CONFIDENCE RELATIONS: AN ORDER-THEORETIC INVESTIGATION

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Abstract. We present a formal study of a special type of relations, which are the carriers of ordinal information. We begin this discussion with the definition of confidence relations on a set of events. Then, the set of the confidence relations defined on an event set is provided with a natural partial order relation. A thorough investigation of the structure of the partially ordered set thus formed leads to a number of interesting conclusions. We show that this structure is an algebraic semilattice, with no greatest element, but containing a set of mutually incomparable maximal elements, which can be interpreted as maxima of ordinal information. Finally, we introduce and study the duality of confidence relations, a notion of central importance in this discussion of ordinal information, which is furthermore intricately linked with absolute certainty.

1. Confidence Relations on an Event Set

In order to study uncertainty, we use the following paradigm. We consider an experiment E, the outcome of which assumes values in a universe X. We also consider a set $\mathcal{V}$ of 'measurable' subsets of X, called *events*. $\mathcal{V}$ is at least a *field of subsets* of X. When the experiment is carried out, an event A in $\mathcal{V}$ is said to *occur* iff it contains the outcome of the experiment E. Of course, the event X must always occur, and is therefore called *certain*; the event $\emptyset$ cannot occur, and is therefore called *impossible*.

Let us now consider the situation where there exists uncertainty about the outcome of the experiment E. In our model, this means that it is not possible for every event A in $\mathcal{V}$ to predict, on the basis of the available information about the experiment E, whether A will occur or not. In many cases, it will nevertheless be possible to express a *relative confidence* in the occurrence of two events. That is, we can define a binary relation R on $\mathcal{V}$ with the following interpretation: ARB iff on the basis of the available information we have at most as much confidence in the occurrence of the event A as in that of the event B. The relation R is a *large-preference relation* [4] that is an *ordinal representation* of the information we have about the outcome of the experiment E. In the following definition, we use the minimal requirements we feel such a preference relation must satisfy, to introduce the notion of a *confidence relation*. In the rest of this paper, we present an order-theoretic study of these important relations, which can be interpreted as the *carriers of ordinal information*.

Definition 1 *Let $\mathcal{V}$ be a field on X. We shall call a binary relation R on $\mathcal{V}$ a confidence relation on $\mathcal{V}$ iff*

(i) R is transitive;

(ii) $(\forall(A,B)\in\mathcal{V}^2)(A\subseteq B \Rightarrow ARB)$;

(iii) $(X,\emptyset)\notin R$.

The couple $(\mathcal{V},R)$ is called a confidence structure. The set of the confidence relations on $\mathcal{V}$ is denoted by $\mathfrak{T}(\mathcal{V})$.

Of course, since a confidence relation R on $\mathcal{V}$ is reflexive and transitive, it is a special partial preorder relation on $\mathcal{V}$. It should be noted that point (iii) of this definition is necessary to make the notion 'confidence relation' meaningful. Indeed, if a relation S on $\mathcal{V}$ satisfies (i) and (ii) but not (iii), it is easily verified that $S=\mathcal{V}^2$. The relation $\mathcal{V}^2$ cannot meaningfully express a relative confidence in the occurrence of elements of $\mathcal{V}$: it is intuitively unacceptable that we should have (at least) as much confidence in an impossible event (such as $\emptyset$) as in a certain one (such as X). Finally, it should be noted that we do *not* demand that a confidence relation R should be *complete*, i.e., that any two events should always be comparable w.r.t. each other. This is done to keep this discussion as general as possible. Of course, if necessary, the completeness of confidence relations can always be imposed afterwards as an additional requirement.

We use the term 'confidence' as an abstract designation for a number of notions that are to a certain extent related to one another, in the sense that the essence of their ordinal characterization is embodied in points (i)–(iii) of the definition above. 'Confidence in' can for instance be replaced by 'probability of', 'possibility of', 'necessity of', 'credibility of', 'plausibility of', etc. These are of course different notions, but they share an ordinal foundation which is reflected in definition 1. Postulating the minimal requirements in this definition in fact amounts to extracting three important common features from the different types of confidence. These common characteristics constitute the starting point of the subsequent abstract study, which will give us valuable insight into the notion of confidence, or, to use a slightly different terminology, uncertainty and information.

At the same time, it is important to note that the relations described by definition 1 are ordinal reflections of the information which is present about the experiment taken into consideration. A change in this information will therefore be reflected in a change in R. This also means that different kinds of information will give rise to relations having different *additional* properties, but still sharing the basic properties expressed in the definition above. To put it differently, if we go from the abstract term 'confidence' towards its more concrete instances, such as probability, possibility, etc., we shall have to impose additional requirements on our relations. But for the present, in this paper, we want to find out just how far we can go by using only definition 1.

To conclude this section, we want to mention that confidence relations can more generally be defined on arbitrary Boolean lattices, and not just on fields of sets. Of course, due to the fact that Boolean lattices and fields of sets are isomorphic notions, the results given in this paper for confidence relations defined on fields of sets are also valid for confidence relations defined on Boolean lattices.

2. An Order-Theoretic Investigation

We now give an order-theoretic investigation of the set of confidence relations defined on an event set, provided with a natural partial order relation. This abstract investigation will give us a deeper insight into the notion of ordinal information. For more information about the order-theoretic notions and techniques used below, we refer to [1].

Definition 2 *In order to distinguish between the inclusion relation on $\mathcal{V}$ and the inclusion relation on $\mathfrak{T}(\mathcal{V})$, we shall also denote the former by R_m, i.e.,*

$$(\forall(A,B)\in\mathcal{V}^2)(AR_mB \Leftrightarrow A\subseteq B).$$

Corollary 1 *R_m is a confidence relation on $\mathcal{V}$, i.e., $R_m\in\mathfrak{T}(\mathcal{V})$. This also implies that $\mathfrak{T}(\mathcal{V})\neq\emptyset$.*

Definition 3 *Consider two arbitrary confidence relations R_1 and R_2 on $\mathfrak{T}(\mathcal{V})$. When $R_1\subseteq R_2$, we shall say that R_2 contains at least as much ordinal information as R_1.*

Let us now take a closer look at the structure $(\mathfrak{T}(\mathcal{V}),\subseteq)$. It is easily verified that the intersection of an arbitrary non-empty family of confidence relations on $\mathfrak{T}(\mathcal{V})$ is also a confidence relation on $\mathfrak{T}(\mathcal{V})$. This leads to the following result.

Proposition 1 *$(\mathfrak{T}(\mathcal{V}),\subseteq)$ is a complete meet-semilattice, with smallest element R_m.*

Let us now try and find out whether $(\mathfrak{T}(\mathcal{V}),\subseteq)$ is a complete lattice. In order to do so, we first look for the maximal elements of $(\mathfrak{T}(\mathcal{V}),\subseteq)$.

Definition 4 *Let $\mathcal{B}$ be an arbitrary subset of $\mathcal{V}$. We introduce the following notation:*

$$R_{\mathcal{B}}\stackrel{\text{def}}{=}(\mathcal{V}\setminus\mathcal{B}\times\mathcal{V})\cup(\mathcal{B}\times\mathcal{B}).$$

Theorem 1 *R is a maximal element of $(\mathfrak{T}(\mathcal{V}),\subseteq)$ if and only if there exists a proper up-set $\mathcal{B}$ of $(\mathcal{V},\subseteq)$ such that $R=R_{\mathcal{B}}$.*

Proposition 2 *R is a maximal element of $(\mathfrak{T}(\mathcal{V}),\subseteq)$ if and only if there exists a proper down-set $\mathcal{B}$ of $(\mathcal{V},\subseteq)$ such that $R=R_{\mathcal{V}\setminus\mathcal{B}}$.*

Theorem 2 *(i) When $\mathcal{V}=\{\emptyset,X\}$, $\mathfrak{T}(\mathcal{V})$ has only one element R_m, which is at the same time the greatest element of $(\mathfrak{T}(\mathcal{V}),\subseteq)$.*

(ii) When $\{\emptyset,X\}\subset\mathcal{V}$, $(\mathfrak{T}(\mathcal{V}),\subseteq)$ has no greatest element.

We are led to the important conclusion that in all cases of interest, the complete meet-semilattice $(\mathfrak{T}(\mathcal{V}),\subseteq)$ has *no greatest element*, and is therefore *not a complete lattice*. This means that there is no absolute maximum of ordinal information, but only a set of mutually incomparable maximal elements (relative maxima of ordinal information). In [2], we have identified two important subclasses of these maximal elements. One of these is associated with classical possibility measures, the other with classical necessity measures. The maximal elements in their intersection describe absolute certainty.

Confidence relations are the mathematical condensation of relative confidence in the occurrence of events, associated with, say, an experiment E. Starting from the information available about this experiment, one practical way to find the associated confidence relation would be to actually carry out the pairwise comparison of all the events in $\mathcal{V}$. But, since $\mathcal{V}$ can be very large, and even infinite, this is not always feasible. If we want to obtain a confidence relation by this process of carrying out the pairwise comparison, we shall always be forced to limit ourselves to comparing only a *finite* number of events. This observation is the starting point for the following results.

Definition 5 *Let Y be an arbitrary finite subset of $\mathcal{V}^2$. We shall call Y confidence consistent iff*

$$\mathcal{D}(Y) \stackrel{\text{def}}{=} \{\, R \mid R \in \mathfrak{T}(\mathcal{V}) \text{ and } Y \subseteq R \,\} \neq \emptyset.$$

An element of $\mathcal{D}(Y)$ is called compatible with Y. The set of the finite, confidence consistent subsets of $\mathcal{V}^2$ will be denoted by $Con(\mathcal{V})$.

Definition 6 *Let Y be an arbitrary, finite, confidence consistent subset of $\mathcal{V}^2$. We shall call*

$$R(Y) \stackrel{\text{def}}{=} \bigcap \mathcal{D}(Y) = \bigcap \{\, R \mid R \in \mathfrak{T}(\mathcal{V}) \text{ and } Y \subseteq R \,\}$$

the confidence relation, generated by Y. Furthermore, we shall call a confidence relation R on $\mathcal{V}$ finitely generated iff there exists a finite, confidence consistent subset Y of $\mathcal{V}^2$, such that $R = R(Y)$.

Proposition 3 *Let Y, Y_1 and Y_2 be arbitrary, finite, confidence consistent subsets of $\mathcal{V}^2$. Then*

(i) $R(Y) \in \mathfrak{T}(\mathcal{V})$;

(ii) $Y \subseteq R(Y)$;

(iii) $R(Y)$ *is the transitive closure of* $R_m \cup Y$;

(iv) $Y_1 \subseteq Y_2 \Rightarrow R(Y_1) \subseteq R(Y_2)$.

We have argued that finitely generated confidence relations are interesting from a practical point of view. Let us now show that they are also important theoretically. In order to do this, we must first say something about *directed sets of confidence relations*, and about *finite confidence relations*. These notions are special instances of the directed sets and finite elements, defined in the theory of algebraic semilattices and complete partially ordered sets [1].

We have shown that in all cases of interest, the complete meet-semilattice $(\mathfrak{T}(\mathcal{V}), \subseteq)$ is not a complete lattice. If we consider an arbitrary nonempty subset $\mathcal{A}$ of $\mathfrak{T}(\mathcal{V})$, this means that the supremum $\sup \mathcal{A}$ of $\mathcal{A}$ in $(\mathfrak{T}(\mathcal{V}), \subseteq)$ does not necessarily exist. But, if it does exist, it is obvious that,

$$\sup \mathcal{A} = \bigcap \{\, R \mid R \in \mathfrak{T}(\mathcal{V}) \text{ and } \bigcup \mathcal{A} \subseteq R \,\},$$

since infimum coincides with intersection in the structure $(\mathfrak{T}(\mathcal{V}), \subseteq)$ and since supremum is the smallest upper bound. Thus, if $\bigcup \mathcal{A}$ is a confidence relation on $\mathcal{V}$, we find that $\sup \mathcal{A}$ exists, and that furthermore $\sup \mathcal{A} = \bigcup \mathcal{A}$. In the following definition, we introduce special subsets of $\mathfrak{T}(\mathcal{V})$, with the interesting property that their union is a confidence relation.

Definition 7 *We shall call a nonempty subset $\mathcal{D}$ of $\mathfrak{T}(\mathcal{V})$ directed iff for every finite subset $\mathcal{E}$ of $\mathcal{D}$ there exists a confidence relation $D \in \mathcal{D}$, such that $D \in \mathcal{E}^u$, i.e., $(\forall S \in \mathcal{E})(S \subseteq D)$.*

Proposition 4 *Let $\mathcal{D}$ be a directed subset of $\mathfrak{T}(\mathcal{V})$. Then $\bigcup \mathcal{D}$ is a confidence relation on $\mathcal{V}$.*

Theorem 3 *$(\mathfrak{T}(\mathcal{V}), \subseteq)$ is an algebraic intersection structure, and $\sup \mathcal{D} = \bigcup \mathcal{D}$ for an arbitrary directed subset $\mathcal{D}$ of $\mathfrak{T}(\mathcal{V})$, where, of course, sup is the supremum in the complete meet-semilattice $(\mathfrak{T}(\mathcal{V}), \subseteq)$.*

Corollary 2 *$(\mathfrak{T}(\mathcal{V}), \subseteq)$ is a CPO (complete partially ordered set), i.e., a partially ordered set satisfying in particular:*

(i) $(\mathfrak{T}(\mathcal{V}), \subseteq)$ *has a smallest element, namely* R_m;

(ii) any directed subset $\mathcal{D}$ of $(\mathfrak{T}(\mathcal{V}), \subseteq)$ has a supremum in $(\mathfrak{T}(\mathcal{V}), \subseteq)$.

Since $(\mathfrak{T}(\mathcal{V}), \subseteq)$ is a CPO, we can apply the definition of a *finite element* in such a complete partially ordered set.

Definition 8 *A confidence relation R on $\mathcal{V}$ is called finite iff for every directed subset $\mathcal{D}$ of $\mathfrak{T}(\mathcal{V})$:*

$$R \subseteq \bigcup \mathcal{D} \Rightarrow (\exists D \in \mathcal{D})(R \subseteq D).$$

The following results tell us that this finiteness of confidence relations in a lattice-theoretic sense has a straightforward interpretation, and is very important from a lattice-theoretic point of view.

Lemma 1 *Let R be an arbitrary confidence relation on $\mathcal{V}$. Consider the subset $FGS(R)$ of $\mathfrak{T}(\mathcal{V})$, defined as follows. For arbitrary D in $\mathfrak{T}(\mathcal{V})$: $D \in FGS(R)$ iff D is finitely generated and $D \subseteq R$. Then $R = \bigcup FGS(R)$ and $FGS(R)$ is a directed subset of $\mathfrak{T}(\mathcal{V})$.*

Theorem 4 *Let R be an arbitrary confidence relation on $\mathfrak{T}(\mathcal{V})$. Then R is finite if and only if R is finitely generated.*

Theorem 5 *$(\mathfrak{T}(\mathcal{V}), \subseteq)$ is an algebraic semilattice, which amongst other things implies that every confidence relation R is the union of the finite (or finitely generated) confidence relations it includes:*

$$R = \bigcup \{\, R(Y) \mid Y \in Con(\mathcal{V}) \text{ and } Y \subseteq R \,\}.$$

3. Duality of Confidence Relations

In this section, we show that confidence relations come in dual pairs. The ordinal aspects of information, represented by our confidence relations, have two different, but mutually related sides. The origin of this duality, is the following *symmetry*: if an event A occurs, its *opposite event* co$A$ does not occur, and *vice versa*. In order to see what are the order-theoretic consequences of this symmetry, let us consider a confidence relation R on an event set $\mathcal{V}$, reflecting the ordinal aspects of a certain type of confidence in – or information about – the outcome of an experiment E. We also consider two arbitrary events A and B in $\mathcal{V}$. If ARB, this means that we have at most as much confidence in the occurrence of A as in that of B. Furthermore, the duality (or dual order-automorphism) co on $(\mathcal{V}, \subseteq)$ leads in a very natural and straightforward way to a duality in the occurrence or otherwise of the events in $\mathcal{V}$. It also provides us with the possibility of taking a first type of confidence and using it to derive a second, *dual* type. This is done by concentrating on the opposite events of the events related to each other by the confidence relation R: whenever we have at most as much confidence in the occurrence of A as in that of B, we shall say that we have at most as much *dual confidence* in the occurrence of co$B$ as in that of co$A$. This leads to the following definition.

Definition 9 *Let R be a confidence relation on $\mathcal{V}$. We shall call the binary relation R^D on $\mathcal{V}$, defined by*

$$(\forall (A, B) \in \mathcal{V}^2)(AR^D B \Leftrightarrow \mathrm{co}B R\, \mathrm{co}A),$$

the dual confidence relation of R on $\mathcal{V}$.

Proposition 5 *Let R be a confidence relation on $\mathcal{V}$. Then R^D is a confidence relation on $\mathcal{V}$ as well. Furthermore, R is the dual confidence relation of R^D, i.e., $(R^D)^D = R$.*

Definition 10 *A confidence relation R on $\mathcal{V}$ is called self-dual iff $R^D = R$.*

Let us now investigate which are the dual confidence relations of the *maximal elements* of $(\mathfrak{T}(\mathcal{V}), \subseteq)$.

Proposition 6 *Let $\mathcal{B}$ be an arbitrary proper upset of $(\mathcal{V}, \subseteq)$ and consider $\mathcal{B}' \stackrel{\text{def}}{=} \{\mathrm{co}A \mid A \in \mathcal{B}\}$. Then $(R_{\mathcal{B}})^D = R_{\mathcal{V}\setminus\mathcal{B}'}$, which implies that the dual confidence relation of a maximal confidence is again a maximal confidence relation. Furthermore, $R_{\mathcal{B}}$ is self-dual if and only if $\mathcal{B} = \mathcal{V} \setminus \mathcal{B}'$.*

4. Conclusion

Let us conclude this paper with a short discussion of the significance of confidence relations. First of all, since a confidence relation R is a partial preorder relation, it can be always completely characterized by a mapping v from $\mathcal{V}$ to a well-chosen partially ordered set $(Q, \leq)$: $(\forall (A, B) \in \mathcal{V}^2)(ARB \Leftrightarrow v(A) \leq v(B))$. The mapping v is isotonic; it is generalization of the *mesures de confiance* (confidence measures) introduced by Dubois and Prade [3]. In this sense, confidence relations provide the order-theoretic foundation for a measure-theoretic account of uncertainty. As is shown in [2], they can in particular be used to provide a foundation for and at the same time a generalization of the existing theory of Sugeno's fuzzy measures [5]. More importantly, we have shown [2] that they also lead to a generalization and justification of the notions 'possibility' and 'necessity', introduced by Zadeh [6], and Dubois and Prade [3] in order to represent the linguistic uncertainty associated with vagueness and imprecision. Finally, our discussion of dual confidence relations provides the basis for the introduction of dual confidence measures [2], and lets us understand why measures describing uncertainty frequently come in pairs: possibility and necessity, plausibility and credibility, etc. Interestingly, it turns out that an important subclass of the maximal confidence relations make up the ordinal basis for the so-called classical possibility measures (assuming only the values 0 and 1). Their duals can be associated with classical necessity measures. If such a maximal confidence relation is self-dual, the associated confidence measure is at the same time a possibility and a necessity measure, and therefore describes absolute certainty.

Confidence relations underlie every mathematical description of uncertainty (and information) that in some way is based on a comparison of confidence in the occurrence of events. The study of these relations is therefore important in its own right, and is bound to lead to new insights into the notion of uncertainty. As an example, we mention the link we have found to exist between absolute certainty (lack of uncertainty) and self-dual maximal confidence relations [2]. We have also shown that these relations provide an order-theoretic foundation to such notions as conditionality and independence. This work will be reported on elsewhere.

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