Ample fields as a basis for possibilistic processes

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Abstract. Ample fields play an important role in possibility theory. These fields of subsets of a universe, which are additionally closed under arbitrary unions, act as the natural domains for possibility measures. A set provided with an ample field is then called an ample space. In this paper we generalise Wang's notions of product ample field and product ample space. We make a topological study of ample spaces and their products, and introduce ample subspaces, extensions and one-point extensions of ample spaces. In this way, a first step towards a mathematical theory of possibilistic processes is made.

Keywords: Product ample space, ample subspace, extension of ample space, one-point extension, possibility theory, measure theory, possibilistic processes.

1 Motivation

Possibility measures are usually defined as supremum preserving set functions on the power class of a non-empty set X [6, 7, 8, 11, 19]. In an effort to define such set functions on more general domains, a number of authors have investigated which set functions, defined on arbitrary classes of subsets, can be extended to possibility measures on the power class [2, 17]. From this study, ample fields emerged as natural domains for possibility measures.

Formally, a subclass $\mathcal{D}$ of the power class $\wp(X)$ of X is called a *plump class* on X iff it is closed under arbitrary unions and intersections [17]. The *atom* of $\mathcal{D}$ containing the element x of X is defined as $[x]_\mathcal{D} = \bigcap\{A \mid A \in \mathcal{D} \text{ and } x \in A\}$. The set of the atoms of $\mathcal{D}$ is denoted by $X_\mathcal{D}$. Clearly, $X_\mathcal{D} \subseteq \mathcal{D}$ and $(\forall x \in X)(x \in [x]_\mathcal{D})$. For any subset A of X, $A \in \mathcal{D} \Leftrightarrow A = \bigcup_{x\in A} [x]_\mathcal{D}$. An *ample field* $\mathcal{R}$ on X is a plump class on X that is closed under complementation [9, 16]. The couple $(X, \mathcal{R})$ is called an *ample space*. Interestingly, for an ample space, the set of atoms $X_\mathcal{R}$ is a partition of X. Ample fields are therefore direct generalisations of the power class, and in that generalisation, their atoms have the same role as the singletons in the power class.

Why ample fields are important in the study of possibility measures can be seen as follows. A possibility measure Π turns an arbitrary union of subsets into the corresponding supremum: $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$. It therefore seems natural to define possibility measures on domains that are closed under arbitrary unions. So, topologies would be candidates for such domains. But if we also want to study the dual necessity measures, which turn arbitrary intersections into the corresponding infima, we must restrict our class of suitable domains to the plump classes. Incidentally, in the above-mentioned work on the extension of set functions to possibility measures, such plump classes indeed emerge as natural candidates for the domains we are looking for. But, there is a third natural condition which makes us turn to ample fields instead; if we want to study the relation between possibility measures Π and necessity measures

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N, as expressed by the duality relation $\mathrm{N}(A) = 1 - \Pi(X \setminus A)$, it seems reasonable to also require our domains to be closed under complements.

Ample fields, then, emerge as natural domains for possibility measures. From the definition given above it is clear that ample spaces are also topological spaces. In fact they are a special kind of pseudo-metric spaces, of which we have collected of number of topological characteristics in Section 2. At this point we want to stress that the mentioned topological results are known but scattered in the literature. For the sake of clarity we have inserted a short motivation for each topological result. In order to set up a sufficient general model for studying possibilistic systems, i.e. systems for which the available information is expressed by possibility measures, we generalise in Section 3 Wang's definition of product ample field. Besides a second alternative approach to introducing this notion we derive some additional results that will be helpful when the information about a system is specified by possibility measures that are defined on finite product ample spaces. In order to deal with ample spaces that are non-compact, we address in Section 4 the issue of their one-point compactification. Furthermore we introduce the notions of ample subspace, extension and one-point extension of an ample space. From a systems-theoretic point of view this study of ample fields and their products is a fundamental prerequisite in the construction of a mathematical (measure-theoretic) theory of possibilistic processes. This construction will be reported on elsewhere.

2 Topological characteristics of an ample space

In this section, we have collected a number of topological properties of ample spaces. In particular, we investigate their compactness and determine for any ample space $(X, \mathcal{R})$ the closed, compact elements of $\mathcal{R}$ in $(X, \mathcal{R})$. In Section 4, we shall use this to construct some special extensions of ample spaces. For the topological notions used in this section, we refer to the basic textbooks [13, 18].

If X is provided with an ample field $\mathcal{R}$, then $\mathcal{R}$ can be regarded as a topology on X. In particular, if $I_\mathcal{R}$ is the binary relation on X, determined by $x I_\mathcal{R} y \Leftrightarrow [x]_\mathcal{R} = [y]_\mathcal{R}$, $(x, y) \in X^2$, then $I_\mathcal{R}$ is an equivalence relation on X. The set determined by $\mathcal{U}_\mathcal{R} = \{U \mid I_\mathcal{R} \subseteq U \subseteq X^2\}$ is a uniformity on X with $\{I_\mathcal{R}\}$ as a base [1, 13, 18]. The uniform topology, i.e., the topology induced by the uniformity $\mathcal{U}_\mathcal{R}$ on X, is precisely $\mathcal{R}$. In fact $\mathcal{U}_\mathcal{R}$ is also generated by the pseudo-metric ρ on X, that is given for any $(x, y) \in X^2$ as follows: $\rho(x, y) = 0$ if $x I_\mathcal{R} y$, and $\rho(x, y) = 1$ otherwise. Moreover, $\mathcal{R}$ is a special case of *a topology associated with a quasi-order relation* [14, 15].

The following proposition collects a number of topological characteristics of ample spaces. Before we can formulate and prove this proposition, we need a number of basic notations. Let U be a binary relation on X. For any x in X, the U-afterset of x is defined as $U[x] = \{y \mid y \in X \text{ and } (x, y) \in U\}$. For any subset A of X, the direct image $U[A]$ of A under U is given by $U[A] = \{y \mid y \in X \text{ and } (\exists x \in A)((x, y) \in U)\} = \bigcup_{x \in A} U[x]$. For any mapping f from a set X to a set Y, we define for $A \subseteq X$ and $B \subseteq Y$: the *direct image* $f(A) = \{f(x) \mid x \in A\}$ of A under f and the *inverse image* $f^{-1}(B) = \{x \mid x \in X \text{ and } f(x) \in B\}$ of B under f.

Proposition 1 *Let $(X, \mathcal{R})$ be an ample space.*

1. *$(X, \mathcal{R})$ is a locally compact space with base $X_\mathcal{R}$. For any x in X:*
   - *(a) $\{[x]_\mathcal{R}\}$ is a local neighbourhood base at x in $(X, \mathcal{R})$;*
   - *(b) $[x]_\mathcal{R}$ is a compact neighbourhood of x.*
2. *Every element of $\mathcal{R}$ is clopen in $(X, \mathcal{R})$.*
3. *$(X, \mathcal{R})$ is a completely regular, completely normal space.*

4. $(X,\mathcal{R})$ *is a locally connected space. In particular, we have for any* x *in* X *that* $[x]_{\mathcal{R}}$ *is the component of* x*. Also, the following statements are equivalent:*

   (a) $(X,\mathcal{R})$ *is a connected space;*

   (b) $(\forall x \in X)([x]_{\mathcal{R}} = X)$*;*

   (c) $\mathcal{R}$ *is the indiscrete topology on* X*, i.e.* $\mathcal{R} = \{\emptyset, X\}$*;*

   (d) $(X,\mathcal{R})$ *is pathwise connected.*

5. *The pseudo-metric* ρ *is a metric on* X *iff* $\mathcal{R}$ *is a* T_0*-space. In particular this is equivalent to each of the following statements.*

   (a) $(X,\mathcal{R})$ *is a Hausdorff space;*

   (b) $(X,\mathcal{R})$ *is a* T_1*-space;*

   (c) $(\forall x \in X)([x]_{\mathcal{R}} = \{x\})$*;*

   (d) $\mathcal{R}$ *is the discrete topology on* X*, i.e.* $\mathcal{R} = \wp(X)$*;*

   (e) $(X,\mathcal{R})$ *is a Tychonov space;*

   (f) $(X,\mathcal{R})$ *is a* T_4*-space;*

Proof. We start with the proof of statement 1. Consider x in X. Since $\{I_{\mathcal{R}}\}$ is a base for the uniformity $\mathcal{U}_{\mathcal{R}}$ on X, it follows that $\{I_{\mathcal{R}}[x]\}$ is a local neighbourhood base at x in $(X,\mathcal{R})$. From the definition of $I_{\mathcal{R}}$, we furthermore deduce that $I_{\mathcal{R}}[x] = [x]_{\mathcal{R}}$. This proves 1a. Let $(O_j \mid j \in J)$ be a family of elements of $\mathcal{R}$ such that $[x]_{\mathcal{R}} \subseteq \bigcup_{j\in J} O_j$. Since $x \in [x]_{\mathcal{R}}$, there exists a $j_x \in J$ such that $x \in O_{j_x}$. Because $O_{j_x} \in \mathcal{R}$ this implies that $[x]_{\mathcal{R}} \subseteq O_{j_x}$. This proves 1b. It follows that $(X,\mathcal{R})$ is locally compact with neighbourhood base $X_{\mathcal{R}}$.

Statement 2 follows from the fact that an ample field is closed under complements. We proceed to prove statement 3. It follows directly from 2 that $(X,\mathcal{R})$ is completely normal. Consider a closed set A in $(X,\mathcal{R})$ and an element x of $X \setminus A$. Then the characteristic mapping χ_A is a continuous mapping from $(X,\mathcal{R})$ to $[0,1]$, provided with the relative Euclidean topology, and $\chi_A(x) = 0$ and $\chi_A(A) = \{1\}$. This proves that $(X,\mathcal{R})$ is completely regular.

Let us continue with the proof of statement 4. Consider x in X, then x has a component C_x, and C_x is a closed set in $(X,\mathcal{R})$ [18]. Since moreover $\mathcal{R}$ is closed under complements, it follows that $C_x \in \mathcal{R}$. Since C_x is a connected subset of $(X,\mathcal{R})$, it follows that a subset A of C_x is clopen iff $A = \emptyset$ or $A = C_x$. Using statements 1 and 2, this implies that $C_x = [x]_{\mathcal{R}}$. Using statement 1a, we obtain that $(X,\mathcal{R})$ is locally connected at x and therefore also locally connected.

We now prove the equivalence of statements 4a, 4b, 4c and 4d. First, assume that $(X,\mathcal{R})$ is connected. Then the only clopen subsets of X are $\emptyset$ and X. This implies that $X_{\mathcal{R}} = \{X\}$, which proves that 4a implies 4b. The implications 4b $\Rightarrow$ 4c and 4c $\Rightarrow$ 4d are trivial. Since every pathwise connected space is connected [18], it also follows that 4d implies 4a. Finally, the proof of 5 is trivial. □

De Cooman [9] has noted that for an ample space $(X,\mathcal{R})$ the $\wp(X)-\mathcal{R}$-mapping $\mathfrak{p}_{\mathcal{R}}$, defined by $\mathfrak{p}_{\mathcal{R}}(A) = \bigcup_{x\in A} [x]_{\mathcal{R}}$, $A \subseteq X$, is the closure operator on X associated with the closure system $\mathcal{R}$ on X [4]. Furthermore, the mapping $\mathfrak{p}_{\mathcal{R}}$ satisfies $\mathfrak{p}_{\mathcal{R}}(A) = \bigcap\{B \mid A \subseteq B \in \mathcal{R}\}$, $A \subseteq X$, which implies that $\mathfrak{p}_{\mathcal{R}}(A)$ is the topological closure of A in $(X,\mathcal{R})$. We use these observations to determine the compact sets in $(X,\mathcal{R})$.

Proposition 2 *Let* $(X,\mathcal{R})$ *be an ample space. Then, for any subset* A *of* X*, the following statements are equivalent;*

1. A *is a totally bounded set in the uniform space* $(X,\mathcal{U}_{\mathcal{R}})$*;*

2. $\mathfrak{p}_{\mathcal{R}}(A)$ *is a finite union of atoms of* $\mathcal{R}$*;*

3. A *is a compact set in the topological space* $(X,\mathcal{R})$*.*

Proof. First, we prove the implication $1 \Rightarrow 2$. Since A is totally bounded, $I_{\mathcal{R}} \in \mathcal{U}_{\mathcal{R}}$ implies that there exists a non-empty, finite subset B of A such that $(I_{\mathcal{R}} \cap A^2)[B] = A$. This implies that for any a in A, there exists a b_a in B such that $(b_a, a) \in I_{\mathcal{R}} \cap A^2$, whence $a \in [b_a]_{\mathcal{R}}$. As a result, we obtain $A \subseteq \bigcup_{b\in B} [b]_{\mathcal{R}}$, whence $\mathfrak{p}_{\mathcal{R}}(A) \subseteq \bigcup_{b\in B} [b]_{\mathcal{R}}$. This means that $\mathfrak{p}_{\mathcal{R}}(A)$ is a subset of a finite union of atoms of $\mathcal{R}$. Because $\mathfrak{p}_{\mathcal{R}}(A) \in \mathcal{R}$ and therefore a union of atoms of $\mathcal{R}$, $\mathfrak{p}_{\mathcal{R}}(A)$ is also a finite union of atoms of $\mathcal{R}$.

Let us now prove that the second statement implies the third. Let $(O_j \mid j \in J)$ be a non-empty family of open sets in $(X,\mathcal{R})$ which covers A. Obviously, $A \subseteq \mathfrak{p}_{\mathcal{R}}(A)$. Because $X_{\mathcal{R}}$ is a partition of X and $\mathfrak{p}_{\mathcal{R}}(A) = \bigcup_{a\in A} [a]_{\mathcal{R}}$, the assumption that $\mathfrak{p}_{\mathcal{R}}(A)$ is a finite union of atoms of the ample space $(X,\mathcal{R})$ implies that $\mathfrak{p}_{\mathcal{R}}(A) = \bigcup_{a\in A'} [a]_{\mathcal{R}}$, where A' is a non-empty, finite subset of A. By definition of a cover, we have that $(\forall a \in A')(\exists j_a \in J)(a \in O_{j_a})$, and consequently $(\forall a \in A')([a]_{\mathcal{R}} \subseteq O_{j_a})$. Therefore $A \subseteq \mathfrak{p}_{\mathcal{R}}(A) = \bigcup_{a\in A'} [a]_{\mathcal{R}} \subseteq \bigcup_{a\in A'} O_{j_a}$, which means that A is a compact set in $(X,\mathcal{R})$.

Finally, we give a proof of the implication $3 \Rightarrow 1$. Let U be an arbitrary element of $\mathcal{U}_{\mathcal{R}}$. Because A is a compact set in $(X,\mathcal{R})$ and $A \subseteq \bigcup_{x\in A} [x]_{\mathcal{R}}$, it follows from $X_{\mathcal{R}} \subseteq \mathcal{R}$ that there exists a finite subset B of A such that $A \subseteq \bigcup_{x\in B} [x]_{\mathcal{R}}$. For any element a of A, this implies that there exists an element b_a of B such that $a \in [b_a]_{\mathcal{R}}$. From the definition of $I_{\mathcal{R}}$, it follows that $(b_a, a) \in I_{\mathcal{R}} \subseteq U$. Because $(b_a, a) \in A^2$, we obtain that $a \in (U \cap A^2)[b_a] \subseteq (U \cap A^2)[B]$. Furthermore, $(U \cap A^2)[B] \subseteq A$, whence $A = (U \cap A^2)[B]$. This shows that A is a totally bounded set in $(X,\mathcal{U}_{\mathcal{R}})$. □

The following corollary follows directly from Proposition 2 and Proposition 1.2.

Corollary 3 *Let* $(X,\mathcal{R})$ *be an ample space, then the following statements are equivalent:*

1. $(X,\mathcal{U}_{\mathcal{R}})$ *is a totally bounded uniform space;*

2. $\mathcal{R}$ *is a finite set;*

3. $X_{\mathcal{R}}$ *is a finite set;*

4. $(X,\mathcal{R})$ *is a compact topological space.*

If A *is a subset of* X*, then* A *is a closed, compact set in* $(X,\mathcal{R})$ *iff* A *is a finite union of atoms of* $\mathcal{R}$*.*

3 Product ample spaces

We now present a number of equivalent ways in which a product ample space can be constructed from a given family of ample spaces. Since an ample field is a special topology, such a family also gives rise to a product topology. We investigate the relation between these product ample spaces and product topological spaces. In particular, we prove that the product ample field is generated by its field of measurable cylinders, which is also a base for the product topology.

Let us recall a number of well-known set-theoretic and topological notions. Let $(X_t \mid t \in T)$ be a non-empty family of sets. The *Cartesian product* of $(X_t \mid t \in T)$ is the set $\times_{t\in T} X_t$ of all $T - \bigcup_{t\in T} X_t$-mappings x such that $x(t) \in X_t$, $t \in T$. In particular, if $A_t \subseteq X_t$, $t \in T$, then $\times_{t\in T} A_t$ is the subset of $\times_{t\in T} X_t$, containing all x in $\times_{t\in T} X_t$, such that $x(t) \in A_t$, $t \in T$. For notational simplicity, we also write $\times_{t\in T} A_t = A_T$, and in particular $\times_{t\in T} X_t = X_T$. Finally, when $X_t = X$, $t \in T$, the Cartesian product $\times_{t\in T} X_t$ is the set of all $T - X$-mappings, and is also given the standard notation X^T.

Using this definition of the Cartesian product of a family of sets we at once come to projection operators. For any $s \in T$, $\mathbf{pr}_{T,s}$ is the $X_T - X_s$-mapping, defined by $\mathbf{pr}_{T,s}(x) = x(s)$, $x \in X_T$, and is called the *s-th projection operator* from X_T onto X_s.

The previous definition enables us to define product mappings. Let A be any set and consider a family $(f_t \mid t \in T)$ of mappings such that $f_t \colon A \to X_t$, $t \in T$. The unique mapping $f \colon A \to X_T$, such that $f_t = \mathbf{pr}_{T,t} \circ f$, $t \in T$, is denoted by $\times_{t \in T} f_t$ and is called the *product mapping* of $(f_t \mid t \in T)$. For notational simplicity, we also write $\times_{t\in T} f_t = f_T$.

We also consider the product topology of a family of ample spaces. In fact, a product topology is a special case of a weak topology induced on a set [13, 18]. Let A be a non-empty set and let $((X_t, \mathfrak{T}_t) \mid t \in T)$ be a non-empty family of topological spaces. Consider the family of mappings $(f_t \mid t \in T)$, where for any t in T, f_t is a $A - X_t$-mapping. The *weak topology* on A induced by the family $(f_t \mid t \in T)$ of mappings is the smallest topology on A making each mapping f_t continuous. The *Tychonov topology* or *product topology* on X_T is the weak topology induced on X_T by the family $\left(\mathbf{pr}_{T,t} \mid t \in T\right)$ of projections, and is denoted by $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$.

Another way of defining a topology is by specifying a subbase or a base for it. This gives us the following result [18], which can also be taken as an equivalent definition for the product topology.

Proposition 4 *Consider a non-empty family $((X_t, \mathfrak{T}_t) \mid t \in T)$ of topological spaces. Let $\mathfrak{S} = \{\mathbf{pr}_{T,t}^{-1}(O_t) \mid t \in T$ and $O_t \in \mathfrak{T}_t\}$ and let $\mathfrak{B}$ be the set of all finite intersections of elements of $\mathfrak{S}$. Then $\mathcal{W}((X_t, \mathfrak{T}_t) \mid t \in T)$ has subbase $\mathfrak{S}$ and base $\mathfrak{B}$.*

We now turn to the definition of product ample fields and spaces. In what follows, for a subset $\mathcal{S}$ of the power class $\wp(X)$, $\tau_X(\mathcal{S})$ denotes the smallest ample field on X which includes $\mathcal{S}$ and is called the ample field *generated by* $\mathcal{S}$ on X. If it is clear from the context on which set the ample field is generated, we simply write $\tau(\mathcal{S})$ instead of $\tau_X(\mathcal{S})$. Consider two ample spaces $(X_1, \mathcal{R}_1)$ and $(X_2, \mathcal{R}_2)$. Wang [16] has defined the *product* of $\mathcal{R}_1$ and $\mathcal{R}_2$ as the ample field on $X_1 \times X_2$ generated by $\{A_1 \times A_2 \mid A_1 \in \mathcal{R}_1 \text{ and } A_2 \in \mathcal{R}_2\}$:

$$\mathcal{R}_1 \times \mathcal{R}_2 = \tau_{X_1 \times X_2}(\{A_1 \times A_2 \mid A_1 \in \mathcal{R}_1 \text{ and } A_2 \in \mathcal{R}_2\}), \tag{1}$$

and he has proven that for any $(x_1, x_2) \in X_1 \times X_2$

$$[(x_1, x_2)]_{\mathcal{R}_1 \times \mathcal{R}_2} = [x_1]_{\mathcal{R}_1} \times [x_2]_{\mathcal{R}_2}. \tag{2}$$

Wang's definition can be generalised to an indexed family of ample fields, without the need for imposing an ordering on the index set. In what follows, we assume that $((X_t, \mathcal{R}_t) \mid t \in T)$ is a non-empty family of ample spaces.

Definition 5 *The product ample field on $\times_{t\in T} X_t$ of the non-empty family of ample fields $(\mathcal{R}_t \mid t \in T)$ is the smallest ample field $\mathcal{H}$ on $\times_{t\in T} X_t$, such that $\mathbf{pr}_{T,s}$ is a $\mathcal{H} - \mathcal{R}_s$-measurable mapping, $s \in T$. It is denoted by $\prod_{t\in T} \mathcal{R}_t$. $(\times_{t\in T} X_t, \prod_{t\in T} \mathcal{R}_t)$ is called the product ample space of the family $((X_t, \mathcal{R}_t) \mid t \in T)$ of ample spaces.*

For notational simplicity, we also write $\prod_{t\in T} \mathcal{R}_t = \mathcal{R}_T$. In case that $X_t = X$ and $\mathcal{R}_t = \mathcal{R}$ for any $t \in T$, where $\mathcal{R}$ is an ample field on X, $\prod_{t\in T} \mathcal{R}_t$ is also denoted by $\mathcal{R}^T$.

Since any ample space is also a topological space, any projection operator $\mathbf{pr}_{T,s}$, $s \in T$ is by definition a $(X_T, \mathcal{R}_T) - (X_s, \mathcal{R}_s)$-continuous mapping. According to the definition of a product topology, this means that $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) \subseteq \mathcal{R}_T$, whence $\tau(\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)) \subseteq \mathcal{R}_T$. In fact, we shall prove at the end of this section that in this last expression the equality holds.

When an ample field $\mathcal{R}$ on X is generated by a subset $\mathcal{S}$ of the power class $\wp(X)$, a simple expression for the atoms of $\mathcal{R}$ can be obtained. This will be very helpful in proving that (2) can be generalised.

Lemma 6 *Let X be a non-empty set and let $\mathcal{S}$ be non-empty collection of subsets of X. Let $\mathcal{S}' = \{X \setminus A \mid A \in \mathcal{S}\}$. Then for any $x \in X$:*

$$[x]_{\tau_X(\mathcal{S})} = \bigcap \{E \mid E \in \mathcal{S} \cup \mathcal{S}' \text{ and } x \in E\}.$$

Proof. For any x in X, let $A_x = \bigcap\{E \mid E \in \mathcal{S} \cup \mathcal{S}' \text{ and } x \in E\}$. Clearly $x \in A_x \subseteq X$, whence $X = \bigcup_{x \in X} A_x$. For any x and y in X, it follows immediately that $x \in A_y \Leftrightarrow A_x \subseteq A_y$ and $y \in A_x \Leftrightarrow A_y \subseteq A_x$. Moreover, since $\mathcal{S} \cup \mathcal{S}'$ is closed under complements, $x \in A_y \Leftrightarrow y \in A_x$. Consequently, $A_x \subseteq A_y \Leftrightarrow A_y \subseteq A_x$, which means that A_x and A_y are either equal or disjoint. As a result, we obtain that $\{A_x \mid x \in X\}$ is a partition of X. Now, let $\mathcal{H}$ denote the ample field on X which has $\{A_x \mid x \in X\}$ as its set of atoms. Furthermore, if $E \in \mathcal{S}$ and $x \in E$, it follows that $x \in A_x \subseteq E$, whence $E = \bigcup_{x \in E} A_x \in \mathcal{H}$. So $\mathcal{S} \subseteq \mathcal{H}$, whence $\tau_X(\mathcal{S}) \subseteq \mathcal{H}$. We have also by definition, $\{A_x \mid x \in X\} \subseteq \tau_X(\mathcal{S})$, whence $\mathcal{H} \subseteq \tau_X(\mathcal{S})$. $\square$

We are now ready to prove that Equations (1) and (2) can be generalised.

Proposition 7 *For any $t \in T$, consider the sets $\mathcal{C}_{T,\{t\}} = \{\mathbf{pr}_{T,t}^{-1}(E) \mid E \in \mathcal{R}_t\}$ and $\mathcal{A}_{T,\{t\}} = \{\mathbf{pr}_{T,t}^{-1}(E) \mid E \in (X_t)_{\mathcal{R}_t}\}$. Furthermore, let $\mathfrak{S}_T = \bigcup_{t \in T} \mathcal{C}_{T,\{t\}}$ and $\mathfrak{S}_T^A = \bigcup_{t \in T} \mathcal{A}_{T,\{t\}}$. The product ample field $\mathcal{R}_T$ satisfies the following equalities:*

$$\mathcal{R}_T = \tau(\mathfrak{S}_T) = \tau(\mathfrak{S}_T^A) = \tau(\{\times_{t \in T} A_t \mid (\forall t \in T)(A_t \in \mathcal{R}_t)\}) \tag{3}$$

The atoms of the product ample field $\mathcal{R}_T$ are characterised by

$$[x]_{\mathcal{R}_T} = \times_{t \in T}[x(t)]_{\mathcal{R}_t} = \bigcap_{t \in T} \mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t}), \quad x \in X_T. \tag{4}$$

Proof. The equality $\mathcal{R}_T = \tau(\mathfrak{S}_T)$ follows directly from the definition of a product ample space. Also $\mathfrak{S}_T^A \subseteq \mathfrak{S}_T$, which implies that $\tau(\mathfrak{S}_T^A) \subseteq \tau(\mathfrak{S}_T)$. Conversely, since any element of $\mathfrak{S}_T$ is a union of elements of $\mathfrak{S}_T^A$, we also have that $\mathfrak{S}_T \subseteq \tau(\mathfrak{S}_T^A)$, whence $\tau(\mathfrak{S}_T) \subseteq \tau(\mathfrak{S}_T^A)$. This proves that $\tau(\mathfrak{S}_T) = \tau(\mathfrak{S}_T^A)$.

Consider A_t in $\mathcal{R}_t$, $t \in T$. Then $\times_{t \in T} A_t = \bigcap_{t \in T} \mathbf{pr}_{T,t}^{-1}(A_t)$, and it follows from the definition of a product ample space that $\times_{t \in T} A_t \in \mathcal{R}_T$, whence $\tau(\{\times_{t \in T} A_t \mid (\forall t \in T)(A_t \in \mathcal{R}_t)\}) \subseteq \mathcal{R}_T$. Conversely, let $s \in T$ and let $A_s \in \mathcal{R}_s$. Then $\mathbf{pr}_{T,s}^{-1}(A_s) = \times_{t \in T} A_t$, where for any $t \in T \setminus \{s\}$, $A_t = X_t$. This implies that $\mathbf{pr}_{T,s}^{-1}(A_s) \in \tau(\{\times_{t \in T} A_t \mid (\forall t \in T)(A_t \in \mathcal{R}_t)\})$. It is now clear from the definition of a product ample space that $\mathcal{R}_T \subseteq \tau(\{\times_{t \in T} A_t \mid (\forall t \in T)(A_t \in \mathcal{R}_t)\})$. This completes the proof of (3).

To prove that (4) holds, consider t in T. Since $\mathcal{R}_t$ is an ample field on X_t, the subset $\mathcal{C}_{T,\{t\}}$ of $\wp(X_T)$ is closed under complements, and so is consequently also $\mathfrak{S}_T$. Using Lemma 6 and $\mathcal{R}_T = \tau(\mathfrak{S}_T)$, it follows that for any x in X_T

$$[x]_{\mathcal{R}_T} = \bigcap \{E \mid E \in \mathfrak{S}_T \text{ and } x \in E\}. \tag{5}$$

Furthermore, consider x in X_T and E in $\mathfrak{S}_T$. There exist $s \in T$ and $A \in \mathcal{R}_s$, such that $E = \mathbf{pr}_{T,s}^{-1}(A)$. Also

$$x \in E \Leftrightarrow \mathbf{pr}_{T,s}(x) \in A \Leftrightarrow x(s) \in A \Leftrightarrow [x(s)]_{\mathcal{R}_s} \subseteq A \Leftrightarrow \mathbf{pr}_{T,s}^{-1}([x(s)]_{\mathcal{R}_s}) \subseteq E. \tag{6}$$

Equations (5) and (6) imply that $\bigcap_{t \in T} \mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t}) \subseteq [x]_{\mathcal{R}_T}$. Since $\{\mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t}) \mid t \in T\} \subseteq \mathfrak{S}_T^A \subseteq \mathfrak{S}_T$, it also follows from (5) that $[x]_{\mathcal{R}_T} \subseteq \bigcap_{t \in T} \mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t})$. The proof of the equality $\times_{t \in T}[x(t)]_{\mathcal{R}_t} = \bigcap_{t \in T} \mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t})$ is trivial. $\square$

Equation (3) assures us that Wang's result (1) can be generalised towards arbitrary products of ample fields. So, as expected, the product ample field $\mathcal{R}_T$ can also be defined as the ample field generated on X_T by $\{\times_{t\in T}A_t \mid (\forall t\in T)(A_t\in\mathcal{R}_t)\}$ on X_T. Furthermore, it follows from (4) that $\mathcal{R}_T$ is precisely the *box topology* [18], which can be defined on X_T using the family $((X_t,\mathcal{R}_t) \mid t\in T)$ of ample spaces.

The following proposition determines the relation between the measurability of a mapping with codomain X_T and the measurability of its components.

Proposition 8 *Let $(X,\mathcal{R})$ be an ample space and let f_t be a $X-X_t$-mapping, $t\in T$. The following statements are equivalent:*

1. *f_T is a $\mathcal{R}-\mathcal{R}_T$-measurable mapping;*
2. *f_t is a $\mathcal{R}-\mathcal{R}_t$-measurable mapping, $t\in T$.*

Proof. First, assume that f_T is a measurable $\mathcal{R}-\mathcal{R}_T$-mapping. Consider s in T. Then, by definition, $f_s=\mathbf{pr}_{T,s}\circ f_T$. From the definition of a product ample space, we deduce that $\mathbf{pr}_{T,s}$ is a $\mathcal{R}_T-\mathcal{R}_s$-measurable mapping. This implies that f_s as the composition of $\mathbf{pr}_{T,s}$ with f_T is a $\mathcal{R}-\mathcal{R}_s$-measurable mapping. Conversely, assume that f_t is a $\mathcal{R}-\mathcal{R}_t$-measurable mapping, $t\in T$. Consider x in X_T. Using Proposition 7 and the definition of f_T, it follows that $f_T^{-1}([x]_{\mathcal{R}_T})=\bigcap_{t\in T}f_t^{-1}([x(t)]_{\mathcal{R}_t})$. Since $[x(t)]_{\mathcal{R}_t}\in\mathcal{R}_t$ and f_t is a $\mathcal{R}-\mathcal{R}_t$-measurable mapping, $t\in T$, it follows that $f_T^{-1}([x]_{\mathcal{R}_T})\in\mathcal{R}$, whence immediately $f_T^{-1}(A)\in\mathcal{R}$, $A\in\mathcal{R}_T$. This means that f_T is $\mathcal{R}-\mathcal{R}_T$-measurable. □

Let us introduce more general projection operators, which will be very useful further on.

Definition 9 *For any non-empty subset S of T, let $\mathbf{pr}_{T,S}$ be the mapping from X_T onto X_S, such that $(\forall x\in X_T)(\mathbf{pr}_{T,S}(x)=x|_S)$, where $x|_S$ is the restriction of the mapping x to the domain S.*

For these mappings, we can derive the following properties, which we shall often (implicitly) return to in what follows. They show how projection operators can be conveniently used to transfer results involving the set T to results involving any of its subsets. They also tell us that for all practical purposes, notions and definitions involving $\{t\}$ can be identified with the corresponding ones involving t, $t\in T$.

Proposition 10 1. *For any non-empty subset S of T, we have that*

   (a) $\mathbf{pr}_{T,S}=\times_{t\in S}\mathbf{pr}_{T,t}$,

   (b) $\mathbf{pr}_{T,S}$ *is a $\mathcal{R}_T-\mathcal{R}_S$-measurable mapping,*

   (c) $\mathcal{R}_S=\{\mathbf{pr}_{T,S}(E)\mid E\in\mathcal{R}_T\}$.

2. *For any $t\in T$, we have that*

   (a) $\mathbf{pr}_{\{t\},t}$ *is a bijection between $X_{\{t\}}$ and X_t,*

   (b) $\mathcal{R}_{\{t\}}=\{\mathbf{pr}_{\{t\},t}^{-1}(E)\mid E\in\mathcal{R}_t\}$ *and* $\mathcal{R}_t=\{\mathbf{pr}_{\{t\},t}(E)\mid E\in\mathcal{R}_{\{t\}}\}$,

   (c) $\mathcal{C}_{T,\{t\}}=\{\mathbf{pr}_{T,\{t\}}^{-1}(E)\mid E\in\mathcal{R}_{\{t\}}\}$,

   (d) $\mathcal{A}_{T,\{t\}}=\{\mathbf{pr}_{T,\{t\}}^{-1}(E)\mid E\in(X_{\{t\}})_{\mathcal{R}_{\{t\}}}\}$,

   (e) $\mathcal{R}_t=\{\mathbf{pr}_{T,t}(E)\mid E\in\mathcal{R}_T\}$.

Proof. Since $(\forall t\in S)(\mathbf{pr}_{T,t}=\mathbf{pr}_{S,t}\circ\mathbf{pr}_{T,S})$, it follows from the definition of a product mapping that $\mathbf{pr}_{T,S}=\times_{t\in S}\mathbf{pr}_{T,t}$. By Proposition 8 it follows that $\mathbf{pr}_{T,S}$ is a $\mathcal{R}_T-\mathcal{R}_S$-measurable mapping. From Proposition 7, we may derive that $\mathbf{pr}_{T,S}([x]_{\mathcal{R}_T})=[x|_S]_{\mathcal{R}_S}$, $x\in X_T$, which implies that $\mathcal{R}_S=\{\mathbf{pr}_{T,S}(E)\mid E\in\mathcal{R}_T\}$.

The proof of statements 2a and 2b is straightforward. Statements 2c and 2d follow directly from 2a and 2b. Finally, statement 2e follows directly from statements 1a, 1c and 2b. □

Inspired by probability theory [3], we define *measurable cylinders* of a product ample space. These cylinders appear naturally in the context of possibilistic processes, where it may be typically assumed that possibilistic information (that is, a possibility measure) is given on (all) finite Cartesian products X_S, where S is a non-empty, finite subset of T, and we want to extend this information to the complete Cartesian product X_T. We shall be concerned with these measurable cylinders in most of what follows. For brevity, we write $A \Subset X$ iff A is a *finite subset* of X.

Definition 11 *For any set S such that $\emptyset \subset S \Subset T$, let*

$$\mathcal{C}_{T,S} = \{\mathbf{pr}_{T,S}^{-1}(E) \mid E \in \mathcal{R}_S\},$$

and let $\mathcal{A}_{T,S}$ be the subset of $\mathcal{C}_{T,S}$ determined by

$$\mathcal{A}_{T,S} = \{\mathbf{pr}_{T,S}^{-1}(E) \mid E \in (X_S)_{\mathcal{R}_S}\}.$$

Any element of $\mathcal{C}_{T,S}$ is called a measurable S-cylinder of $(X_T, \mathcal{R}_T)$, and any element of $\mathcal{A}_{T,S}$ is called a measurable atomic S-cylinder of $(X_T, \mathcal{R}_T)$. Furthermore, let $\mathcal{C}_T = \bigcup_{\emptyset \subset S \Subset T} \mathcal{C}_{T,S}$ and $\mathcal{A}_T = \bigcup_{\emptyset \subset S \Subset T} \mathcal{A}_{T,S}$. $\mathcal{C}_T$ is the set of all measurable cylinders of $(X_T, \mathcal{R}_T)$ and $\mathcal{A}_T$ is the set of all atomic measurable cylinders of $(X_T, \mathcal{R}_T)$.

Note that Proposition 10 assures that the notations introduced in this definition are consistent with the ones introduced in Proposition 7.

We need the following property.

Lemma 12 *Let X be any set, $(Y, \mathcal{R}_Y)$ an ample space and f a $X - Y$-mapping. If $\mathcal{R}_X = \{f^{-1}(E) \mid E \in \mathcal{R}_Y\}$, then $\mathcal{R}_X$ is an ample field on X, and $[x]_{\mathcal{R}_X} = f^{-1}([f(x)]_{\mathcal{R}_Y})$, $x \in X$. Furthermore, $X_{\mathcal{R}_X} = \{f^{-1}(E) \mid E \in Y_{\mathcal{R}_Y}$ and $E \cap f(X) \neq \emptyset\}$.*

Proof. Since the inverse image under a mapping preserves complements, and arbitrary unions and intersections, it follows that $\mathcal{R}_X$ is an ample field on X. Consider an element $x \in X$. Then, by definition, the atom of $\mathcal{R}_X$ containing x is given by

$$\begin{aligned}[x]_{\mathcal{R}_X} &= \bigcap \{f^{-1}(M) \mid M \in \mathcal{R}_Y \text{ and } x \in f^{-1}(M)\} \\ &= f^{-1}\big(\bigcap \{M \mid M \in \mathcal{R}_Y \text{ and } f(x) \in M\}\big) \\ &= f^{-1}([f(x)]_{\mathcal{R}_Y}).\end{aligned}$$

The remaining part of the lemma is now trivial. □

If we take a closer look at the different kinds of measurable cylinders, the following results can be derived. Let $\widehat{\cdot}$ be the $\wp(T) - \wp(T)$-mapping such that $\widehat{S} = \{t \mid t \in S \text{ and } \mathcal{R}_t \neq \{\emptyset, X_t\}\}$ for any subset S of T.

Proposition 13 *1. For any non-empty, finite subset S of T, $\mathcal{C}_{T,S}$ is an ample field on X_T with set of atoms $\mathcal{A}_{T,S}$, and $\mathcal{C}_{T,S} = \tau(\bigcup_{t \in S} \mathcal{C}_{T,\{t\}}) = \tau(\bigcup_{t \in S} \mathcal{A}_{T,\{t\}})$.*

2. $\widehat{\cdot}$ is a dual closure operator on T, and $\widehat{S} = \widehat{T} \cap S$, $S \subseteq T$.

3. If S_1 and S_2 are non-empty, finite subsets of T such that $S_1 \subseteq S_2$, then $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$.

4. For any non-empty, finite subset S of T, the following statements are equivalent:

(a) $\widehat{S} = \emptyset$,

(b) $\mathcal{C}_{T,S} = \{\emptyset, X_T\}$,

(c) $\mathcal{A}_{T,S} = \{X_T\}$;

and if $\widehat{S} \neq \emptyset$, *then* $\mathcal{C}_{T,S} = \mathcal{C}_{T,\widehat{S}}$ *and* $\mathcal{A}_{T,S} = \mathcal{A}_{T,\widehat{S}}$.

5. *If* S_1 *and* S_2 *are non-empty, finite subsets of* T, *then:*

(a) $\widehat{S_1} \subseteq \widehat{S_2}$ *iff* $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$,

(b) $\widehat{S_1} \subset \widehat{S_2}$ *iff* $\mathcal{C}_{T,S_1} \subset \mathcal{C}_{T,S_2}$,

(c) $\widehat{S_1} = \widehat{S_2}$ *iff* $\mathcal{C}_{T,S_1} = \mathcal{C}_{T,S_2}$ *iff* $\mathcal{A}_{T,S_1} = \mathcal{A}_{T,S_2}$.

6. $\mathcal{C}_T = \{\emptyset, X_T\}$ *iff* $\mathcal{A}_T = \{X_T\}$ *iff* $\widehat{T} = \emptyset$.

Proof. First, we prove statement 1. Consider a set S such that $\emptyset \subset S \Subset T$. It follows directly from Lemma 12 and the surjectivity of $\mathbf{pr}_{T,S}$ that $\mathcal{C}_{T,S}$ is an ample field on X_T with set of atoms $\mathcal{A}_{T,S}$. Furthermore, let $t \in S$ and $A \in \mathcal{C}_{T,\{t\}}$. By definition, there is an E in $\mathcal{R}_t$ such that $A = \mathbf{pr}_{T,t}^{-1}(E)$. Then $\mathbf{pr}_{T,t} = \mathbf{pr}_{S,t} \circ \mathbf{pr}_{T,S}$ implies that $A = \mathbf{pr}_{T,S}^{-1}(\mathbf{pr}_{S,t}^{-1}(E))$, where $\mathbf{pr}_{S,t}^{-1}(E) \in \mathcal{R}_S$, since $E \in \mathcal{R}_t$. Consequently $A \in \mathcal{C}_{T,S}$. Hence $\bigcup_{t\in S} \mathcal{C}_{T,\{t\}} \subseteq \mathcal{C}_{T,S}$, which implies that $\tau(\bigcup_{t\in S} \mathcal{C}_{T,\{t\}}) \subseteq \mathcal{C}_{T,S}$. Conversely, let A be an atom of $\mathcal{C}_{T,S}$, then, by Lemma 12, there is an $x \in X_S$ such that $A = \mathbf{pr}_{T,S}^{-1}([x]_{\mathcal{R}_S})$. This, together with $\mathbf{pr}_{T,t} = \mathbf{pr}_{S,t} \circ \mathbf{pr}_{T,S}$, $t \in S$, gives rise to $A = \mathbf{pr}_{T,S}^{-1}(\bigcap_{t\in S} \mathbf{pr}_{S,t}^{-1}([x(t)]_{\mathcal{R}_t})) = \bigcap_{t\in S} \mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t}) \in \tau(\bigcup_{t\in S} \mathcal{C}_{T,\{t\}})$. This implies that also $\mathcal{C}_{T,S} \subseteq \tau(\bigcup_{t\in S} \mathcal{C}_{T,\{t\}})$. The proof of the equality $\tau(\bigcup_{t\in S} \mathcal{C}_{T,\{t\}}) = \tau(\bigcup_{t\in S} \mathcal{A}_{T,\{t\}})$ is trivial. So is the proof of statement 2. Statement 3 follows directly from statement 1. We continue with the proof of 4. Let S be a non-empty, finite subset of T. If $\widehat{S} \neq \emptyset$, it follows from statement 3 that $\mathcal{C}_{T,\widehat{S}} \subseteq \mathcal{C}_{T,S}$. Conversely, let $A \in \mathcal{C}_{T,S}$. Then there is an element $E \in \mathcal{R}_S$, such that $A = \mathbf{pr}_{T,S}^{-1}(E)$. Using Proposition 10.1c, it follows that $\mathbf{pr}_{S,\widehat{S}}(E) \in \mathcal{R}_{\widehat{S}}$. Since $E = \mathbf{pr}_{S,\widehat{S}}^{-1}(\mathbf{pr}_{S,\widehat{S}}(E))$, it follows that $A = \mathbf{pr}_{T,\widehat{S}}^{-1}(\mathbf{pr}_{S,\widehat{S}}(E))$, whence $A \in \mathcal{C}_{T,\widehat{S}}$. Therefore $\mathcal{C}_{T,S} \subseteq \mathcal{C}_{T,\widehat{S}}$. The remaining part of statement 4 is immediate.

We now prove statement 5. Let S_1 and S_2 be two non-empty, finite subsets of T. Assume that $\widehat{S_1} \subseteq \widehat{S_2}$. In case $\widehat{S_1} = \emptyset$, it follows from statements 1 and 4 that $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$. In case $\widehat{S_1} \neq \emptyset$, it follows from statements 3 and 4 that $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$. Conversely, let $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$. Assume *ex absurdo* that $\widehat{S_1} \not\subseteq \widehat{S_2}$. This is equivalent to $\widehat{S_1} \setminus \widehat{S_2} \neq \emptyset$. Consider an element $t \in \widehat{S_1} \setminus \widehat{S_2}$. Since $t \in \widehat{S_1}$, there exists an element $A \in (X_t)_{\mathcal{R}_t}$, such that $A \subset X_t$. Furthermore, it follows from statement 1 that $\mathbf{pr}_{T,t}^{-1}(A) \in \mathcal{C}_{T,S_1}$. Using the assumption $\mathcal{C}_{T,S_1} \subseteq \mathcal{C}_{T,S_2}$, it follows that $\mathbf{pr}_{T,t}^{-1}(A) \in \mathcal{C}_{T,S_2}$. Consequently, there exists a $B \in \mathcal{R}_{S_2}$ such that $\mathbf{pr}_{T,t}^{-1}(A) = \mathbf{pr}_{T,S_2}^{-1}(B)$. Since, by assumption $t \notin \widehat{S_2}$, we obtain that $A = \mathbf{pr}_{T,t}(\mathbf{pr}_{T,t}^{-1}(A)) = \mathbf{pr}_{T,t}(\mathbf{pr}_{T,S_2}^{-1}(B)) = X_t$, a contradiction. We find that $\widehat{S_1} \subseteq \widehat{S_2}$. Statement 5c can be deduced from statements 1 and 5a. Statement 5b can be easily deduced from statements 5a and 5c. The proof of statement 6 is immediate. □

It follows that in general the set of all measurable cylinders of a product ample field is a field.

Proposition 14 *$\mathcal{C}_T$ is a field on X_T.*

Proof. Let S be a non-empty, finite subset of T, then $\mathcal{C}_{T,S} \subseteq \mathcal{C}_T$. By Proposition 13.1 $\mathcal{C}_{T,S}$ is an ample field on X_T. Therefore, $\mathcal{C}_{T,S}$ and hence also $\mathcal{C}_T$ contains $\emptyset$ and X_T. Let A_1 and A_2 be elements of $\mathcal{C}_T$. Then, by Proposition 13.3 there is a non-empty, finite subset S of T such that A_1 and A_2 belong to $\mathcal{C}_{T,S}$. By Proposition 13.1, $A_1 \cap A_2 \in \mathcal{C}_{T,S} \subseteq \mathcal{C}_T$ and

$A_1 \cup A_2 \in \mathcal{C}_{T,S} \subseteq \mathcal{C}_T$. Finally, if $A \in \mathcal{C}_T$, there is a non-empty, finite subset S of T, such that $A \in \mathcal{C}_{T,S}$. By Proposition 13.1, it follows that $X_T \setminus A \in \mathcal{C}_{T,S} \subseteq \mathcal{C}_T$. This proves that $\mathcal{C}_T$ is a field on X_T. □

Clearly, since $\mathcal{R}_T$ is an ample field, it is particular a topology on X_T, for which, by definition, all the projection operators $\mathbf{pr}_{T,t}$ are continuous, given the topologies $\mathcal{R}_t$ on X_t, $t \in T$. On the other hand, we may consider the *smallest* topology on X_T for which the mappings $\mathbf{pr}_{T,t}$, $t \in T$ are continuous. By definition this is the *Tychonov topology* or *product topology* of the topologies $\mathcal{R}_t$ on X_t, $t \in T$, and is denoted by $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$. Note that $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) \subseteq \mathcal{R}_T$. The following proposition relates these topologies and the set of measurable cylinders and atomic cylinders.

Proposition 15 *Let $\mathfrak{B}_T$ be the family of all finite intersections of elements of $\mathfrak{S}_T$, then the product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T has $\mathfrak{S}_T$ as a subbase and $\mathfrak{B}_T$ as a base. Furthermore, $\mathfrak{B}_T \subseteq \mathcal{C}_T \subseteq \mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$, which implies that $\mathcal{C}_T$ is also a base for $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$, and $\tau(\mathfrak{S}_T) = \tau(\mathcal{C}_T) = \tau(\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)) = \mathcal{R}_T$. The product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T also has $\mathfrak{S}_T^A$ as a subbase and $\mathcal{A}_T$ as a base, and $\tau(\mathfrak{S}_T^A) = \tau(\mathcal{A}_T) = \mathcal{R}_T$.*

Proof. It follows at once from Proposition 4 that the product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T has $\mathfrak{S}_T$ as a subbase and $\mathfrak{B}_T$ as a base. By definition it follows that $\mathfrak{S}_T \subseteq \mathcal{C}_T$. Since $\mathcal{C}_T$ is a field on X_T, we also obtain that $\mathfrak{B}_T \subseteq \mathcal{C}_T$. Since by definition also $\mathcal{C}_T \subseteq \mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$, it follows that $\mathcal{C}_T$ is a base for $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$, and $\mathfrak{S}_T \subseteq \mathcal{C}_T \subseteq \mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) \subseteq \mathcal{R}_T$, whence, by Proposition 7, $\mathcal{R}_T = \tau(\mathfrak{S}_T) \subseteq \tau(\mathcal{C}_T) \subseteq \tau(\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)) \subseteq \mathcal{R}_T$. Consider an element $A \in \mathcal{C}_T$. Then, by definition, there exist a non-empty, finite subset S of T and an element E of $\mathcal{R}_S$ such that $A = \mathbf{pr}_{T,S}^{-1}(E) = \bigcup_{x \in E} \mathbf{pr}_{T,S}^{-1}([x]_{\mathcal{R}_S})$. This implies that any element of $\mathcal{C}_T$ is a union of elements of $\mathcal{A}_T$. Since $\mathcal{A}_T \subseteq \mathcal{C}_T$, it follows that $\mathcal{A}_T$ is also a base for the product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T. The remaining part of the proposition follows directly from Proposition 7, which implies that any element of $\mathcal{A}_T$ is a finite intersection of elements of $\mathfrak{S}_T^A$. □

To end this section, we investigate in the following proposition under what condition the field $\mathcal{C}_T$ is also an ample field on X_T.

Proposition 16 *The following statements are equivalent.*

1. *$\mathcal{C}_T$ is an ample field on X_T.*
2. *$\mathcal{C}_T$ is a plump class on X_T.*
3. *$\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ is an ample field on X_T.*
4. *$\widehat{T} \Subset T$.*
5. *There exists a set S such that $\emptyset \subset S \Subset T$ and $\mathcal{C}_T = \mathcal{C}_{T,S}$.*
6. *$\mathcal{C}_T = \mathcal{R}_T$.*
7. *$\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T) = \mathcal{R}_T$.*

Moreover, if T is a finite set, then $\mathcal{C}_T$ is an ample field on X_T.

Proof. The equivalence of statements 1 and 2 follows directly from the fact that $\mathcal{C}_T$ is a field on X_T, and therefore closed under complements. The equivalence of statements 1 and 6 and the equivalence of 3 and 7 follow directly from Proposition 15.

Let us therefore prove the equivalence of statements 1, 3, 4 and 5. The implication 1 ⇒ 3 follows directly from Proposition 15. We continue with the proof of the implication 3 ⇒ 4. In case $\widehat{T} = \emptyset$, then $\widehat{T} \Subset T$. Assume $\widehat{T} \neq \emptyset$. By definition we have $(\forall t \in \widehat{T})(\exists A_t \in (X_t)_{\mathcal{R}_t})(A_t \subset X_t)$. Let $A = \times_{t \in T} A_t$, where $(\forall t \in T \setminus \widehat{T})(A_t = X_t)$. Then $A = \bigcap_{t \in \widehat{T}} \mathbf{pr}_{T,t}^{-1}(A_t)$ and, since $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ is an ample field, it follows that $A \in \mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$. Because $\mathcal{C}_T$ is a base of $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ according to Proposition 15, there exists a non-empty element $B \in \mathcal{C}_T$ such that $B \subseteq A$. Also, by definition, there exists a non-empty, finite subset S of T such that $B \in \mathcal{C}_{T,S}$. This implies that

$$(\forall t \in T \setminus S)(X_t = \mathbf{pr}_{T,t}(B) \subseteq \mathbf{pr}_{T,t}(A) = A_t \subseteq X_t),$$

whence $(\forall t \in T \setminus S)(A_t = X_t)$. Then, it follows from the definition of A that $\widehat{T} \subseteq S \Subset T$, which implies that $\widehat{T} \Subset T$. We now prove the implication 4 ⇒ 5. In case $\widehat{T} = \emptyset$, it follows from Proposition 13.6 that $\mathcal{C}_T = \{\emptyset, X_T\}$, which implies that $\mathcal{C}_T = \mathcal{C}_{T,S}$ for any non-empty finite subset S of T. Assume therefore that $\widehat{T} \neq \emptyset$. Then, for any non-empty finite subset S of T such that $\widehat{S} \neq \emptyset$, it follows from Proposition 13 that $\mathcal{C}_{T,S} = \mathcal{C}_{T,\widehat{S}} \subseteq \mathcal{C}_{T,\widehat{T}} \subseteq \mathcal{C}_T$. On the other hand, it is clear that, by Proposition 13, $\mathcal{C}_T = \bigcup_{\emptyset \subset S \Subset T} \mathcal{C}_{T,S} = \bigcup_{\emptyset \subset S \Subset T, \widehat{S} \neq \emptyset} \mathcal{C}_{T,\widehat{S}} \subseteq \mathcal{C}_{T,\widehat{T}}$, whence $\mathcal{C}_T = \mathcal{C}_{T,\widehat{T}}$. Finally, the implication 5 ⇒ 1 follows directly from Proposition 13.1. The rest of the proof is now trivial. □

4 Ample subspaces and extensions of ample spaces

In this section, we define *ample subspaces* and *extensions* of an ample space. In particular, by taking the compactness of an ample space $(X, \mathcal{R})$ into account, we introduce **-extensions* of $(X, \mathcal{R})$, which are called *one-point extensions* of $(X, \mathcal{R})$ if $(X, \mathcal{R})$ is a non-compact topological space. Furthermore, we investigate the relation between the product ample space of a family $((X_t, \mathcal{R}_t) \mid t \in T)$ of ample spaces and the product ample space of a family $((X_t^*, \mathcal{R}_t^*) \mid t \in T)$ of associated *-extensions.

The results of this section will allow us to fairly naturally embed non-compact (product) spaces into compact ones, and still preserve a number of desirable properties. Since compactness is a very useful property in many contexts, the relevance of this section is fairly evident.

Definition 17 *Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be ample spaces. Then $(X, \mathcal{R}_X)$ is called an ample subspace of $(Y, \mathcal{R}_Y)$, and we write $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$, iff $\mathcal{R}_X = \{E \cap X \mid E \in \mathcal{R}_Y\}$.*

It follows immediately that $X \subseteq Y$. Furthermore, it is easily verified that $\{E \cap X \mid E \in \mathcal{R}_Y\}$ is a ample field on X, so that our definition of an ample subspace is meaningful. Since ample spaces are topological spaces, it follows from the definition that $(X, \mathcal{R}_X)$ is an ample subspace of $(Y, \mathcal{R}_Y)$ iff $\mathcal{R}_X$ is the *relativisation* of the topology $\mathcal{R}_Y$ to the subset X of Y.

We also introduce *extensions* of an ample space, which provide a generalisation of the notion of *coarseness* [9].

Definition 18 *Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be ample spaces. Then $(Y, \mathcal{R}_Y)$ is called an extension of the ample space $(X, \mathcal{R}_X)$, and we write $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$, iff $\mathcal{R}_X \subseteq \mathcal{R}_Y$.*

It follows from this definition that $X \subseteq Y$, and if $X = Y$, then $\mathcal{R}_X$ is coarser than $\mathcal{R}_Y$. The following property can easily be proven.

Proposition 19 *$\sqsubseteq$ is a partial order on the class of ample subspaces of any ample space. $\preceq$ is a partial order on the class of extensions of any ample space.*

In the following proposition, we compare an ample space and its extensions by their atoms.

Proposition 20 *Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be ample spaces and let $X \subseteq Y$.*

1. $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$ *iff* $(\forall x \in X)([x]_{\mathcal{R}_X} = [x]_{\mathcal{R}_Y} \cap X)$.
2. *If* $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$, *then* $(\forall x \in X)([x]_{\mathcal{R}_X} \subseteq [x]_{\mathcal{R}_Y} = \mathfrak{p}_{\mathcal{R}_Y}([x]_{\mathcal{R}_X}))$.
3. $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$ *iff* $(\forall x \in X)([x]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X})$.
4. *If* $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$, *then* $(\forall x \in X)([x]_{\mathcal{R}_X} = \mathfrak{p}_{\mathcal{R}_X}([x]_{\mathcal{R}_Y}))$. *In particular,* $\mathcal{R}_X = \{\mathfrak{p}_{\mathcal{R}_X}(E \cap X) \mid E \in \mathcal{R}_Y\}$.
5. $\tau_Y(X_{\mathcal{R}_X} \cup \{Y \setminus X\}) = \tau_Y(X_{\mathcal{R}_X}) = \tau_Y(\mathcal{R}_X)$.
6. $\mathcal{R}_Y = \tau_Y(\mathcal{R}_X)$ *iff* $Y_{\mathcal{R}_Y} = X_{\mathcal{R}_X} \cup \{Y \setminus X\}$.
7. *The following statements are equivalent.*
   (a) $(\forall x \in X)([x]_{\mathcal{R}_X} = [x]_{\mathcal{R}_Y})$.
   (b) $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$ *and* $X \in \mathcal{R}_Y$.
   (c) $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$ *and* $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$.

Proof. Suppose that $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$. Consider an element x of X, then $[x]_{\mathcal{R}_Y} \cap X \in \mathcal{R}_X$, and since $x \in [x]_{\mathcal{R}_Y} \cap X$, it follows that $[x]_{\mathcal{R}_X} \subseteq [x]_{\mathcal{R}_Y} \cap X$. Furthermore, because $[x]_{\mathcal{R}_X} \in \mathcal{R}_X$, it follows from the assumption $\mathcal{R}_X = \{E \cap X \mid E \in \mathcal{R}_Y\}$ that there is a set $A \in \mathcal{R}_Y$ such that $[x]_{\mathcal{R}_X} = A \cap X$, which implies that $[x]_{\mathcal{R}_Y} \subseteq A$, whence $[x]_{\mathcal{R}_Y} \cap X \subseteq [x]_{\mathcal{R}_X}$. Conversely, assume that $(\forall x \in X)([x]_{\mathcal{R}_X} = [x]_{\mathcal{R}_Y} \cap X)$, whence $\mathcal{R}_X \subseteq \{E \cap X \mid E \in \mathcal{R}_Y\}$. Consider now an element $B \in \mathcal{R}_Y$. For any $x \in B \cap X$, it follows that $x \in [x]_{\mathcal{R}_X} = [x]_{\mathcal{R}_Y} \cap X \subseteq B \cap X$. This implies that $B \cap X = \bigcup_{x \in B \cap X} [x]_{\mathcal{R}_X} \in \mathcal{R}_X$. This proves statement 1.

Statement 2 follows directly from statement 1. To prove statements 3 and 4, we first show that

$$(\forall x \in X)([x]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X}) \Rightarrow (\forall x \in X)([x]_{\mathcal{R}_X} = \mathfrak{p}_{\mathcal{R}_Y}([x]_{\mathcal{R}_X})). \tag{7}$$

Assume that $(\forall x \in X)([x]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X})$ and consider an element $x \in X$. Then for any $y \in [x]_{\mathcal{R}_X}$, it follows that $[y]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X}$, since $[y]_{\mathcal{R}_X} = [x]_{\mathcal{R}_X}$. Hence $[x]_{\mathcal{R}_X} = \bigcup_{y \in [x]_{\mathcal{R}_X}} \{y\} \subseteq \bigcup_{y \in [x]_{\mathcal{R}_X}} [y]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X}$, which proves (7).

Now, we continue with the proof of statement 3. Assume that $(\forall x \in X)([x]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X})$. Formula (7) implies that $X_{\mathcal{R}_X} \subseteq \mathcal{R}_Y$, and, since $\mathcal{R}_Y$ is an ample field on Y, it follows that $\mathcal{R}_X \subseteq \mathcal{R}_Y$ or $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$. Conversely, if $(X, \mathcal{R}_X) \preceq (Y, \mathcal{R}_Y)$, then it follows easily that $(\forall x \in X)([x]_{\mathcal{R}_Y} \subseteq [x]_{\mathcal{R}_X})$. This proves statement 3.

Statement 4 follows directly from statement 3 and (7). We continue with statement 5. Because $X \in \tau_Y(X_{\mathcal{R}_X})$, it follows that $Y \setminus X \in \tau_Y(X_{\mathcal{R}_X})$. From $X_{\mathcal{R}_X} \subseteq \tau_Y(X_{\mathcal{R}_X}) \subseteq \tau_Y(X_{\mathcal{R}_X} \cup \{Y \setminus X\})$, we obtain that $\tau_Y(X_{\mathcal{R}_X} \cup \{Y \setminus X\}) = \tau_Y(X_{\mathcal{R}_X})$. Furthermore, it follows from $X_{\mathcal{R}_X} \subseteq \mathcal{R}_X$ that $\tau_Y(X_{\mathcal{R}_X}) \subseteq \tau_Y(\mathcal{R}_X)$. Using $\mathcal{R}_X \subseteq \tau_Y(X_{\mathcal{R}_X})$, we obtain that $\tau_Y(X_{\mathcal{R}_X}) = \tau_Y(\mathcal{R}_X)$.

Statement 6 can be derived as follows. Suppose $\mathcal{R}_Y = \tau_Y(\mathcal{R}_X)$. Then, by statement 5, $\mathcal{R}_Y = \tau_Y(X_{\mathcal{R}_X} \cup \{Y \setminus X\})$, and, because $X_{\mathcal{R}_X} \cup \{Y \setminus X\}$ is a partition of Y, it follows from Lemma 6 that $Y_{\mathcal{R}_Y} = X_{\mathcal{R}_X} \cup \{Y \setminus X\}$. Conversely, if $Y_{\mathcal{R}_Y} = X_{\mathcal{R}_X} \cup \{Y \setminus X\}$, it follows from statement 5 that $\mathcal{R}_Y = \tau_Y(\mathcal{R}_X)$.

We begin the proof of statement 7 with the implication 7a $\Rightarrow$ 7b. It follows immediately from 7a that $(\forall x \in X)([x]_{\mathcal{R}_X} = [x]_{\mathcal{R}_Y} \cap X)$. Using statement 1, we obtain that $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$. Since $X \in \mathcal{R}_X$, we have also that $X = \bigcup_{x \in X} [x]_{\mathcal{R}_Y} \in \mathcal{R}_Y$. The implication 7b $\Rightarrow$ 7c can be derived as follows. Using statement 1 and $X \in \mathcal{R}_Y$, we find that $(\forall x \in X)([x]_{\mathcal{R}_X} = [x]_{\mathcal{R}_Y})$. Then, from 3 we obtain 7c. Implication 7c $\Rightarrow$ 7a follows directly from statements 1 and 3. □

The following proposition gives two other topological characterisations of ample subspaces.

Proposition 21 *Let* $(X, \mathcal{R}_X)$ *and* $(Y, \mathcal{R}_Y)$ *be ample spaces, then the following statements are equivalent:*

1. $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$;
2. $I_{\mathcal{R}_Y} \cap X^2 = I_{\mathcal{R}_X}$;
3. *The uniform space* $(X, \mathcal{U}_{\mathcal{R}_X})$ *is a subspace of the uniform space* $(Y, \mathcal{U}_{\mathcal{R}_Y})$.

Proof. Suppose $(X, \mathcal{R}_X) \sqsubseteq (Y, \mathcal{R}_Y)$. Then it follows from Proposition 20.1 that $I_{\mathcal{R}_Y} \cap X^2 = I_{\mathcal{R}_X}$. This proves the implication $1 \Rightarrow 2$.

Assume that $I_{\mathcal{R}_Y} \cap X^2 = I_{\mathcal{R}_X}$. Let $\mathcal{U}_{\mathcal{R}_Y, X} = \{U \cap X^2 \mid U \in \mathcal{U}_{\mathcal{R}_Y}\}$. Since $\{I_{\mathcal{R}_Y} \cap X^2\}$ is a base of the uniformity $\mathcal{U}_{\mathcal{R}_Y, X}$ on X and $\{I_{\mathcal{R}_X}\}$ is a base of the uniformity $\mathcal{U}_{\mathcal{R}_X}$, it follows that $(X, \mathcal{U}_{\mathcal{R}_X})$ is a uniform subspace of $(Y, \mathcal{U}_{\mathcal{R}_Y})$. This proves the implication $2 \Rightarrow 3$.

The implication $3 \Rightarrow 1$ follows directly from the fact that the uniform topology $\mathcal{R}_X$, induced by the uniformity $\mathcal{U}_{\mathcal{R}_X}$ on X, is the relativisation of the uniform topology $\mathcal{R}_Y$ to X [13]. □

We now demonstrate how an extension of an ample space $(X, \mathcal{R})$ can be constructed by taking the compactness of $(X, \mathcal{R})$ into account. If we consider an ample space $(X, \mathcal{R})$, then Corollary 3 tells us that $(X, \mathcal{R})$ is not a compact topological space unless $X_{\mathcal{R}}$ is a finite set. However [13], a non-compact topological space $(X, \mathcal{R})$ can always be embedded in a compact topological space. In particular, if we let

$$\mathfrak{T}^* = \mathcal{R} \cup \{X^* \setminus G \mid G \text{ is a closed, compact set in } (X, \mathcal{R})\} \tag{8}$$

where

$$X^* = X \cup \{\infty\}, \tag{9}$$

where $\infty \notin X$, then $(X^*, \mathfrak{T}^*)$ is a compact space. $(X^*, \mathfrak{T}^*)$ is called *a one-point compactification* of $(X, \mathcal{R})$. If $(X, \mathcal{R})$ is a compact topological space, then we define for notational reasons $X^* = X$ and $\mathfrak{T}^* = \mathcal{R}$. Furthermore, if we let $\mathcal{R}^* = \tau_{X^*}(\mathfrak{T}^*)$, then according to Definition 18, the ample space $(X^*, \mathcal{R}^*)$ is an extension of $(X, \mathcal{R})$. This leads to the following definition.

Definition 22 *Let* $(X, \mathcal{R})$ *be an ample space. Then* $(X^*, \mathcal{R}^*)$ *is called a *-extension of the ample space* $(X, \mathcal{R})$. $(X^*, \mathcal{R}^*)$ *is called a one-point extension of* $(X, \mathcal{R})$ *iff* $(X, \mathcal{R})$ *is a non-compact topological space. For a one-point extension* $(X^*, \mathcal{R}^*)$ *of* $(X, \mathcal{R})$, $(X^*, \mathfrak{T}^*)$ *is called the one-point compactification associated with* $(X^*, \mathcal{R}^*)$.

*-extensions have the following properties.

Proposition 23 *Let* $(X, \mathcal{R})$ *be an ample space and let* $(X^*, \mathcal{R}^*)$ *be a *-extension of* $(X, \mathcal{R})$. *Then* $(X, \mathcal{R}) \sqsubseteq (X^*, \mathcal{R}^*)$ *and* $(X, \mathcal{R}) \preceq (X^*, \mathcal{R}^*)$. *In particular, if* $(X^*, \mathcal{R}^*)$ *is a one-point extension of* $(X, \mathcal{R})$, *i.e. if* $(X, \mathcal{R})$ *is not compact, then* $\mathcal{R}^* = \tau_{X^*}(\mathcal{R})$, $X^*_{\mathcal{R}^*} = X_{\mathcal{R}} \cup \{\{\infty\}\}$ *and* $\mathcal{R} \subset \mathfrak{T}^* \subset \mathcal{R}^*$. *If* $(X, \mathcal{R})$ *is compact,* $\mathcal{R} = \mathfrak{T}^* = \mathcal{R}^*$ *and* $X^*_{\mathcal{R}^*} = X_{\mathcal{R}}$.

Proof. If $(X, \mathcal{R})$ is a compact topological space, the proof is trivial. Assume therefore that $(X^*, \mathcal{R}^*)$ is a one-point extension of $(X, \mathcal{R})$. By definition, this means that $X^* = X \cup \{\infty\}$ with $\infty \notin X$. It follows from $\mathcal{R} \subseteq \mathfrak{T}^*$ that $\tau_{X^*}(\mathcal{R}) \subseteq \tau_{X^*}(\mathfrak{T}^*) = \mathcal{R}^*$. Conversely, let K be a closed, compact set in $(X, \mathcal{R})$. Then $X \setminus K \in \mathcal{R}$, which implies that

$$X^* \setminus K = (X \setminus K) \cup \{\infty\} \in \tau_{X^*}(\mathcal{R} \cup \{\{\infty\}\}) = \tau_{X^*}(\mathcal{R}),$$

whence clearly $\mathfrak{T}^* \subseteq \tau_{X^*}(\mathcal{R})$. This in turn implies that $\mathcal{R}^* \subseteq \tau_{X^*}(\mathcal{R})$. Hence, it follows from Proposition 20.6 that ${X^*}_{\mathcal{R}^*} = X_{\mathcal{R}} \cup \{\{\infty\}\}$. This implies that $(\forall x \in X)([x]_{\mathcal{R}^*} = [x]_{\mathcal{R}})$, and, by

Proposition 20.7, it follows that $(X,\mathcal{R}) \sqsubseteq (X^*,\mathcal{R}^*)$ and $(X,\mathcal{R}) \preceq (X^*,\mathcal{R}^*)$. Since $\infty \notin X$, it follows that $X^* \notin \mathcal{R}$, which implies that $\mathcal{R} \subset \mathfrak{T}^*$. Furthermore, because $(X,\mathcal{R})$ is not compact, $\{\infty\} \in \mathcal{R}^* \setminus \mathfrak{T}^*$, whence $\mathfrak{T}^* \subset \mathcal{R}^*$. □

If $(X,\mathcal{R})$ is a non-compact topological space, we at once deduce from Proposition 23 that $\mathfrak{T}^*$ is not an ample field on X^*. In the following proposition, we determine the compact sets in $(X^*,\mathfrak{T}^*)$.

Proposition 24 *Let $(X,\mathcal{R})$ be a non-compact ample space and $(X^*,\mathfrak{T}^*)$ be a one-point compactification of $(X,\mathcal{R})$, so $X^* = X \cup \{\infty\}$ and $\infty \notin X$. Any subset of X^* which contains ∞ is compact in $(X^*,\mathfrak{T}^*)$. If $K \subseteq X$, then K is compact in $(X^*,\mathfrak{T}^*)$ iff K is compact in $(X,\mathcal{R})$.*

Proof. Let $B \subseteq X^*$ such that $\infty \in B$. Consider an open cover $(O_j \mid j \in J)$ of B in $(X^*,\mathfrak{T}^*)$. Since $\infty \in B$, there exists a $j_\infty \in J$ such that $\infty \in O_{j_\infty}$. By definition, it follows that $O_{j_\infty} = X^* \setminus K$, where K is a closed and compact set in $(X,\mathcal{R})$. Then, by Proposition 2, $K = \bigcup_{A\in\mathcal{S}} A$, where $\mathcal{S}$ is a finite subset of $X_{\mathcal{R}}$. Now, let $\widetilde{\mathcal{S}} = \{A \mid A \in \mathcal{S} \text{ and } A \cap B \neq \emptyset\}$. If $\widetilde{\mathcal{S}} = \emptyset$, it follows that $B \subseteq O_{j_\infty}$, so that B is indeed compact in $(X^*,\mathfrak{T}^*)$. We therefore aasume that $\widetilde{\mathcal{S}} \neq \emptyset$. Consider any element A of $\widetilde{\mathcal{S}}$. There exists an element $x_A \in A \cap B$. Since $(O_j \mid j \in J)$ is an open cover of B in $(X^*,\mathfrak{T}^*)$, there is an element $j_A \in J$ such that $x_A \in A = [x_A]_{\mathcal{R}} \subseteq O_{j_A}$. Hence $B \cap K \subseteq \bigcup_{A\in\widetilde{\mathcal{S}}} A \subseteq \bigcup_{A\in\widetilde{\mathcal{S}}} O_{j_A}$, and, from $O_{j_\infty} = X^* \setminus K$, it follows that $B \subseteq \bigcup_{A\in\widetilde{\mathcal{S}}} O_{j_A} \cup O_{j_\infty}$. We therefore again obtain that B is a compact in $(X^*,\mathfrak{T}^*)$.

Consider a subset K of X. If K is compact in $(X^*,\mathfrak{T}^*)$, if follows from $K \subseteq X \subseteq X^*$ and $\mathcal{R} \subseteq \mathfrak{T}^*$ that K is also compact in $(X,\mathcal{R})$. Conversely, suppose K is compact in $(X,\mathcal{R})$. Then, since $i\colon X \to X^*\colon x \to x$ is a continuous mapping from $(X,\mathcal{R})$ into $(X^*,\mathfrak{T}^*)$, it follows that $K = i(K)$ is compact in $(X^*,\mathfrak{T}^*)$. □

We find that the set of compact sets in the one-point compactification $(X^*,\mathfrak{T}^*)$ of the topological space $(X,\mathcal{R})$ contains the compact sets in $(X,\mathcal{R})$ and $(X^*,\mathcal{R}^*)$, where $\mathcal{R}^* = \tau_{X^*}(\mathfrak{T}^*)$.

Consider now a family $((X_t,\mathcal{R}_t) \mid t \in T)$ of ample spaces with non-empty index set T and any element $t \in T$. Then we can associate a *-extension $(X_t^*,\mathcal{R}_t^*)$ with $(X_t,\mathcal{R}_t)$. In particular, if $(X_t,\mathcal{R}_t)$ is a non-compact topological space, then $(X_t^*,\mathcal{R}_t^*)$ is a one-point extension of $(X_t,\mathcal{R}_t)$, with associated one-point compactification $(X_t^*,\mathfrak{T}_t^*)$, such that $X_t^* = X_t \cup \{\infty_t\}$, $\infty_t \notin X_t$ and $\mathcal{R}_t^* = \tau_{X_t^*}(\mathcal{R}_t)$. In case $(X_t,\mathcal{R}_t)$ is a compact topological space, then $(X_t^*,\mathcal{R}_t^*)$ coincides with $(X_t,\mathcal{R}_t)$, and for notational uniformity, we also denote $\mathcal{R}_t$ by $\mathfrak{T}_t^*$. By means of the family $((X_t^*,\mathcal{R}_t^*) \mid t \in T)$ of associated *-extensions, we intend to prove that the product ample space $(\times_{t\in T} X_t^*, \prod_{t\in T}\mathcal{R}_t^*)$ is an extension of $(\times_{t\in T} X_t, \prod_{t\in T}\mathcal{R}_t)$, which inherits most properties of a *-extension.

To this end, let us introduce the following notations and definitions. For notational simplicity, let $\times_{t\in T} X_t^* = X_T^*$. Consider an element t of T. Then $\mathbf{pr}_{T,t}^*$ is the $X_T^* - X_t^*$-mapping such that $(\forall x \in X_T^*)(\mathbf{pr}_{T,t}^*(x) = x(t))$. $\prod_{t\in T}\mathcal{R}_t^*$ is the smallest ample field $\mathcal{H}$ on X_T^*, such that $\mathbf{pr}_{T,s}^*$ is a $\mathcal{H} - \mathcal{R}_s^*$-measurable mapping, $s \in T$. For notational simplicity, let $\prod_{t\in T}\mathcal{R}_t^* = \mathcal{R}_T^*$. If S is a non-empty subset of T, $\mathbf{pr}_{T,S}^*$ denotes the mapping from X_T^* onto X_S^*, such that $\mathbf{pr}_{T,S}^*(x) = x\mid_S$, $x \in X_T^*$. This allows us to define for any non-empty, finite subset S of T:

$$\mathcal{C}_{T,S}^* = \{\mathbf{pr}_{T,S}^{*-1}(E) \mid E \in \mathcal{R}_S^*\} \text{ and } \mathcal{C}_T^* = \bigcup_{\emptyset\subset S\Subset T} \mathcal{C}_{T,S}^*.$$

Then $\mathcal{C}_T^*$ is, according to Definition 11 and Proposition 14, the field of all measurable cylinders of $(X_T^*,\mathcal{R}_T^*)$. Finally, let

$$\mathcal{O}_T^* = \bigcup_{\emptyset\subset S\Subset T} \{\mathbf{pr}_{T,S}^{*-1}(O) \mid O \in \mathcal{W}((X_t^*,\mathfrak{T}_t^*) \mid t \in S)\},$$

$$\widetilde{\mathcal{C}}_T = \bigcup_{\emptyset\subset S\Subset T} \{\mathbf{pr}_{T,S}^{*-1}(E) \mid E \in \mathcal{R}_S\}.$$

In the next proposition, we have collected a number of important properties of the ample space $(X^*_T, \mathcal{R}^*_T)$ and the subclasses $\widetilde{\mathcal{C}}_T$, $\mathcal{O}^*_T$, $\mathcal{C}^*_T$ of the power class $\wp(X^*_T)$, introduced above.

Proposition 25 *1.* $(X_T, \mathcal{R}_T) \sqsubseteq (X^*_T, \mathcal{R}^*_T)$ *and* $(X_T, \mathcal{R}_T) \preceq (X^*_T, \mathcal{R}^*_T)$.

2. For any non-empty subset S *of* T,

$$\mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S) \subseteq \mathcal{W}((X^*_t, \mathcal{R}^*_t) \mid t \in S) \subseteq \mathcal{R}^*_S \text{ and } \mathcal{R}_S \subseteq \mathcal{R}^*_S.$$

3. For any non-empty, finite subset S *of* T,

$$\mathcal{R}_S \subseteq \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S) \subseteq \mathcal{W}((X^*_t, \mathcal{R}^*_t) \mid t \in S) = \mathcal{R}^*_S.$$

4. $\mathcal{O}^*_T$ *is a base for the product topology* $\mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in T)$ *on* X^*_T.

5. $\mathcal{C}^*_T$ *is a base for the product topology* $\mathcal{W}((X^*_t, \mathcal{R}^*_t) \mid t \in T)$ *on* X^*_T.

6. $\widetilde{\mathcal{C}_T} \subseteq \mathcal{O}^*_T \subseteq \mathcal{C}^*_T$.

7. The set $\mathcal{O}^*_T$ *is closed for finite unions and finite intersections and contains* $\emptyset$ *and* X^*_T.

8. $\mathcal{R}^*_T = \tau_{X^*_T}(\mathcal{C}^*_T) = \tau_{X^*_T}(\mathcal{W}((X^*_t, \mathcal{R}^*_t) \mid t \in T)) = \tau_{X^*_T}(\mathcal{O}^*_T) = \tau_{X^*_T}(\widetilde{\mathcal{C}}_T)$.

Proof. First, we prove statement 1. For any non-empty subset S of T, it follows directly from Propositions 7 and 23 that

$$(\forall x \in X_S)([x]_{\mathcal{R}_S} = [x]_{\mathcal{R}^*_S}). \tag{10}$$

Then from Proposition 20.7 for $S = T$, we obtain directly statement 1, since $X_T \subseteq X^*_T$.

We now turn to the proof of statement 2. Let S be a non-empty subset of T. Then, by definition $\mathfrak{T}^*_t \subseteq \mathcal{R}^*_t$ for any $t \in S$, which implies that $\mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S) \subseteq \mathcal{W}((X^*_t, \mathcal{R}^*_t) \mid t \in S)$. From Proposition 15, it follows that $\mathcal{W}((X^*_t, \mathcal{R}^*_t) \mid t \in S) \subseteq \mathcal{R}^*_S$, and from (10) it follows that $\mathcal{R}_S \subseteq \mathcal{R}^*_S$.

We continue with the proof of statement 3. Let S be a non-empty, finite subset of T. Then, for any $x \in X_S$, it follows from (10) and Propositions 7 that

$$[x]_{\mathcal{R}_S} = [x]_{\mathcal{R}^*_S} = \bigcap_{t \in S} \mathbf{pr}^{*-1}_{T,t}([x(t)]_{\mathcal{R}^*_t}) = \bigcap_{t \in S} \mathbf{pr}^{*-1}_{T,t}([x(t)]_{\mathcal{R}_t}).$$

Since $(\forall t \in S)([x(t)]_{\mathcal{R}_t} \in \mathcal{R}_t \subseteq \mathfrak{T}^*_t)$ and S is finite, it follows that $[x]_{\mathcal{R}_S} \in \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S)$. This implies that $\mathcal{R}_S \subseteq \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S)$. The remaining part follows directly from Proposition 16 and statement 2.

Statement 5 follows directly from Proposition 15. Statement 6 follows directly from 3. To prove statement 7, let $n \in \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of all strictly positive natural numbers, and let $\{O_i \mid 1 \leq i \leq n\} \subseteq \mathcal{O}^*_T$. Then there is a non-empty, finite subset S_i of T and an element $B_i \in \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S_i)$ such that $O_i = \mathbf{pr}^{*-1}_{T,S_i}(B_i)$, $i \in \{1, \ldots, n\}$. Let $S_o = \bigcup_{i=1}^n S_i$, and let $\widetilde{O}_i = \mathbf{pr}^{*-1}_{S_o,S_i}(B_i)$, $i \in \{1, \ldots, n\}$. Since $\mathbf{pr}^*_{S_o,S_i}$ is a continuous mapping from $(X^*_{S_o}, \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S_o))$ onto $(X^*_{S_i}, \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S_i))$, $i \in \{1, \ldots, n\}$, it follows that $\{\widetilde{O}_i \mid i \in \{1, \ldots, n\}\} \subseteq \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S_o)$, whence $\bigcap_{i=1}^n \widetilde{O}_i \in \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S_o)$ and $\bigcup_{i=1}^n \widetilde{O}_i \in \mathcal{W}((X^*_t, \mathfrak{T}^*_t) \mid t \in S_o)$. It also follows that $O_i = \mathbf{pr}^{*-1}_{T,S_i}(B_i) = \mathbf{pr}^{*-1}_{T,S_o}(\widetilde{O}_i)$, $i \in \{1, \ldots, n\}$, which implies that $\bigcap_{i=1}^n O_i$ and $\bigcup_{i=1}^n O_i$ belong to $\mathcal{O}^*_T$. This proves the first part of statement 7. The second part is trivial.

In order to prove 4, let

$$\mathcal{H}^*_T = \{\bigcap_{t \in S} \mathbf{pr}^{*-1}_{T,t}(O_t) \mid \emptyset \subset S \Subset T \text{ and } (\forall t \in S)(O_t \in \mathfrak{T}^*_t)\}.$$

Proposition 4 tells us that $\mathcal{H}_T^*$ is a base for $\mathcal{W}((X_t^*, \mathfrak{T}_t^*) \mid t \in T)$. Using statement 7 and the obvious fact that for any $s \in T$ and $O_s \in \mathfrak{T}_s^*$, $\mathbf{pr}_{T,s}^{*-1}(O_s) \in \mathcal{O}_T^*$, it follows that $\mathcal{H}_T^* \subseteq \mathcal{O}_T^* \subseteq \mathcal{W}((X_t^*, \mathfrak{T}_t^*) \mid t \in T)$, which implies that $\mathcal{O}_T^*$ is also a base of $\mathcal{W}((X_t^*, \mathfrak{T}_t^*) \mid t \in T)$.

To prove statement 8, let $t \in T$. Then obviously

$$(\forall E \in \mathcal{R}_t)(\mathbf{pr}_{T,t}^{*-1}(E) \in \widetilde{\mathcal{C}}_T). \tag{11}$$

Consider an element x of X_t. By Proposition 23, $[x]_{\mathcal{R}_t^*} = [x]_{\mathcal{R}_t} \in \mathcal{R}_t$. It follows therefore from (11) that $\mathbf{pr}_{T,t}^{*-1}([x]_{\mathcal{R}_t^*}) \in \widetilde{\mathcal{C}}_T$. If $(X_t^*, \mathcal{R}_t^*)$ is a one-point extension of $(X_t, \mathcal{R}_t)$, then $X_t^* = X_t \cup \{\infty_t\}$ where $\infty_t \notin X_t$, and, by Proposition 23, $[\infty_t]_{\mathcal{R}_t^*} = \{\infty_t\} = X_t^* \setminus X_t$. Since $X_t \in \mathcal{R}_t$ it follows from (11) that $\mathbf{pr}_{T,t}^{*-1}([\infty_t]_{\mathcal{R}_t^*}) = \mathbf{pr}_{T,t}^{*-1}(X_t^* \setminus X_t) = X_T^* \setminus \mathbf{pr}_{T,t}^{*-1}(X_t) \in \tau_{X_T^*}(\widetilde{\mathcal{C}_T})$. Therefore, Proposition 7 tells us that $(X_T^*)_{\mathcal{R}_T^*} \subseteq \tau_{X_T^*}(\widetilde{\mathcal{C}}_T)$, which implies that $\mathcal{R}_T^* \subseteq \tau_{X_T^*}(\widetilde{\mathcal{C}}_T)$. Using statements 2, 4 and 6, and Proposition 15, we obtain statement 8. □

5 Conclusion

We defined the product $\mathcal{R}_T$ of a family of ample fields $((X_t, \mathcal{R}_t) \mid t \in T)$ on the Cartesian product X_T of $(X_t \mid t \in T)$ as the smallest ample field on X_T making each projection operator $\mathbf{pr}_{T,t}$, $t \in T$, measurable. Of course, a more direct way to introduce this notion is by simply defining $\mathcal{R}_T$ as the smallest ample field on X_T generated by $\{\times_{t \in T} A_t \mid (\forall t \in T)(A_t \in \mathcal{R}_t)\}$. We proved that both approaches are equivalent. Moreover, they generalise Wang's original definition of product ample field [16], and lead to a similar characterisation of the atoms, namely, $[x]_{\mathcal{R}_T} = \times_{t \in T}[x(t)]_{\mathcal{R}_t} = \cap_{t \in T} \mathbf{pr}_{T,t}^{-1}([x(t)]_{\mathcal{R}_t})$ where $x \in X_T$. It immediately follows from these equations that the measurability of a mapping with codomain X_T depends on the measurability of all its components, and vice versa. Using De Cooman's notion of possibilistic variable [5, 6], it is obvious that a similar result also holds for possibilistic variables. We investigated a special subset of $\mathcal{R}_T$, namely the field $\mathcal{C}_T$ of all measurable cylinders of $(X_T, \mathcal{R}_T)$. We derived that $\mathcal{C}_T$ generates $\mathcal{R}_T$ on X_T and acts as a basis for the product topology $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ on X_T. Finally, we investigated when $\mathcal{W}((X_t, \mathcal{R}_t) \mid t \in T)$ and $\mathcal{C}_T$ coincide with $\mathcal{R}_T$.

In a forthcoming paper [12] we will interpret the collection of ample spaces $((X_t, \mathcal{R}_t) \mid t \in T)$ as state spaces. More precisely, we will regard any set X_t, $t \in T$, as the collection of all states that a system may assume at time t of a given time set T. In particular we will restrict ourselves to systems for which possibilistic information on all finite Cartesian products $(X_S, \mathcal{R}_S)$, $\emptyset \subset S \Subset T$, is given by a $(L, \leq)$-possibility measure Π_S. For these systems we will solve the following fundamental problem: is it possible to represent the possibilistic information by means of a family of possibilistic variables in the corresponding *state spaces* $((X_t, \mathcal{R}_t) \mid t \in T)$, sharing a common $(L, \leq)$-possibility space $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ as basic space? The results of the foregoing section and our topological study of ample spaces will be very helpful in deriving this result, which is a possibilistic counterpart of the classical probabilistic Daniell-Kolmogorov theorem [10].

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