),(y,Q)) \mid (x,y)\in R'_{(P,Q)} \,\},$$

where

$$R'_{(P,Q)} \stackrel{\text{def}}{=} (Z_P\times Z_Q)\cup(T_P\times E_Q)\cup(\text{co}E_P\times X).$$

Also,

$$\begin{aligned} S'_{\mathcal{E}} &= \bigcup_{(P,Q)\in\mathcal{E}^2} \{\, ((x,P),(y,Q)) \mid (x,y)\in S'_{(P,Q)} \,\} \\ I'_{\mathcal{E}} &= \bigcup_{(P,Q)\in\mathcal{E}^2} \{\, ((x,P),(y,Q)) \mid (x,y)\in I'_{(P,Q)} \,\} \\ O'_{\mathcal{E}} &= \emptyset, \end{aligned}$$

where, obviously

$$\begin{cases} S'_{(P,Q)} \stackrel{\text{def}}{=} (T_P \times Z_Q) \cup (\text{co}E_P \times E_Q) \\ I'_{(P,Q)} \stackrel{\text{def}}{=} (Z_P \times Z_Q) \cup (T_P \times T_Q) \cup (\text{co}E_P \times \text{co}E_Q). \end{cases}$$

Introducing the following notations

$$\begin{cases} \alpha \stackrel{\text{def}}{=} \bigcup_{P\in\mathcal{E}} \text{co}E_P \times \{P\} \\ \beta \stackrel{\text{def}}{=} \bigcup_{P\in\mathcal{E}} T_P \times \{P\} \\ \gamma \stackrel{\text{def}}{=} \bigcup_{P\in\mathcal{E}} Z_P \times \{P\}, \end{cases}$$

we also find that

$$X \times \mathcal{E}/_{I'_{\mathcal{E}}} = \{\alpha, \beta, \gamma\}$$

$$q'_{\mathcal{E}}(x,P) = \begin{cases} \alpha & ; \quad (x,P) \in \alpha, \text{ or equivalently, } x \in \text{co}E_P \\ \beta & ; \quad (x,P) \in \beta, \text{ or equivalently, } x \in T_P \\ \gamma & ; \quad (x,P) \in \gamma, \text{ or equivalently, } x \in Z_P. \end{cases}$$

The partial-order relation $\rho_{\mathcal{E}}$ is given by

$$\rho'_{\mathcal{E}} = \{(\alpha,\alpha), (\alpha,\beta), (\alpha,\gamma), (\beta,\beta), (\beta,\gamma), (\gamma,\gamma)\}.$$

The approximate evaluation set $(X \times \mathcal{E}/_{I'_{\mathcal{E}}}, \rho'_{\mathcal{E}})$ is a chain of length 3 as soon as one property in $\mathcal{E}$ is fuzzy. The approximation itself consists in giving the same treatment to all elements of T_G. An important consequence of this approximation is that $O'_{\mathcal{E}} = \emptyset$. There is no incomparability and any two properties in $\mathcal{E}$ are completely commensurable in this crudest of approximations.

Let us now return to the original large-preference relation $R_{\mathcal{E}}$ in order to investigate whether Zadeh's model is able to describe this situation. Since G is a quasi-order relation on $T_G = \beta$, we can apply the methods of section 2. Putting

$$\begin{cases} I_G \stackrel{\text{def}}{=} G \cap G^{-1} \\ O_G \stackrel{\text{def}}{=} (T_G^2 \backslash G) \cap (T_G^2 \backslash G)^{-1}, \end{cases}$$

we define the quotient set

$$T_G/_{I_G} \stackrel{\text{def}}{=} \{\, (x,P)/_{I_G} \mid P \in \mathcal{E} \text{ and } x \in T_P \,\},$$

the quotient mapping

$$q_G\colon T_G \to T_G/_{I_G}\colon (x,P) \mapsto (x,P)/_{I_G}$$

and the partial-order relation ρ_G on $T_G/_{I_G}$ in the by now familiar way.

After a fair amount of manipulations, we find

$$I_{\mathcal{E}} = \bigcup_{(P,Q)\in\mathcal{E}^2} \{\, ((x,P),(y,Q)) \mid (x,y) \in I_{(P,Q)} \,\} \cup I_G$$

$$O_{\mathcal{E}} = O_G,$$

where, of course,

$$I_{(P,Q)} \stackrel{\text{def}}{=} (Z_P \times Z_Q) \cup (\text{co}E_P \times \text{co}E_Q).$$

Also

$$X \times \mathcal{E}/_{I_{\mathcal{E}}} = \{\alpha, \gamma\} \cup T_G/_{I_G}$$

$$q_{\mathcal{E}}(x,P) = \begin{cases} \alpha & ; \quad (x,P) \in \alpha, \text{ or equivalently, } x \in \text{co}E_P \\ q_G(x,P) & ; \quad (x,P) \in T_G, \text{ or equivalenty, } x \in T_P \\ \gamma & ; \quad (x,P) \in \gamma, \text{ or equivalently, } x \in Z_P. \end{cases}$$

Finally

$$\rho_{\mathcal{E}} = \{(\alpha,\alpha),(\alpha,\gamma),(\gamma,\gamma)\} \cup \{\,(\alpha,\eta) \mid \eta \in T_G/_{I_G}\,\} \\ \cup \{\,(\eta,\gamma) \mid \eta \in T_G/_{I_G}\,\} \cup \rho_G.$$

We easily deduce from these formulae that the poset $(X \times \mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$ is bounded, with smallest element α and greatest element γ. The ordering between these extremes is determined by the partial-order relation ρ_G on $T_G/_{I_G}$. We can therefore interpret the partial mappings $q_{\mathcal{E}}(\,\cdot\,, P)$ $(P \in \mathcal{E})$ as $X \times \mathcal{E}/_{I_{\mathcal{E}}}$-fuzzy sets (following Goguen), associated with these properties.
A discussion similar to the one followed in example 4 will reveal that only in the case $O_{\mathcal{E}} = O_G = \emptyset$, the bounded chain $([0,1], \leq)$ is a perfectly qualified evaluation set. When $O_{\mathcal{E}} = O_G$ is not empty, $([0,1], \leq)$ can only be an approximate evaluation set, because it does not allow any representation of incomparability. By starting from $([0,1], \leq)$ as an evaluation set the way Zadeh does, we *assume* that any two of the properties taken into consideration are completely commensurable in the universe considered.

4 Elementary Combinations of Properties

In natural language, it is possible on the one hand to modify properties (e.g. by weakening, strengthening or negating them) and on the other hand to combine several properties into a new property. In particular, a property P can be modified to form the properties *not* P, *very* P, *more or less* P, ... and two properies P_1 and P_2 can be combined to form the properties P_1 *and* P_2, P_1 *or* P_2, ... In this section, we intend to study some of the more frequent *modifications* and *combinations*, using the methods discussed in the previous sections. Let us start with a more or less general discussion.

4.1 General Considerations

In this subsection, we derive a few general and abstract results that constitute the basis on which the more specific discussion of the following subsections will be built. We start with a universe X and an arbitrary set $\mathcal{E}$ of properties that are crisp or fuzzy in X. We also consider an arbitrary $n \in \mathbb{N}^*$ and an arbitrary $\mathcal{E}^n - \mathcal{E}$ mapping F, called *combinator* (of arity n) in $\mathcal{E}$. This combinator maps an arbitrary n-tuple of properties $(P_1, \ldots, P_n)$ in $\mathcal{E}^n$ to a *unique* combined property $F(P_1, \ldots, P_n)$. When $n = 1$ we shall also call F a *modifier* in $\mathcal{E}$. Inspired by the preceding sections, we also consider a consistent partial-preorder relation $R_{\mathcal{E}}$ on $X \times \mathcal{E}$. Finally, in this subsection we denote by $(L, \leq)$ an arbitrary evaluation set of X under the properties in $\mathcal{E}$ and by $\epsilon_{\mathcal{E}}$ the associated evaluation mapping. Of course, the substructure $(\epsilon_{\mathcal{E}}(X \times \mathcal{E}), \leq)$ of $(L, \leq)$ is order-isomorphic to $(X \times \mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$. First, we give a few preliminary definitions.

Definition 9
Let F be a combinator in $\mathcal{E}$ of arity 1. Then F is called involutive in $(X, \mathcal{E})$ iff

$$(\forall P \in \mathcal{E})(F(F(P)) =_X P).$$

Definition 10
Let F be a combinator in $\mathcal{E}$ of arity 2. Then F is called symmetrical (or commutative) in $(X, \mathcal{E})$ iff

$$(\forall (P, Q) \in \mathcal{E}^2)(F(P, Q) =_X F(Q, P)).$$

Definition 11
Let F be a combinator in $\mathcal{E}$ or arity 2. Then F is called associative in $(X, \mathcal{E})$ iff

$$(\forall (P_1, P_2, P_3) \in \mathcal{E}^3)(F(F(P_1, P_2), P_3) =_X F(P_1, F(P_2, P_3))).$$

Definition 12
Let F be a combinator in $\mathcal{E}$ of arity 2. Then F is called idempotent in $(X, \mathcal{E})$ iff

$$(\forall P \in \mathcal{E})(F(P, P) =_X P).$$

Definition 13
Let F_1 and F_2 be combinators in $\mathcal{E}$ of arity 2. Then F_2 is called distributive w.r.t. F_1 in $(X, \mathcal{E})$ iff

$$(\forall (P_1, P_2, P_3) \in \mathcal{E}^3)(F_2(F_1(P_1, P_2), P_3)) =_X F_1(F_2(P_1, P_3), F_2(P_2, P_3)).$$

and

$$(\forall (P_1, P_2, P_3) \in \mathcal{E}^3)(F_2(P_1, F_1(P_2, P_3)) =_X F_1(F_2(P_1, P_2), F_2(P_1, P_3)).$$

Definition 14
Let F_1 and F_2 be combinators in $\mathcal{E}$ of arity 2 and let F be a combinator in $\mathcal{E}$ of arity 1. Then F, F_1 and F_2 satisfy the laws of de Morgan in $(X, \mathcal{E})$ iff

$$(\forall (P, Q) \in \mathcal{E}^2)(F(F_1(P, Q)) =_X F_2(F(P), F(Q)))$$

and

$$(\forall (P, Q) \in \mathcal{E}^2)(F(F_2(P, Q)) =_X F_1(F(P), F(Q))).$$

We now face a very important problem. Is it generally possible to represent the action of F on an arbitrary n-tuple of properties $(P_1, \ldots, P_n)$ in $\mathcal{E}$ using a function from L^n to L? More specifically, *can we find a function from L^n to L, such that the degree to which an arbitrary object x of X satisfies the combination $F(P_1, \ldots, P_n)$ of an arbitrary n-tuple of properties in $\mathcal{E}$ is the image under this function of the degrees to which the object x satisfies the properties $P_1, \ldots, P_n$?* The first step towards the solution of this problem is the definition of a binary *combinator relation* F_L between L^n and L, i.e. , $F_L \subseteq L^n \times L$, with an immediate interpretation, since it relates the degrees to which an arbitrary x in X satisfies an arbitary n-tuple of properties $(P_1, \ldots, P_n)$ to the degree to which x satisfies the combination $F(P_1, \ldots, P_n)$.

Definition 15

$$F_L \stackrel{\text{def}}{=} \{\, ((\epsilon_{\mathcal{E}}(x, P_1), \ldots, \epsilon_{\mathcal{E}}(x, P_n)), \epsilon_{\mathcal{E}}(x, F(P_1, \ldots, P_n))) \mid x \in X \text{ and } (P_1, \ldots, P_n) \in \mathcal{E}^n \,\}.$$

Corollary 6 *Whenever the combinator relation F_L is functional, we have, with obvious notations,*

$$(\forall (P_1, \ldots, P_n) \in \mathcal{E}^n)(\forall x \in X)$$
$$(\epsilon_{\mathcal{E}}(x, F(P_1, \ldots, P_n)) = F_L(\epsilon_{\mathcal{E}}(x, P_1), \ldots, \epsilon_{\mathcal{E}}(x, P_n))),$$

which means that the action of F on an arbitrary n-tuple of properties in $\mathcal{E}$ can be represented using the function F_L from L^n to L.

The solution to the problem mentioned above is therefore closely connected with the combinator relation F_L being functional or not. One easily verifies that if this relation is functional, this will also be the case for the combinator relations defined using the other possible evaluation sets and mappings of X under the properties in $\mathcal{E}$. This functionality is therefore invariant under the change of evaluation set and depends solely on the properties of the relation $R_{\mathcal{E}}$. A formal proof for this statement appears in the following theorem.

Theorem 1 *The combinator relation F_L is functional iff*

$$(\forall (P_1, \ldots, P_n) \in \mathcal{E}^n)(\forall (Q_1, \ldots, Q_n) \in \mathcal{E}^n)(\forall x \in X)(\forall y \in X)$$
$$((\forall k \in \{1, \ldots, n\})((x, P_k) I_{\mathcal{E}} (y, Q_k)) \Rightarrow$$
$$(x, F(P_1, \ldots, P_n)) I_{\mathcal{E}} (y, F(Q_1, \ldots, Q_n))). \tag{12}$$

Proof. On the one hand, assume that F_L is functional. Choose arbitrary x and y in X, and $(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$ in $\mathcal{E}^n$. If

$$(\forall k \in \{1, \ldots, n\})((x, P_k) I_{\mathcal{E}} (y, Q_k))$$

holds, we have, taking into account proposition 2, that

$$(\forall k \in \{1, \ldots, n\})(\epsilon_{\mathcal{E}}(x, P_k) = \epsilon_{\mathcal{E}}(y, Q_k))$$

and, since F_L is assumed functional, that

$$F_L(\epsilon_{\mathcal{E}}(x, P_1), \ldots, \epsilon_{\mathcal{E}}(x, P_n)) = F_L(\epsilon_{\mathcal{E}}(y, Q_1), \ldots, \epsilon_{\mathcal{E}}(y, Q_n)).$$

Corollary 6 then enables us to write this as

$$\epsilon_{\mathcal{E}}(x, F(P_1, \ldots, P_n)) = \epsilon_{\mathcal{E}}(y, F(Q_1, \ldots, Q_n)),$$

and again taking into account proposition 2, we may therefore conclude that

$$(x, F(P_1, \ldots, P_n)) I_{\mathcal{E}} (y, F(Q_1, \ldots, Q_n)).$$

On the other hand, assume that (12) holds and let us show that this implies the functionality of F_L. Indeed, choose an arbitrary $(\alpha_1, \ldots, \alpha_n)$ in L^n and an arbitrary β_1 and β_2 in L, and suppose that both $((\alpha_1, \ldots, \alpha_n), \beta_1)$ and $((\alpha_1, \ldots, \alpha_n), \beta_2)$ belong to F_L. From the definition of F_L we then deduce that there exist x and y in X, and $(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$ in $\mathcal{E}^n$ satisfying

$$\begin{cases} \alpha_k = \epsilon_{\mathcal{E}}(x, P_k);\ k = 1(1)n \\ \beta_1 = \epsilon_{\mathcal{E}}(x, F(P_1, \ldots, P_n)) \end{cases} \quad \text{and} \quad \begin{cases} \alpha_k = \epsilon_{\mathcal{E}}(y, Q_k);\ k = 1(1)n \\ \beta_2 = \epsilon_{\mathcal{E}}(y, F(Q_1, \ldots, Q_n)). \end{cases}$$

Since (12) is assumed to hold, this immediately implies that $\beta_1 = \beta_2$. □

This theorem enables us to introduce the following concept.

Definition 16 (Truth-Functionality)
We shall call a combinator F in $\mathcal{E}$ truth-functional in X iff the combinator relation F_L is functional, or equivalently, iff (12) holds.

Whenever F_L is a function, we can look for ways to restrict it so as to turn it into a mapping. It is easy to prove that the domain of the combinator relation F_L of a combinator F with arity n is given by

$$D^n_{\mathcal{E}} \stackrel{\text{def}}{=} \{ (\epsilon_{\mathcal{E}}(x, P_1), \ldots, \epsilon_{\mathcal{E}}(x, P_n)) \mid x \in X \text{ and } (P_1, \ldots, P_n) \in \mathcal{E}^n \}.$$

The subsets $D^n_{\mathcal{E}}$ of L^n satisfy the following properties.

Proposition 4

(i) $D^1_{\mathcal{E}} = \epsilon_{\mathcal{E}}(X \times \mathcal{E})$. *For any combinator F in $\mathcal{E}$ of arity 1 that is truth-functional in X, the combinator relation F_L is a transformation of $\epsilon_{\mathcal{E}}(X \times \mathcal{E})$.*

(ii) *For arbitrary $n \in \mathbb{N}^*$ with $n > 1$, we have $D^n_{\mathcal{E}} \subseteq \epsilon_{\mathcal{E}}(X \times \mathcal{E})^n$.*

(iii) *If*

$$(\forall \alpha \in \epsilon_{\mathcal{E}}(X \times \mathcal{E}))(\forall x \in X)(\exists P \in \mathcal{E})(\alpha = \epsilon_{\mathcal{E}}(x, P)) \tag{13}$$

we also have for arbitrary n in $\mathbb{N}^*$ that $D^n_{\mathcal{E}} = \epsilon_{\mathcal{E}}(X \times \mathcal{E})^n$.

Proof. Trivial. □

4.2 Modifications of a property

Let us once again consider a universe X and a property P that is fuzzy or crisp in X. We have already indicated that the evaluation sets $(X/_{I_P}, \rho_P)$ respectively $(X/_{I_{very\,P}}, \rho_{very\,P})$ of X under P respectively *very* P are order-isomorphic. Of course, the same holds for the properties P and *more or less* P. The preference structures corresponding with P, *very* P and *more or less* P are identical and the modifiers associated with these properties belong to the same class, namely, the class of modifiers that to a certain extent affirm the property they modify.

In a complete analogous way we can consider a second class of modifiers, the members of which to a certain extent negate the property they modify. The preference structure corresponding with each individual member of this class will also be identical. An important member of this class is the modifier *not*. Let us make a intuitively acceptable assumption about this modifier.

Assumption V
With every property P that is crisp or fuzzy in X there corresponds a unique property not P. This last property is also crisp or fuzzy in X.

In the remainder of this subsection, we shall denote by $N(P)$ the unique property *not* P of a property P that is fuzzy or crisp in X.

Definition 17 *Let $\mathcal{E}$ be a set of properties that are crisp or fuzzy in X. We call $\mathcal{E}$ closed under negation iff $(\forall P \in \mathcal{E})(N(P) \in \mathcal{E})$.*

In the rest of this subsection we shall mean by $\mathcal{E}$ an arbitrary set of properties that are crisp or fuzzy in X, closed under negation. This implies that the negation N can be considered as a transformation of $\mathcal{E}$ and therefore as a combinator in $\mathcal{E}$ of arity 1. Also, the results of the previous subsection can be applied, putting $F = N$ and $n = 1$. We shall furthermore start from a large-preference relation $R_{\mathcal{E}}$, representing amongst other things the ordinal aspects of the behaviour of N in $\mathcal{E}$ in natural language. When we denote by $(L, \leq)$ an arbitrary evaluation set of X under the properties in $\mathcal{E}$ and by $\epsilon_{\mathcal{E}}$ the associated evaluation mapping, we have in particular

$$N_L = \{\, (\epsilon_{\mathcal{E}}(x, P), \epsilon_{\mathcal{E}}(x, N(P))) \mid x \in X \text{ and } P \in \mathcal{E} \,\}.$$

Of course, there are also the special cases of the general results proven earlier.

Corollary 7 (Truth-Functionality)
The negation N in $\mathcal{E}$ is truth-functional in X iff

$$(\forall (P, Q) \in \mathcal{E}^2)(\forall (x, y) \in X^2)((x, P) I_{\mathcal{E}} (y, Q) \Rightarrow (x, N(P)) I_{\mathcal{E}} (y, N(Q))).$$

Corollary 8 *Whenever the negation N in $\mathcal{E}$ is truth-functional in X, we have that*

$$(\forall P \in \mathcal{E})(\forall x \in X)(\epsilon_{\mathcal{E}}(x, N(P)) = N_L(\epsilon_{\mathcal{E}}(x, P))).$$

In the next few propositions a number of possible properties of N_L are related to possible properties of N, as reflected in $R_{\mathcal{E}}$. These possible properties of N (or $R_{\mathcal{E}}$) are a reflection of the possible meaning of negation in natural language. In this sense, the propositions below allow us on the one hand to draw conclusions regarding N_L from certain assumptions about N and on the other hand to verify what is implicitly assumed about the behaviour of N when certain properties of N_L are taken to hold.

Proposition 5 (Symmetry)
If N is involutive in $(X, \mathcal{E})$ then N_L is a symmetrical relation.

Proof. Assume that N is indeed involutive in $(X, \mathcal{E})$ and choose arbitrary α and β in L. Assuming that $(\alpha, \beta) \in N_L$, we have

$$(\exists x \in X)(\exists P \in \mathcal{E})(\alpha = \epsilon_{\mathcal{E}}(x, P) \text{ and } \beta = \epsilon_{\mathcal{E}}(x, N(P))).$$

Since N is involutive in $(X, \mathcal{E})$, this implies

$$(\exists x \in X)(\exists P \in \mathcal{E})(\alpha = \epsilon_{\mathcal{E}}(x, N(N(P))) \text{ and } \beta = \epsilon_{\mathcal{E}}(x, N(P))),$$

and also, putting $Q = N(P)$, since $\mathcal{E}$ is closed under negation,

$$(\exists x \in X)(\exists Q \in \mathcal{E})(\alpha = \epsilon_{\mathcal{E}}(x, N(Q)) \text{ and } \beta = \epsilon_{\mathcal{E}}(x, Q)).$$

Hence, $(\beta, \alpha) \in N_L$. $\square$

Proposition 6 (Involutivity)
The negation relation N_L is an involutive permutation of the subset $\epsilon_{\mathcal{E}}(X \times \mathcal{E})$ of L iff the negation N is involutive in $(X, \mathcal{E})$ and

$$(\forall (P, Q) \in \mathcal{E}^2)(\forall (x, y) \in X^2)((x, P) I_{\mathcal{E}} (y, Q) \Rightarrow (x, N(P)) I_{\mathcal{E}} (y, N(Q))), \tag{14}$$

i.e., the negation N in $\mathcal{E}$ is truth-functional in X.

Proof. Assume on the one hand that N_L is an involutive permutation of $\epsilon_{\mathcal{E}}(X \times \mathcal{E})$. In particular N_L is a functional relation on L, whence (14), using corollary 7. Involutivity of N_L implies $N_L \circ N_L = \mathbf{1}_L$, where $\mathbf{1}_L$ is the identical permutation of L. Now choose an arbitrary P in $\mathcal{E}$. Since $\mathcal{E}$ is closed under negation, $N(P)$ and $N(N(P))$ are elements of $\mathcal{E}$. Hence, for arbitrary x in X, taking into account the functionality of N_L and corollary 8

$$\begin{aligned}\epsilon_{\mathcal{E}}(x, N(N(P))) &= N_L(\epsilon_{\mathcal{E}}(x, N(P))) \\ &= N_L(N_L(\epsilon_{\mathcal{E}}(x, P))) \\ &= (N_L \circ N_L)(\epsilon_{\mathcal{E}}(x, P)) \\ &= \epsilon_{\mathcal{E}}(x, P),\end{aligned}$$

which means that N is involutive in $(X, \mathcal{E})$.
Suppose on the other hand that N is involutive in $(X, \mathcal{E})$ and that (14) holds. On the basis of corollary 7 we conclude from (14) that N_L is a functional relation on L and on the basis of proposition 4 (i) that N_L is a transformation of $\epsilon_{\mathcal{E}}(X \times \mathcal{E})$. Since N is involutive in $\mathcal{E}$, proposition 5 assures the symmetry of N_L. A symmetrical transformation is a permutation and therefore an involution. □

Proposition 7 (Antitonicity and Involutivity)
The negation N_L is an antitone and involutive permutation of the substructure $(\epsilon_{\mathcal{E}}(X \times \mathcal{E}), \leq)$ of $(L, \leq)$ iff the negation N is involutive in $(X, \mathcal{E})$ and

$$(\forall (P, Q) \in \mathcal{E}^2)(\forall (x, y) \in X^2)((x, P) R_{\mathcal{E}} (y, Q) \Rightarrow (y, N(Q)) R_{\mathcal{E}} (x, N(P))). \tag{15}$$

Proof. Assume on the one hand that N_L is an antitone and involutive permutation of $(\epsilon_{\mathcal{E}}(X \times \mathcal{E}), \leq)$. From the proposition above it follows already that N is involutive in $(X, \mathcal{E})$. In order to show that (15) holds, we choose arbitrary P and Q in $\mathcal{E}$, and x and y in X. Assume that $(x, P) R_{\mathcal{E}} (y, Q)$ holds. Taking into account proposition 2, we may write

$$\epsilon_{\mathcal{E}}(x, P) \leq \epsilon_{\mathcal{E}}(y, Q).$$

Since N_L is antitone, this implies,

$$N_L(\epsilon_{\mathcal{E}}(y, Q)) \leq N_L(\epsilon_{\mathcal{E}}(x, P)),$$

which can be rewritten as

$$\epsilon_{\mathcal{E}}(y, N(Q)) \leq \epsilon_{\mathcal{E}}(x, N(P)),$$

according to corollary 8. Again using proposition 2, this implies that

$$(y, N(Q)) R_{\mathcal{E}} (x, N(P)).$$

Assume on the other hand that N is involutive in $(X, \mathcal{E})$ and that (15) holds. This immediately implies

$$(\forall (P, Q) \in \mathcal{E}^2)(\forall (x, y) \in X^2)((x, P) I_{\mathcal{E}} (y, Q) \Rightarrow (y, N(Q)) I_{\mathcal{E}} (x, N(P))).$$

Using proposition 6 we conclude that N_L is an involutive permutation of $\epsilon_{\mathcal{E}}(X \times \mathcal{E})$. It remains to show that this permutation is antitone. To this effect, choose arbitrary α and β in $\epsilon_{\mathcal{E}}(X \times \mathcal{E})$. Then there exist x and y in X, and P and Q in $\mathcal{E}$ such that $\alpha = \epsilon_{\mathcal{E}}(x, P)$ and $\beta = \epsilon_{\mathcal{E}}(y, Q)$. Assume that $\alpha \leq \beta$. Taking into account proposition 2, this implies

$$(x, P) R_{\mathcal{E}} (y, Q),$$

whence, using (15),

$$(y, N(Q)) R_{\mathcal{E}} (x, N(P)).$$

Once again using proposition 2, we may write

$$\epsilon_{\mathcal{E}}(y, N(Q)) \leq \epsilon_{\mathcal{E}}(x, N(P)),$$

whence, from corollary 8,

$$N_L(\epsilon_{\mathcal{E}}(y, Q)) \leq N_L(\epsilon_{\mathcal{E}}(x, P)).$$

This means that $N_L(\beta) \leq N_L(\alpha)$. □

We conclude this subsection with a few examples in order to clarify the foregoing discussion.

Example 8 (Classical Sets)
Let P be a property that has a classical representation in X. Then $N(P)$ has a classical representation in X as well. This follows from the following observations.

- When an object x in X satisfies P, it does not satisfy $N(P)$.
- When an object x in X does not satisfy P, it satisfies $N(P)$.

This implies that

$$A_{N(P)} = \mathrm{co}A_P, \tag{16}$$

using the notations of example 2. $\mathrm{co}A_P$ therefore contains the objects that satisfy $N(P)$. Also

$$\begin{aligned} R_{N(P)} &= (A_{N(P)} \times A_{N(P)}) \cup (\mathrm{co}A_{N(P)} \times X) \\ &= (\mathrm{co}A_P \times \mathrm{co}A_P) \cup (A_P \times X) \\ &= (\mathrm{co}A_P \times \mathrm{co}A_P) \cup (A_P \times A_P) \cup (A_P \times \mathrm{co}A_P) \\ &= (X \times \mathrm{co}A_P) \cup (A_P \times A_P) \\ &= R_P^{-1}, \end{aligned}$$

which means that negation N reverses the ordering induced by P on X. Furthermore, since $N(P)$ also has a classical representation in X

$$A_{N(N(P))} = \mathrm{co}A_{N(P)} = A_P,$$

which implies that P coincides with $N(N(P))$ in X, i.e. , $N(N(P)) =_X P$. Now let $\mathcal{E}$ be a set of properties that are crisp in X, which is furthermore closed under negation. On the basis of the results of example 6 we have an expression for the large-preference relation $R_\mathcal{E}$ associated with the evaluation problem in X under the properties in $\mathcal{E}$. Let us investigate what can be deduced from $R_\mathcal{E}$, also taking into account formula (16). Further exploiting the results of example 6 and also borrowing the notations introduced there, we find that there exists a permutation n of $X \times \mathcal{E}/_{I_\mathcal{E}}$, defined by

$$\begin{cases} n(\alpha) \stackrel{\text{def}}{=} \beta \\ n(\beta) \stackrel{\text{def}}{=} \alpha, \end{cases}$$

satisfying

$$(\forall P \in \mathcal{E})(\forall x \in X)(q_\mathcal{E}(x, N(P)) = n(q_\mathcal{E}(x, P))).$$

We remark that n is involutive and antitone. It is the unique complement operator on the Boolean chain $(X \times \mathcal{E}/_{I_\mathcal{E}}, \rho_\mathcal{E})$ of length 2. These results are in complete accordance with the fact that negation is truth-functional in classical two-valued logic.

We could have derived the same result using the following more abstract line of reasoning. To start with, we already know that for an arbitrary P in $\mathcal{E}$, P coincides with $N(N(P))$ in X, which implies that N is involutive in $(X, \mathcal{E})$. Furthermore, consider arbitrary x and y in X, and P and Q in $\mathcal{E}$, and assume that $(x, P) R_\mathcal{E} (y, Q)$. Again borrowing the results and notations of example 6, this implies that

$$(x, y) \in (A_P \times A_Q) \cup (\mathrm{co}A_P \times X),$$

whence

$$(y, x) \in (A_Q \times A_P) \cup (X \times \mathrm{co}A_P),$$

or equivalently

$$(y, x) \in (A_Q \times A_P) \cup (A_Q \times \mathrm{co}A_P) \cup (\mathrm{co}A_Q \times \mathrm{co}A_P),$$

leading to

$$(y, x) \in (A_Q \times X) \cup (\mathrm{co}A_Q \times \mathrm{co}A_P).$$

Taking into account (16), this implies that

$$(y, x) \in (\mathrm{co}A_{N(Q)} \times X) \cup (A_{N(Q)} \times A_{N(P)}).$$

Since $\mathcal{E}$ is closed under negation, this can also be written as

$$(y, N(Q)) R_\mathcal{E} (x, N(P)).$$

From proposition 7 it now follows that $n = N_{X \times \mathcal{E}/_{I_\mathcal{E}}}$ is an involutive and antitone permutation of $(X \times \mathcal{E}/_{I_\mathcal{E}}, \rho_\mathcal{E})$.

Example 9 (Flou Sets and Fuzzy Sets)
Consider the more general case of a set $\mathcal{E}$ of properties that are crisp or fuzzy in X, closed under negation. For arbitrary P in $\mathcal{E}$ we use the notations of examples 4 and 7 and we furthermore assume that neither Z_P nor $\text{co}E_P$ are empty. Analogous results can be deduced in the other cases. Taking into account the remarks made in the beginning of this subsection, it seems reasonable to assume that the modifiers of the negating class reverse the ordering on X, induced by the property P in $\mathcal{E}$ they act upon. In particular, this assumption leads to

$$(\forall P \in \mathcal{E})(R_{N(P)} = R_P^{-1}). \tag{17}$$

Also, it is reasonable to assume that the negation N is involutive in $(X, \mathcal{E})$. Considering the meaning of negation in natural language, we may indeed infer that the modifier *not* is characterized in the class of negating modifiers by the fact that applying it twice to a property leaves this property unchanged. Let us investigate what can be deduced about the operator N in $\mathcal{E}$ on the basis of these two assumptions. From formula (17) we deduce for arbitrary P in $\mathcal{E}$,

$$\begin{cases} Z_{N(P)} = \text{co}E_P \\ T_{N(P)} = T_P \\ \text{co}E_{N(P)} = Z_P \end{cases} \tag{18}$$

or equivalently,

$$\begin{aligned}(Z_{N(P)}, E_{N(P)}) &= (\text{co}E_P, \text{co}Z_P) \\ &= \text{co}_G(Z_P, E_P),\end{aligned}$$

where co_G is the complement operator for flou sets, introduced by Gentilhomme (1968). As a first step, we investigate the crudest approximation $R'_{\mathcal{E}}$. Together with the results from example 7, formula (18) implies the existence of an antitone and involutive permutation n' of $(X \times \mathcal{E}/_{I'_{\mathcal{E}}}, \rho'_{\mathcal{E}})$ satisfying

$$(\forall P \in \mathcal{E})(\forall x \in X)(q'_{\mathcal{E}}(x, N(P)) = n'(q'_{\mathcal{E}}(x, P))).$$

n' is defined by

$$\begin{cases} n'(\alpha) \stackrel{\text{def}}{=} \gamma \\ n'(\beta) \stackrel{\text{def}}{=} \beta \\ n'(\gamma) \stackrel{\text{def}}{=} \alpha. \end{cases}$$

Amongst other things, this implies that in the crudest possible approximation the negation N in $\mathcal{E}$ is truth-functional in X, a result that may be arrived at in a different manner. Indeed, consider arbitrary x and y in X, and P and Q in $\mathcal{E}$. Assume that $(x, P)R'_{\mathcal{E}}(y, Q)$. From the results of example 7, it follows that

$$(x, y) \in (Z_P \times Z_Q) \cup (T_P \times E_Q) \cup (\text{co}E_P \times X),$$

whence

$$(y, x) \in (Z_Q \times Z_P) \cup (E_Q \times T_P) \cup (X \times \text{co}E_P),$$

or equivalently,

$$\begin{aligned}(y, x) \in &(Z_Q \times Z_P) \cup (Z_Q \times T_P) \cup (T_Q \times T_P) \cup \\ &\cup (Z_Q \times \text{co}E_P) \cup (T_Q \times \text{co}E_P) \cup (\text{co}E_Q \times \text{co}E_P),\end{aligned}$$

leading to

$$(y, x) \in (Z_Q \times X) \cup (T_Q \times \text{co}Z_P) \cup (\text{co}E_Q \times \text{co}E_P).$$

Taking into account (18) and the fact that $\mathcal{E}$ is closed under negation, this implies

$$(y, x) \in (\text{co}E_{N(Q)} \times X) \cup (T_{N(Q)} \times E_{N(P)}) \cup (Z_{N(Q)} \times Z_{N(P)}),$$

whence finally $(y, N(Q))R'_{\mathcal{E}}(x, N(P))$. Together with the involutivity of N in $(X, \mathcal{E})$ this leads to the conclusion that $n' = N_{X\times\mathcal{E}/_{I'_P}}$ is an involutive and antitone permutation of $(X \times \mathcal{E}/_{I'_{\mathcal{E}}}, \rho'_{\mathcal{E}})$, also taking into account proposition 7. We also want to remark that in an arbitrary chain of length 3 there exists only one involutive and antitone permutation, which can be identified with the negation operator in three-valued logics (Rescher, 1969).

We now leave this approximation and return to the study of the large-preference relation $R_{\mathcal{E}}$. Assumption IV together with formula (17) leads to

$$(\forall P \in \mathcal{E})(\forall (x, y) \in X^2)((x, P)R_{\mathcal{E}}(y, P) \Rightarrow (y, N(P))R_{\mathcal{E}}(x, N(P))).$$

One readily verifies that this condition is necessary but not sufficient for (15) to hold. In the next example we will show that the truth-functionality in X of the negation in $\mathcal{E}$ does not necessarily follow from (17) and the involutivity of N in $(X, \mathcal{E})$. We nevertheless do have that for arbitrary x in X and P in $\mathcal{E}$

$$\begin{cases} q_{\mathcal{E}}(x, P) = \alpha \Leftrightarrow q_{\mathcal{E}}(x, N(P)) = \gamma \\ q_{\mathcal{E}}(x, P) = \gamma \Leftrightarrow q_{\mathcal{E}}(x, N(P)) = \alpha \\ q_{\mathcal{E}}(x, P) \in T_G/_{I_G} \Leftrightarrow q_{\mathcal{E}}(x, N(P)) \in T_G/_{I_G}, \end{cases}$$

meaning that the restriction of $N_{X\times\mathcal{E}/_{I_{\mathcal{E}}}}$ to $\{\alpha, \gamma\}^2$ is an involutive and antitone permutation.

It appears that (17) and the involutivity of N in $(X, \mathcal{E})$ are not sufficient to assure that N is an involutive and antitone permutation of $(X \times \mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$. If, on the other hand, we assume that N is involutive in $(X, \mathcal{E})$ and also satisfies the more stringent condition (15) in stead of (17), (17) can be shown to hold as well. Furthermore, in this case the negation relation $N_{X\times\mathcal{E}/_{I_{\mathcal{E}}}}$, also satisfies

$$\begin{cases} N_{X\times\mathcal{E}/_{I_{\mathcal{E}}}}(\alpha) = \gamma \\ N_{X\times\mathcal{E}/_{I_{\mathcal{E}}}}(\gamma) = \alpha, \end{cases}$$

where, of course,

$$\begin{cases} \alpha = \bigcup_{P\in\mathcal{E}} \mathrm{co}E_P \times \{P\} \\ \gamma = \bigcup_{P\in\mathcal{E}} Z_P \times \{P\} \end{cases}$$

are respectively the smallest and the greatest element of $(X \times \mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$.

Example 10

Let X be a universe and consider a property P that is fuzzy in X, together with its unique negation $N(P)$. Assume that $P \neq N(P)$ and that $N(N(P)) = P$, which implies that $N(N(P)) =_X P$. This means that we can consider the set of properties $\mathcal{E} = \{P, N(P)\}$ of properties that are fuzzy in X, which is furthermore closed under negation. We assume that with P there can be associated a large-preference relation R_P on X, such that $I_P = \{Z_P, \alpha, \beta, \mathrm{co}E_P\}$ and that the Hasse diagram of $(X/_{I_P}, \rho_P)$ is represented by figure 4 (a). When we associate with $N(P)$ the large-preference relation $R_{N(P)} = R_P^{-1}$, we may conclude that $I_{N(P)} = I_P$ and that the Hasse diagram of $(X/_{I_{N(P)}}, \rho_{N(P)})$ is represented by figure 4 (b). Putting, for notational simplicity

$$\begin{cases} 1 \stackrel{\text{def}}{=} Z_P \times \{P\} \cup \mathrm{co}E_P \times \{N(P)\} \\ \gamma_1 \stackrel{\text{def}}{=} \alpha \times \{P\} \\ \gamma_2 \stackrel{\text{def}}{=} \beta \times \{N(P)\} \\ \gamma_3 \stackrel{\text{def}}{=} \beta \times \{P\} \cup \alpha \times \{N(P)\} \\ 0 \stackrel{\text{def}}{=} \mathrm{co}E_P \times \{P\} \cup Z_P \times \{N(P)\} \end{cases}$$

and choosing $R_{\mathcal{E}}$ such that $I_{\mathcal{E}} = \{0, \gamma_1, \gamma_2, \gamma_3, 1\}$ and that the Hasse diagram of $(X \times \mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$ is represented by figure 4 (c), we have that this $\mathcal{E}$ and $R_{\mathcal{E}}$ satisfy assumptions III, IV (and V) and also that N is involutive in $(X, \mathcal{E})$ and that (17) holds.

Moreover

$$q_{\mathcal{E}}(x,P) = \begin{cases} 1 & ; \quad x \in Z_P \\ \gamma_1 & ; \quad x \in \alpha \\ \gamma_3 & ; \quad x \in \beta \\ 0 & ; \quad x \in \mathrm{co}E_P \end{cases}$$

$$q_{\mathcal{E}}(x,N(P)) = \begin{cases} 0 & ; \quad x \in Z_P \\ \gamma_3 & ; \quad x \in \alpha \\ \gamma_2 & ; \quad x \in \beta \\ 1 & ; \quad x \in \mathrm{co}E_P. \end{cases}$$

Therefore, the negation relation

$$N_{X\times\mathcal{E}/_{I_{\mathcal{E}}}} = \{(1,0),(0,1),(\gamma_1,\gamma_3),(\gamma_3,\gamma_1),(\gamma_3,\gamma_2),(\gamma_2,\gamma_3)\}$$

is not a functional relation. This example shows that in general the involutivity of N in $(X,\mathcal{E})$ and (17) are not sufficient to assure the truth-functionality in X of the negation in $\mathcal{E}$.

Figure 4: (a) Hasse diagram of the bounded lattice $(X/_{I_P}, \rho_P)$ of example 10
(b) Hasse diagram of the bounded lattice $(X/_{I_{N(P)}}, \rho_{N(P)})$ of example 10
(c) Hasse diagram of the bounded lattice $(X \times \mathcal{E}/_{I_{\mathcal{E}}}, \rho_{\mathcal{E}})$ of example 10, together with a few relevant arrows of the relation $N_{X\times\mathcal{E}/_{I_{\mathcal{E}}}}$

4.3 Combinations of two properties

Besides negation, there are two important combinations of properties that play an important role in natural language, namely the conjunction (*and*) and the disjunction (*or*) of two properties. We start by making an intuitively acceptable assumption about these combinations.

Assumption VI
With any two properties P_1 and P_2 that are crisp or fuzzy in X, there correspond a unique property P_1 and *P_2 and a unique property P_1* or *P_2. The latter are also crisp or fuzzy in X.*

In what follows we shall denote by $E(P_1,P_2)$ the property P_1 *and* P_2 and by $O(P_1,P_2)$ the property P_1 *or* P_2.

Definition 18 *Let $\mathcal{E}$ be an arbitrary set of properties that are crisp or fuzzy in X. We call $\mathcal{E}$*

(i) *closed under conjunction iff* $(\forall(P_1, P_2) \in \mathcal{E}^2)(E(P_1, P_2) \in \mathcal{E})$;

(ii) *closed under disjunction iff* $(\forall(P_1, P_2) \in \mathcal{E}^2)(O(P_1, P_2) \in \mathcal{E})$.

In the remainder of this subsection we shall mean by $\mathcal{E}$ a set of properties that are crisp or fuzzy in X. We also consider the consistent partial-preorder relation $R_{\mathcal{E}}$ on $X \times \mathcal{E}$. By $(L, \leq)$ we mean an arbitrary but fixed evaluation set of X under the properties in $\mathcal{E}$ and by $\epsilon_{\mathcal{E}}$ the evaluation mapping associated with it.

- When $\mathcal{E}$ is closed under conjunction, E can be considered as a combinator in $\mathcal{E}$ of arity 2. We can therefore borrow the results from subsection 4.1, putting $F = E$ and $n = 2$. In particular, we have

$$E_L \stackrel{\text{def}}{=} \{ ((\epsilon_{\mathcal{E}}(x, P_1), \epsilon_{\mathcal{E}}(x, P_2)), \epsilon_{\mathcal{E}}(x, E(P_1, P_2))) \mid (x, (P_1, P_2)) \in X \times \mathcal{E}^2 \}.$$

- When $\mathcal{E}$ is closed under disjunction, O can be considered as a combinator in $\mathcal{E}$ of arity 2. We can therefore borrow the results from subsection 4.1, putting $F = O$ and $n = 2$. In particular, we have

$$O_L \stackrel{\text{def}}{=} \{ ((\epsilon_{\mathcal{E}}(x, P_1), \epsilon_{\mathcal{E}}(x, P_2)), \epsilon_{\mathcal{E}}(x, O(P_1, P_2))) \mid (x, (P_1, P_2)) \in X \times \mathcal{E}^2 \}.$$

There are amongst other things the following particular cases of general results proven in subsection 4.1.

Corollary 9

(i) *Let $\mathcal{E}$ be closed under conjunction. Then the conjunction E in $\mathcal{E}$ is truth-functional in X iff*

$$(\forall(P_1, P_2, Q_1, Q_2) \in \mathcal{E}^4)(\forall(x, y) \in X^2)$$
$$((x, P_1)I_{\mathcal{E}}(y, Q_1) \textit{ and } (x, P_2)I_{\mathcal{E}}(y, Q_2) \Rightarrow (x, E(P_1, P_2))I_{\mathcal{E}}(y, E(Q_1, Q_2))). \quad (19)$$

(ii) *Let $\mathcal{E}$ be closed under disjunction. The disjunction O in $\mathcal{E}$ is truth-functional in X iff*

$$(\forall(P_1, P_2, Q_1, Q_2) \in \mathcal{E}^4)(\forall(x, y) \in X^2)$$
$$((x, P_1)I_{\mathcal{E}}(y, Q_1) \textit{ and } (x, P_2)I_{\mathcal{E}}(y, Q_2) \Rightarrow (x, O(P_1, P_2))I_{\mathcal{E}}(y, O(Q_1, Q_2))). \quad (19')$$

Corollary 10

(i) *Let $\mathcal{E}$ be closed under conjunction. When the conjunction E in $\mathcal{E}$ is truth-functional in X, we have that*

$$(\forall(P_1, P_2) \in \mathcal{E}^2)(\forall x \in X)(\epsilon_{\mathcal{E}}(x, E(P_1, P_2)) = E_L(\epsilon_{\mathcal{E}}(x, P_1), \epsilon_{\mathcal{E}}(x, P_2))).$$

(ii) *Let $\mathcal{E}$ be closed under disjunction. When the disjunction O in $\mathcal{E}$ is truth-functional in X, we have that*

$$(\forall(P_1, P_2) \in \mathcal{E}^2)(\forall x \in X)(\epsilon_{\mathcal{E}}(x, O(P_1, P_2)) = O_L(\epsilon_{\mathcal{E}}(x, P_1), \epsilon_{\mathcal{E}}(x, P_2))).$$

In the propositions below, we investigate the relationship between on the one hand certain possible properties of E_L, O_L and N_L, and on the other hand certain possible properties of E, O and N as reflected in $R_{\mathcal{E}}$, the latter being a representation of the possible meaning of conjunction, disjunction and negation in natural language, together with their possible interactions. The results that follow will enable us on the one hand to immediately draw conclusions about the behaviour of E_L, O_L and N_L from assumptions (or observations) about E, O and N in natural language, and on the other hand to verify what is assumed about the conjunction, disjunction and negation in natural language when E_L, O_L and N_L are taken to satisfy certain properties. Whenever possible, we shall restrict ourselves to only mentioning and proving results for the conjunction. From these results one easily derives analogous results for the disjunction.

Proposition 8
Let $\mathcal{E}$ be closed under conjunction. Moreover let the conjunction E in $\mathcal{E}$ be truth-functional in X. Then E is symmetric (or commutative) in $(X, \mathcal{E})$ iff

$$(\forall(\alpha, \beta) \in D^2_{\mathcal{E}})(E_L(\alpha, \beta) = E_L(\beta, \alpha)). \tag{20}$$

Proof. Let us first assume that E is symmetric in $(X, \mathcal{E})$. Choose an arbitrary $(\alpha, \beta) \in D^2_{\mathcal{E}}$. By definition of $D^2_{\mathcal{E}}$, we know there exist an x in X, and P and Q in $\mathcal{E}$, such that $\alpha = \epsilon_{\mathcal{E}}(x, P)$ and $\beta = \epsilon_{\mathcal{E}}(x, Q)$. From the assumption it follows that

$$\epsilon_{\mathcal{E}}(x, E(P, Q)) = \epsilon_{\mathcal{E}}(x, E(Q, P)).$$

Moreover, since the conjunction E in $\mathcal{E}$ is assumed to be truth-functional in X, corollary 10 now implies that

$$E_L(\epsilon_{\mathcal{E}}(x, P), \epsilon_{\mathcal{E}}(x, Q)) = E_L(\epsilon_{\mathcal{E}}(x, Q), \epsilon_{\mathcal{E}}(x, P))$$

and therefore also $E_L(\alpha, \beta) = E_L(\beta, \alpha)$.
Conversely, assume that (20) holds. Choose arbitrary P and Q in $\mathcal{E}$. From the assumption and the definition of $D^2_{\mathcal{E}}$ we know that, for arbitrary x in X

$$E_L(\epsilon_{\mathcal{E}}(x, P), \epsilon_{\mathcal{E}}(x, Q)) = E_L(\epsilon_{\mathcal{E}}(x, Q), \epsilon_{\mathcal{E}}(x, P)).$$

Taking into account the truth-functionality in X of the conjunction E in $\mathcal{E}$ and corollary 10, this implies

$$\epsilon_{\mathcal{E}}(x, E(P, Q)) = \epsilon_{\mathcal{E}}(x, E(Q, P)).$$

Hence, E is commutative in $(X, \mathcal{E})$. □

Proposition 9
Let $\mathcal{E}$ be closed under conjunction. Moreover let the conjunction E in $\mathcal{E}$ be truth-functional in X. Then E is idempotent in $(X, \mathcal{E})$ iff

$$(\forall\alpha \in \epsilon_{\mathcal{E}}(X \times \mathcal{E}))(E_L(\alpha, \alpha) = \alpha). \tag{21}$$

Proof. Analogous to the proof of the proposition above. □

Proposition 10
Let $\mathcal{E}$ be closed under conjunction. Moreover let the conjunction E in $\mathcal{E}$ be truth-functional in X. Then E is associative in $(X, \mathcal{E})$ iff

$$(\forall(\alpha, \beta, \gamma) \in D^3_{\mathcal{E}})(E_L(E_L(\alpha, \beta), \gamma) = E_L(\alpha, E_L(\beta, \gamma))). \tag{22}$$

Proof. Analogous to the proof of proposition 8. □

Proposition 11
Let $\mathcal{E}$ be closed under conjunction and disjunction and let these combinators in $\mathcal{E}$ be truth-functional in X. Then

(i) *O is distributive w.r.t. E in $(X, \mathcal{E})$ iff*

$$\begin{cases} (\forall(\alpha, \beta, \gamma) \in D^3_{\mathcal{E}})(O_L(E_L(\alpha, \beta), \gamma) = E_L(O_L(\alpha, \gamma), O_L(\beta, \gamma))) \\ (\forall(\alpha, \beta, \gamma) \in D^3_{\mathcal{E}})(O_L(\alpha, E_L(\beta, \gamma)) = E_L(O_L(\alpha, \beta), O_L(\alpha, \gamma))); \end{cases} \tag{23}$$

(ii) *E is distributive w.r.t. O in $(X, \mathcal{E})$ iff*

$$\begin{cases} (\forall(\alpha, \beta, \gamma) \in D^3_{\mathcal{E}})(E_L(O_L(\alpha, \beta), \gamma) = O_L(E_L(\alpha, \gamma), E_L(\beta, \gamma))) \\ (\forall(\alpha, \beta, \gamma) \in D^3_{\mathcal{E}})(E_L(\alpha, O_L(\beta, \gamma)) = O_L(E_L(\alpha, \beta), E_L(\alpha, \gamma))). \end{cases} \tag{23'}$$

Proof. Analogous to the proof of proposition 8. □

Proposition 12
Let $\mathcal{E}$ be closed under conjunction, disjunction and negation and let these combinators in $\mathcal{E}$ be truth-functional in X. Then N, E and O satisfy de Morgan's laws in $(X, \mathcal{E})$ iff

$$(\forall(\alpha,\beta) \in D^2_{\mathcal{E}})(N_L(E_L(\alpha,\beta)) = O_L(N_L(\alpha), N_L(\beta))) \tag{24}$$

and

$$(\forall(\alpha,\beta) \in D^2_{\mathcal{E}})(N_L(O_L(\alpha,\beta)) = E_L(N_L(\alpha), N_L(\beta))). \tag{24'}$$

Proof. Analogous to the proof of proposition 8. □

Proposition 13
Let $\mathcal{E}$ be closed under conjunction. Then E_L is a functional relation and

$$\begin{gathered}(\forall(\alpha_1,\alpha_2) \in D^2_{\mathcal{E}})(\forall(\beta_1,\beta_2) \in D^2_{\mathcal{E}})\\ ((\alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2) \Rightarrow E_L(\alpha_1,\alpha_2) \leq E_L(\beta_1,\beta_2))\end{gathered} \tag{25}$$

iff

$$\begin{gathered}(\forall(P_1,P_2,Q_1,Q_2) \in \mathcal{E}^4)(\forall(x,y) \in X^2)\\ (((x,P_1)R_{\mathcal{E}}(y,Q_1) \text{ and } (x,P_2)R_{\mathcal{E}}(y,Q_2)) \Rightarrow\\ (x,E(P_1,P_2))R_{\mathcal{E}}(y,E(Q_1,Q_2))).\end{gathered} \tag{26}$$

Proof. Let us first assume that E_L is functional and that (25) holds. We have to show that (26) is valid. Choose to this effect an arbitrary $(P_1,P_2,Q_1,Q_2) \in \mathcal{E}^4$ and $(x,y) \in X^2$ and assume that $(x,P_1)R_{\mathcal{E}}(y,Q_1)$ and $(x,P_2)R_{\mathcal{E}}(y,Q_2)$. From proposition 2,

$$\epsilon_{\mathcal{E}}(x,P_1) \leq \epsilon_{\mathcal{E}}(y,Q_1) \text{ and } \epsilon_{\mathcal{E}}(x,P_2) \leq \epsilon_{\mathcal{E}}(y,Q_2)$$

and taking into account (25)

$$E_L(\epsilon_{\mathcal{E}}(x,P_1), \epsilon_{\mathcal{E}}(x,P_2)) \leq E_L(\epsilon_{\mathcal{E}}(y,Q_1), \epsilon_{\mathcal{E}}(y,Q_2)).$$

Since E_L is functional, corollary 10 now implies

$$\epsilon_{\mathcal{E}}(x,E(P_1,P_2)) \leq \epsilon_{\mathcal{E}}(y,E(Q_1,Q_2)),$$

whence using proposition 2

$$(x,E(P_1,P_2))R_{\mathcal{E}}(y,E(Q_1,Q_2)).$$

Conversely, assume that (26) holds, which immediately implies that (19) holds. Hence, E_L is a functional relation. It only remains to be shown that (25) is valid. To this effect, choose arbitrary (α_1,α_2) and (β_1,β_2) in $D^2_{\mathcal{E}}$. By definition of $D^2_{\mathcal{E}}$, there exist on the one hand an $x \in X$ and a $(P_1,P_2) \in \mathcal{E}^2$ such that $\alpha_1 = \epsilon_{\mathcal{E}}(x,P_1)$ and $\alpha_2 = \epsilon_{\mathcal{E}}(x,P_2)$, and on the other hand an $y \in X$ and a $(Q_1,Q_2) \in \mathcal{E}^2$ such that $\beta_1 = \epsilon_{\mathcal{E}}(y,Q_1)$ and $\beta_2 = \epsilon_{\mathcal{E}}(y,Q_2)$. Assume that $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. Then, taking into account proposition 2

$$(x,P_1)R_{\mathcal{E}}(y,Q_1) \text{ and } (x,P_2)R_{\mathcal{E}}(y,Q_2)$$

and also, using (26)

$$(x,E(P_1,P_2))R_{\mathcal{E}}(y,E(Q_1,Q_2)).$$

Again using proposition 2, this implies

$$\epsilon_{\mathcal{E}}(x,E(P_1,P_2)) \leq \epsilon_{\mathcal{E}}(y,E(Q_1,Q_2))$$

and since E_L is functional, also taking into account corollary 10

$$E_L(\epsilon_{\mathcal{E}}(x,P_1), \epsilon_{\mathcal{E}}(x,P_2)) \leq E_L(\epsilon_{\mathcal{E}}(y,Q_1), \epsilon_{\mathcal{E}}(y,Q_2)),$$

whence finally

$$E_L(\alpha_1,\alpha_2) \leq E_L(\beta_1,\beta_2). \quad \square$$

We end this section with an example about crisp properties.

Example 11 (Classical Sets)
Let X be a universe and let P_1 and P_2 be two properties that have a classical representation in X. One readily verifies that, borrowing the notations from example 2,

$$\begin{cases} A_{E(P_1,P_2)} = A_{P_1} \cap A_{P_2} \\ A_{O(P_1,P_2)} = A_{P_1} \cup A_{P_2}. \end{cases} \tag{27}$$

Let us now consider a set $\mathcal{E}$ of properties that have a classical representation in X, which is furthermore closed under negation, disjunction and conjunction and borrow the results and notations from example 6. This means that we have an expression for the large-preference relation $R_{\mathcal{E}}$ associated with $\mathcal{E}$. We shall try and find out what can be deduced about the negation-, conjunction- and disjunction relations on the basis of this expression for $R_{\mathcal{E}}$ and formula (27), together with formula (16). In particular, we remind the reader of the following notations

$$\begin{cases} \alpha = \bigcup_{P \in \mathcal{E}} \mathrm{co}A_P \times \{P\} \\ \beta = \bigcup_{P \in \mathcal{E}} A_P \times \{P\}. \end{cases}$$

Choose an arbitrary x in X and an arbitrary P in $\mathcal{E}$. Either $x \in A_P$ and then $q_{\mathcal{E}}(x,P) = \beta$ and $q_{\mathcal{E}}(x,N(P)) = \alpha$, or $x \in \mathrm{co}A_P$ and then $q_{\mathcal{E}}(x,P) = \alpha$ and $q_{\mathcal{E}}(x,N(P)) = \beta$. This implies that (13) holds. It therefore follows from proposition 4 that, with an obvious adaptation of the notations

$$(\forall n \in \mathbb{N}^*)(D^n_{\mathcal{E}} = (X \times \mathcal{E}/_{I_{\mathcal{E}}})^n). \tag{28}$$

We now prove a few important properties of negation, conjunction and disjunction of properties that are crisp in X, using the propositions proven in this subsection. Although our approach here is fairly abstract as well as indirect, it is preferred above the more direct methods, because we are mainly concerned here with the illustration of the ideas, techniques and results introduced before. We feel that the best way to achieve some familiarity with these is to apply them to the one concrete case most scientists know about. For a start, let us show that $E_{X \times \mathcal{E}/_{I_{\mathcal{E}}}}$ and $O_{X \times \mathcal{E}/_{I_{\mathcal{E}}}}$ are isotone $(X \times \mathcal{E}/_{I_{\mathcal{E}}})^2 - X \times \mathcal{E}/_{I_{\mathcal{E}}}$ mappings, which also implies that the conjunction and the disjunction in $\mathcal{E}$ are truth-functional in X. To prove this, we use proposition 13. Restricting ourselves to the proof for the conjunction, it appears from this proposition that we must show (26) to hold. Explicitly, we must prove that

$$\begin{aligned} &(\forall (P_1,P_2,Q_1,Q_2) \in \mathcal{E}^4)(\forall (x,y) \in X^2) \\ &(((x,P_1)R_{\mathcal{E}}(y,Q_1) \text{ and } (x,P_2)R_{\mathcal{E}}(y,Q_2)) \Rightarrow \\ &\qquad (x,E(P_1,P_2))R_{\mathcal{E}}(y,E(Q_1,Q_2))). \end{aligned}$$

To this effect, choose an arbitrary (P_1,P_2,Q_1,Q_2) in $\mathcal{E}^4$ and an arbitrary (x,y) in X^2. Taking into account (27) and the results of example 6, we may write

$$\begin{cases} (x,P_1)R_{\mathcal{E}}(y,Q_1) \Leftrightarrow (x,y) \in M_1 \\ (x,P_2)R_{\mathcal{E}}(y,Q_2) \Leftrightarrow (x,y) \in M_2 \\ (x,E(P_1,P_2))R_{\mathcal{E}}(y,E(Q_1,Q_2)) \Leftrightarrow (x,y) \in M, \end{cases}$$

where, for notational simplicity,

$$\begin{cases} M_1 \stackrel{\text{def}}{=} (A_{P_1} \times A_{Q_1}) \cup (\mathrm{co}A_{P_1} \times X) \\ M_2 \stackrel{\text{def}}{=} (A_{P_2} \times A_{Q_2}) \cup (\mathrm{co}A_{P_2} \times X) \end{cases}$$

and also

$$\begin{aligned} M &\stackrel{\text{def}}{=} (A_{E(P_1,P_2)} \times A_{E(Q_1,Q_2)}) \cup (\mathrm{co}A_{E(P_1,P_2)} \times X) \\ &= ((A_{P_1} \cap A_{P_2}) \times (A_{Q_1} \cap A_{Q_2})) \cup ((\mathrm{co}A_{P_1} \cup \mathrm{co}A_{P_2}) \times X). \end{aligned}$$

To prove (26), we must show that $M_1 \cap M_2 \subseteq M$, which is now easily verified.
From (27) it follows furthermore that E and O are idempotent, symmetric, associative and mutually distributive in $(X, \mathcal{E})$. Also taking into account (16) we know that N, E and O satisfy de Morgan's laws in $(X, \mathcal{E})$. Propositions 8–12 therefore assure us of the idempotency, symmetry, associativity and mutual distributivity of the $(X \times \mathcal{E}/_{I_\mathcal{E}})^2 - X \times \mathcal{E}/_{I_\mathcal{E}}$ mappings $E_{X\times\mathcal{E}/_{I_\mathcal{E}}}$ and $O_{X\times\mathcal{E}/_{I_\mathcal{E}}}$, that together with the involutive and antitone transformation $N_{X\times\mathcal{E}/_{I_\mathcal{E}}}$ of $X \times \mathcal{E}/_{I_\mathcal{E}}$ also satisfy de Morgan's laws.
Finally, choose an arbitrary x in X and (P, Q) in $\mathcal{E}^2$. Taking into account (27) $q_\mathcal{E}(x, E(P, Q)) = \beta$ is equivalent with $x \in A_P \cap A_Q$ and therefore also with $q_\mathcal{E}(x, P) = \beta$ and $q_\mathcal{E}(x, Q) = \beta$. Hence, from the preceding discussion it follows that

$$\begin{cases} E_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\alpha, \alpha) = \alpha \\ E_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\alpha, \beta) = \alpha \\ E_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\beta, \alpha) = \alpha \\ E_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\beta, \beta) = \beta. \end{cases}$$

Analogously

$$\begin{cases} O_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\alpha, \alpha) = \alpha \\ O_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\alpha, \beta) = \beta \\ O_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\beta, \alpha) = \beta \\ O_{X\times\mathcal{E}/_{I_\mathcal{E}}}(\beta, \beta) = \beta. \end{cases}$$

This means that $E_{X\times\mathcal{E}/_{I_\mathcal{E}}}$ and $O_{X\times\mathcal{E}/_{I_\mathcal{E}}}$ correspond with the conjunction respectively disjunction operator of classical logic.

5 Conclusion

Starting from the theory of preference relations, we have tried to give a solid foundation to the theory of fuzzy sets. In order to represent the solution of an evaluation problem in a universe X under a property P, we started from a binary relation R_P on X, reflecting a preference among the objects of the universe on the basis of their satisfying the property P. The reason for this approach is obvious: whenever the property P is fuzzy in the universe considered, an absolute approach to the evaluation problem associated with it becomes impossible. It is not generally possible to answer the basic question of whether an object in X satisfies P or not. A twofold classification on the basis of the property P is impossible. Let us therefore change the basic question and ask to what extent an arbitrary object in X satisfies P. A fundamental step towards the solution of that problem is made when we consider an ordering of the objects in X on the basis of their satisfying P. This ordering is mathematically represented by the relation R_P. In order to keep this discussion as general as possible, we did not demand that R_P be complete. We did not want to preclude incomparability on beforehand. On the other hand, we did make the assumption that the relation R_P is reflexive as well as transitive, which means that R_P is at least a partial-preorder relation.
The relation R_P contains all the information about the solution of the evaluation problem at hand, that can be obtained on the basis of ordering alone. This does not mean, however, that this information cannot be represented in a more transparant way. The search for other representations leads to the notions of evaluation set and mapping. Precisely these notions can in general be identified with the L-fuzzy sets of Goguen and also with the fuzzy sets of Zadeh whenever the relation R_P is complete. That our approach also leads to the classical representation of well-posed evaluation problems should come as no surprise. To the picture thus obtained we may add more detail by taking into account other aspects of the property P besides ordering. We have however in this contribution solely taken into consideration the order-theoretic aspects of the meaning of properties as a possible basis for further refinements, as already discussed in section 2.4.
We can also study evaluation problems in a universe X under a set $\mathcal{E}$ of properties that are crisp or fuzzy in X, which enables us to concentrate on the relationships between the properties in $\mathcal{E}$. Using the analogy with evaluation problems under a single property, we represent the order-theoretic aspect of the solution of this broader type of problem by a (binary) partial-preorder relation $R_\mathcal{E}$ on $X \times \mathcal{E}$. The search

for a more transparent representation of the information contained in $R_{\mathcal{E}}$ again leads to evaluation sets an mappings. Goguen's L-fuzzy sets and also Zadeh's fuzzy sets, if $R_{\mathcal{E}}$ is complete, can be interpreted as partial mappings of these evaluation mappings, keeping the properties fixed. The main advantage of this approach is that the properties in $\mathcal{E}$ can be evaluated using a *single* evaluation set, which is impossible when these properties are considered separately.

We want to stress that the approach given above is not only sufficient but also necessary in order to be able to introduce Goguen's L-fuzzy sets and (with the restriction of completeness) Zadeh's fuzzy sets. Indeed, when we evaluate a single property in a universe X using a mapping from this universe into a partially ordered set $(L, \leq)$, we can immediately infer the existence of a large-preference relation R_P, which is furthermore a partial-preorder relation on X. Also, when we evaluate a set $\mathcal{E}$ of properties in X using a mapping from X to $(L, \leq)$ for every property in $\mathcal{E}$, we can immediately infer the existence of a large-preference relation $R_{\mathcal{E}}$, which is furthermore a partial-preorder relation on $X \times \mathcal{E}$. Incommensurability of the properties in $\mathcal{E}$ cannot be excluded. If, as is the case with the Zadeh fuzzy sets, a chain is taken to be the common evaluation set for the properties in $\mathcal{E}$, all these properties are supposed to be mutually completely commensurable. According to Zadeh it must be perfectly possible for instance to say that Roland is less intelligent than Maud is, or that Roland is more shy than Maud is rebellious, because he invariably uses the chain $([0, 1], \leq)$ as the common evaluation set for properties that are fuzzy or crisp in X. Once again, we did not want to postulate the complete commensurability of fuzzy properties, which explains why we have not made the completeness of the large-preference relation $R_{\mathcal{E}}$ a *conditio sine qua non.*

The introduction of common evaluation sets and mappings enables us to study combinations and modifications of properties, since a natural prerequisite for such a study is the ability to represent the relationship between the combination and the properties that combine into it. Truth-functionality is the central theme in this study, since it leads to the introduction of complementation (negation), union and intersection operators for (L)-fuzzy sets.

Let us consider as an example the conjunction of two properties, closely connected with the intersection of the (L)-fuzzy sets (evaluation mappings) associated with these properties. Suppose that these properties can be evaluated in X using the fuzzy sets h and g. According to Zadeh the intersection of these fuzzy sets is the fuzzy set corresponding with the conjunction of the properties P and Q, and is given by $\min \circ (h, g)$, min being the well-known minimum operator on the unit interval $[0, 1]$ (Zadeh, 1965). This is generalized in the literature by assuming this intersection to be equal to $f \circ (h, g)$, where f is a $[0, 1]^2 - [0, 1]$ mapping satisfying a number of natural properties (monotonicity, commutativity, associativity, ...) The mapping f is called *intersection operator* (Kerre, 1988). Following Goguen, the notion of intersection operator can be further generalized in the theory of L-fuzzy sets, where it is taken to mean an $L^2 - L$ mapping, also satisfying a number of natural properties. Section 4 tells us however that the existence of such an intersection operator is not altogether obvious. Indeed, a certain number of conditions must be satisfied.

(i) Firstly, the conjunction operator in the set $\mathcal{E}$ of the properties considered must be truth-functional in the universe X, or equivalently, the conjunction relation must be functional. On the basis of corollary 9, we know that to this effect formula (19) must be satisfied. Since the large-preference relation $R_{\mathcal{E}}$ amongst other things represents the ordinal characteristics of the conjunction in natural language, formula (19) is the condition this operator in $\mathcal{E}$ must satisfy in natural language in order to be truth-functional in X. Whether (19) is satisfied for particular X and $\mathcal{E}$ can only be determined by experiments.

(ii) The second condition concerns the evaluation set $(L, \leq)$ of the evaluation problem in X under the properties in $\mathcal{E}$. It must indeed be possible to define a binary operator on L (i.e., a $L^2 - L$ *mapping*) that coincides with the conjunction relation on its domain $D^2_{\mathcal{E}}$. Then, and only then can the behaviour of the conjunction in $\mathcal{E}$ be modelled in X using an $L^2 - L$ mapping. The smallest substructure of $(L, \leq)$ that still is an evaluation set of X under the properties in $\mathcal{E}$ is given by $(\epsilon_{\mathcal{E}}(X \times \mathcal{E}), \leq)$. There is however no guarantee that $\epsilon_{\mathcal{E}}(X \times \mathcal{E})^2 = D^2_{\mathcal{E}}$, which means that the existence of a $\epsilon_{\mathcal{E}}(X \times \mathcal{E})^2 - \epsilon_{\mathcal{E}}(X \times \mathcal{E})$ mapping serving as an intersection operator is not really self-evident. Analogous observations can be made concerning the properties an intersection operator might satisfy. Consider for instance the commutativity of intersection operators. If the conjunction in $\mathcal{E}$ is commutative in $(X, \mathcal{E})$ as well as truth-functional in X proposition 8 assures the commutativity of the functional conjunction relation within $D^2_{\mathcal{E}}$. Once again, without further investigation we cannot be certain that a commutative $L^2 - L$ mapping can be found that coincides with this commutative functional conjunction relation on $D^2_{\mathcal{E}}$.

Another point of interest are the modifications of properties in $\mathcal{E}$ (such as *very*, *more or less*, *not* …). Whenever the set $\mathcal{E}$ of properties considered is closed under such a modification, this modification is a unary operator in $\mathcal{E}$ that can be treated in very much the same way as we treated the negation in subsection 4.2. We stress that a necessary and sufficient condition for the functionality of the combinator relations corresponding with these modifications is expressed in general by formula (12). This condition imposes certain restrictions on the relation $R_{\mathcal{E}}$ that is the representation of the ordinal aspects of the meaning of these modifications in natural language. Whether this condition is satisfied, can once again only be determined by experiments. This remark is important for the theory of L-fuzzy sets because these combinator relations, whenever functional, can be identified with the notion of *hedge modificators* (or *linguistic hedges*) of fuzzy set theory (see part I of this volume).

Finally, let us discuss the originality of the methods and results presented here. To our knowledge, we are the first to try and give a solid basis for a general theory of (L-)fuzzy sets along the lines of the discussion above, although we have made use of mathematical techniques that are fairly common nowadays. We mentioned earlier that Norwich and Turksen (1982) in their series of articles about the meaning of membership degrees in fuzzy sets, give a definition that is related to our definition 2. However, what they derive on the basis of this definition is on the one hand less general and on the other hand follows a different direction. Moreover, they only explicitly concern themselves with evaluation problems under a single property and do not make the important transition to evaluation problems under a set of properties. It is nevertheless this transition that renders possible the investigation of the origin of hedge modificators and set theoretic operators for (L-)fuzzy sets. Although the literature deals extensively with these operators (see for instance Kerre 1988, Zimmermann 1985), they are always introduced in an *ad hoc* manner as more or less appropriate extensions of the well-known operators for classical sets. Up to now, the theoretical research into the origin of operators for (L-)fuzzy sets has not gone much further than to stress and exploit the analogy with the operators for crisp sets.

Notes

1. We do not exclude the possibility that a property that has a classical representation in a universe X has no classical representation in another universe Y. The discussions in this paper always start from a certain well-defined universe and are therefore relative to this universe.

2. In the first instance, because the models proposed furtheron are mathematical models and are therefore based on classical set theory.

3. It is obvious from the definition of R_P that it is reflexive. For a discussion of the transivity of relations like R_P in the related field of preference logic, we refer to (Apostel, 1986), where more references to related topics can be found.

4. In the sequel of this paper, we shall denote a generic poset by $(L, \leq)$. Since the partial-order relation $\leq$ is a special case of a quasi-order relation and can also be interpreted as a large-preference relation, we may look for the indifference, strict-preference and incomparability relations associated with $\leq$. Since furthermore $\leq$ is antisymmetric (whereas a general quasi-order relation is not), the indifference relation coincides with the diagonal of L^2, or equivalently, with the equality relation $=$ on L. The strict preference relation will be denoted $<$ and the incomparability relation $\|$. We shall also use the notation $\geq$ for the inverse relation $\leq^{-1}$ of $\leq$. We nevertheless want to remark that the symbols $\leq$, $\geq$ and $<$ will also be used for the special case of the natural ordering of the reals. The reader should in all cases be able to resolve this ambiguity using clues in the context.

5. This is not really a requirement of a mathematical nature but rather of methodological consistency in deriving large-preference relations.

6. An analogous discussion can be made when more than one universe is considered. We shall not consider extensions of this kind in order not to complicate this discussion beyond necessity.

References

Apostel, L.; Vandamme, F., 1982, *Formele logika, deel 2: Niet-klassieke en toegepaste logika*, Communication & Cognition, De Sikkel, Gent.

Dienes, Z.P., 1949, On an Implication Function In Many-Valued Systems of Logic, *The Journal of Symbolic Logic*, vol. 14, pp. 95–97.

Gentilhomme, Y., 1968, *Les ensembles flous en linguistique*, Cahiers de linguistique théorique et appliquée, pp. 47–65, Editions de l'Académie de la République Socialiste de Roumanie.

Goguen, J.A., 1967, L-fuzzy Sets, *Journal of Mathematical Analysis and Applications*, vol. 18, pp.145–174.

Kerre, E.E., 1988, *Fuzzy Sets and Approximate Reasoning*, Lecture Notes for the Summer Course "Special Topics in Computer Science," University of Nebraska, Lincoln, Private publication.

Kleene, S.C., 1952, Introduction to Metamathematics, Van Nostrand, New York.

Norwich A.M.; Turksen, I.B., 1982, The Fundamental Measurement of Fuzziness, in *Fuzzy Set and Possibility Theory: Recent Developments*, ed. R.R. Jager, Pergamon Press, New York, pp. 49–60.

Rescher, N., 1968, The Logic of Preference, in *Topics in Philosophical Logic*, D. Reidel Publishing Company, Dordrecht, pp. 287–320.

Rescher, N., 1969, *Many-Valued Logic*, McGraw-Hill, New York.

Roubens, M.; Vincke, P., 1985, *Preference Modelling*, Springer Verlag, Berlin.

Von Wright, G.H., 1963, *The Logic of Preference*, Edinburgh.

Zadeh, L.A., 1965, Fuzzy Sets, *Information and Control*, vol. 8, pp. 338–353.

Zimmermann, H.-J., 1985, *Fuzzy Set Theory—and Its Applications*, Kluwer-Nijhoff Publishing, Dordrecht.