First Results for a Mathematical Theory of Possibilistic Processes

H.J. Janssen, G. de Cooman* and **E.E. Kerre**
Universiteit Gent
Vakgroep Toegepaste Wiskunde en Informatica
Krijgslaan 281, B-9000 Gent, Belgium
{ Hugo.Janssen, Gert.deCooman, Etienne.Kerre}@rug.ac.be

Abstract

This paper provides the measure theoretic basis for a theory of possibilistic processes. We generalize the definition of a product τ-field to an indexed family of τ-fields, without imposing an ordering on the index set. We also introduce the notion 'measurable cylinder' and show that any product τ-field can be generated by its associated field of measurable cylinders. Furthermore, we introduce and study the notions 'τ-subspace', 'extension of a τ-space' and 'one-point extension of a τ-space'. Using these notions, we prove that for any family of possibility distributions $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$, satisfying a natural consistency condition, a family $(f_t \mid t \in T)$ of possibilistic variables can be constructed such that the possibilistic variable $\times_{t\in T'} f_t$ (with $\emptyset \subset T' \Subset T$) has $\pi_{T'}$ as a possibility distribution. As a special case we obtain a possibilistic analogon of the probabilistic Daniell-Kolmogorov theorem, a cornerstone for the theory of stochastic processes.

1 Preliminary notions

In this paper, we develop the mathematical and topological apparatus necessary for proving a possibilistic analogon for the well-known theorem of Daniell-Kolmogorov [Doob, 1967]. This theorem is the cornerstone for the mathematical theory of stochastic processes. In short, it tells us that, given a family of real-valued functions on finite Cartesian powers of a sample space that satisfy natural consistency conditions, there exists a basic space, a probability measure on that basic space, and a family of stochastic variables that have these real-valued functions as their probability distribution functions. The results in this paper are the possibilistic counterparts.

*Postdoctoral Fellow of the Belgian National Fund for Scientific Research (NFWO).

In this section, we give a number of basic definitions, which are needed in the following sections.

A subset $\mathcal{L}$ of the power class $\mathcal{P}(X)$ of a nonempty set X is called a *plump field* [Wang and Klir, 1992] on X iff it is closed under arbitrary unions and intersections. The *atom* of $\mathcal{L}$ containing the element $x \in X$ is defined as $[\,x\,]_{\mathcal{L}} \stackrel{\text{def}}{=} \bigcap\{A \mid A \in \mathcal{L} \text{ and } x \in A\}$. Furthermore, a subset A of X is an atom of $\mathcal{L}$ iff $(\exists x \in X)(A = [\,x\,]_{\mathcal{L}})$. The set of the atoms of $\mathcal{L}$ is denoted by $X_{\mathcal{L}}$. It is readily verified that $X_{\mathcal{L}} \subseteq \mathcal{L}$ and that $(\forall x \in X)(x \in [\,x\,]_{\mathcal{L}})$. Finally, for any subset A of X, $A \in \mathcal{L} \Leftrightarrow A = \bigcup_{x\in A} [\,x\,]_{\mathcal{L}}$.

A *τ-field* or *ample field* [Wang, 1982] $\mathcal{R}$ on a nonempty set X is a plump field on X that is closed under complementation. The couple $(X, \mathcal{R})$ is called a τ-space. Finally [Wang and Klir, 1992], for a nonempty set X and a subset $\mathcal{A}$ of its power class $\mathcal{P}(X)$, $\tau_X(\mathcal{A})$ denotes the smallest τ-field on X which includes $\mathcal{A}$.

Furthermore, if A is a subset of a nonempty set X, then we write $A \Subset X$ iff A is a finite subset of X.

Throughout this paper, X denotes a nonempty set, $\mathcal{R}$ is a τ-field on X, $(L, \leq)$ denotes a complete lattice with greatest element 1 and smallest element 0, and $\mathbf{1}_L$ is the identical permutation of L.

2 Possibilistic Processes

De Cooman and Kerre [1993] have generalized Zadeh's original definition of a possibility measure as follows: if $(L, \leq)$ is a complete lattice and $(X, \mathcal{R})$ is a τ-space, then a $\mathcal{R}-L$ mapping Π is a $(L, \leq)$-*possibility measure* on $(X, \mathcal{R})$ iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j). \qquad (1)$$

The triple $(X, \mathcal{R}, \Pi)$ is called a $(L, \leq)$-*possibility space*.

Furthermore, if $\mathcal{A}$ is a subset of the power class $\mathcal{P}(X)$ of X and Π is any $\mathcal{A}-L$ mapping, then a $X-L$ mapping π is called a *distribution* of Π iff for any A in

$\mathcal{A}$:

$$\Pi(A) = \sup_{x \in A} \pi(x). \tag{2}$$

In particular, De Cooman has proven that every $(L, \leq)$-possibility measure Π on a τ-space $(X, \mathcal{R})$ possesses a unique distribution π that is constant on the atoms of $\mathcal{R}$, and is given by

$$(\forall x \in X)(\pi(x) = \Pi([\, x\,]_{\mathcal{R}})). \tag{3}$$

Finally, if $\mathcal{A}_1 \subseteq \mathcal{P}(X_1)$ and $\mathcal{A}_2 \subseteq \mathcal{P}(X_2)$, where X_1 and X_2 are nonempty sets, then a $X_1 - X_2$ mapping f is called $\mathcal{A}_1 - \mathcal{A}_2$ *measurable* iff $(\forall B \in \mathcal{A}_2)(f^{-1}(B) \in \mathcal{A}_1)$.

In particular [De Cooman, 1993], if $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ is a $(L, \leq)$-possibility space and $(X, \mathcal{R})$ is a τ-space, then a $\Omega - X$ mapping f is a *possibilistic variable* in $(X, \mathcal{R})$ iff f is a $\mathcal{R}_\Omega - \mathcal{R}$ measurable mapping. The τ-space $(X, \mathcal{R})$ in which the possibilistic variable takes its values, is called the *sample space* of f, and $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ is called the *basic space* of f. Throughout this paper, we shall assume that any possibilistic variable has the $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as its basic space. Finally, if π_Ω represents the distribution of Π_Ω and f is a possibilistic variable in $(X, \mathcal{R})$, then the $X - L$ mapping π_f, which is given for any $x \in X$ by $\pi_f(x) = \sup_{w \in f^{-1}([\, x\,]_{\mathcal{R}})} \pi_\Omega(\omega)$, is called the possibility distribution of f.

Using the previous notions, we define a *family of possibilistic variables*, which is more general than a *possibilistic process*.

Definition 1 (Family of possibilistic variables) *Let T be a nonempty set. A family $(f_t \mid t \in T)$, such that $(\forall t \in T)(f_t$ is a possibilistic variable in a τ-space $(X_t, \mathcal{R}_t))$, is called a family of possibilistic variables in the family $((X_t, \mathcal{R}_t) \mid t \in T)$ of τ-spaces, with index set T.*

A possibilistic process will be defined as a family of possibilistic variables, such that the possibilistic variables of this family have the same sample space. This leads to the following definition.

Definition 2 (Possibilistic process) *Let T be a nonempty set. A family $(f_t \mid t \in T)$, such that $(\forall t \in T)(f_t$ is a possibilistic variable in a τ-space $(X, \mathcal{R}))$, is a possibilistic process in the τ-space $(X, \mathcal{R})$, with index set T. If the index set T is countable, then $(f_t \mid t \in T)$ is called a discrete possibilistic process. A process $(f_t \mid t \in T)$, for which the index set T is a real interval, is called a continuous possibilistic process.*

3 Product τ-spaces

First, we introduce the notions 'Cartesian product of a family of sets', 'projection mapping from a Cartesian product' and 'product mapping'.

Definition 3 *Let $(X_t \mid t \in T)$ be a family of nonempty sets with nonempty index set T. Then, the Cartesian product of $(X_t \mid t \in T)$ is the set $\times_{t \in T} X_t$ of all $T - \bigcup_{t \in T} X_t$ mappings x, such that $(\forall t \in T)(x(t) \in X_t)$. In particular, if $(\forall t \in T)(A_t \subseteq X_t)$, then $\times_{t \in T} A_t$ is the subset of $\times_{t \in T} X_t$, which contains all the elements x of $\times_{t \in T} X_t$, such that $(\forall t \in T)(x(t) \in A_t)$. If $(\forall t \in T)(X_t = X)$, then the Cartesian product $\times_{t \in T} X_t$ is also denoted by X^T.*

For any $s \in T$, $\mathbf{pr}_{T,s}$ is the $\times_{t \in T} X_t - X_s$ mapping, defined by $(\forall x \in \times_{t \in T} X_t)(\mathbf{pr}_{T,s}(x) = x(s))$, and is called the s-th projection mapping from $\times_{t \in T} X_t$ onto X_s.

Finally, let A be any set. Then, for any family $(f_t \mid t \in T)$ of $A - X_t$ mappings f_t, the unique mapping $f : A \to \times_{t \in T} X_t$, such that $(\forall t \in T)(f_t = \mathbf{pr}_{T,t} \circ f)$, is denoted by $\times_{t \in T} f_t$ and is called the product mapping of $(f_t \mid t \in T)$.

Using the definitions above we can define the notion 'Tychonov topology' or 'product topology', which is a special case of a weak topology induced on a set (see [Kelley, 1959; Willard, 1970]).

Definition 4 *Let X be any set and $((X_t, \mathcal{T}_t) \mid t \in T)$ a family of topological spaces with nonempty index set T. The Tychonov topology or product topology on $\times_{t \in T} X_t$ is the weak topology induced on $\times_{t \in T} X_t$ by the family $(\mathbf{pr}_{T,t} \mid t \in T)$ of projections, and is denoted by $\mathcal{W}((X_t, \mathcal{T}_t) \mid t \in T)$.*

Throughout this section, T is a nonempty set and $((X_t, \mathcal{R}_t) \mid t \in T)$ is a family of τ-spaces with index set T. Furthermore, the closure operator $\tau_{\times_{t \in T} X_t}$ is abbreviated by τ.

Let $(X_1, \mathcal{R}_1)$ and $(X_2, \mathcal{R}_2)$ be τ-spaces. Wang [1982] has defined *the product of $\mathcal{R}_1$ and $\mathcal{R}_2$* as the τ-field on $X_1 \times X_2$

$$\mathcal{R}_1 \times \mathcal{R}_2 \stackrel{\text{def}}{=} \tau(\{A_1 \times A_2 \mid A_1 \in \mathcal{R}_1 \text{ and } A_2 \in \mathcal{R}_2\}) \tag{4}$$

where $\tau = \tau_{X_1 \times X_2}$, and he has proven that for any $(x_1, x_2) \in X_1 \times X_2$

$$[\,(x_1, x_2)\,]_{\mathcal{R}_1 \times \mathcal{R}_2} = [\, x_1\,]_{\mathcal{R}_1} \times [\, x_2\,]_{\mathcal{R}_2}. \tag{5}$$

The following definition generalizes Wang's original definition towards the product of an indexed family of τ-fields, without imposing an ordering on the index set.

Definition 5 *$\prod_{t \in T} \mathcal{R}_t$ denotes the smallest τ-field on $\times_{t \in T} X_t$, such that for any $s \in T$*

$$\mathbf{pr}_{T,s} \textit{ is a } \prod_{t \in T} \mathcal{R}_t - \mathcal{R}_s \textit{ measurable mapping.}$$

$\prod_{t \in T} \mathcal{R}_t$ is called the product τ-field on $\times_{t \in T} X_t$ of the family $(\mathcal{R}_t \mid t \in T)$ of τ-fields and

$(\times_{t\in T} X_t, \prod_{t\in T} \mathcal{R}_t)$ *is called the product* τ*-space of the family* $((X_t, \mathcal{R}_t) \mid t \in T)$ *of* τ*-spaces. In case* $(\forall t \in T)(\mathcal{R}_t = \mathcal{R})$*, where* $\mathcal{R}$ *is a* τ*-field on* X*,* $\prod_{t\in T} \mathcal{R}_t$ *is also denoted by* $\mathcal{R}^T$*.*

The following theorem generalizes (4) and (5).

Theorem 6 *The product* τ*-field* $\prod_{t\in T} \mathcal{R}_t$ *satisfies:*

$$\prod_{t\in T} \mathcal{R}_t = \tau(\{ \times_{t\in T} A_t \mid (\forall t \in T)(A_t \in \mathcal{R}_t)\}) \qquad (6)$$

The atoms of the product τ*-field* $\prod_{t\in T} \mathcal{R}_t$ *are characterized by*

$$[\, x\,]_{\prod_{t\in T} \mathcal{R}_t} = \times_{t\in T} [\, x(t)\,]_{\mathcal{R}_t} = \bigcap_{t\in T} \mathbf{pr}_{T,t}^{-1}([\, x(t)\,]_{\mathcal{R}_t}), \qquad (7)$$

where $x \in \times_{t\in T} X_t$*.*

The following definition introduces the notion of measurable cylinder of a product τ-space.

Definition 7 *For any set* T' *such that* $\emptyset \subset T' \Subset T$*, let* $\mathbf{pr}_{T,T'}$ *be the mapping from* $\times_{t\in T} X_t$ *onto* $\times_{t\in T'} X_t$*, such that* $(\forall x \in \times_{t\in T} X_t)(\mathbf{pr}_{T,T'}(x) = x|_{T'})$*, where* $x|_{T'}$ *is the restriction of the mapping* x *to the domain* T'*. Then, let*

$$\mathcal{C}_{T,T'} = \{\mathbf{pr}_{T,T'}^{-1}(E) \mid E \in \prod_{t\in T'} \mathcal{R}_t\}.$$

Any element of $\mathcal{C}_{T,T'}$ *is called a measurable* T'*-cylinder of* $(\times_{t\in T} X_t, \prod_{t\in T} \mathcal{R}_t)$*. Furthermore, let* $\mathcal{C}_T = \bigcup_{\emptyset\subset T'\Subset T} \mathcal{C}_{T,T'}$*. Then any element of* $\mathcal{C}_T$ *is a measurable cylinder of* $(\times_{t\in T} X_t, \prod_{t\in T} \mathcal{R}_t)$*.*

The following properties can be proven.

Proposition 8

1. $\mathcal{C}_T$ *is a field on* $\times_{t\in T} X_t$*.*
2. $\mathcal{C}_T$ *is a base for the product topology* $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ *on* $\times_{t\in T} X_t$*.*
3. $\tau(\mathcal{C}_T) = \tau(\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)) = \prod_{t\in T} \mathcal{R}_t$*.*
4. *The following statements are equivalent.*
   (a) $\mathcal{C}_T$ *is a* τ*-field on* $\times_{t\in T} X_t$*.*
   (b) $\mathcal{C}_T$ *is a plump field on* $\times_{t\in T} X_t$*.*
   (c) *There exists a set* T' *such that* $\emptyset \subset T' \Subset T$ *and* $\mathcal{C}_T = \mathcal{C}_{T,T'}$*.*
   (d) $\mathcal{C}_T = \prod_{t\in T} \mathcal{R}_t$*.*
5. *Moreover, if* T *is a finite set, then* $\mathcal{C}_T$ *is a* τ*-field on* $\times_{t\in T} X_t$*.*

4 τ-subspaces and extensions of τ-spaces

We first define the notions '*τ-subspace*' and '*extension of a τ-space*'.

Definition 9 *Let* $(X, \mathcal{R}_X)$ *and* $(Y, \mathcal{R}_Y)$ *be* τ*-spaces. Then* $(X, \mathcal{R}_X)$ *is called a* τ*-subspace of* $(Y, \mathcal{R}_Y)$*, and we write* $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$ *iff*

$$\mathcal{R}_X = \{E \cap X \mid E \in \mathcal{R}_Y\}. \qquad (8)$$

It follows immediately from (8) that $X \subseteq Y$. Furthermore, it is easily verified that $\{E \cap X \mid E \in \mathcal{R}_Y\}$ is a τ-field on X, so that definition 9 is meaningful. We now introduce the notion of *an extension of a τ-space*, which generalizes the notion of *coarseness* [De Cooman and Kerre, 1993].

Definition 10 *Let* $(X, \mathcal{R}_X)$ *and* $(Y, \mathcal{R}_Y)$ *be* τ*-spaces. Then* $(Y, \mathcal{R}_Y)$ *is called an extension of* $(X, \mathcal{R}_X)$*, and we write* $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$ *iff*

$$\mathcal{R}_X \subseteq \mathcal{R}_Y. \qquad (9)$$

Condition (9) implies that $X \subseteq Y$, and if $X = Y$, then $\mathcal{R}_X$ is coarser than $\mathcal{R}_Y$. The following property can easily be proven.

Proposition 11 $\sqsubseteq$ *is a partial order on the set of* τ*-subspaces of any* τ*-space and* $\preceq$ *is a partial order on the set of extensions of any* τ*-space.*

Definition 9 gives rise to the notion '*extension of a $(L, \leq)$-possibility space*' as follows.

Definition 12 *A* $(L, \leq)$*-possibility space* $(Y, \mathcal{R}_Y, \Pi_Y)$ *is called an extension of the* $(L, \leq)$*-possibility space* $(X, \mathcal{R}_X, \Pi_X)$ *iff* $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$ *and* $\Pi_Y|_{\mathcal{R}_X} = \Pi_X$*.*

Since any τ-space $(X, \mathcal{R})$ is a topological space, the the following results can be shown to hold.

Proposition 13 *Let* $(X, \mathcal{R})$ *be a* τ*-space. Then* $(X, \mathcal{R})$ *is a compact topological space iff* $X_{\mathcal{R}}$ *is finite. A subset* A *of* X *is closed and compact in* $(X, \mathcal{R})$ *iff* A *is a finite union of atoms of* $\mathcal{R}$*.*

By the previous proposition, a τ-space $(X, \mathcal{R})$ is not always a compact topological space, However [Kelley, 1959], a noncompact topological space $(X, \mathcal{R})$ can always be embedded in a compact topological space. In particular, if we let $\mathcal{T}^*$ be the set

$$\mathcal{R} \cup \{X^* \setminus G \mid G \text{ is a closed, compact set in } (X, \mathcal{R})\} \qquad (10)$$

where $X^* = X \cup \{\infty\}$ and $\infty \notin X$, then $(X^*, \mathcal{T}^*)$ is a compact topological space, and is called *a one-point compactification of* $(X, \mathcal{R})$. If $(X, \mathcal{R})$ is a compact topological space, then let $X^* = X$ and $\mathcal{T}^* = \mathcal{R}$. Furthermore, if $\mathcal{R}^* = \tau_{X^*}(\mathcal{T}^*)$, then, according to definition 10, $(X^*, \mathcal{R}^*)$ is an extension of $(X, \mathcal{R})$. This leads to the following definition.

Definition 14 *Let $(X, \mathcal{R})$ be a τ-space. Then $(X^*, \mathcal{R}^*)$ is called a *-extension of $(X, \mathcal{R})$. $(X^*, \mathcal{R}^*)$ is called a one-point extension of $(X, \mathcal{R})$ iff $(X, \mathcal{R})$ is a noncompact topological space. For a one-point extension $(X^*, \mathcal{R}^*)$ of $(X, \mathcal{R})$, $(X^*, \mathcal{T}^*)$ is called the one-point compactification associated with $(X^*, \mathcal{R}^*)$.*

Using the previous definition, we obtain the following proposition.

Proposition 15 *Let $(X, \mathcal{R})$ be a τ-space and let $(X^*, \mathcal{R}^*)$ be a *-extension of $(X, \mathcal{R})$. Then $(X, \mathcal{R}) \sqsubseteq (X^*, \mathcal{R}^*)$ and $(X, \mathcal{R}) \preceq (X^*, \mathcal{R}^*)$. In particular,*

1. *if $(X, \mathcal{R})$ is compact, then $\mathcal{R} = \mathcal{T}^* = \mathcal{R}^*$ and $X^*_{\mathcal{R}^*} = X_{\mathcal{R}}$,*
2. *if $(X^*, \mathcal{R}^*)$ is a one-point extension of $(X, \mathcal{R})$, then $\mathcal{R}^* = \tau_{X^*}(\mathcal{R})$, $X^*_{\mathcal{R}^*} = X_{\mathcal{R}} \cup \{\{\infty\}\}$, and $\mathcal{R} \subset \mathcal{T}^* \subset \mathcal{R}^*$.*

5 Main result

In this section, let $((X_t, \mathcal{R}_t) \mid t \in T)$ be a family of τ-spaces with nonempty index set T and let $s \in T$. We can associate a *-extension $(X^*_s, \mathcal{R}^*_s)$ with the τ-space $(X_s, \mathcal{R}_s)$. According to definition 14, if $(X^*_s, \mathcal{R}^*_s)$ is a one-point extension of $(X_s, \mathcal{R}_s)$, then $X^*_s = X_s \cup \{\infty_s\}$, where $\infty_s \notin X_s$ and using proposition 15, $\mathcal{R}^*_s = \tau_{X^*_s}(\mathcal{T}^*_s) = \tau_{X^*_s}(\mathcal{R}_s)$, in which $(X^*_s, \mathcal{T}^*_s)$ is the one-point compactification associated with $(X^*_s, \mathcal{R}^*_s)$. In case $(X_s, \mathcal{R}_s)$ is a compact topological space, $(X^*_s, \mathcal{R}^*_s)$ coincides with $(X_s, \mathcal{R}_s)$. Furthermore, if $\mathbf{pr}^*_{T,s}$ is the s-th projection mapping from $\times_{t\in T} X^*_t$ onto X^*_s, then $\prod_{t\in T} \mathcal{R}^*_t$ is the smallest τ-field on $\times_{t\in T} X^*_t$, such that $(\forall s \in T)(\mathbf{pr}^*_{T,s}$ is a $\prod_{t\in T} \mathcal{R}^*_t - \mathcal{R}^*_s$ measurable mapping).

Also, if T' is a nonempty, finite subset of T, $\mathbf{pr}^*_{T,T'}$ denotes the mapping from $\times_{t\in T} X^*_t$ onto $\times_{t\in T'} X^*_t$, such that $(\forall x \in \times_{t\in T} X^*_t)(\mathbf{pr}^*_{T,T'}(x) = x|_{T'})$.

For any nonempty, finite subset T' of T, we define $\mathcal{C}^*_{T,T'} = \{\mathbf{pr}^{*\,-1}_{T,T'}(E) \mid E \in \prod_{t\in T'} \mathcal{R}^*_t\}$. Then $\mathcal{C}^*_T = \bigcup_{\emptyset \subset T' \Subset T} \mathcal{C}^*_{T,T'}$ is, in accordance with definition 7 and proposition 8, the field of all measurable cylinders of $(\times_{t\in T} X^*_t, \prod_{t\in T} \mathcal{R}^*_t)$. Finally, let

$$\mathcal{O}^*_T = \bigcup_{\emptyset \subset T' \Subset T} \{\mathbf{pr}^{*\,-1}_{T,T'}(O) \mid O \in \mathcal{W}((X^*_t, \mathcal{T}^*_t) \mid t \in T')\}, \tag{11}$$

$$\widetilde{\mathcal{C}}_T = \bigcup_{\emptyset \subset T' \Subset T} \{\mathbf{pr}^{*\,-1}_{T,T'}(E) \mid E \in \prod_{t\in T'} \mathcal{R}_t\}. \tag{12}$$

The following properties can be proven.

Proposition 16

1. $\widetilde{\mathcal{C}_T} \subseteq \mathcal{O}^*_T \subseteq \mathcal{C}^*_T$
2. $$\prod_{t\in T} \mathcal{R}^*_t = \tau_{\times_{t\in T} X^*_t}(\mathcal{C}^*_T) = \tau_{\times_{t\in T} X^*_t}(\mathcal{W}((X^*_t, \mathcal{T}^*_t) \mid t \in T)) = \tau_{\times_{t\in T} X^*_t}(\mathcal{O}^*_T).$$
3. *For any set T' such that $\emptyset \subset T' \subseteq T$, it follows that*
$$(\times_{t\in T'} X_t, \prod_{t\in T'} \mathcal{R}_t) \sqsubseteq (\times_{t\in T'} X^*_t, \prod_{t\in T'} \mathcal{R}^*_t)$$
and
$$(\times_{t\in T'} X_t, \prod_{t\in T'} \mathcal{R}_t) \preceq (\times_{t\in T'} X^*_t, \prod_{t\in T'} \mathcal{R}^*_t).$$

We are now ready to proceed to the main topic of this paper, namely, finding a possibilistic counterpart for the probabilistic Daniell-Kolmogorov theorem. We want to prove that, if we have a family of L-valued functions on finite Cartesian powers of a sample space, and if these functions satisfy natural consistency conditions, then we can always find a basic space with possibility measure, and a family of possibilistic variables that have these L-valued functions as their possibility distribution functions. As a matter of fact, we prove a more general result, of which the possibilistic Daniell-Kolmogorov theorem turns out to be a special case.

Let $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ be a family of distributions such that $\pi_{T'}$ is the distribution of a $(L, \leq)$-possibility measure $\Pi_{T'}$ on the τ-space $(\times_{t\in T'} X_t, \prod_{t\in T'} \mathcal{R}_t)$ for any nonempty, finite subset T' of T. For such family of distributions, we introduce the following consistency condition.

Definition 17 $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ *is called consistent iff for any two sets T_1 and T_2 such that $\emptyset \subset T_1 \subseteq T_2 \Subset T$, we have:*

$$(\forall x \in \times_{t\in T_1} X_t)\left(\pi_{T_1}(x) = \sup_{\mathbf{pr}_{T_2,T_1}(y)=x} \pi_{T_2}(y)\right). \tag{13}$$

Furthermore, for any set T' such that $\emptyset \subset T' \Subset T$, let $\pi^*_{T'}$ denote the $\times_{t\in T'} X^*_t - L$ mapping, given for any $x \in \times_{t\in T'} X^*_t$ by

$$\pi^*_{T'}(x) = \begin{cases} \pi_{T'}(x) & \text{if } x \in \times_{t\in T'} X_t \\ 0 & \text{otherwise} \end{cases}. \tag{14}$$

If $\Pi^*_{T'}$ denotes the $(L, \leq)$-possibility measure on $(\times_{t\in T'} X^*_t, \prod_{t\in T'} \mathcal{R}^*_t)$ with distribution $\pi^*_{T'}$, then $\Pi^*_{T'} |_{\prod_{t\in T'} \mathcal{R}_t} = \Pi_{T'}$ and using proposition 16.3, it follows that $(\times_{t\in T'} X^*_t, \prod_{t\in T'} \mathcal{R}^*_t, \Pi^*_{T'})$ is an extension of the $(L, \leq)$-possibility space $(\times_{t\in T'} X_t, \prod_{t\in T'} \mathcal{R}_t, \Pi_{T'})$. With the above notations, we obtain directly the following result.

Proposition 18 *The family $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent iff the family $(\pi^*_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent.*

Recently, Boyen et al. [1995] have generalized Wang's definition [Wang, 1985; Wang and Klir, 1992] of P-consistency for set mappings as follows.

Definition 19 *Let X be a nonempty set, $\mathcal{A}$ a family of subsets of X and $(L, \leq)$ a complete lattice. A $\mathcal{A}-L$ mapping Π is called P-consistent iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{A}$ and any element A of $\mathcal{A}$: $A \subseteq \bigcup_{j\in J} A_j \Rightarrow \Pi(A) \leq \sup_{j\in J} \Pi(A_j)$.*

The notion of P-consistency was introduced by Wang [Wang, 1985; Wang and Klir, 1992] in the context of the extension of set mappings with $([0,1], \leq)$ as codomain to $([0,1], \leq)$-possibility measures. The notion of extendability is generalized by Boyen et al. [1995] as follows.

Definition 20 *Let X be a nonempty set, $\mathcal{A}$ a family of subsets of X and $(L, \leq)$ a complete lattice. A $\mathcal{A} - L$ mapping Π is extendable to a $(L, \leq)$-possibility measure on a τ-space $(X, \mathcal{R})$ iff there exists a $(L, \leq)$-possibility measure Π' on $(X, \mathcal{R})$ such that $(\forall A \in \mathcal{A})(\Pi(A) = \Pi'(A))$. Π is called extendable to a $(L, \leq)$-possibility measure iff there exists an ample field $\mathcal{R}$ on X such that Π is extendable to a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$.*

The following theorem [Boyen et al., 1995] is needed to prove the theorems that follow.

Theorem 21 *Let X be a nonempty set, $\mathcal{A}$ be a family of subsets of X and $(L, \leq)$ a complete lattice. Then, for any P-consistent $\mathcal{A} - L$ mapping Π, any of the following conditions is sufficient for the extendability of Π.*

(E_1) *$(L, \leq)$ is a complete chain.*

(E_2) *$\mathcal{A}$ is a plump field.*

(E_3) *$(L, \leq) = (\mathcal{B}, \supseteq)$, where $\mathcal{B}$ is a plump field on some set Y.*

Moreover, the complete lattice $(L, \leq)$ can always be embedded using a supremum preserving mapping ϕ into a second complete lattice $(L', \leq')$, in such a way that for any P-consistent $\mathcal{A} - L$ mapping Π, $\phi \circ \Pi$ is a P-consistent $\mathcal{A} - L'$ mapping which is extendable to a $(L', \leq')$-possibility measure.

Now, we are ready to construct a possibility measure on the product τ-space $(\times_{t\in T} X_t^*, \prod_{t\in T} \mathcal{R}_t^*)$ by means of the family $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$. Therefore, we need the following result.

Theorem 22 *Suppose the family $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent, then the $\mathcal{C}_T^* - L$ mapping Π^*, such that*

$$(\forall B \in \mathcal{C}_T^*)(\Pi^*(B) = \Pi_{T'}^*(A)), \tag{15}$$

in which T' satisfies $\emptyset \subset T' \Subset T$ and $A \in \prod_{t\in T'} \mathcal{R}_t^*$ such that $B = \mathbf{pr}_{T,T'}^{*\,-1}(A)$ is well defined. Furthermore, Π^* has the following properties:

1. *Π^* preserves finite suprema, and therefore is an isotone mapping from $(\mathcal{C}_T^*, \subseteq)$ to $(L, \leq)$.*
2. *If T' is a nonempty, finite subset of T, then the restriction $\Pi^*|_{\mathcal{C}_{T,T'}^*}$ is a $(L, \leq)$-possibility measure on $(\times_{t\in T} X_t^*, \mathcal{C}_{T,T'}^*)$.*
3. *If $\mathcal{C}_T^*$ is a τ-field on $\times_{t\in T} X_t^*$, then Π^* is a $(L, \leq)$-possibility measure on $(\times_{t\in T} X_t^*, \mathcal{C}_T^*)$.*
4. *If $(\forall t \in T)((X_t^*, \mathcal{R}_t^*)$ is a compact topological space $)$, then Π^* is P-consistent.*
5. *$\Pi^*|_{\mathcal{O}_T^*}$ is P-consistent.*

Using theorem 22.5 and theorem 21, we can construct a possibility measure on $(\times_{t\in T} X_t^*, \prod_{t\in T} \mathcal{R}_t^*)$ as follows.

Theorem 23 *Suppose the family $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent. Then there exist a complete lattice $(L', \leq')$, a supremum preserving order-embedding ϕ from $(L, \leq)$ to $(L', \leq')$ and a $(L', \leq')$-possibility measure Π' on $(\times_{t\in T} X_t^*, \prod_{t\in T} \mathcal{R}_t^*)$, such that*

$$\Pi'|_{\mathcal{O}_T^*} = \phi \circ (\Pi^*|_{\mathcal{O}_T^*}),$$

in which Π^* is the mapping constructed in theorem 22. If the mapping $\Pi^*|_{\mathcal{O}_T^*}$ satisfies at least one of the sufficient conditions for extendability $(E_1), (E_2), (E_3)$, then one can take $(L, \leq)$ for $(L', \leq')$ and $\mathbf{1}_L$ for ϕ.

This theorem can be translated into the following result, which provides the starting point for a measure-theoretic treatment of possibilistic processes.

Theorem 24 *Suppose the family $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ is consistent. Then there exist a $(L', \leq')$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, where $(L', \leq')$ is a complete lattice in which $(L, \leq)$ is embedded using a supremum preserving mapping ϕ from $(L, \leq)$ to $(L', \leq')$, and a family $(f_t \mid t \in T)$ of $(L', \leq')$-possibilistic variables in the family $((X_t^*, \mathcal{R}_t^*) \mid t \in T)$, with $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as basic space, such that*

$$\pi_{\underset{t\in T'}{\times} f_t}(x) = (\phi \circ \pi_{T'})(x)$$

for any $x \in \times_{t\in T'} X_t$, and

$$\Pi_{\underset{t\in T'}{\times} f_t}(\underset{t\in T'}{\times} X_t^*) = (\phi \circ \Pi_{T'})(\underset{t\in T'}{\times} X_t),$$

for any nonempty, finite subset T' of T. If the mapping $\Pi^*|_{\mathcal{O}_T^*}$, constructed in theorem 22, satisfies at least one of the sufficient conditions for extendability $(E_1), (E_2), (E_3)$, then one can take $(L, \leq)$ for $(L', \leq')$ and $\mathbf{1}_L$ for ϕ. If the field $\mathcal{C}_T$ is a τ-field on X_T, then there exist a $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ and a family $(f_t \mid t \in T)$ of $(L, \leq)$-possibilistic variables in the family $((X_t, \mathcal{R}_t) \mid t \in T)$ with $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as basic space, such that $\pi_{T'}$ is the possibility distribution of $\times_{t\in T'} f_t$ for any nonempty, finite subset T' of T.

From the previous theorem we can immediately derive a possibilistic Daniell-Kolmogorov theorem [Doob, 1967].

Corollary 25 *Suppose the family* $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ *is consistent. Furthermore, assume that any* τ*-space* $(X_t, \mathcal{R}_t)$ *coincides with a* τ*-space* $(X, \mathcal{R})$ *for any* $t \in T$*. Then there exist a* $(L', \leq')$*-possibility space* $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$*, where* $(L', \leq')$ *is a complete lattice in which* $(L, \leq)$ *is embedded using a supremum preserving mapping* ϕ *from* $(L, \leq)$ *to* $(L', \leq')$*, and a possibilistic process* $(f_t \mid t \in T)$ *in a *-extension* $(X^*, \mathcal{R}^*)$ *of* $(X, \mathcal{R})$*, with* $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ *as basic space, such that*

$$\pi_{\underset{t \in T'}{\times} f_t}|_{X^{T'}} = \phi \circ \pi_{T'},$$

and

$$\Pi_{\underset{t \in T'}{\times} f_t}((X^*)^{T'}) = (\phi \circ \Pi_{T'})(X^{T'}),$$

for any nonempty, finite subset T' *of* T*. If the mapping* $\Pi^*|_{\mathcal{O}^*_T}$*, constructed in theorem 22, satisfies at least one of the sufficient conditions for extendability* $(E_1), (E_2), (E_3)$*, then one can take* $(L, \leq)$ *for* $(L', \leq')$ *and* $\mathbf{1}_L$ *for* ϕ*. If the field* $\mathcal{C}_T$ *of measurable cylinders of* $(X^T, \mathcal{R}^T)$ *is a* τ*-field on* X^T*, then there exist a* $(L, \leq)$*-possibilitity space* $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ *and a family* $(f_t \mid t \in T)$ *of* $(L, \leq)$*-possibilistic variables in the* τ*-space* $(X, \mathcal{R})$ *with* $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ *as basic space, such that* $\pi_{T'}$ *is the possibility distribution of* $\times_{t \in T'} f_t$ *for any nonempty, finite subset* T' *of* T*.*

Note that this is not a perfect analogon of the probabilistic Daniell-Kolmogorov theorem. In case of a noncompact sample space $(X, \mathcal{R})$, we obtain a possibilistic process $(f_t \mid t \in T)$ in a one-point extension $(X^*, \mathcal{R}^*)$. Although this sample space $(X^*, \mathcal{R}^*)$ is larger than the given τ-space $(X, \mathcal{R})$, we obtain for any nonempty, finite subset T' of T that the restriction $\pi_{\underset{t \in T'}{\times} f_t}|_{X^{T'}}$ of the joint distribution of the finite subfamily $(f_t \mid t \in T')$ of the possibilistic process $(f_t \mid t \in T)$ to the finite Cartesian power $X^{T'}$ of the given sample space $(X, \mathcal{R})$ is completely determined by the given distribution $\pi_{T'}$. Furthermore, $\Pi_{\times_{t \in T'} f_t}((X^*)^{T'}) = (\phi \circ \Pi_{T'})(X^{T'}) = \Pi_{\times_{t \in T'} f_t}(X^{T'})$ for any nonempty, finite subset T' of T. So, by extending the sample space $(X, \mathcal{R})$ to a one-point extension $(X^*, \mathcal{R}^*)$, the possibility of a finite Cartesian product of the sample space $(X^*, \mathcal{R}^*)$ still equals the possibility of the corresponding Cartesian product of the given space $(X, \mathcal{R})$. In case the given space $(X, \mathcal{R})$ is compact, we have a true analogon of the probabilistic Daniell-Kolmogorov theorem.

If the index set T is finite, we obtain the following result.

Corollary 26 *Suppose the family* $(\pi_{T'} \mid \emptyset \subset T' \Subset T)$ *is consistent. If* T *is finite, then there exist a* $(L, \leq)$*-possibility space* $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ *and a family* $(f_t \mid t \in T)$ *of* $(L, \leq)$*-possibilistic variables in the family* $((X_t, \mathcal{R}_t) \mid t \in T)$ *with* $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ *as basic space, such that* $\pi_{T'}$ *is the possibility distribution of* $\times_{t \in T'} f_t$ *for any nonempty subset* T' *of* T*.*

If any τ-space $(X_t, \mathcal{R}_t)$ coincides with a τ-space $(X, \mathcal{R})$ for any $t \in T$, this corollary is a special case of the possibilistic Daniell-Kolmogorov theorem, which is a true analogon of its probabilistic counterpart.

References

[Boyen et al., 1995] L. Boyen, G. de Cooman and E.E. Kerre. On the extension of P-consistent mappings. In De Cooman, G. and Ruan, D. and Kerre, E. E., editor, *Foundations and Applications of Possibility Theory*, pages 88–98, Singapore, 1995. World Scientific.

[De Cooman and Kerre, 1993] G. de Cooman and E. E. Kerre. Ample Fields. *Simon Stevin*, 67:235–244, 1993.

[De Cooman, 1993] G. de Cooman. *Evaluatieverzamelingen en -afbeeldingen - Een ordetheoretische benadering van vaagheid en onzekerheid [Evaluation Sets and Mappings - An Order-Theoretic Approach to Vagueness and Uncertainty].* PhD thesis, Universiteit Gent, Ghent, 1993.

[Doob, 1967] J. L. Doob. *Stochastic Processes.* John Wiley & Sons, New York, 1967.

[Kelley, 1959] J. L. Kelley. *General Topology.* D. Van Nostrand, Princeton, New Jersey, 1959.

[Wang and Klir, 1992] Z. Wang and G. J. Klir. *Fuzzy Measure Theory.* Plenum Press, New York, 1992.

[Wang, 1982] P.-Z. Wang. Fuzzy contactibility and fuzzy variables. *Fuzzy Sets and Systems*, 8:81–92, 1982.

[Wang, 1985] Z. Wang. Extension of possibility measures defined on an arbitrary nonempty class of sets. In *Proceedings of the First IFSA Congress*, Palma de Mallorca, Spain, 1985.

[Willard, 1970] S. Willard. *General Topology.* Addison-Wesley, Reading, Massachusetts, 1970.