On the coherence of supremum preserving upper previsions

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Abstract

We study the relation between possibility measures and the theory of imprecise probabilities. It is shown that a possibility measure is a coherent upper probability iff it is normal. We also prove that a possibility measure is the restriction to events of the natural extension of a special kind of upper probability, defined on a class of nested sets. Next, we go from upper probabilities to upper previsions. We show that if a coherent upper prevision defined on the convex cone of all positive gambles is supremum preserving, then it must take the form of a Shilkret integral associated with a possibility measure. But at the same time, we show that a supremum preserving upper prevision is not necessarily coherent! This makes us look for alternative extensions of possibility measures that are not necessarily supremum preserving, through natural extension.

1 INTRODUCTION

Supremum preserving set functions have popped up in the literature under a number of different guises and names, and in a diversity of contexts. To give only a few examples, they are join-morphisms and therefore play an important part in order (or lattice) theory [1, 4]; in a modified form they appear in Shackle's logic of surprise [13]; they were studied in a measure-theoretic context by Shilkret [15]; they appear as limiting cases in Shafer's theory of belief functions [14] and they were associated with fuzzy sets by Zadeh [22]. Zadeh called them *possibility measures* because, in his view, they model graded possibility.

In this paper, we investigate how these supremum preserving set functions, or possibility measures, adopting Zadeh's nomenclature for the sake of continuity, fit into the theory of imprecise probabilities, as it was elegantly formulated by Walley [19]. To a certain extent, we also generalize results by Dubois and Prade [11] from a finitary towards a more general context.

2 PRELIMINARY DEFINITIONS AND RESULTS

In what follows, we consider a non-empty *universe of discourse* Ω. Note that Ω need not be finite. A real-valued mapping X on Ω will be called a *gamble* on Ω iff it is bounded, i.e., if $\sup X = \sup\{X(\omega) \mid \omega \in \Omega\}$ and $\inf X = \inf\{X(\omega) \mid \omega \in \Omega\}$ are finite real numbers. The set of the gambles on Ω is a linear space under pointwise addition and multiplication with real numbers, and will be denoted by $\mathcal{L}(\Omega)$. If X and Y are gambles on Ω, we write $X \leq Y$ iff $(\forall \omega \in \Omega)(X(\omega) \leq Y(\omega))$. Also, a *constant gamble* will be denoted by the unique value it assumes.

There is a special class of gambles which assume only values in $\{0, 1\}$. If X is such a mapping, then clearly it is the characteristic function (or indicator) of the subset $A = \{\omega \mid X(\omega) = 1\}$ of Ω, and it will also be denoted as χ_A. A subset of Ω will be called an *event*, and the set of events will be denoted by $\wp(\Omega)$. We shall often identify an event A with its characteristic function χ_A. It should be clear from the context whether A denotes an event (a set) or a gamble (an indicator).

With a gamble X on Ω we may associate a class of events, the *dual cut sets* of X at level x, defined as $D_x^X = \{\omega \mid X(\omega) \leq x\}$, $x \in \mathbb{R}$.

An *ample field* $\mathcal{R}$ on Ω [5, 20] is a class of subsets of Ω that is closed under arbitrary unions and complements. It is therefore also closed under arbitrary intersections. For any ω in Ω, the *atom* $[\omega]_\mathcal{R}$ of $\mathcal{R}$ containing ω is defined as $[\omega]_\mathcal{R} = \bigcap\{A \in \mathcal{R} \mid \omega \in A\}$. The atoms of $\mathcal{R}$ make up a partition of Ω. Interestingly, for any subset A of Ω we have that $A \in \mathcal{R}$ iff $A = \bigcup_{\omega \in A}[\omega]_\mathcal{R}$. An element of $\mathcal{R}$ is called $\mathcal{R}$*-measurable*. A gamble X

on Ω is called *$\mathcal{R}$-measurable* iff its dual cut sets are $\mathcal{R}$-measurable, or equivalently, iff it is constant on the atoms of $\mathcal{R}$.

A *possibility measure* Π on $(\Omega, \mathcal{R})$ is a complete join-morphism [4] between the complete lattices $(\mathcal{R}, \subseteq)$ and $([0,1], \leq)$. In other words, for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j \in J} A_j) = \sup_{j \in J} \Pi(A_j)$. Note that $\Pi(\emptyset) = 0$. Π is called *normal* iff $\Pi(\Omega) = 1$. A *distribution* for Π is a $\mathcal{R}$-measurable $\Omega - [0,1]$ mapping π for which for any A in $\mathcal{R}$: $\Pi(A) = \sup_{\omega \in A} \pi(\omega)$. Clearly, such a distribution is unique and completely determined by $\pi(\omega) = \Pi([\,\omega\,]_{\mathcal{R}})$, $\omega \in \Omega$. Conversely, a possibility measure is uniquely determined by its distribution. For more information about possibility measures, we refer to [5, 6, 7, 9, 10, 22].

We conclude this section with a number of notions from the theory of imprecise probabilities. An *upper prevision* $\overline{P}$ [19] can be formally defined as a real-valued function defined on a class of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. In order to identify its universe of discourse and domain, we also denote $\overline{P}$ as $(\Omega, \mathcal{K}, \overline{P})$. The *corresponding lower prevision* $\underline{P}$ is defined on $-\mathcal{K} = \{-X \mid X \in \mathcal{K}\}$ by $\underline{P}(X) = -\overline{P}(-X)$. When the domain of an upper (lower) prevision is a class of events – indicators –, it will also be called an upper (lower) *probability*.

A *linear prevision* P on $\mathcal{L}(\Omega)$ [19] is a positive linear functional with unit norm ($P(1) = 1$) on the linear space $\mathcal{L}(\Omega)$. Its restriction to $\wp(\Omega)$ is a finitely additive probability. We shall denote the set of all linear previsions on $\mathcal{L}(\Omega)$ by $\mathcal{P}(\Omega)$. Given an upper probability $(\Omega, \mathcal{K}, \overline{P})$, we define the set $\mathcal{M}(\overline{P})$ by

$$\mathcal{M}(\overline{P}) = \{P \in \mathcal{P}(\Omega) \mid (\forall X \in \mathcal{K})(P(X) \leq \overline{P}(X))\}.$$

Define, for any gamble X on Ω, $G(X) = X - \underline{P}(X)$, whence $G(-X) = \overline{P}(X) - X$. An upper prevision $(\Omega, \mathcal{K}, \overline{P})$ is said to *avoid sure loss* [19] iff for any X_1, ..., X_n in $\mathcal{K}$, $n \geq 1$,

$$\sup \sum_{k=1}^{n} G(-X_k) \geq 0.$$

Walley [19] has shown that $(\Omega, \mathcal{K}, \overline{P})$ avoids sure loss iff $\mathcal{M}(\overline{P}) \neq \emptyset$.

An upper prevision $(\Omega, \mathcal{K}, \overline{P})$ is said to be *coherent* [19] iff for any $m \geq 0$, $n \geq 0$ and X_o, X_1, ..., X_n in $\mathcal{K}$,

$$\sup \Big(\sum_{k=1}^{n} G(-X_k) - mG(-X_o) \Big) \geq 0.$$

It has been proven by Walley [19] that $(\Omega, \mathcal{K}, \overline{P})$ is coherent iff it is the upper envelope of $\mathcal{M}(\overline{P})$, i.e.,

$$(\forall X \in \mathcal{K})(\overline{P}(X) = \sup\{P(X) \mid P \in \mathcal{M}(\overline{P})\}).$$

If the upper prevision $(\Omega, \mathcal{K}, \overline{P})$ avoids sure loss, then its *natural extension* [19] $(\Omega, \mathcal{L}(\Omega), \overline{E})$ is the maximal coherent upper prevision on $\mathcal{L}(\Omega)$ that is dominated by $\overline{P}$ on $\mathcal{K}$. It is then given by, for any $X \in \mathcal{L}(\Omega)$,

$$\overline{E}(X) = \max\{P(X) \mid P \in \mathcal{M}(\overline{P})\}).$$

This discussion of upper and lower previsions, coherence and natural extension is necessarily very limited. For a detailed exposition of the theory of imprecise probabilities, we refer to Walley's book [19].

3 POSSIBILITY MEASURES AND FAMILIES OF NESTED SETS

Let us consider a possibility measure Π with distribution π, defined on an ample field $\mathcal{R}$ of subsets of Ω. We turn our attention to the family of subsets $(D_x^\pi \mid x \in [0,1])$ of Ω. For any x and y in $[0,1]$, if $x \leq y$ then $D_x^\pi \subseteq D_y^\pi$, which implies that the sets D_x^π are *nested*. Moreover, for any ω in Ω, we find that $\pi(\omega) = \inf\{x \mid \omega \in D_x^\pi\}$, and that $\omega \in D_{\pi(\omega)}^\pi$. Since π is $\mathcal{R}$-measurable, it is obvious that $D_x^\pi \in \mathcal{R}$, $x \in [0,1]$. We may therefore also consider the $[0,1]-[0,1]$-mapping κ, defined by $\kappa(x) = \Pi(D_x^\pi) = \sup_{\pi(\omega) \leq x} \pi(\omega)$, $x \in [0,1]$. Note that κ is increasing and that $\kappa(x) = \kappa(y)$ whenever $D_x^\pi = D_y^\pi$. Since $D_1^\pi = \Omega$ it also follows that $\kappa(1) = 1$ iff Π is a normal possibility measure.

So, given the possibility measure Π, we are in a very natural way able to define a family of nested subsets $(D_x^\pi \mid x \in [0,1])$ of Ω and a mapping κ specifying the values which the possibility measure Π assumes on those subsets. In what follows, we want to answer the opposite question: *given a nested family of subsets of Ω and a mapping specifying the 'possibility' of these subsets, is it possible to construct*[1] *a possibility measure that is compatible with this information?*

We are thus led to consider a mapping Γ from $[0,1]$ to $\wp(\Omega)$, increasing in the following sense:

$$(\forall (x,y) \in [0,1]^2)(x \leq y \Rightarrow \Gamma(x) \subseteq \Gamma(y)). \tag{1}$$

Using the so-called *multivalued mapping* Γ and drawing inspiration from the discussion above, we also define the $\Omega - [0,1]$-mapping α as follows:

$$\alpha(\omega) = \inf\{x \mid \omega \in \Gamma(x)\},$$

for any ω in Ω. We want to stress here that the mapping α need not be surjective. Our next assumption is a kind of continuity condition imposed on Γ, also inspired by the foregoing discussion:

$$(\forall \omega \in \Omega)(\omega \in \Gamma(\alpha(\omega))). \tag{2}$$

[1] This is a special case of a question treated by Wang [21] and in a more general context by Boyen et al. [2].

Interestingly, it is easily verified that conditions (1) and (2) also ensure that

$$\Gamma(1) = \Omega. \tag{3}$$

Besides Γ we also consider a mapping ξ from $[0,1]$ to $[0,1]$ satisfying the following isotonicity condition:

$$(\forall (x,y) \in [0,1]^2)(x \leq y \Rightarrow \xi(x) \leq \xi(y)). \tag{4}$$

Furthermore, ξ should be constant wherever Γ is:

$$(\forall (x,y) \in [0,1]^2)(\Gamma(x) = \Gamma(y) \Rightarrow \xi(x) = \xi(y)). \tag{5}$$

Note that at this point we do *not* require that $\xi(1) = 1$.

We may interpret ξ as an assignment of possibility to the nested sets $\Gamma(x)$, $x \in [0,1]$. The question we want to answer here is whether there exists a possibility measure Π, defined on some ample field $\mathcal{R}$ of subsets of Ω such that $(\forall x \in [0,1])(\Pi(\Gamma(x)) = \xi(x))$, or equivalently, $\Pi \circ \Gamma = \xi$.

The requirements (4) and (5) imposed on ξ are necessary conditions for the existence of such a possibility measure. The requirement $\Pi \circ \Gamma = \xi$ also presupposes that $\Gamma([0,1]) \subseteq \mathcal{R}$, with of course, $\Gamma([0,1]) = \{\Gamma(x) \mid x \in [0,1]\}$. Moreover, it follows from (3) that Π will be normal iff $\xi(1) = 1$. Also, suppose we can find a possibility measure fulfilling the above requirement, which is defined on an ample field $\mathcal{R}$ on Ω. Given any other ample field $\mathcal{R}'$ on Ω that includes $\mathcal{R}$, we can easily find a possibility measure Π' defined on $\mathcal{R}'$ and satisfying $\Pi' \circ \Gamma = \xi$ by extending Π in the familiar way. For any B in $\mathcal{R}'$:

$$\begin{aligned} \Pi'(B) &= \inf\{\Pi(A) \mid A \in \mathcal{R} \text{ and } B \subseteq A\} \\ &= \Pi(\bigcup_{\omega \in B} [\,\omega\,]_{\mathcal{R}}). \end{aligned}$$

Remark that Π' and Π have the same distribution!

It follows from these remarks that it is natural to try and solve the problem for the ample field

$$\mathcal{R}_\Gamma = \bigcap\{\mathcal{R} \mid \mathcal{R} \text{ ample field on } \Omega \text{ and } \Gamma([0,1]) \subseteq \mathcal{R}\},$$

i.e., the smallest ample field for which the sets $\Gamma(x)$, $x \in [0,1]$, are all measurable. In order to characterize the atoms of this ample field, it will be convenient to introduce a new class of subsets of Ω. For any x in $[0,1]$:

$$\Delta(x) = \Gamma(x) \setminus \bigcup_{y<x} \Gamma(y) \tag{6}$$

The following propositions tell us that there is a very interesting relation between the $\Omega - [0,1]$-mapping α and the sets $\Gamma(x)$ and $\Delta(x)$, $x \in [0,1]$. They ensure that for any x in $[0,1]$, $\Gamma(x) = D_x^\alpha$, or in other words, that the $\Gamma(x)$ are dual cut sets for the mapping α.

Proposition 1 *Let x and ω be arbitrary elements of the sets $[0,1]$ and Ω respectively. Then*

1. *$\omega \in \Gamma(x) \Leftrightarrow \alpha(\omega) \leq x$;*
2. *$\omega \in \Delta(x) \Leftrightarrow \alpha(\omega) = x$.*

Proposition 2 *The atoms of the ample field $\mathcal{R}_\Gamma$ are given by $[\,\omega\,]_{\mathcal{R}_\Gamma} = \Delta(\alpha(\omega))$, $\omega \in \Omega$.*

In the following theorem, we formulate a necessary and sufficient condition that ξ must satisfy besides (4) and (5) in order that the set mapping assuming the values $\xi(x)$ on $\Gamma(x)$, $x \in [0,1]$, would be extendable to a possibility measure on $(\Omega, \mathcal{R}_\Gamma)$.

Theorem 3 *Let Γ be a $[0,1] - \wp(\Omega)$-mapping satisfying (1) and (2). Let ξ be a $[0,1] - [0,1]$-mapping satisfying (4) and (5). Then there exists a possibility measure Π on $(\Omega, \mathcal{R}_\Gamma)$ that is a solution of $\Pi \circ \Gamma = \xi$ iff for any x in $[0,1]$:*

$$\xi(x) = \sup_{\alpha(\omega) \leq x} \xi(\alpha(\omega)). \tag{7}$$

In that case, the greatest such possibility measure has distribution $\xi \circ \alpha$. Moreover, any solution Π will be normal iff $\xi(1) = 1$.

It can be shown that conditions (1), (2), (4) and (5) are in general not sufficient for (7) to hold, unless Ω is a finite set. Interestingly, (7) is satisfied in the case $\Gamma(\cdot) = D^\pi_\cdot$ and $\xi = \kappa$ considered in the beginning of this section. It also holds if Γ and therefore also ξ assume only a finite number of different values.

4 POSSIBILITY MEASURES AS NATURAL EXTENSIONS

Let us now consider the upper probability $\overline{P}_\Gamma$ on the set of events $\Gamma([0,1])$, defined as follows:

$$(\forall x \in [0,1])(\overline{P}_\Gamma(\Gamma(x)) = \xi(x)).$$

In the previous section, we investigated under what conditions this upper probability can be extended to a possibility measure. Here, we determine the *natural extension* $\overline{E}_\Gamma$ of the upper probability $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ on the set of events $\wp(\Omega)$.

First of all, we want $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ to avoid sure loss[2].

Proposition 4 *The upper probability $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ avoids sure loss iff $\xi(1) = 1$.*

[2]If an upper probability does not avoid sure loss, then its natural extension assumes the value $-\infty$ everywhere [19].

It will therefore from now on be assumed that $\xi(1) = 1$.

Since $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ avoids sure loss, its natural extension to $\mathcal{L}(\Omega)$ (and therefore also the restriction to $\wp(\Omega)$) is a coherent upper prevision (probability). If A is any subset of $\mathcal{R}_\Gamma$, then we must determine [19]

$$\overline{E}_\Gamma(A) = \inf_{\substack{\lambda_k \geq 0; x_k \in [0,1] \\ k=1,\ldots,n; n \geq 0}} \sup Y(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n)$$

where

$$Y(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n) = A + \sum_{k=1}^{n} \lambda_k G(-\Gamma(x_k))$$

and for any x in $[0,1]$, $G(-\Gamma(x)) = \xi(x) - \Gamma(x)$.

Theorem 5 *Let* Γ *be a* $[0,1] - \wp(\Omega)$*-mapping satisfying (1) and (2). Let* ξ *be a* $[0,1] - [0,1]$*-mapping satisfying (4), (5) and* $\xi(1) = 1$. *Let* $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ *be the upper probability defined by* $\overline{P}_\Gamma(\Gamma(x)) = \xi(x)$, $x \in [0,1]$. *Let* $\overline{E}_\Gamma$ *be the natural extension of* $\overline{P}_\Gamma$. *Then* $\overline{E}_\Gamma(\emptyset) = 0$ *and for any non-empty subset* A *of* Ω,

$$\overline{E}_\Gamma(A) = \inf_{A \subseteq \Gamma(x)} \xi(x). \tag{8}$$

Theorem 6 *Let* Γ *be a* $[0,1] - \wp(\Omega)$*-mapping satisfying (1) and (2). Let* ξ *be a* $[0,1] - [0,1]$*-mapping satisfying (4), (5) and* $\xi(1) = 1$. *Let* $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ *be the upper probability defined by* $\overline{P}_\Gamma(\Gamma(x)) = \xi(x)$, $x \in [0,1]$. *Let* $\overline{E}_\Gamma$ *be the natural extension of* $\overline{P}_\Gamma$. *Then the upper probability* $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ *is coherent. Moreover the restriction of* $\overline{E}_\Gamma$ *to* $\wp(\Omega)$ *is a possibility measure on* $(\Omega, \wp(\Omega))$ *iff*

$$(\forall \emptyset \subset A \subseteq \Omega)(\sup \xi(\alpha(A)) = \xi(\sup \alpha(A))). \tag{9}$$

In that case, the distribution of this possibility measure is given by $\xi \circ \alpha$.

Since by Propositions 1 and 2 α is constant on the atoms of $\mathcal{R}_\Gamma$, (9) is equivalent to

$$(\forall A \in \mathcal{R}_\Gamma \setminus \{\emptyset\})(\sup \xi(\alpha(A)) = \xi(\sup \alpha(A))),$$

which also implies that the restriction of $\overline{E}_\Gamma$ to $\wp(\Omega)$ is a possibility measure on $(\Omega, \wp(\Omega))$ iff its restriction to $\mathcal{R}$ is a possibility measure on $(\Omega, \mathcal{R})$, where $\mathcal{R}$ is any ample field on Ω with $\mathcal{R}_\Gamma \subseteq \mathcal{R}$.

From the theorem above, we may also deduce the following.

Theorem 7 *Let* $\mathcal{R}$ *be an ample field on* Ω. *Consider an arbitrary possibility measure* Π *on* $(\Omega, \mathcal{R})$. *Then* $(\Omega, \mathcal{R}, \Pi)$ *is a coherent upper probability iff* Π *is normal.*

This result also implies that any possibility measure is an upper envelope of a class of additive probabilities.

Proposition 8 *Let* $\mathcal{R}$ *be an ample field on* Ω. *Consider an arbitrary normal possibility measure* Π *on* $(\Omega, \mathcal{R})$. *Consider the set*

$$\mathcal{M}(\Pi) = \{P \in \mathcal{P}(\Omega) \mid (\forall A \in \mathcal{R})(P(A) \leq \Pi(A))\}$$

Then Π *is the upper envelope of* $\mathcal{M}(\Pi)$, *i.e., for any* A *in* $\mathcal{R}$:

$$\Pi(A) = \max\{P(A) \mid P \in \mathcal{M}(\Pi)\}.$$

To conclude this section, let us consider the special case where Γ and therefore also ξ assume only a *finite number of different values*. We have already mentioned in the previous section that in this case the equation $\Pi \circ \Gamma = \xi$ always has a solution. Moreover, since $\alpha(\Omega)$ is now a finite set, condition (9) is trivially satisfied, which means that the restriction of the natural extension $\overline{E}_\Gamma$ to $\wp(\Omega)$ is indeed a possibility measure. Our discussion therefore includes and at the same time generalizes the finite version, treated by Dubois and Prade [11].

5 UPPER PREVISIONS ASSOCIATED WITH POSSIBILITY MEASURES

Let us again consider a normal possibility measure Π, defined on an ample field $\mathcal{R}$ of subsets of Ω. We denote its distribution by π. We know from Theorem 7 that $(\Omega, \mathcal{R}, \Pi)$ is a coherent upper probability. In this section, we first look for coherent extensions of Π to the set $\mathcal{K}(\mathcal{R})$ of the $\mathcal{R}$-measurable gambles on Ω.

First of all, note that for any A and B in $\mathcal{R}$, since Π is increasing and $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$,

$$\begin{aligned} \Pi(A \cap B) \quad + \quad & \Pi(A \cup B) \\ \leq \quad & \min(\Pi(A), \Pi(B)) + \max(\Pi(A), \Pi(B)) \\ = \quad & \Pi(A) + \Pi(B), \end{aligned}$$

which, together with $\Pi(\emptyset) = 0$, $\Pi(\Omega) = 1$ and $(\forall A \in \mathcal{R})(\Pi(A) \leq 1)$ implies that the upper probability $(\Omega, \mathcal{R}, \Pi)$ is *2-alternating*, or equivalently, a *Choquet capacity of order 2* [3, 18, 19][3]. Let us define the *lower distribution function* $\underline{F}_X : \mathbb{R} \to [0,1]$ of X under Π by, for any x in $\mathbb{R}$:

$$\begin{aligned} \underline{F}_X(x) &= 1 - \Pi(\mathrm{co}D_x^X) \\ &= 1 - \Pi(\{\omega \mid X(\omega) > x\}) \\ &= 1 - \sup_{X(\omega) > x} \pi(\omega). \end{aligned}$$

[3]This simple observation, together with the fact that 2-alternating upper probabilities are always coherent [12, 18], provides an alternative proof for Theorem 7.

Walley [18, 19] has shown that, since Π is 2-alternating, the natural extension $\overline{E}_\Pi$ of Π on $\mathcal{K}(\mathcal{R})$ is given by

$$\overline{E}_\Pi(X) = \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x) \tag{10}$$

for any X in $\mathcal{K}(\mathcal{R})$, where the integral is a Riemann-Stieltjes integral with integrator $\underline{F}_X$. Moreover, since $\mathcal{K}(\mathcal{R})$ is a linear space containing all the constant gambles, we deduce from [19] that for the natural extension $\overline{E}_\Pi$ of Π on the set $\mathcal{L}(\Omega)$ of all gambles on Ω:

$$\begin{aligned} \overline{E}_\Pi(X) &= \inf\{\overline{E}_\Pi(Y) \mid X \leq Y \text{ and } Y \in \mathcal{K}(\mathcal{R})\} \\ &= \overline{E}_\Pi(X^\#) \end{aligned}$$

for any gamble X on Ω, where, for any ω in Ω, $X^\#(\omega) = \sup_{\nu \in [\,\omega\,]_\mathcal{R}} X(\nu)$.

It can be verified that in general, the natural extension of a possibility measure Π on $(\Omega, \mathcal{R})$ as determined by (10) need not be supremum preserving (on the convex cone of all positive gambles), even though it coincides with Π on $\mathcal{R}$. Indeed, in the rest of this section, we shed more light on this problem. For a start, it is natural when considering supremum preservation to restrict ourselves to gambles which are uniformly non-negative.

Theorem 9 *Let $(\Omega, \mathcal{C}, \overline{P})$ be a coherent upper prevision, where $\mathcal{C} \subset \mathcal{L}(\Omega)$ is the convex cone of all positive gambles on Ω: $\mathcal{C} = \{X \in \mathcal{L}(\Omega) \mid X \geq 0\}$. Remark that $\mathcal{C}$ is closed under arbitrary suprema. Then for any X in $\mathcal{C}$:*

$$\overline{P}(X) \geq \sup_{\omega \in \Omega} X(\omega)\overline{P}(\{\omega\}).$$

Moreover, $\overline{P}$ is supremum preserving on $\mathcal{C}$ iff for any X in $\mathcal{C}$:

$$\overline{P}(X) = \sup_{\omega \in \Omega} X(\omega)\overline{P}(\{\omega\}) \tag{11}$$

This result, and in particular (11), tells us that if a coherent prevision $(\Omega, \mathcal{C}, \overline{P})$ is supremum preserving, it must take the form of a Shilkret[4] integral [15] associated with the possibility measure which is the restriction of $\overline{P}$ to $\wp(\Omega)$.

We are therefore led to study upper previsions $(\Omega, \mathcal{C}, \overline{P}_\pi)$ of the form:.

$$\overline{P}_\pi(X) = \sup_{\omega \in \Omega} X(\omega)\pi(\omega), \quad X \in \mathcal{C},$$

where π is any $\Omega - [0,1]$-mapping[5]. More in particular, we want to find out whether such upper previsions are necessarily coherent. Note that for any $A \subseteq \Omega$, $\overline{P}_\pi(A) = \sup_{\omega \in A} \pi(\omega)$, which means that the restriction of $\overline{P}_\pi$ to events is a possibility measure with distribution π, and is, by Theorem 7, coherent iff $\sup \pi = 1$. That $\sup \pi = 1$ (sup-normality of π) is therefore a necessary condition for the coherence of $(\Omega, \mathcal{C}, \overline{P}_\pi)$.

Generally it can be shown [19] that any upper prevision $(\Omega, \mathcal{C}, \overline{P})$ is coherent iff it satisfies the following three conditions, for arbitrary X and Y in $\mathcal{C}$, for arbitrary λ in $\mathbb{R}$ with $\lambda > 0$ and for arbitrary μ in $\mathbb{R}$:

$$\overline{P}(\lambda X) = \lambda \overline{P}(X) \tag{12}$$

$$\overline{P}(X + Y) \leq \overline{P}(X) + \overline{P}(Y) \tag{13}$$

$$X \geq Y + \mu \Rightarrow \overline{P}(X) \geq \overline{P}(Y) + \mu \tag{14}$$

It is easily verified that $\overline{P}_\pi$ satisfies (12) and (13). However, as the following counterexample shows, even if $\sup \pi = 1$, $\overline{P}_\pi$ is not necessarily coherent. Together with Theorem 9, it also tells us that the natural extension of a possibility measure is not necessarily supremum preserving on $\mathcal{C}$.

Example 10 Let $\Omega = [0,1]$, $\pi(\omega) = \omega$ and $Y(\omega) = 1 - \omega$, $\omega \in [0,1]$. Then clearly $\sup \pi = 1$ and $Y \geq 0$. For any real μ, consider $X = Y + \mu$. If $\mu \geq 0$ then $X \geq Y + \mu \geq 0$. Moreover,

$$\overline{P}_\pi(X) = \sup_{\omega \in [0,1]} \omega(1 - \omega + \mu) = \left(\frac{1+\mu}{2}\right)^2$$

whereas $\overline{P}_\pi(Y) + \mu = 1/4 + \mu$. Clearly, $\overline{P}_\pi(X) \geq \overline{P}_\pi(Y) + \mu$ iff $\mu \leq 0$ or $\mu \geq 2$. If we choose $\mu = 1$, we see that (14) is not satisfied. Note on the other hand that $\overline{E}_\Pi(Y) = 1/2$ and $\overline{E}_\Pi(Y + \mu) = 1/2 + \mu$.

There is a special case, however, in which $(\Omega, \mathcal{C}, \overline{P}_\pi)$ is always coherent, as the following proposition tells us.

Proposition 11 *Let π be any $\Omega - [0,1]$-mapping and let Π be the possibility measure on $(\Omega, \wp(\Omega))$ with distribution π. If π can only assume the values 0 and 1, and $\sup \pi = 1$ then $(\Omega, \mathcal{C}, \overline{P}_\pi)$ is coherent and coincides on $\mathcal{C}$ with the natural extension of $(\Omega, \wp(\Omega), \Pi)$.*

6 CONCLUSION

The results in this paper show that it is possible to incorporate possibility measures, or supremum preserving set functions, into the framework of imprecise probabilities. Particularly interesting is the fact that any possibility measure is a coherent upper probability measure iff it is normal; and that any possibility measure can be retrieved as the restriction to events of the natural extension of an upper probability defined on a class of nested sets, and *vice versa* (under some restrictions).

[4]Actually, in a different context Shilkret proved a somewhat stronger result, because he only used the preservation of countable suprema, and imposed conditions which are weaker than coherence.

[5]A coherent upper *probability* can only assume values in $[0,1]$ [19].

It is on the other hand surprising that the preservation of suprema does not carry over from events to (positive) gambles: supremum preserving upper previsions on $\mathcal{C}$ are not necessarily coherent, and natural extensions of possibility measures are not necessarily supremum preserving on $\mathcal{C}$. There therefore seems to be a potential incompatibility between the Shilkret integral [15] – which is, by the way, the only form of the fuzzy integral [8, 16, 17] which may lead to coherent extension – and the notion of coherence.

For a more detailed exposition, with the proofs of the results given here, and a discussion of necessity measures, we refer to a forthcoming paper.

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