Application of Homotopy Perturbation Method (HPM) to solve particle population balance model of a circulating fluidized bed

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1. INTRODUCTION

In recent years, the application of Homotopy Perturbation Method (HPM) in solving a wide range of complex engineering problems has increased manifold. Ranging from non-linear wave equations to gas dynamics equation, from magneto-hydrodynamic (MHD) flow to couette and poiseuille flows; HPM is being widely used by scientists and engineers to provide an analytical approximate solution to various linear and nonlinear problems. It is done by continuously deforming a difficult problem under study to a simple problem which becomes easier to solve. The homotopy perturbation method, first proposed by He (1999), was developed and improved by He (2006).

In this paper, the applicability of homotopy perturbation method is extended to solve a particle population balance model of a circulating fluidized bed. The unsteady-state population balance model consists of a first order partial integro-differential equation, incorporating particle size reduction through attrition and breakage. The numerical results are then compared to a *true* solution of a standard finite difference scheme. While an analytical solution of the equation without the integral term is well known, the presence of the integral term makes impossible to find an analytical solution for the whole equation. The use of HPM allows to approximate the analytical solution of the whole equation by means of an infinite sum of terms, where each term is obtained solving the non-integral equation having as input a function of the previously evaluated terms. The HPM solution is shown to reach a very good approximation of the true solution already after a few iteration steps.

2. POPULATION BALANCE MODEL

To study the effect of particle size reduction on gas-solids hydrodynamics in a circulating fluidized bed (CFB), a transient analysis of the bed particle size distribution (PSD) is made by incorporating attrition and breakage (also referred to as fragmentation) in the overall mass balance. The population balance model, developed for particle size, is expressed as follows (Saastamoinen, 2003):

$$\frac{\partial}{\partial t}P(r,t)+R\frac{\partial}{\partial r}P(r,t)+\left(\varsigma(r)+\psi(r)-\frac{3R}{r}+B(r)\right)P(r,t)-\int_{r}^{r_{\max}}P(r_x,t)B(r_x)w(r,r_x)dr_x-\bar{m}_0 p_0(r,t)=0. \quad (1)$$

Eq. (1) must be solved with some boundary conditions and is represented by:

$$P(r,0)=0 \text{ for every } r\in[0,r_{\max}] \text{ and } P(r_{\max},t)=0 \text{ for every } t\geq 0. \quad (2)$$

The functional forms of the breakage $w(r,r_x)$ and selection $B(r)$ functions can be written as

$$w(r,r_x)=K\left(\frac{m}{r_x}\right)\left(\frac{r}{r_x}\right)^{m-1}+(1-K)\left(\frac{n}{r_x}\right)\left(\frac{r}{r_x}\right)^{n-1} \quad ; \quad 0\leq K\leq 1 \text{ and } m,n\geq 0 \quad (3)$$

$$B(r)=L\left(\frac{r}{r_{\max}}\right)^{\beta}. \quad (4)$$

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Parameter	Value
$\overline{m}_0$	0.58734 kg/s
R	-4.4281×10^{-9} m/s
$\varsigma(r)$	1.929×10^{-5} 1/s
$\psi(r)$	4.9946×10^{-6} 1/s
K	0.85
L	0.63
m	0.00598
n	3.5
β	0.62

Table 1. Values of the unknown parameters *K*, *L*, *m*, *n* and β (Dutta *et al.*, unpublished)

3. HOMOTOPY PERTUBATION METHOD

To illustrate the basic ideas of this method, we consider the differential equation (He, 1999):

$$A(u) - f(x) = 0, \quad x \in \Omega \tag{5}$$

In Eq. (5), x is a variable belonging to a certain domain Ω: in our case (see Eq. 1) $x = (r,t)$ with $t \geq 0$ and $0 < r \leq r_{\max}$. u is an unknown function defined on x, and is the quantity that has to be found by solving Eq. (5) i.e. $u = P(r,t)$. *A* is a general differential operator acting on the function u and *f* is a known analytical function of *x*. Considering the population balance model (Eq. 1), *A* is given by

$$A[P(r,t)] = \frac{\partial}{\partial t}P(r,t) + R\frac{\partial}{\partial r}P(r,t) + \left(\varsigma(r) + \psi(r) - \frac{3R}{r} + B(r)\right)P(r,t) - \int_r^{r_{\max}} P(r_x,t)B(r_x)w(r,r_x)dr_x \tag{6}$$

while the known function *f* is given by

$$f(r,t) = \overline{m}_0 p_0(r,t) \tag{7}$$

Usually HPM is applied when *A* is a nonlinear operator, and can be divided in two parts *L* and *N*, where *L* is linear while *N* is nonlinear (He, 1999). However, in our case the operator *A* is linear, but it is still possible to apply HPM by dividing it in two parts *L* and *N*, where *L* is *easy* to solve while *N* is *difficult* to solve. Considering Eq. (6), *L* and *N* are given by

$$L[P(r,t)] = \frac{\partial}{\partial t}P(r,t) + R\frac{\partial}{\partial r}P(r,t) + \left(\varsigma(r) + \psi(r) - \frac{3R}{r} + B(r)\right)P(r,t) \tag{8}$$

$$N[P(r,t)] = -\int_r^{r_{\max}} P(r_x,t)B(r_x)w(r,r_x)dr_x \tag{9}$$

Eq. (5) can be rewritten as follows:

$$L(u) + N(u) - f(x) = 0 \text{ where } L(u) + N(u) = A(u). \tag{10}$$

Applying HPM to Eq. (10) means introducing a new parameter *p*, which varies continuously from 0 to 1, and by means of this parameter, defining a new function called homotopy as follows:

$$H(v,p) = (1-p)\cdot(L(v) - L(u_0)) + p\cdot(A(v) - f(x)) = 0. \tag{11}$$

The first approximation u_0, to the solution of Eq. (5) satisfying the given boundary conditions, is defined in Eq. (11).

$$H(v,0) = L(v) - L(u_0) = 0 \text{ for } p = 0\text{, and has a simple solution } v = u_0 \tag{12}$$

$$H(v,1) = A(v) - f(x) = 0 \text{ for } p = 1\text{, and has a solution } v = u \tag{13}$$

The solution of Eq. (13) becomes identical to the original equation (5) at *p* = 1. The solution *v* of Eq. (11) can then be expressed as a series expansion in *p*:

$$v = v_0 + p\cdot v_1 + p^2\cdot v_2 + \ldots \tag{14}$$

and in the limit $p \to 1$ the approximate solution becomes

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \quad (15)$$

To obtain the coefficients v_i it is enough to equate the similar powers of *p* in Eq. (9), which can be rewritten as:

$$H(v, p) = L(v_0 + p \cdot v_1 + p^2 \cdot v_2 + ...) - L(u_0) + p \cdot \Big(L(u_0) + N\big(v_0 + p \cdot v_1 + p^2 \cdot v_2 + ...\big) - f(x)\Big) = 0 \quad (16)$$

Remembering that both *L* and *N* are linear we obtain the following equalities:

$$L(v_0) - L(u_0) = 0 \qquad 0^{th} \text{ order} \quad (17a)$$

$$L(v_1) + L(u_0) + N(v_0) - f(x) = 0 \qquad 1^{st} \text{ order} \quad (17b)$$

$$L(v_2) + N(v_1) = 0 \qquad 2^{nd} \text{ order} \quad (17c)$$

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$$L(v_i) + N(v_{i-1}) = 0. \qquad i^{th} \text{ order} \quad (17d)$$

Eqs. (17a-d) are useful because the *difficult* operator must never be solved; it must only be applied to previous coefficients of the series expansion. Considering Eq. (8)

$$L[P(r,t)] = \frac{\partial}{\partial t} P(r,t) + R \frac{\partial}{\partial r} P(r,t) - C(r) P(r,t) = g(r,t) \quad (18)$$

The input function $g(r,t)$ in general does not satisfy to the boundary conditions of Eq. (2). The solution to Eq. (18) can be found as a special case of the general solution discussed in Section 1.3 under first-order partial differential linear equations in Polyanin *et al.* (2002). At this stage, a new variable $\mathrm{u} = r - R \cdot t$ is defined. Following the general solution, the solution to Eq. (18) is given by

$$P(r,t) = F(t,\mathrm{u}) \left(\Phi(\mathrm{u}) + \int_{t_0}^{t} \frac{g(\mathrm{u} + R \cdot t', t')}{F(t', \mathrm{u})} dt' \right) \quad (19)$$

where $F(t,\mathrm{u}) = \exp \int_0^t C(\mathrm{u} + R \cdot t') dt'$. (20)

Therefore from the condition that $g(r,t)$ is different from zero only in the domain $t \geq 0$ and $0 < r \leq r_{\max}$, we deduce $t_0 = \max(0, t - \frac{r_{\max} - r}{-R})$. The solution to Eq. (18) thus becomes:

$$P(r,t) = F(t,\mathrm{u}) \int_{t_0}^{t} \frac{g(\mathrm{u} + R \cdot t', t')}{F(t', \mathrm{u})} dt'. \quad (21)$$

Finally, the terms of the HPM expansions can be written as:

$$P_1(r,t) = -P_0(r,t) + F(t,\mathrm{u}) \int_{t_0}^{t} \frac{g_0(\mathrm{u} + R \cdot t', t')}{F(t', \mathrm{u})} dt' \quad (22a)$$

where $g_0(r,t) = -N[P_0(r,t)] + f(r,t)$. (22b)

The next coefficients are then simply obtained by applying Eq. (17d)

$$P_{i+1}(r,t) = F(t,\mathrm{u}) \int_{t_0}^{t} \frac{g_i(\mathrm{u} + R \cdot t', t')}{F(t', \mathrm{u})} dt' \quad (23a)$$

where $g_i(r,t) = -N[P_i(r,t)]$. (23b)

The above expressions completely determine the HPM expansion for the unsteady-state population balance equation.

4. ANALYSIS

Applying Eqs. (17a-d) to find the HPM expansion

$$P(r,t) = P_0(r,t) + P_1(r,t) + P_2(r,t) + ... \quad (24)$$

a reasonable value should be assigned to $P_0(r,t)$ as a good starting approximation to the HPM expansion. This is given by the following:

$$P_0(r,t) = P_\infty(r)\cdot\left(1-\exp\left(C(r)\cdot t\right)\right). \tag{25}$$

Fig. 1 indicates the particle paper is obtained for only a normal distribution (□ $=100\times10^{-6}$ m, □ = 10×10^{-6} m) for the feed inlet. Fig. 1 shows the bed particle size distribution $p(r,t)$ at t = 2s using both HPM and finite difference method. Figs. 2 show the time evolution of the particle size 10 □m and 100 □m respectively. It is seen that although the initial approximation P_0 overestimates the *true* solution, the subsequent terms (P_1, P_2 and P_3) correct this initial error and a good approximation of the finite difference solution is reached. A reason for the high initial term P_0 could be that the initial guess P_0 is very good for high values of t, but it is not so good for small values of t and of r. It is noteworthy to mention that as few as 8 iterations are required to get the desired convergence in this method. This proves the efficiency and robustness of HPM in solving partial differential equations.

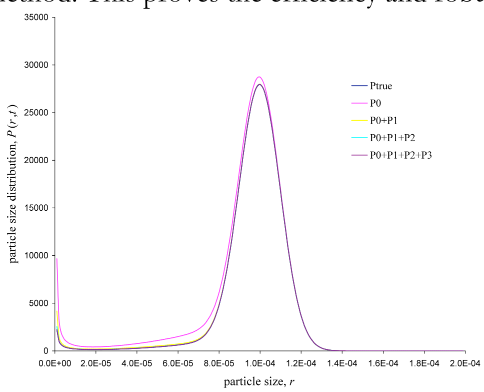

Figure 1. HPM solution at t = 2 s.

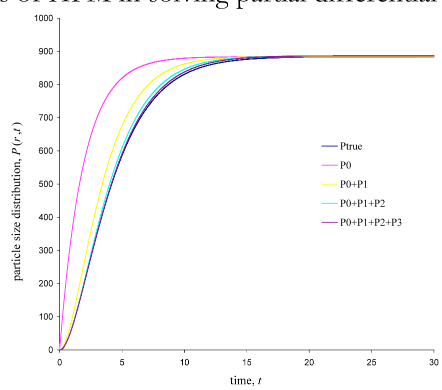

Figure 2. Time evolution of size r = 10 □m.

CONCLUSION

In this paper, the homotopy perturbation method (HPM) is successfully applied to solve the particle population balance model of a circulating fluidized bed. The obtained results are compared to the results of a standard finite difference method (true solution) and are found to be similar. The results indicate that HPM is a powerful and efficient technique in solving linear systems (as is the case of our present study). MATLAB 7.5.0 has been used for computations in this paper.

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