A DANIELL-KOLMOGOROV THEOREM FOR SUPREMUM PRESERVING UPPER PROBABILITIES

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Abstract. Possibility measures are interpreted as upper probabilities that are in particular supremum preserving. We define a possibilistic process as a special family of possibilistic variables, and show how its possibility distribution functions can be constructed. We introduce and study the notions of inner and outer regularity for possibility measures. Using these notions, we prove an analogon for possibilistic processes (and possibility measures) of the well-known probabilistic Daniell-Kolmogorov theorem, in the important special case that the variables assume values in compact spaces, and that the possibility measures involved are regular.

1. Introduction

A process p can be informally defined as a variable which changes in time. If we consider a time set T, and denote by X the set in which the variable p may take its values, then formally, a *process* p in X is a $T-X$-mapping, and $p(t)$ is the *value of the process* at time t, $t \in T$. In many practical applications, processes are *real*, that is, they assume real values: $X = \mathbb{R}$.

A process p is called *uncertain* if there is not enough information available in order to specify its value $p(t)$ in X unequivocally for all times $t \in T$. Such processes can be modelled by considering a collection P of processes in X, one of which, but unknown to us, is the actual process. Information about the process essentially leads to a restriction on the candidate processes in P, which can take a number of forms. It could take the form of a subset of P, or of an additive probability on P. In the latter case, we may speak of a *stochastic* process. Arguably the most general case, which encompasses the cases just mentioned, is when the available information is represented by an imprecise probability model [17] on P.

We are interested in the values that an uncertain process may assume at given times. This leads to the consideration of a family of mappings $f_t \colon P \to X$, indexed by the time set T. Each f_t maps a process p in P to its corresponding value at time t: $f_t(p) = p(t)$. Of course, these mappings (and their products) may be used to transport the given imprecise probability model on P to imprecise probability models on X, that is, to information about the values which the process may assume at the various corresponding times.

A very general formal model for an uncertain process therefore consists of a nonempty set Ω, called *basic space*, a nonempty set X, called *sample space*, and a family of $\Omega - X$-mappings $(f_t \mid t \in T)$, indexed by a nonempty time set T. Information about the process is represented by an imprecise probability model on the basic space.

The paper provides a closer look at uncertain processes of a special type, namely for which the given imprecise probability model is a possibility measure, that is, a supremum preserving upper probability (see the next section for a more precise definition). For such *possibilistic processes*, we intend to show that it is possible to prove a counterpart to a fundamental result in the theory of *stochastic* processes, namely the Daniell-Kolmogorov theorem.

Key words and phrases. Possibilistic process, Daniell-Kolmogorov theorem, regularity, upper semicontinuity, possibility measure, upper probability.

2. Preliminary results and definitions

In this section, we have collected the preliminary material, requisite for understanding the formal developments in the rest of the paper.

2.1. **Plump classes, ample fields and measurability.** Throughout we shall denote by X a nonempty set. A subset $\mathcal{D}$ of the power set $\wp(X)$ of X is called a *plump class* on X iff it is closed under arbitrary unions and intersections [20]. The *atom* $[x]_{\mathcal{D}}$ of $\mathcal{D}$ containing the element x of X is defined as $[x]_{\mathcal{D}} = \bigcap\{A \mid A \in \mathcal{D} \text{ and } x \in A\}$, and a subset A of X is a called an atom of $\mathcal{D}$ iff $(\exists x \in X)(A = [x]_{\mathcal{D}})$. The set of the atoms of $\mathcal{D}$ is denoted by $X_{\mathcal{D}}$. Note that $X_{\mathcal{D}} \subseteq \mathcal{D}$ and that $(\forall x \in X)(x \in [x]_{\mathcal{D}})$. Also, for any subset A of X, $A \in \mathcal{D} \Leftrightarrow A = \bigcup_{x \in A} [x]_{\mathcal{D}}$.

A plump class is in particular a topology. Let $\mathcal{S}$ be a nonempty collection of subsets of X, then $\mathfrak{T}_{\mathcal{S}}$ denotes the topology on X with subbase $\mathcal{S}$, i.e. the smallest topology on X that includes $\mathcal{S}$, and $\mathcal{D}_{\mathcal{S}}$ denotes the *plump class generated by* $\mathcal{S}$ on X, i.e. the smallest plump class on X including $\mathcal{S}$.

An *ample field* $\mathcal{R}$ on X is a plump class on X that is closed under complementation [9, 18]. The couple $(X, \mathcal{R})$ is called *ample space*. The set of the atoms $X_{\mathcal{R}}$ constitutes a partition of X. For a subset $\mathcal{S}$ of the power set $\wp(X)$, $\tau_X(\mathcal{S})$ denotes the smallest ample field on X which includes $\mathcal{S}$ and is called *the ample field generated by* $\mathcal{S}$ (on X).

An ample field is a special topology, for which all open sets are closed, and vice versa. It can be verified that an element of this ample field is compact iff it is a finite union of atoms. More information about the topological aspects of ample fields can be found in [13].

We continue with a number of measurability definitions. If $\mathcal{S} \subseteq \wp(X)$, and $A \subseteq X$, then we call A $\mathcal{S}$*-measurable* iff $A \in \mathcal{S}$. If $\mathcal{S}_1 \subseteq \wp(X_1)$ and $\mathcal{S}_2 \subseteq \wp(X_2)$, where X_1 and X_2 are nonempty sets, then a $X_1 - X_2$-mapping f is called $\mathcal{S}_1 - \mathcal{S}_2$*-measurable* iff $(\forall B \in \mathcal{S}_2)(f^{-1}(B) \in \mathcal{S}_1)$.

2.2. **Products of sets, ample fields and mappings.** In this subsection, we have collected a number of basic definitions and results concerning products of sets, mappings and ample fields. A much more detailed discussion of this topic, with explicit proofs of the results mentioned here, can be found in [13].

Let $(X_t \mid t \in T)$ be a nonempty family of nonempty sets. The *Cartesian product* $\times_{t \in T} X_t$ of $(X_t \mid t \in T)$ is the set of all $T - \bigcup_{t \in T} X_t$-mappings x such that $x(t) \in X_t$, $t \in T$. For simplicity of notation, we denote this Cartesian product by X_T. If all the sets X_t, $t \in T$, are equal to a given set X, the Cartesian product is the set of all the $T - X$-mappings, which is also given the standard notation X^T.

We need the following projection operators. For any $t \in T$, $\mathbf{pr}_{T,t}$ is the *projection mapping* from X_T onto X_t, defined by $\mathbf{pr}_{T,t}(x) = x(t)$, $x \in X_T$. If S is a nonempty subset of T, then $\mathbf{pr}_{T,S}$ is the $X_T - X_S$-mapping such that for any $x \in X_T$, $\mathbf{pr}_{T,S}(x) = x|_S$ is the restriction of the mapping x to S.

This brings us to the notion of product mappings. Let f_t be a mapping from a set X to X_t, $t \in T$. The *product mapping* $\times_{t \in T} f_t$ of the family of mappings $(f_t \mid t \in T)$ is the unique $X - X_T$-mapping f such that $f_t = \mathbf{pr}_{T,t} \circ f$, $t \in T$. For notational simplicity, it will also be denoted by f_T.

Projections allow us to define product ample fields. Let $\mathcal{R}_t$ be an ample field on X_t, $t \in T$. The *product* $\prod_{t \in T} \mathcal{R}_t$ of the family $(\mathcal{R}_t \mid t \in T)$ of ample fields is the smallest ample field $\mathcal{H}$ on X_T such that $\mathbf{pr}_{T,t}$ is a $\mathcal{H} - \mathcal{R}_t$-measurable mapping, $t \in T$. For notational simplicity, it is also denoted by $\mathcal{R}_T$. $(X_T, \mathcal{R}_T)$ is called the *product ample space* of the family of ample spaces $((X_t, \mathcal{R}_t) \mid t \in T)$.

Proposition 1. *Let S be a nonempty subset of T, and let $x \in X_S$. Then the following statements hold:*

1. $\mathbf{pr}_{T,S} = \times_{t\in S}\mathbf{pr}_{T,t}$;
2. $\mathbf{pr}_{T,S}$ *is a* $\mathcal{R}_T - \mathcal{R}_S$*-measurable mapping;*
3. $[x]_{\mathcal{R}_S} = \bigcap_{t\in S} \mathbf{pr}_{S,t}^{-1}([x(t)]_{\mathcal{R}_t}) = \times_{t\in S}[x(t)]_{\mathcal{R}_t}$.

Moreover, let $(X, \mathcal{R})$ be an ample space and consider the mappings $f_t \colon X \to X_t$, $t \in S$. Then the product mapping f_S is $\mathcal{R} - \mathcal{R}_S$-measurable iff f_t is $\mathcal{R} - \mathcal{R}_t$-measurable, $t \in S$.

Inspired by probability theory, we introduce measurable cylinders. Consider a *finite* subset S of T – we denote this as $S \Subset T$ – which is furthermore nonempty. We define $\mathcal{C}_{T,S} = \{\mathbf{pr}_{T,S}^{-1}(E) \mid E \in \mathcal{R}_S\}$ and $\mathcal{C}_T = \bigcup_{\emptyset\subset S\Subset T} \mathcal{C}_{T,S}$. Note that $C_T \subseteq \mathcal{R}_T$, taking into account Proposition 1.2. An element of $\mathcal{C}_T$ is called a *measurable cylinder* of $(X_T, \mathcal{R}_T)$. It will also be useful to consider special subsets of the measurable cylinders: $\mathcal{A}_{T,S} = \{\mathbf{pr}_{T,S}^{-1}(A) \mid A \in (X_S)_{\mathcal{R}_S}\}$ and $\mathcal{A}_T = \bigcup_{\emptyset\subset S\Subset T} \mathcal{A}_{T,S}$. Any element of $\mathcal{A}_T$ is called an *atomic* measurable cylinder of $(X_T, \mathcal{R}_T)$.

Proposition 2. *Let S, S_1 and S_2 be nonempty, finite subsets of T. Then the following statements hold:*

1. $\mathcal{C}_{T,S}$ *is an ample field on* X_T*, with set of atoms* $\mathcal{A}_{T,S}$*;*
2. *if* $S_1 \subseteq S_2$*, then* $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$.

Moreover, $\mathcal{C}_T$ is a field on X_T.

Clearly, since $\mathcal{R}_T$ is an ample field, it is particular a topology on X_T, for which, by definition, all the projection operators $\mathbf{pr}_{T,t}$ are continuous, given the topologies $\mathcal{R}_t$ on X_t, $t \in T$. On the other hand, we may consider the *smallest* topology on X_T for which the mappings $\mathbf{pr}_{T,t}$, $t \in T$ are continuous. By definition this is the *product topology* of the topologies $\mathcal{R}_t$ on X_t, $t \in T$, and is denoted by $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$. Note that $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) \subseteq \mathcal{R}_T$. The following proposition relates these topologies and the set of measurable cylinders.

Proposition 3. *$\mathcal{C}_T$ is a base for the product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T. Moreover, $\tau_{X_T}(\mathcal{C}_T) = \tau_{X_T}(\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)) = \mathcal{R}_T$.*

Let us make clear under what conditions the field of measurable cylinders $\mathcal{C}_T$ is also an ample field on X_T.

Proposition 4. *The following statements are equivalent:*

1. $\mathcal{C}_T$ *is an ample field on* X_T*;*
2. $\mathcal{C}_T = \mathcal{R}_T$*;*
3. *there exists a nonempty, finite subset* S *of* T *such that* $\mathcal{C}_T = \mathcal{C}_{T,S}$*;*
4. $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ *is an ample field on* X_T*;*
5. $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) = \mathcal{R}_T$.

Moreover, if T is a finite set, then $\mathcal{C}_T$ is an ample field on X_T.

2.3. **Product lattices.** Throughout, $(L, \leq)$ denotes a complete lattice with top 1_L and bottom 0_L. Furthermore, if $\big((L_j, \leq_j) \mid j \in J\big)$ is a nonempty family of lattices, then we can equip the Cartesian product $\times_{j\in J}L_j$ with the usual product order $\times_{j\in J}\leq_j$, i.e. if λ and μ are elements of $\times_{j\in J}L_j$, then $\lambda \times_{j\in J}\leq_j \mu$ iff $\lambda(j) \leq_j \mu(j)$, $j \in J$. The lattice $(\times_{j\in J}L_j, \times_{j\in J}\leq_j)$ is called the *product* of the lattices $\big((L_j, \leq_j) \mid j \in J\big)$ [4]. If $(L_j, \leq_j)$ is a complete lattice, $j \in J$, then it is well known that $(\times_{j\in J}L_j, \times_{j\in J}\leq_j)$ is also a complete lattice. Let $\mathbf{pr}_k$ denote the projection mapping from $\times_{j\in J}L_j$ onto L_k, $k \in J$. Then, for any subset A of $\times_{j\in J}L_j$ and $k \in J$, it follows that $\mathbf{pr}_k(\sup A) = \sup \mathbf{pr}_k(A)$ and $\mathbf{pr}_k(\inf A) = \inf \mathbf{pr}_k(A)$. In the special case that all the lattices $(L_j, \leq_j)$ coincide with the complete chain $([0,1], \leq)$, we denote the product complete lattice $(\times_{j\in J} L_j, \times_{j\in J} \leq_j)$ by $([0,1]^J, \leq^J)$.

2.4. **Possibility measures.** In the context of this paper, we need a definition of a possibility measure that slightly generalises Zadeh's original notion [21]. A set function Π defined on an ample field $\mathcal{R}$ and taking values in a complete lattice $(L, \leq)$ is called a $(L, \leq)$-*possibility measure* on $(X, \mathcal{R})$ iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$ [9]. The triple $(X, \mathcal{R}, \Pi)$ is then called a $(L, \leq)$-*possibility space*. A *distribution* for Π is a $X - L$-mapping π that is $\mathcal{R}-\wp(L)$-measurable, i.e. constant on the atoms of $\mathcal{R}$, and satisfies $\Pi(A) = \sup_{x\in A} \pi(x)$, $A \in \mathcal{R}$. Clearly, such a distribution is unique and completely determined by $\pi(x) = \Pi([x]_{\mathcal{R}})$, $x \in X$.

3. Formal definition of a possibilistic process

As a first step towards the formal introduction of a possibilistic process, we introduce the notion of a possibilistic variable. Informally, this is a variable for which the available information about the values it may assume, takes the form of a possibility measure. In a formal approach, we consider a *basic space* Ω, provided with an ample field $\mathcal{R}_\Omega$, and a *sample space* X, provided with an ample field $\mathcal{R}$. The available information is represented by a $(L, \leq)$-possibility measure Π_Ω on the basic space $(\Omega, \mathcal{R}_\Omega)$ with distribution π_Ω. A $\Omega - X$-mapping f that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, is called a *possibilistic variable* in $(X, \mathcal{R})$. The $(L, \leq)$-possibility measure Π_f on $(X, \mathcal{R})$, defined by $\Pi_f(B) = \Pi_\Omega(f^{-1}(B))$, $B \in \mathcal{R}$, represents the available information about the values that the variable f may assume in X. Π_f is completely characterised by its distribution π_f, given by

$$\pi_f(x) = \Pi_f([x]_{\mathcal{R}}) = \Pi_\Omega(f^{-1}([x]_{\mathcal{R}})) = \sup_{\omega\in f^{-1}([x]_{\mathcal{R}})} \pi_\Omega(\omega),$$

which is called the *possibility distribution function* of f. More information about possibilistic variables and their distribution functions can be found in [5, 6, 7, 8]. Unless stated to the contrary, it will be implicitly assumed that $(\Omega, \mathcal{R}_\Omega)$ is the basic space for any possibilistic variable we consider in this paper. Knowledge of Π_Ω then enables us to determine the possibility distribution functions of any possibilistic variable considered.

A possibilistic process will now be formally defined as a family of possibilistic variables having the same sample space.

Definition 5. *Let T be a nonempty set and $(X, \mathcal{R})$ an ample space. A family $(f_t \mid t \in T)$ such that f_t is a possibilistic variable in $(X, \mathcal{R})$, $t \in T$, is called a possibilistic process in $(X, \mathcal{R})$ with index set T. A possibilistic process is called discrete iff its index set is countable, and continuous iff its index set is a real nondegenerate interval.*

T is also called time set. f_t is called the *value of the process* at time t, $t \in T$. A possibilistic process $(f_t \mid t \in T)$ in $(X, \mathcal{R})$ can also be viewed as a family of mappings, which for a fixed element ω of Ω relates any $t \in T$ to the value $f_t(\omega)$ of the possibilistic variable f_t in ω. The $T - X$-mapping f_ω, such that $f_\omega(t) = f_t(\omega)$, $t \in T$, is called a *sample mapping* or *realisation* of the possibilistic process. If the index set T is countable, the sample mappings are called *sample sequences*.

Consider an arbitrary family of possibilistic variables $(f_t \mid t \in T)$, where f_t is a possibilistic variable in the sample space $(X_t, \mathcal{R}_t)$, $t \in T$. If S is any subset of T, we may consider the product mapping f_S of the family $(f_t \mid t \in S)$. It follows from Proposition 1 that f_S is a possibilistic variable in $(X_S, \mathcal{R}_S)$, and we may therefore consider its possibility distribution function

$\pi_{f_S}\colon X_S \to L$, given for any x in X_S by

$$\pi_{f_S}(x) = \sup_{f_S(\omega)\in[x]_{\mathcal{R}_S}} \pi_\Omega(\omega) = \sup_{(\forall t\in S)(f_t(\omega)\in[x(t)]_{\mathcal{R}_t})} \pi_\Omega(\omega).$$

In the last equality, we have made use of Proposition 1.3. π_{f_S} is also called the *joint* possibility distribution function of the possibilistic variables f_t, $t \in S$, and it completely characterises the values that these variables may assume jointly, that is as a product, in the set X_S.

So, when a basic space $(\Omega, \mathcal{R}_\Omega)$ and the possibilistic information Π_Ω are present, we are able to calculate the joint possibility distribution function of any family of possibilistic variables. In particular, we are able to calculate the joint possibility distribution functions of the values of a possibilistic process at any collection of times.

In this paper, we investigate an important and practical special case of the general converse problem: given a collection of mappings $\pi_S\colon X_S \to L$, where S belongs to a collection Λ of subsets of T, does there exist a basic space with possibility measure, and a possibilistic process with this basic space, which has the given functions as the corresponding joint possibility distribution functions? The special case we consider here is when Λ is the collection of all finite subsets of the time set T. We call the corresponding problem the *possibilistic Daniell-Kolmogorov problem*, as it is the possibilistic counterpart of a problem solved in probability theory by Daniell and Kolmogorov [3].

Intuitively, it is easily understood from the foregoing discussion that specifying a collection of mappings $\{\pi_S \mid S \in \Lambda\}$ and interpreting them as joint possibility distribution functions amounts to specifying the values of a set function on a collection of subsets of a basic space. A crucial question will therefore be whether this set function can be extended to a possibility measure on the basic space. In the next section, we present a brief overview of the existing general results about extending set functions to possibility measures. In Section 5, we devote some attention to the notion of regularity for possibility measures. We use these results in Section 6 to solve the possibilistic Daniell-Kolmogorov problem in a number of important special cases, namely when the sets X_t are compact, $t \in T$, and the collection of mappings $\{\pi_S \mid S \in \Lambda\}$ leads to regular possibility measures.

4. P-consistency of set mappings

Consider a nonempty collection $\mathcal{S}$ of subsets of the nonempty set X, and a mapping μ defined on $\mathcal{S}$ and taking values in a complete lattice $(L, \leq)$. We can ask ourselves if the set function μ is *extendable to a* $(L, \leq)$*-possibility measure*, that is, if there exists an ample field $\mathcal{R}$ on X and a $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$, such that Π coincides with μ on $\mathcal{S}$: $(\forall A \in \mathcal{S})(\Pi(A) = \mu(A))$. It is clear that we must at least have that $\tau_X(\mathcal{S}) \subseteq \mathcal{R}$.

This so-called *possibilistic extension problem* was solved by Wang [19, 20] for $[0,1]$-valued set functions. Quite recently, Boyen *et al.* [2] considered and partially solved the more general problem for $(L, \leq)$-valued set functions. In this section, we briefly recall the most relevant results in their study.

Boyen *et al.* have generalised Wang's definition of *P-consistency* for set mappings as follows.

Definition 6. *Let X be a nonempty set and let $\mathcal{S}$ be a nonempty collection of subsets of X. A $\mathcal{S}-L$-mapping μ is called P-consistent iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{S}$ and any element A of $\mathcal{S}$:*

$$A \subseteq \bigcup_{j\in J} A_j \Rightarrow \mu(A) \leq \sup_{j\in J} \mu(A_j).$$

They have shown that P-consistency is a necessary condition for extendability to a possibility measure. Moreover, they have proven the following result, which tells us that it is also a sufficient condition in a number of special cases.

Theorem 7. *Let X be a nonempty set and let $\mathcal{S}$ be a nonempty collection of subsets of X. Then, for a P-consistent $\mathcal{S}-L$-mapping μ, any of the following conditions is sufficient for the extendability of μ to a $(L,\leq)$-possibility measure.*

(E_1) *$(L,\leq)$ is a complete chain.*
(E_2) *$\mathcal{S}$ is a plump class.*
(E_3) *$(L,\leq)=(\mathcal{B},\supseteq)$, where $\mathcal{B}$ is a plump class on some set Y.*

In the following sections, we shall often associate a special possibility measure Π^g_μ with a $\mathcal{S}-L$-mapping μ. Π^g_μ is the possibility measure on $(X,\wp(X))$ with distribution π^g_μ, defined by

$$\pi^g_\mu(x)=\inf_{A\in\mathcal{S},x\in A}\mu(A),\quad x\in X.$$

It is straightforward to show that Π^g_μ is the greatest possibility measure that is dominated on $\mathcal{S}$ by μ. Moreover, obviously μ is extendable to a $(L,\leq)$-possibility measure iff Π^g_μ is the greatest possibility measure that coincides with μ on $\mathcal{S}$. In case that $\mathcal{S}$ is an ample field on X and μ is a $(L,\leq)$-possibility measure on $(X,\mathcal{S})$ with distribution π, then $\pi^g_\mu=\pi$. Note also that these remarks remain valid if we substitute for $\wp(X)$ any ample field $\mathcal{R}$ on X satisfying $\tau_X(\mathcal{S})\subseteq\mathcal{R}\subseteq\wp(X)$. In particular, one may verify that for any such $\mathcal{R}$, π^g_μ is $\mathcal{R}-\wp(L)$-measurable, i.e. constant on the atoms of $\mathcal{R}$, and that $\pi^g_\mu(x)=\Pi^g_\mu([x]_\mathcal{R})$, $x\in X$.

We may add a fourth case in which P-consistency is sufficient for extendability to a possibility measure. Assume that $(L,\leq)$ is the product complete lattice $(\times_{j\in J}L_j,\times_{j\in J}\leq_j)$ of the nonempty family of complete lattices $\big((L_j,\leq_j)\mid j\in J\big)$. The following results are immediate and their proof is therefore omitted.

Proposition 8. *Let X be a nonempty set, let $\mathcal{S}$ be a nonempty collection of subsets of X, and let μ be a $\mathcal{S}-\times_{j\in J}L_j$-mapping. Then μ is P-consistent iff $\mathbf{pr}_j\circ\mu$ is P-consistent for any $j\in J$. μ is extendable to a $(\times_{j\in J}L_j,\times_{j\in J}\leq_j)$-possibility measure iff $\mathbf{pr}_j\circ\mu$ is extendable to a $(L_j,\leq_j)$-possibility measure, $j\in J$.*

Therefore, using Theorem 7, we obtain that a P-consistent $\mathcal{S}-L$-mapping μ is extendable to a $(L,\leq)$-possibility measure if

(E_4) $(L,\leq)=(\times_{j\in J}L_j,\times_{j\in J}\leq_j)$, where for all $j\in J$, $(L_j,\leq_j)$ satisfies (E_1) or (E_3).

For $([0,1],\leq)$-possibility measures condition (E_1) is satisfied, and this means that in this case P-consistency is a necessary and sufficient condition for possibilistic extendability. This is precisely the result found by Wang [19, 20]. In general, for set functions valued on a complete lattice, P-consistency is only necessary and not sufficient for extendability to a possibility measure. However, Boyen *et al.* also proved the following result, which provides an interesting way to circumvent this problem.

Theorem 9. *Let X be a nonempty set and let $\mathcal{S}$ be a nonempty collection of subsets of X. The complete lattice $(L,\leq)$ can always be embedded using a supremum preserving mapping ϕ in a second complete lattice $(L',\leq')$, in such a way that for any P-consistent $\mathcal{S}-L$-mapping μ, $\phi\circ\mu$ is a P-consistent $\mathcal{S}-L'$-mapping, which is furthermore extendable to a $(L',\leq')$-possibility measure.*

5. Regularity

We now introduce and study inner and outer regularity of possibility measures. Unless explicitly stated otherwise, X will be a nonempty set, and $\mathfrak{T}$ a topology on X.

Let us first extend the well-known notions of inner and outer regularity [12] towards set mappings which have a complete lattice $(L, \leq)$ as their codomain.

Definition 10. *Let $\mathcal{S}$ be a nonempty collection of subsets of X and let μ be a $\mathcal{S} - L$-mapping. Let A be an element of $\mathcal{S}$.*

1. *μ is called inner regular with respect to $\mathfrak{T}$ in A iff*
$$\mu(A) = \sup \{\mu(C) \mid A \supseteq C \in \mathcal{S} \text{ and } C \text{ is compact in } (X, \mathfrak{T})\},$$
and inner regular with respect to $\mathfrak{T}$ iff this equality holds for any A in $\mathcal{S}$.
2. *μ is called outer regular with respect to $\mathfrak{T}$ in A iff*
$$\mu(A) = \inf \{\mu(O) \mid A \subseteq O \in \mathcal{S} \text{ and } O \text{ is open in } (X, \mathfrak{T})\},$$
and outer regular with respect to $\mathfrak{T}$ iff this equality holds for any A in $\mathcal{S}$.
3. *μ is called regular with respect to $\mathfrak{T}$ in A iff μ is both inner and outer regular with respect to $\mathfrak{T}$ in A.*
4. *μ is called regular with respect to $\mathfrak{T}$ iff μ is both inner and outer regular with respect to $\mathfrak{T}$.*

If it is clear from the context which topology is used to express inner and outer regularity, we will not mention it explicitly and simply speak of inner and outer regularity of μ in an element of its domain $\mathcal{S}$.

If the codomain $(L, \leq)$ of the set function μ is a product complete lattice, then it is obvious that the inner and outer regularity of μ depend on the inner and outer regularity of the components of μ.

Proposition 11. *Let $(L, \leq)$ be the product complete lattice of the nonempty family of complete lattices $\big((L_j, \leq_j) \mid j \in J\big)$. Let $\mathcal{S}$ be a nonempty collection of subsets of X and let μ be a $\mathcal{S} - L$-mapping. Consider an arbitrary A in $\mathcal{S}$.*

1. *μ is inner regular with respect to $\mathfrak{T}$ in A iff $(\forall j \in J)(\mathbf{pr}_j \circ \mu$ is inner regular with respect to $\mathfrak{T}$ in $A)$.*
2. *μ is outer regular with respect to $\mathfrak{T}$ in A iff $(\forall j \in J)(\mathbf{pr}_j \circ \mu$ is outer regular with respect to $\mathfrak{T}$ in $A)$.*
3. *μ is regular with respect to $\mathfrak{T}$ in A iff $(\forall j \in J)(\mathbf{pr}_j \circ \mu$ is regular with respect to $\mathfrak{T}$ in $A)$.*

In what follows, we are primarily interested in the inner and outer regularity of possibility measures.

Outer regularity of a set function essentially means that its domain contains enough open sets, so that it can be approximated from above on these open sets. Similarly, inner regularity means that the domain of the set function contains enough compact sets, so that it can be approximated from below on these compact sets. Any ample space is in particular also a topological space, for which all open sets are closed, and vice versa. Moreover, finite unions of atoms are closed compact sets in this topological space. The following proposition therefore gives an immediate, but rather weak sufficient condition for regularity.

Proposition 12. *Let $\mathcal{R}$ be an ample field on X. A $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ is (inner and outer) regular with respect to $\mathcal{R}$.*

It turns out that the possibility measures which are inner regular with respect to a given topology can be easily characterised. Let $\mathcal{S}$ be a plump class on X, and assume that Π is a supremum preserving $\mathcal{S} - L$-mapping. For any element A of $\mathcal{S}$, we know that $A = \bigcup_{x \in A} [x]_{\mathcal{S}}$, whence $\Pi(A) = \sup_{x \in A} \Pi([x]_{\mathcal{S}})$. If we assume that for any x in A such that $\Pi([x]_{\mathcal{S}}) > 0_L$, $[x]_{\mathcal{S}}$ is compact in $(X, \mathfrak{T})$, then Π is obviously inner regular with respect to $\mathfrak{T}$ in A.

Conversely, let $\mathcal{S}$ in particular be an ample field on X and assume that the supremum preserving mapping Π on $\mathcal{S}$ – a possibility measure – is inner regular with respect to $\mathfrak{T}$. Consider an element x of X such that $\Pi([x]_{\mathcal{S}}) > 0_L$. By assumption, there exists a nonempty C in $\mathcal{S}$ that is compact in $(X, \mathfrak{T})$, such that $C \subseteq [x]_{\mathcal{S}}$. Hence $[x]_{\mathcal{S}} = C$ is compact in $(X, \mathfrak{T})$.

This gives rise to the following characterisation of inner regularity for possibility measures.

Proposition 13. *Let $\mathcal{R}$ be an ample field on X and let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π. Let $\mathfrak{T}$ be a topology on X. Then Π is inner regular with respect to $\mathfrak{T}$ iff*

$$(\forall x \in X)(\pi(x) > 0_L \Rightarrow [x]_{\mathcal{R}} \text{ is compact in } (X, \mathfrak{T})). \tag{1}$$

We can restate this by using Suzuki's notion of atom of a (not necessarily additive) positive set mapping [15, 16]. Recall that, for a $(L, \leq)$-valued mapping μ defined on a nonempty collection $\mathcal{S}$ of subsets of some set X, an element A of $\mathcal{S}$ with $\mu(A) > 0_L$ is called an atom of μ iff for any element B of $\mathcal{S}$ such that $B \subset A$ it follows that $\mu(B) = 0_L$ or that both $\mu(A) = \mu(B)$ and $\mu(A \setminus B) = 0_L$. If we reconsider the possibility measure Π of proposition 13, then it is easily verified that for an atom E of $\mathcal{R}$ to be an atom of Π it is necessary and sufficient that $\Pi(E) > 0_L$. In fact, it turns out that these elements of $\mathcal{R}$ are precisely the minimal atoms of Π. Moreover, a $\mathcal{R}$-measurable subset A of X is an atom of Π iff there exists an element x of A such that $\Pi(A) = \pi(x) > 0_L$ and $\Pi(A \setminus [x]_{\mathcal{R}}) = 0_L$. From this characterisation it is easily deduced that the maximal atoms of Π are obtained by adding the $\mathcal{R}$-measurable set $\{x \mid x \in X \text{ and } \pi(x) = 0_L\}$ to the minimal atoms of Π. Consequently, we can reformulate Proposition 13 as follows: *a $(L, \leq)$-possibility measure Π on an ample space $(X, \mathcal{R})$ is inner regular with respect to a topology $\mathfrak{T}$ on X iff the minimal atoms of Π are compact in the topological space $(X, \mathfrak{T})$.*

It should be noted that condition (1) holds in a number of interesting special cases. First of all, if $\mathfrak{T} \subseteq \mathcal{R}$ (and in particular if $\mathcal{R} = \tau_X(\mathfrak{T})$), any atom of $\mathcal{R}$ will be compact in $(X, \mathfrak{T})$. Let $\mathcal{O}$ be an open cover of $[x]_{\mathcal{R}}$, i.e. $[x]_{\mathcal{R}} \subseteq \bigcup \mathcal{O}$. Since $x \in [x]_{\mathcal{R}}$, this implies that there exists a O in $\mathcal{O} \subseteq \mathcal{R}$ such that $x \in O$, whence $[x]_{\mathcal{R}} \subseteq O$. $\{O\}$ is a finite subcover of $[x]_{\mathcal{R}}$, and $[x]_{\mathcal{R}}$ is therefore compact.

Next, condition (1) obviously also holds if $(\forall x \in X)(\Pi([x]_{\mathcal{R}}) > 0_L \Rightarrow [x]_{\mathcal{R}} = \{x\})$, and in particular if $\mathcal{R} = \wp(X)$. This leads to the following conclusion.

Corollary 14. *Let $\mathcal{R}$ be an ample field on X and let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$. Let $\mathfrak{T}$ be a topology on X. If $\mathcal{R} = \wp(X)$ or more generally $\mathfrak{T} \subseteq \mathcal{R}$, then Π is inner regular with respect to $\mathfrak{T}$. In particular, any possibilistic extension of Π to $\wp(X)$ is always inner regular with respect to $\mathfrak{T}$.*

The last statement in this corollary tells us that inner regularity is a very natural property for possibility measures: any possibility measure can be made inner regular by extending it to a possibility measure on $\wp(X)$, which is always possible, as we have seen in Section 4. Remember by the way that if π is the distribution of a possibility measure μ on $(X, \mathcal{R})$, then the greatest possibilistic extension Π_μ^g of μ to $\wp(X)$ has the same distribution as μ, i.e. $\pi_\mu^g = \pi$.

On the other hand, possibility measures are generally not outer regular, even if we make their domains as large as possible, as the following counterexample tells us.

Example 15. *Consider $x_o \in \mathbb{R}$ and let Π be the $([0,1],\leq)$-possibility measure on $(\mathbb{R},\wp(\mathbb{R}))$ with distribution π given for any $x \in \mathbb{R}$ by*

$$\pi(x) = \begin{cases} 1 & ; \quad x \neq x_o \\ 0 & ; \quad x = x_o. \end{cases}$$

Then Π is not outer regular with respect to the Euclidean topology on $\mathbb{R}$ in $\{x_o\}$.

However, we now show that a special class of possibility measures is always outer regular (with respect to a specific topology) in the atoms of the ample fields on which they are defined.

Proposition 16. *Let $\mathcal{S}$ be a nonempty collection of subsets of X, and let μ be a $\mathcal{S}-L$-mapping. Consider the $(L,\leq)$-possibility measure Π^g_μ on $(X,\wp(X))$ with distribution π^g_μ given by $\pi^g_\mu(x) = \inf_{A\in\mathcal{S},x\in A}\mu(A)$, $x \in X$. Let $\mathcal{D}_\mathcal{S}$ be the smallest plump class and $\tau_X(\mathcal{S})$ the smallest ample field including $\mathcal{S}$. Finally, let $\mathfrak{T}_\mathcal{S}$ be the topology with subbase $\mathcal{S}$. Then Π^g_μ is outer regular with respect to $\mathfrak{T}_\mathcal{S}$ in any $E \in \wp(X)$ for which there exists an $x \in X$ such that $x \in E \subseteq [x]_{\mathcal{D}_\mathcal{S}}$. In particular, the restriction $\Pi^g_\mu|_{\tau_X(\mathcal{S})}$ of Π^g_μ to $\tau_X(\mathcal{S})$ is outer regular with respect to $\mathfrak{T}_\mathcal{S}$ in any atom of $\tau_X(\mathcal{S})$ and $\mathcal{D}_\mathcal{S}$.*

Proof. Consider an element x of X and let y be an element of $[x]_{\mathcal{D}_\mathcal{S}}$. It follows from the definition of a plump class that $(\forall A \in \mathcal{S})(x \in A \Rightarrow y \in A)$, whence $\inf_{A\in\mathcal{S},y\in A}\mu(A) \leq \inf_{A\in\mathcal{S},x\in A}\mu(A)$, or equivalently, $\pi^g_\mu(y) \leq \pi^g_\mu(x)$. Since $x \in [x]_{\mathcal{D}_\mathcal{S}}$, this implies that $\Pi^g_\mu([x]_{\mathcal{D}_\mathcal{S}}) = \sup_{y\in[x]_{\mathcal{D}_\mathcal{S}}}\pi^g_\mu(y) = \pi^g_\mu(x)$. So, if E is a subset of X such that $x \in E \subseteq [x]_{\mathcal{D}_\mathcal{S}}$, then it follows from the monotonicity of Π^g_μ that

$$\pi^g_\mu(x) = \Pi^g_\mu(E) = \Pi^g_\mu([x]_{\mathcal{D}_\mathcal{S}}). \tag{2}$$

Furthermore, since Π^g_μ is monotone, $\pi^g_\mu(x) \leq \inf_{O\in\mathfrak{T}_\mathcal{S},x\in O}\Pi^g_\mu(O)$, and since on the other hand $\mathcal{S} \subseteq \mathfrak{T}_\mathcal{S}$, $\inf_{O\in\mathfrak{T}_\mathcal{S},x\in O}\Pi^g_\mu(O) \leq \inf_{A\in\mathcal{S},x\in A}\Pi^g_\mu(A)$. Consider $E \in \mathcal{S}$, then clearly $\Pi^g_\mu(E) = \sup_{x\in E}\inf_{A\in\mathcal{S},x\in A}\mu(A) \leq \mu(E)$, which tells us that μ dominates Π^g_μ on $\mathcal{S}$. As a consequence, $\inf_{O\in\mathfrak{T}_\mathcal{S},x\in O}\Pi^g_\mu(O) \leq \inf_{A\in\mathcal{S},x\in A}\mu(A) = \pi^g_\mu(x)$, whence $\pi^g_\mu(x) = \Pi^g_\mu([x]_{\tau_X(\mathcal{S})}) = \Pi^g_\mu([x]_{\mathcal{D}_\mathcal{S}}) = \inf_{O\in\mathfrak{T}_\mathcal{S},x\in O}\Pi^g_\mu(O)$. Since an ample field is a plump class, and a plump class a topology, it is obvious that $\mathfrak{T}_\mathcal{S} \subseteq \mathcal{D}_\mathcal{S} \subseteq \tau_X(\mathcal{S})$. As a consequence, we find for any $O \in \mathfrak{T}_\mathcal{S}$ that $x \in O \Leftrightarrow [x]_{\mathcal{D}_\mathcal{S}} \subseteq O \Leftrightarrow [x]_{\tau_X(\mathcal{S})} \subseteq O$, whence, also using (2),

$$\Pi^g_\mu([x]_{\mathcal{D}_\mathcal{S}}) = \Pi^g_\mu(\{x\}) = \inf\{\Pi^g_\mu(O) \mid x \in O \in \mathfrak{T}_\mathcal{S}\} = \inf\{\Pi^g_\mu(O) \mid [x]_{\mathcal{D}_\mathcal{S}} \subseteq O \in \mathfrak{T}_\mathcal{S}\}$$

So Π^g_μ is outer regular with respect to $\mathfrak{T}_\mathcal{S}$ in $[x]_{\mathcal{D}_\mathcal{S}}$. If E is a subset of X such that $x \in E \subseteq [x]_{\mathcal{D}_\mathcal{S}}$, then, again by (2), we get

$$\Pi^g_\mu(E) \leq \inf\{\Pi^g_\mu(O) \mid E \subseteq O \in \mathfrak{T}_\mathcal{S}\} \leq \inf\{\Pi^g_\mu(O) \mid [x]_{\mathcal{D}_\mathcal{S}} \subseteq O \in \mathfrak{T}_\mathcal{S}\} = \Pi^g_\mu([x]_{\mathcal{D}_\mathcal{S}}) = \Pi^g_\mu(E).$$

Hence, Π^g_μ is outer regular with respect to $\mathfrak{T}_\mathcal{S}$ in E. The rest of the proof is now obvious. $\square$

The previous proposition assures us that the greatest possibilistic extension of a $(L,\leq)$-valued set mapping μ on an arbitrary collection $\mathcal{S}$ of subsets of some set X is always outer regular in the atoms of its domain with respect to a topology $\mathfrak{T}$ on X that includes $\mathcal{S}$.

Since it appears that it is fairly natural for a possibility measure to be outer regular in the atoms of its domain, it seems natural to ask the following question: given that a possibility measure is outer regular in the atoms of its domain, can we infer that it is outer regular in all the elements of its domain? The next proposition gives a sufficient condition on the codomain which allows us to indeed make that step.

Proposition 17. *Let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$, and let $\mathfrak{T}$ be a topology on X. Assume that $(L, \leq)$ is an order-dense*[1]*, complete chain. Then Π is outer regular with respect to $\mathfrak{T}$ iff Π is outer regular with respect to $\mathfrak{T}$ in the atoms of $\mathcal{R}$.*

Proof. Assume that Π is outer regular with respect to $\mathfrak{T}$ in the atoms of $\mathcal{R}$. Consider an element A of $\mathcal{R}$. By the monotonicity of Π we always have that

$$\Pi(A) \leq \inf \{\Pi(O) \mid A \subseteq O \in \mathfrak{T} \cap \mathcal{R}\}.$$

Obviously, if $\Pi(A) = 1_L$, then the equality holds in the previous expression and we obtain that Π is outer regular with respect to $\mathfrak{T}$ in A. Suppose that $\Pi(A) < 1_L$. Assume *ex absurdo* that $\Pi(A) < \lambda = \inf \{\Pi(O) \mid A \subseteq O \in \mathfrak{T} \cap \mathcal{R}\}$. Since $(L, \leq)$ is order-dense, there exists an element γ of L such that $\Pi(A) < \gamma < \lambda$. Consequently, for any element x of A, we deduce from the assumptions that there is an element O_x of $\mathfrak{T} \cap \mathcal{R}$ such that $[x]_{\mathcal{R}} \subseteq O_x$ and $\Pi([x]_{\mathcal{R}}) \leq \Pi(O_x) < \gamma$. Let $O = \bigcup_{x \in A} O_x$. Then $O \in \mathfrak{T} \cap \mathcal{R}$, $A \subseteq O$ and $\Pi(A) \leq \Pi(O) \leq \gamma < \lambda$, which is impossible. This implies that

$$\Pi(A) = \inf \{\Pi(O) \mid A \subseteq O \in \mathfrak{T} \cap \mathcal{R}\}.$$

The converse result is immediate, and obviously does not require that the complete lattice $(L, \leq)$ should be order-dense. □

There is an interesting link between the outer regularity of a possibility measure and the upper semicontinuity of its distribution. This is what we set out to prove in the following two propositions.

Consider a mapping f from a set X to a complete lattice $(L, \leq)$. With f and an element λ of L, we may associate the following sets:

$$\overline{D}^f_\lambda = \{x \mid x \in X \text{ and } f(x) < \lambda\} \quad \text{the strict dual cut set of } f \text{ at level } \lambda;$$
$$D^f_\lambda = \{x \mid x \in X \text{ and } f(x) \leq \lambda\} \quad \text{the dual cut set of } f \text{ at level } \lambda.$$

If f is a $\mathcal{R}-\wp(L)$-measurable mapping where $\mathcal{R}$ is an ample field on X, then f has $\mathcal{R}$-measurable (strict) dual cut sets.

The next proposition tells us that, if Π is a possibility measure on $(X, \mathcal{R})$ and $A \in \mathcal{R}$, the possibility $\Pi(A)$ can be expressed as the infimum of the possibilities of the (strict) dual cut sets of the distribution π of Π which include A.

Proposition 18. *Let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π, and let $E \in \mathcal{R}$.*

1. $\Pi(E) = \inf \{\Pi(D^\pi_\lambda) \mid \lambda \in [\Pi(E), 1_L]\} = \inf \{\Pi(D^\pi_\lambda) \mid \lambda \in L,\, E \subseteq D^\pi_\lambda\}$.
2. *If $(L, \leq)$ is an order-dense complete chain, then*

$$\Pi(E) = \inf \{\Pi(\overline{D}^\pi_\lambda) \mid \lambda \in L,\, E \subseteq \overline{D}^\pi_\lambda\}.$$

Proof. Consider an element E of $\mathcal{R}$. If $\lambda \in L$, then $E \subseteq D^\pi_\lambda$ iff $\Pi(E) \leq \lambda$. This makes the proof of the first statement fairly obvious. Let us therefore proceed with the proof of statement 2. It follows from the monotonicity of Π that $\Pi(E) \leq \inf \{\Pi(\overline{D}^\pi_\lambda) \mid \lambda \in L,\, E \subseteq \overline{D}^\pi_\lambda\}$. Suppose that $\Pi(E) < 1_L$. Assume *ex absurdo* that $\Pi(E) < \delta = \inf \{\Pi(\overline{D}^\pi_\lambda) \mid \lambda \in L,\, E \subseteq \overline{D}^\pi_\lambda\}$. Since L is

[1] A complete chain $(L, \leq)$ is *order-dense* or *dense-in-itself* iff for any $x, y \in L$ such that $x < y$ there exists a $z \in L$ such that $x < z < y$ [1]. If $(L, \leq)$ is an order-dense, complete chain with $0_L \neq 1_L$, then $(L, \leq)$ is isomorphic to $([0, 1], \leq)$ iff $(L, \leq)$ has a countable order-dense subset [1].

order-dense, there exists $\gamma \in L$ such that $\Pi(E) < \gamma < \delta$. This implies that $E \subseteq \overline{D}^{\pi}_{\gamma}$, and $\delta \leq \Pi(\overline{D}^{\pi}_{\gamma}) \leq \gamma < \delta$, which is impossible. Therefore

$$\Pi(E) = \inf \{\Pi(\overline{D}^{\pi}_{\lambda}) \mid \lambda \in L,\, E \subseteq \overline{D}^{\pi}_{\lambda}\}.$$

In case $\Pi(E) = 1_L$, the proof of the equality above is immediate. □

Proposition 18.2 indicates that a $(L, \leq)$-possibility measure Π will be outer regular with respect to a topology $\mathfrak{T}$ on X when $(L, \leq)$ is an order-dense, complete chain and the strict dual cut sets of its distribution π are open in $(X, \mathfrak{T})$.

Suppose $(L, \leq)$ is a complete chain, let f be a L-valued mapping on a set X, and let $\mathfrak{T}$ be a topology on X. We can provide L with the *Scott topology* on the complete chain $(L, \geq)$, i.e. the topology given by the collection of sets $\mathfrak{T}_L = \{[0_L, \lambda[\mid \lambda \in L\} \cup \{L\}$ [11]. We call f *upper semicontinuous* in $x \in X$ with respect to $\mathfrak{T}$ on X iff f is continuous in x with respect to $\mathfrak{T}$ on X and $\mathfrak{T}_L$ on L. Obviously, f is always upper semicontinuous with respect to $\mathfrak{T}$ in the elements of $f^{-1}(\{1_L\})$. Furthermore, f is called upper semicontinuous with respect to $\mathfrak{T}$ iff f is upper semicontinuous in all $x \in X$. Of course, this is equivalent to requiring that the strict dual cut sets of f should be open in $(X, \mathfrak{T})$.

This brings us to the following characterisation of outer regularity for possibility measures assuming values in a complete chain.

Proposition 19. *Assume that $(L, \leq)$ is a complete chain. Let Π be a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π, let $\mathfrak{T}$ be a topology on X, and let $x \in X$.*

1. *If Π is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$, then π is upper semicontinuous with respect to $\mathfrak{T}$ in x. If Π is outer regular with respect to $\mathfrak{T}$ in the atoms of $\mathcal{R}$, then π is upper semicontinuous with respect to $\mathfrak{T}$.*
2. *Let the complete chain $(L, \leq)$ be order-dense. If $\mathfrak{T} \subseteq \mathcal{R}$ and π is upper semicontinuous with respect to $\mathfrak{T}$ in x, then Π is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$. If π is upper semicontinuous with respect to $\mathfrak{T}$, then Π is outer regular with respect to $\mathfrak{T}$.*

Proof. Consider $x \in X$. Without loss of generality we may assume that $\pi(x) < 1_L$. Let $\lambda \in L$ such that $\pi(x) < \lambda$. If π is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$, then $\pi(x) = \inf\{\Pi(O) \mid x \in O \in \mathfrak{T} \cap \mathcal{R}\}$. Hence, there exists an open neighbourhood O_x of x in $(X, \mathfrak{T})$ such that $\pi(x) \leq \Pi(O_x) < \lambda$. We now find that, for any $y \in O_x$, $\pi(y) \leq \Pi(O_x) < \lambda$, whence $y \in \overline{D}^{\pi}_{\lambda}$. This means that π is upper semicontinuous with respect to $\mathfrak{T}$ in x. The second part of statement 1 now follows at once.

Let us proceed with the proof of statement 2. Assume that π is upper semicontinuous with respect to $\mathfrak{T}$ in $x \in X$. Again, we may assume that $\pi(x) < 1_L$. If $\lambda \in L$ such that $\pi(x) < \lambda$, then there is an open neighbourhood U of x in $(X, \mathfrak{T})$ such that $U \subseteq \overline{D}^{\pi}_{\lambda}$. Since $\mathfrak{T} \subseteq \mathcal{R}$, $U \in \mathcal{R} \cap \mathfrak{T}$, and, since $(L, \leq)$ is assumed to be order-dense, Proposition 18.2 implies that Π is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$. The second part of statement 2 follows directly from Proposition 18.2. □

We conclude this section with two propositions that will be useful in the next section.

Consider a $(L, \leq)$-possibility measure Π on an ample space $(X, \mathcal{R})$, whose distribution is the pointwise infimum of the distributions of a given collection $(\Pi_j \mid j \in J)$ of $(L, \leq)$-possibility measures on $(X, \mathcal{R})$. The following result connects the inner and outer regularity of the Π_j with that of Π.

Proposition 20. *Let $(\Pi_j \mid j \in J)$ be a nonempty family of $(L, \leq)$-possibility measures on an ample space $(X, \mathcal{R})$. For any $j \in J$, let π_j be the distribution of Π_j. Let Π be the $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ with distribution π given for any $x \in X$ by $\pi(x) = \inf_{j \in J} \pi_j(x)$. Let $\mathfrak{T}$ be a topology on X and let $x \in X$.*

1. *If each Π_j is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$, then Π is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$.*
2. *Let $(L, \leq)$ be an order-dense, complete chain. If each Π_j is outer regular with respect to $\mathfrak{T}$, then Π is outer regular with respect to $\mathfrak{T}$.*
3. *If there exists $i \in J$ such that Π_i is inner regular with respect to $\mathfrak{T}$, then Π is inner regular with respect to $\mathfrak{T}$.*

Proof. Assume that, for all $j \in J$, Π_j is outer regular with respect to $\mathfrak{T}$ in $[x]_{\mathcal{R}}$. Then, since Π is in particular monotone,

$$\begin{aligned}\pi(x) &= \inf_{j \in J} \pi_j(x) \\ &= \inf_{j \in J} \inf \{\Pi_j(E) \mid x \in E \in \mathcal{R} \cap \mathfrak{T}\} \\ &= \inf \{\inf_{j \in J} \Pi_j(E) \mid x \in E \in \mathcal{R} \cap \mathfrak{T}\} \\ &\geq \inf \{\Pi(E) \mid x \in E \in \mathcal{R} \cap \mathfrak{T}\} \\ &\geq \pi(x),\end{aligned}$$

which proves statement 1. Statement 2 follows directly from statement 1 and Proposition 17. Statement 3 is a direct consequence of Proposition 13. □

Let us investigate whether there is a link between the regularity of a L-valued mapping μ on an arbitrary collection $\mathcal{S}$ and the regularity of its corresponding greatest possibilistic extension Π^g_μ.

Proposition 21. *Let $\mathcal{S}$ be a nonempty collection of subsets of X, and let μ be a $\mathcal{S} - L$-mapping. Consider the $(L, \leq)$-possibility measure Π^g_μ on $(X, \wp(X))$ with distribution π^g_μ given by $\pi^g_\mu(x) = \inf_{A \in \mathcal{S}, x \in A} \mu(A)$, $x \in X$. Let $\mathfrak{T}$ be a topology on X.*

1. *Π^g_μ is inner regular with respect to $\mathfrak{T}$.*
2. *Assume that μ is extendable to a $(L, \leq)$-possibility measure, i.e. that Π^g_μ coincides with μ on $\mathcal{S}$. Let $A \in \mathcal{S}$. If μ is outer regular with respect to $\mathfrak{T}$ in A then so is Π^g_μ.*

Proof. The first statement follows directly from Proposition 13. We prove the second statement. Consider $A \in \mathcal{S}$, then it follows from the assumptions that

$$\Pi^g_\mu(A) = \mu(A) = \inf \{\mu(O) \mid A \subseteq O \in \mathfrak{T} \cap \mathcal{S}\} \geq \inf \{\Pi^g_\mu(O) \mid A \subseteq O \in \mathfrak{T}\} \geq \Pi^g_\mu(A),$$

which indeed implies that Π^g_μ is outer regular with respect to $\mathfrak{T}$ in A. □

6. A possibilistic Daniell-Kolmogorov theorem

In this final section, we derive a possibilistic analogon for the probabilistic Daniell-Kolmogorov theorem [10]. For a given family of $(L, \leq)$-valued mappings on finite Cartesian powers of a sample space that satisfy a natural consistency condition, we want to prove that there always exist a basic space with a possibility measure and a family of possibilistic variables having the given $(L, \leq)$-valued mappings as their possibility distribution functions.

Throughout this section, T denotes a nonempty set and $((X_t, \mathcal{R}_t) \mid t \in T)$ a family of ample spaces. Furthermore, $(\pi_S \mid \emptyset \subset S \Subset T)$ is a family of mappings, such that $\pi_S \colon X_S \to L$ is $\mathcal{R}_S - \wp(L)$-measurable, $S \Subset T$. π_S is interpreted as the distribution of a $(L, \leq)$-possibility measure Π_S on the ample space $(X_S, \mathcal{R}_S)$, for any nonempty, finite subset S of T. For such families of distributions, we introduce the following consistency condition.

Definition 22. *$(\pi_S \mid \emptyset \subset S \Subset T)$ is called consistent iff for any two sets S_1 and S_2 such that $\emptyset \subset S_1 \subseteq S_2 \Subset T$, and for any $x \in X_{S_1}$:*

$$\pi_{S_1}(x) = \sup_{\mathbf{pr}_{S_2,S_1}(y)=x} \pi_{S_2}(y).$$

It can easily be verified that $(\pi_S \mid \emptyset \subset S \Subset T)$ is consistent iff for any two sets S_1 and S_2 such that $\emptyset \subset S_1 \subseteq S_2 \Subset T$, Π_{S_1} is the marginal possibility measure of Π_{S_2} on $(X_{S_1}, \mathcal{R}_{S_1})$, i.e. $\Pi_{S_1} = \Pi_{S_2} \circ \mathbf{pr}^{-1}_{S_2,S_1}$. It is obvious that the family of finite joint distribution functions $(\pi_{f_S} \mid \emptyset \subset S \Subset T)$, that can be derived from a given family of possibilistic variables $(f_t \mid t \in T)$ with corresponding sample spaces $((X_t, \mathcal{R}_t) \mid t \in T)$ and sharing a common basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, is always consistent. So, consistency is a necessary condition for a family of distributions to be representable as the family of finite joint distribution functions of a family of possibilistic variables with a common basic space.

We now show that, if the family of mappings $(\pi_S \mid \emptyset \subset S \Subset T)$ is consistent, it allows us to define in a very natural way a $(L, \leq)$-valued set function $\mathfrak{M}$ on the field $\mathcal{C}_T$ of measurable cylinders of the product ample space $(X_T, \mathcal{R}_T)$.

Let S_1 be a nonempty, finite subset of T. Any element A_1 of $\mathcal{R}_{S_1}$ generates a measurable cylinder $B = \mathbf{pr}^{-1}_{T,S_1}(A_1)$, and it is tempting to let $\mathfrak{M}(B) = \Pi_{S_1}(A_1)$. But, if there is another nonempty, finite subset S_2 of T and a A_2 in $\mathcal{R}_{S_2}$ such that $B = \mathbf{pr}^{-1}_{T,S_2}(A_2)$, then our construction method for $\mathfrak{M}$ would at the same time have us write that $\mathfrak{M}(B) = \Pi_{S_2}(A_2)$. It therefore leads to inconsistencies if $\Pi_{S_1}(A_1) \neq \Pi_{S_2}(A_2)$. However, since $\mathbf{pr}_{T,S_1} = \mathbf{pr}_{S_1\cup S_2,S_1} \circ \mathbf{pr}_{T,S_1\cup S_2}$, it also follows that $B = \mathbf{pr}^{-1}_{T,S_1\cup S_2}(\mathbf{pr}^{-1}_{S_1\cup S_2,S_1}(A_1))$, and similarly $B = \mathbf{pr}^{-1}_{T,S_1\cup S_2}(\mathbf{pr}^{-1}_{S_1\cup S_2,S_2}(A_2))$. Since the projection mappings involved are surjective, we find that $\mathbf{pr}^{-1}_{S_1\cup S_2,S_1}(A_1) = \mathbf{pr}^{-1}_{S_1\cup S_2,S_2}(A_2)$. It then follows from the consistency of the family $(\pi_S \mid \emptyset \subset S \Subset T)$ that

$$\begin{aligned}\Pi_{S_1}(A_1) &= \sup_{x\in A_1} \pi_{S_1}(x) = \sup_{x\in A_1} \sup_{\mathbf{pr}_{S_1\cup S_2,S_1}(y)=x} \pi_{S_1\cup S_2}(y)\\ &= \sup_{y\in \mathbf{pr}^{-1}_{S_1\cup S_2,S_1}(A_1)} \pi_{S_1\cup S_2}(y) = \Pi_{S_1\cup S_2}(\mathbf{pr}^{-1}_{S_1\cup S_2,S_1}(A_1)).\end{aligned}$$

Similarly, we find that $\Pi_{S_2}(A_2) = \Pi_{S_1\cup S_2}(\mathbf{pr}^{-1}_{S_1\cup S_2,S_2}(A_2))$, whence $\Pi_{S_1}(A_1) = \Pi_{S_2}(A_2)$. This implies that our construction method allows us to define a mapping $\mathfrak{M}$ on $\mathcal{C}_T$ in a consistent way.

Definition 23. *Let the family $(\pi_S \mid \emptyset \subset S \Subset T)$ be consistent. Then $\mathfrak{M}$ is the $\mathcal{C}_T - L$-mapping, given for an element B of $\mathcal{C}_T$ by $\mathfrak{M}(B) = \Pi_S(A)$, where S is a nonempty, finite subset of T, and A is an element of $\mathcal{R}_S$ such that $B = \mathbf{pr}^{-1}_{T,S}(A)$.*

We now investigate if the set function $\mathfrak{M}$ on $\mathcal{C}_T$ can be extended to a possibility measure. Recall that $\tau_{X_T}(\mathcal{C}_T) = \mathcal{R}_T$ is the smallest ample field on which a possibilistic extension of $\mathfrak{M}$, if any, may be defined. If we find a possibilistic extension of $\mathfrak{M}$ to $\mathcal{R}_T$, possibilistic extension to any ample field including $\mathcal{C}_T$ – and therefore also $\mathcal{R}_T$ – is straightforward: the greatest possibilistic extension of $\mathfrak{M}$ on any ample field including $\mathcal{C}_T$ will then have distribution $\pi^g_{\mathfrak{M}}$. In Section 4, we saw that a necessary condition for possibilistic extendability is that $\mathfrak{M}$ should be P-consistent on $\mathcal{C}_T$. This is investigated in the next proposition.

Proposition 24. *Let the family $(\pi_S \mid \emptyset \subset S \Subset T)$ be consistent, so that the mapping $\mathfrak{M}$ is well defined. It then has the following properties.*

1. *$\mathfrak{M}$ preserves finite suprema, and is therefore monotone.*
2. *If S is a nonempty, finite subset of T, then the restriction $\mathfrak{M}|_{\mathcal{C}_{T,S}}$ of $\mathfrak{M}$ to $\mathcal{C}_{T,S}$ is a $(L,\leq)$-possibility measure on the ample space $(X_T,\mathcal{C}_{T,S})$.*
3. *If $\mathcal{C}_T$ is an ample field on X_T, then $\mathfrak{M}$ is a $(L,\leq)$-possibility measure on $(X_T,\mathcal{C}_T)$.*
4. *If $\mathfrak{M}$ is P-consistent on $\mathcal{A}_T$, then $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.*
5. *If $(X_t,\mathcal{R}_t)$ is a compact topological space for any $t\in T$, then $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.*

Proof. We begin with the first statement. Let n be a strictly positive natural number, and consider a finite family $(B_i \mid i\in\{1,\dots,n\})$ of elements of $\mathcal{C}_T$. Then, by Proposition 2.2, there exists a nonempty, finite subset S of T, such that $(\forall i\in\{1,\dots,n\})(\exists A_i\in\mathcal{R}_S)(B_i=\mathbf{pr}_{T,S}^{-1}(A_i))$. Since $\mathcal{R}_S$ is an ample field on X_S, it follows that $\bigcup_{i=1}^n A_i$ belongs to $\mathcal{R}_S$ and that $\bigcup_{i=1}^n B_i = \mathbf{pr}_{T,S}^{-1}(\bigcup_{i=1}^n A_i)$ belongs to $\mathcal{C}_{T,S}$. Then, since Π_S is a $(L,\leq)$-possibility measure on $\mathcal{R}_S$, the definition of $\mathfrak{M}$ implies that $\mathfrak{M}(\bigcup_{i=1}^n B_i)=\Pi_S(\bigcup_{i=1}^n A_i)=\sup_{i=1}^n \Pi_S(A_i)=\sup_{i=1}^n \mathfrak{M}(B_i)$. This proves that $\mathfrak{M}$ preserves finite suprema, and is therefore monotone.

To prove statement 2, consider a nonempty, finite subset S of T. Then $\mathcal{C}_{T,S}\subseteq\mathcal{C}_T$. Since Π_S is a $(L,\leq)$-possibility measure on the ample field $\mathcal{R}_S$, Proposition 2.1 implies that the restriction $\mathfrak{M}|_{\mathcal{C}_{T,S}}$ is a $(L,\leq)$-possibility measure on the ample field $\mathcal{C}_{T,S}$. We continue with the proof of the third statement. In the special case that $\mathcal{C}_T$ is an ample field on X_T, Proposition 4 tells us that there exists a nonempty, finite subset S of T such that $\mathcal{C}_T=\mathcal{C}_{T,S}$. Therefore, using statement 2, we infer the validity of the statement 3.

To prove statement 4, assume that $\mathfrak{M}$ is P-consistent on $\mathcal{A}_T$, consider $B\in\mathcal{C}_T$, and a family $(B_j\mid j\in J)\subseteq\mathcal{C}_T$ such that $B\subseteq\bigcup_{j\in J}B_j$. Then, there exists a nonempty, finite subset S of T and a subfamily $(A^i\mid i\in I)$ of $\mathcal{A}_{T,S}$ such that $B=\bigcup_{i\in I}A^i$. Similarly, for any $j\in J$, there exist a nonempty, finite subset S_j of T and a subfamily $(A_j^i\mid i\in I_j)$ of $\mathcal{A}_{T,S_j}$ such that $B_j=\bigcup_{i\in I_j}A_j^i$. Let $k\in I$, then clearly $A^k\subseteq\bigcup_{j\in J}\bigcup_{i\in I_j}A_j^i$. Since $\mathfrak{M}$ is P-consistent on $\mathcal{A}_T$, it follows that $\mathfrak{M}(A^k)\leq\sup_{j\in J}\sup_{i\in I_j}\mathfrak{M}(A_j^i)$. Taking into account that $\mathfrak{M}(B)=\sup_{i\in I}\mathfrak{M}(A^i)$ and $\mathfrak{M}(B_j)=\sup_{i\in I_j}\mathfrak{M}(A_j^i)$ for any $j\in J$, we obtain that $\mathfrak{M}(B)\leq\sup_{j\in J}\mathfrak{M}(B_j)$, which tells us that $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.

Finally, assume that $(X_t,\mathcal{R}_t)$ is a compact topological space for any $t\in T$. Let us denote by $\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T)$ the product topology induced on X_T by the family of projections $(\mathbf{pr}_{T,t}\mid t\in T)$. Using the Tychonov theorem [14], it follows that $(X_T,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T))$ is a compact topological space. Consider an element B of $\mathcal{C}_T$ and a family $(B_j\mid j\in J)$ of elements of $\mathcal{C}_T$ that covers B, i.e. $B\subseteq\bigcup_{j\in J}B_j$. Then, there exists a nonempty, finite subset S of T, such that $B=\mathbf{pr}_{T,S}^{-1}(E)$, where E belongs to $\mathcal{R}_S$. Furthermore, since $\mathcal{R}_S$ is closed under complementation, E is a closed set in the topological space $(X_S,\mathcal{R}_S)$. Since by Proposition 4, $\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in S)=\mathcal{R}_S$ and since $\mathbf{pr}_{T,S}$ is a continuous mapping from $(X_T,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T))$ onto $(X_S,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in S))$, we find that B is a closed set in the compact topological space $(X_T,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T))$, which implies that B is a compact set in $(X_T,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T))$. By Proposition 3, $\mathcal{C}_T\subseteq\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T)$. Therefore $(B_j\mid j\in J)$ forms an open cover of B in $(X_T,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T))$. Since B is a compact set in $(X_T,\mathcal{W}((X_t,\mathcal{R}_t)\mid t\in T))$, there exists a finite subset J' of J, such that $B\subseteq\bigcup_{j\in J'}B_j$. Because $\mathfrak{M}$ is monotone and preserves finite suprema, it follows that $\mathfrak{M}(B)\leq\mathfrak{M}(\bigcup_{j\in J'}B_j)=\sup_{j\in J'}\mathfrak{M}(B_j)\leq\sup_{j\in J}\mathfrak{M}(B_j)$. Hence, $\mathfrak{M}$ is a P-consistent mapping on $\mathcal{C}_T$. □

So, we know that if the ample spaces $(X_t,\mathcal{R}_t)$, $t\in T$ are compact topological spaces, our set function $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$. Going from P-consistency to extendability is now but a small step.

Theorem 25. *Let $(X_t,\mathcal{R}_t)$ be compact for any $t\in T$, and assume that the family $(\pi_S\mid\emptyset\subset S\Subset T)$ is consistent. In general, there exist a complete lattice $(L',\leq')$, a supremum preserving order-embedding ϕ from $(L,\leq)$ to $(L',\leq')$ and a $(L',\leq')$-possibility measure Π' on $(X_T,\mathcal{R}_T)$, such that $\Pi'|_{\mathcal{C}_T}=\phi\circ\mathfrak{M}$.*

If the mapping $\mathfrak{M}$ satisfies at least one of the conditions (E_1), (E_2), (E_3) or (E_4), then there exists a $(L,\leq)$-possibility measure Π on $(X_T,\mathcal{R}_T)$ – and therefore also on $(X_T,\wp(X_T))$ –, such that $\Pi|_{\mathcal{C}_T}=\mathfrak{M}$. The greatest such possibility measure $\Pi^g_{\mathfrak{M}}$ has distribution $\pi^g_{\mathfrak{M}}$, determined by

$$\pi^g_{\mathfrak{M}}(x)=\inf_{\emptyset\subset S\Subset T}\pi_S(\mathbf{pr}_{T,S}(x)),\quad x\in X_T.$$

Proof. By Proposition 24.5 $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$. By Theorem 9, there exists a complete lattice $(L',\leq')$, in which $(L,\leq)$ can be embedded using a supremum preserving mapping ϕ, in such a way that $\phi\circ\mathfrak{M}$ is a P-consistent $\mathcal{C}_T-L'$-mapping, which is extendable to a $(L',\leq')$-possibility measure Π' on $(X_T,\mathcal{R}_T)$. The remaining part of the theorem follows from Theorem 7, and the discussion in Section 4. In particular, we find for the distribution $\pi^g_{\mathfrak{M}}$ of the greatest possibilistic extension $\Pi^g_{\mathfrak{M}}$ of $\mathfrak{M}$ that, for any x in X_T:

$$\pi^g_{\mathfrak{M}}(x)=\inf_{A\in\mathcal{C}_T,x\in A}\mathfrak{M}(A)=\inf_{\emptyset\subset S\Subset T}\inf_{B\in\mathcal{R}_S,x\in\mathbf{pr}^{-1}_{T,S}(B)}\Pi_S(B)=\inf_{\emptyset\subset S\Subset T}\pi_S(\mathbf{pr}_{T,S}(x)),$$

where the last equality holds because $\pi^g_{\Pi_S}=\pi_S$. □

It goes without saying that the compactness of the topological spaces $(X_t,\mathcal{R}_t)$ is a very strong requirement. Indeed, $(X_t,\mathcal{R}_t)$ is compact iff X_t has a finite number of atoms. The above theorem therefore states that there is possibilistic extendability for $\mathfrak{M}$ if all the sets X_t, $t\in T$ are essentially finite. This result is not strong enough for many practical purposes. We shall therefore prove a stronger result for the case that the $(X_t,\mathcal{R}_t)$, $t\in T$, are not necessarily compact, by imposing a number of additional conditions on the distributions $(\pi_S\mid\emptyset\subset S\Subset T)$. This is precisely where regularity comes in. Because compactness is essential when we have to prove that $\mathfrak{M}$ is P-consistent, we still require that the X_t should be compact, but with respect to a topology $\mathfrak{T}_t$ which is not necessarily the topology $\mathcal{R}_t$, $t\in T$. Both classes of topologies will then be implicitly linked by imposing regularity conditions on the possibility measures $(\Pi_S\mid\emptyset\subset S\Subset T)$.

Proposition 26. *Let $\mathfrak{T}_t$ be a topology on X_t such that the topological space $(X_t,\mathfrak{T}_t)$ is compact, $t\in T$. Let $(L,\leq)$ be a complete chain or the product of a nonempty collection of complete chains provided with the product order. Let the family of distributions $(\pi_S\mid\emptyset\subset S\Subset T)$ be consistent, and assume that for any finite, nonempty subset S of T the corresponding $(L,\leq)$-possibility measure Π_S is regular with respect to the product topology $\mathcal{W}((X_t,\mathfrak{T}_t)\mid t\in S)$ in the atoms of $\mathcal{R}_S$. Then $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$.*

Proof. Let us first deal with the case that $(L,\leq)$ is a complete chain. By Proposition 24.4 it is sufficient to prove that $\mathfrak{M}$ is P-consistent on $\mathcal{A}_T$. Consider a B in $\mathcal{A}_T$ that is covered by a family $(B_i\mid i\in I)$ of elements of $\mathcal{A}_T$, i.e. $B\subseteq\bigcup_{i\in I}B_i$. Assume that $\mathfrak{M}(B)>0_L$. There exists a nonempty, finite subset S of T and an atom A of $\mathcal{R}_S$ such that $B=\mathbf{pr}^{-1}_{T,S}(A)$. It follows that $\Pi_S(A)=\mathfrak{M}(B)>0_L$. Since Π_S is inner regular with respect to $\mathcal{W}((X_t,\mathfrak{T}_t)\mid t\in S)$ in A, it follows from Proposition 13 that A is compact in $(X_S,\mathcal{W}((X_t,\mathfrak{T}_t)\mid t\in S))$. We know from Proposition 1.3 that $A=\times_{t\in S}A_t$, where A_t is an atom of $\mathcal{R}_t$, $t\in S$. Since the projection mapping $\mathbf{pr}_{S,t}$ is by construction continous with respect to the topologies $\mathcal{W}((X_t,\mathfrak{T}_t)\mid t\in S)$ on X_S and $\mathfrak{T}_t$ on X_t, it follows that A_t is a compact set in $(X_t,\mathfrak{T}_t)$, $t\in T$. From $B=\mathbf{pr}^{-1}_{T,S}(A)$ it also follows that B is essentially a Cartesian product of the sets A_t, $t\in S$ and X_t, $t\in T\setminus S$.

Also taking into account that $(X_t, \mathfrak{T}_t)$ is compact for any $t \in T \setminus S$, it follows from the Tychonov theorem [14] that B is compact in $(X_T, \mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T))$.

Assume *ex absurdo* that $\sup_{i\in I} \mathfrak{M}(B_i) < \mathfrak{M}(B)$. Let $i \in I$. There exists a nonempty, finite subset S_i of T and an atom A_i of $\mathcal{R}_{S_i}$ such that $B_i = \mathbf{pr}_{T,S_i}^{-1}(A_i)$. Note that $\Pi_{S_i}(A_i) = \mathfrak{M}(B_i) < \mathfrak{M}(B)$. Since Π_{S_i} is outer regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S_i)$ in A_i, there exists a $O_i \in \mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S_i) \cap \mathcal{R}_{S_i}$ such that $A_i \subseteq O_i$ and $\Pi_{S_i}(A_i) \leq \Pi_{S_i}(O_i) < \mathfrak{M}(B)$. Note that $\tilde{O}_i = \mathbf{pr}_{T,S_i}^{-1}(O_i) \in \mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$ and $B_i \subseteq \tilde{O}_i$, whence $B \subseteq \bigcup_{i\in I} \tilde{O}_i$. Using the compactness of B, there is a nonempty, finite subset I' of I such that $B \subseteq \bigcup_{i\in I'} \tilde{O}_i$. According to Proposition 24, $\mathfrak{M}$ is monotone and preserves finite suprema, whence

$$\mathfrak{M}(B) \leq \sup_{i\in I'} \mathfrak{M}(\tilde{O}_i) = \sup_{i\in I'} \Pi_{S_i}(O_i) < \mathfrak{M}(B),$$

since I' is finite, a contradiction. We may therefore conclude that $\mathfrak{M}(B) \leq \sup_{i\in I} \mathfrak{M}(B_i)$, or, in other words, that $\mathfrak{M}$ is P-consistent on $\mathcal{A}_T$, and therefore also on $\mathcal{C}_T$.

We continue with the case $(L, \leq) = (\times_{j\in J} L_j, \times_{j\in J} \leq_j)$, where, for any j of the nonempty set J, $(L_j, \leq_j)$ is a complete chain. Let $j \in J$. For any $\emptyset \subset S \Subset T$, $\mathbf{pr}_j \circ \Pi_S$ is a $(L_j, \leq_j)$-possibility measure on $(X_S, \mathcal{R}_S)$ with distribution $\mathbf{pr}_j \circ \pi_S$, which is according to Proposition 11, regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S)$ in any atom of $\mathcal{R}_S$. Furthermore, the family $(\mathbf{pr}_j \circ \pi_S \mid \emptyset \subset S \Subset T)$ is consistent. This enables us to define a set mapping $\mathfrak{M}_j$ from $\mathcal{C}_T$ to L_j as follows. Let $B \in \mathcal{C}_T$, then $\mathfrak{M}_j(B) = \mathbf{pr}_j \circ \Pi_S(E)$, where $\emptyset \subset S \Subset T$ and $E \in \mathcal{R}_S$ such that $B = \mathbf{pr}_{T,S}^{-1}(E)$. Furthermore, $\mathfrak{M}_j = \mathbf{pr}_j \circ \mathfrak{M}$, and using the result above, we have that $\mathbf{pr}_j \circ \mathfrak{M}$ is P-consistent on $\mathcal{C}_T$. Since this holds for any $j \in J$, it follows from Proposition 8 that $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$. □

Note that if $\mathfrak{T}_t = \mathcal{R}_t$ for any $t \in T$, we recover the result mentioned in Proposition 24.5 for the case that $(L, \leq)$ is the product of a nonempty collection of complete chains provided with the product order, by also taking into account Proposition 12.

Consider a nonempty, finite subset S of T. Since Π_S is a possibility measure on $(X_S, \mathcal{R}_S)$, it is always extendable to a possibility measure on $(X_S, \wp(X_S))$. The greatest such possibilistic extension $\Pi_{\Pi_S}^g$ has distribution $\pi_{\Pi_S}^g = \pi_S$. Proposition 21 leads us to suspect that outer regularity of the extension $\Pi_{\Pi_S}^g$ with respect to the corresponding product topology $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S)$ in the singletons of the Cartesian product X_S, $\emptyset \subset S \Subset T$, is sufficient for the P-consistency of $\mathfrak{M}$. This leads to the following even stronger result.

Theorem 27. *Let $\mathfrak{T}_t$ be a topology on X_t such that the topological space $(X_t, \mathfrak{T}_t)$ is compact, $t \in T$. Let $(L, \leq)$ be the product of a nonempty collection of complete chains provided with the product order, or let $(L, \leq)$ be a complete chain. Let the family of distributions $(\pi_S \mid \emptyset \subset S \Subset T)$ be consistent. Assume that the greatest possibilistic extension $\Pi_{\Pi_S}^g$ of Π_S to $\wp(X_S)$ is outer regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S)$ in the atoms of $\wp(X_S)$, $\emptyset \subset S \Subset T$. Then $\mathfrak{M}$ is extendable to a $(L, \leq)$-possibility measure on $(X_T, \mathcal{R}_T)$ – and therefore also on $(X_T, \wp(X_T))$. The greatest possibilistic extension $\Pi_{\mathfrak{M}}^g$ of $\mathfrak{M}$ has distribution $\pi_{\mathfrak{M}}^g$ is given by*

$$\pi_{\mathfrak{M}}^g(x) = \inf_{\emptyset\subset S\Subset T} \pi_S(\mathbf{pr}_{T,S}(x)), \quad x \in X_T.$$

$\Pi_{\mathfrak{M}}^g$ is outer regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$ in the atoms of $\wp(X_T)$ and is inner regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$.

Proof. Let $\emptyset \subset S \Subset T$. It follows from the assumptions and Proposition 14 that $\Pi_{\Pi_S}^g$ is regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S)$ in the singletons of X_S. Since the distribution $\pi_{\Pi_S}^g$ of $\Pi_{\Pi_S}^g$ is equal to π_S, it follows that $(\pi_{\Pi_S}^g \mid \emptyset \subset S \Subset T)$ is consistent. Furthermore, let

$\tilde{\mathcal{C}}_T = \{\mathbf{pr}_{T,S}^{-1}(E) \mid \emptyset \subset S \Subset T \text{ and } E \in \wp(X_S)\}$, and let $\tilde{\mathfrak{M}}$ be the $\tilde{\mathcal{C}}_T - L$-mapping, given for any $B \in \tilde{\mathcal{C}}_T$ by $\tilde{\mathfrak{M}}(B) = \Pi^g_{\Pi_S}(E)$, where $\emptyset \subset S \Subset T$ and $E \in \wp(X_S)$ such that $B = \mathbf{pr}_{T,S}^{-1}(E)$. By Proposition 26, $\tilde{\mathfrak{M}}$ is P-consistent on $\tilde{\mathcal{C}}_T$. Since $\mathcal{C}_T \subseteq \tilde{\mathcal{C}}_T$ and $\tilde{\mathfrak{M}}|_{\mathcal{C}_T} = \mathfrak{M}$, it follows that $\mathfrak{M}$ is P-consistent on $\mathcal{C}_T$. Since $\mathfrak{M}$ satisfies condition (E_4), it follows that $\mathfrak{M}$ is extendable to a $(L, \leq)$-possibility measure on $(X_T, \mathcal{R}_T)$. Using the results in Section 4, it indeed follows for the distribution $\pi^g_{\mathfrak{M}}$ of the greatest such possibility measure $\Pi^g_{\mathfrak{M}}$ that, for any x in X_T:

$$\pi^g_{\mathfrak{M}}(x) = \inf_{A \in \mathcal{C}_T, x \in A} \mathfrak{M}(A) = \inf_{\emptyset \subset S \Subset T} \inf_{B \in \mathcal{R}_S, x \in \mathbf{pr}_{T,S}^{-1}(B)} \Pi_S(B) = \inf_{\emptyset \subset S \Subset T} \pi_S(\mathbf{pr}_{T,S}(x)),$$

where the last equality holds because $\pi^g_{\Pi_S} = \pi_S$. For any $\emptyset \subset S \Subset T$, let $\widetilde{\Pi}_S$ be the $(L, \leq)$-possibility measure on $(X_T, \wp(X_T))$ with distribution $\pi_S \circ \mathbf{pr}_{T,S}$. It is easily verified that $\widetilde{\Pi}_S$ is outer regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$ in the atoms of $\wp(X_T)$. By Proposition 20 it follows that $\Pi^g_{\mathfrak{M}}$ is outer regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$ in the atoms of $\wp(X_T)$. Since $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T) \subseteq \wp(X_T)$, it follows from Corollary 14 that $\Pi^g_{\mathfrak{M}}$ is also inner regular with respect to the product topology $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$. □

We are now ready to prove the main result of this paper.

Theorem 28. *Let the family $(\pi_S \mid \emptyset \subset S \Subset T)$ be consistent. Consider the following conditions.*

(C_1) *$(X_t, \mathcal{R}_t)$ is compact for any $t \in T$.*

(C_2) *$\mathcal{C}_T$ is an ample field on X_T.*

(C_3) *$(L, \leq)$ is the product of a nonempty collection of complete chains provided with the product order, or $(L, \leq)$ is a complete chain. For any $t \in T$, $\mathfrak{T}_t$ is a topology on X_t such that X_t is compact, and for any $\emptyset \subset S \Subset T$, the greatest possibilistic extension $\Pi^g_{\Pi_S}$ is outer regular with respect to $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in S)$ in the atoms of $\wp(X_S)$.*

If condition (C_1) holds, then there exist a $(L', \leq')$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, where $(L', \leq')$ is a complete lattice in which $(L, \leq)$ is embedded using a supremum preserving mapping ϕ, and a family $(f_t \mid t \in T)$ of possibilistic variables in $((X_t, \mathcal{R}_t) \mid t \in T)$, with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, such that for any nonempty, finite subset S of T, $\pi_{f_S} = \phi \circ \pi_S$. If the mapping $\mathfrak{M}$ moreover satisfies at least one of the sufficient conditions for extendability (E_1), (E_2), (E_3) or (E_4), then one can take $(L, \leq)$ for $(L', \leq')$ and the identical transformation $\mathbf{1}_L$ of L for ϕ.

If (C_2) or (C_3) holds, then there exist a $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ and a family $(f_t \mid t \in T)$ of possibilistic variables in $((X_t, \mathcal{R}_t) \mid t \in T)$ with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, such that for any nonempty, finite subset S of T, $\pi_{f_S} = \pi_S$.

Proof. Let us first deal with the case that $(X_t, \mathcal{R}_t)$ is compact for all $t \in T$. Let $\Omega = X_T$ and $\mathcal{R}_\Omega = \mathcal{R}_T$. Then using Theorem 25, there exist a complete lattice $(L', \leq')$, a supremum preserving order-embedding ϕ from $(L, \leq)$ to $(L', \leq')$ and a $(L', \leq')$-possibility measure Π_Ω on $\mathcal{R}_\Omega$, such that for any nonempty, finite subset S of T and for any element $A \in \mathcal{R}_S$

$$\Pi_\Omega(\mathbf{pr}_{T,S}^{-1}(A)) = (\phi \circ \Pi)(\mathbf{pr}_{T,S}^{-1}(A)) = (\phi \circ \Pi_S)(A) \tag{3}$$

and if Π satisfies at least one of the sufficient conditions for extendability $(E_1), (E_2), (E_3), (E_4)$, then one can take $(L, \leq)$ for $(L', \leq')$ and the identical transformation $\mathbf{1}_L$ of L for ϕ.

For any $t \in T$, let $f_t = \mathbf{pr}_{T,t}$. Clearly, f_t is a possibilistic variable in $(X_t, \mathcal{R}_t)$ with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, $t \in T$. Furthermore, consider a nonempty, finite subset S of T. By Proposition 1, it follows that $f_S = \times_{t \in S} f_t = \mathbf{pr}_{T,S}$, and that f_S is a possibilistic variable in $(X_S, \mathcal{R}_S)$ with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$. Consider an element $A \in \mathcal{R}_S$. It follows from (3) that $\Pi_{f_S}(A) = (\phi \circ \Pi_S)(A)$. Hence, we obtain that $\pi_{f_S} = \phi \circ \pi_S$.

The proof of the remaining part is similar, now using Proposition 24.3 or Theorem 27 instead of Theorem 25. □

From the previous theorem we can immediately derive a possibilistic Daniell-Kolmogorov theorem [10].

Corollary 29. *Assume that for any $t \in T$, $(X_t, \mathcal{R}_t)$ coincides with a given ample space $(X, \mathcal{R})$. Assume that the family $(\pi_S \mid \emptyset \subset S \Subset T)$ is consistent.*

If condition (C_1) holds, then there exist a $(L', \leq')$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, where $(L', \leq')$ is a complete lattice in which $(L, \leq)$ is embedded using a supremum preserving mapping ϕ, and a possibilistic process $(f_t \mid t \in T)$ in $(X, \mathcal{R})$ with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, such that $\pi_{f_S} = \phi \circ \pi_S$ for any nonempty, finite subset S of T. If the mapping $\mathfrak{M}$ satisfies at least one of the sufficient conditions for extendability (E_1), (E_2), (E_3) or (E_4), then one can take $(L, \leq)$ for $(L', \leq')$ and the identical transformation $\mathbf{1}_L$ of L for ϕ.

If (C_2) or (C_3) holds, then there exist a $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ and a possibilistic process $(f_t \mid t \in T)$ in $(X, \mathcal{R})$ with basic space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$, such that $\pi_{f_S} = \pi_S$ for any nonempty, finite subset S of T.

It should also be noted that condition (C_2) holds if the index set T is finite.

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