In Dov Gabbay, Shahid Rahman, John Symons, and Jean Paul Van Bendegem, editors, *Logic, Epistemology and the Unity of Science*, pages 459–485. Kluwer Academic Publishers, Dordrecht, 2004.

The Need for Adaptive Logics in Epistemology*

Diderik Batens
Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
Diderik.Batens@UGent.be

Abstract

After it is argued that philosophers of science have lost interest in logic because they applied the wrong type of logics, examples are given of the forms of dynamic reasoning that are central for philosophy of science and epistemology. Adaptive logics are presented as a means to understand and explicate those forms of reasoning. All members of a specific (large) set of adaptive logics are proved to have a number of properties that warrant their formal decency and their suitability with respect to understanding and explicating dynamic forms of reasoning. Most of the properties extend to other adaptive logics.

1 Aim of this Paper

In the first half of the twentieth century epistemology largely reduced to the philosophy of science and logic played a central role in it. We are here interested in the last half of the previous sentence. It raises at once the question why logic lost its central role in epistemology, including the philosophy of science. We all know when this happened—the Vienna Circle was succeeded by people like Hanson, Kuhn, Lakatos, Feyerabend, and Laudan, just to name a few central ones, who hardly ever mention logic. But *why* did it happen?

Of these philosophers of science, only Feyerabend made some explicit claims on the topic. While arguing, in [31], that inconsistencies often occur in episodes of the history of the sciences, he remarks that 'logic' cannot handle inconsistencies, and comments that this is a problem for logic, not for the history and philosophy of science. I do not think that this diagnosis is correct. For one thing, logics that can handle inconsistencies had been around for a while in those days—see, for example, [29] and many other papers on paraconsistent[1] logics (by da Costa and associates); even some of Jaśkowski's work had been translated into English, for example [33].

*The research for this paper was financed by the Fund for Scientific Research – Flanders, by the Research Fund of Ghent University, and indirectly by the Flemish Minister responsible for Science and Technology (contract BIL98/73). I am indebted to Kristof De Clercq for locating several mistakes and misprints in a former draft of this paper, and to the referees for helpful comments and corrections.

[1]A logic is paraconsistent iff (if and only if) it does not validate ex falso quodlibet ($A, {\sim}A \vdash B$).

The correct diagnosis, it seems to me, is that philosophers of science had been applying the wrong *type* of logic. They had mainly been applying **CL** (Classical Logic). Even the few that recurred to 'non-standard logics' applied modal logic or intuitionistic logic—systems that share their central properties with **CL**. I do not claim that there is anything wrong with those logics. I only claim that they had been used for purposes for which they are unfit.

That the wrong type of logics was chosen was not merely an accident. Mainstream Western epistemology has always been foundational. Notwithstanding some occasional remarks concerning the revision of (mainly) theories, for example Neurath's idea that we have to rebuild our ship in the open sea, the Vienna Circle fitted perfectly within this tradition. 'Protokollsätze' provided the absolute basis. The only epistemological role for logic was to relate Protokollsätze to theories, mainly by deriving consequences from theories together with Protokollsätze. Needless to say, **CL** performs that role in an excellent way, especially as the Vienna Circle's view on the relation between observation and theory was simple and one-dimensional: saving the phenomena.[2]

When Hanson, Kuhn, and the others came around, it soon became clear that the most interesting aspects of scientific reasoning do not concern the relation between observations and theories, *and* that the Vienna Circle view on the relation was simplistic and mistaken—see especially [35]. This led to renewed attention for discovery processes, which was greatly activated by [41] and [42]. Not long thereafter, the study of explanation as a logical relation was replaced by a study of the *process* of explanation: how does one, given an *explanandum* E and a theory, arrive at 'initial conditions' to explain E?[3] And these are just a few examples.

In Section 2, I shall consider some reasoning processes that are essential to epistemology and philosophy of science, and that display an internal and possibly also an external dynamics. Such reasoning clearly cannot be handled by logics of the same type as **CL**. In Section 3, I shall introduce adaptive logics, the type of logics that is able to handle such reasoning processes. Characterizing such logics semantically will enable me, in Section 4, to prove that the consequence relations of these logics have the required properties. Much more important, however, is the dynamic proof theory of adaptive logics, which I shall discuss in Section 5. Unlike the semantics, which characterizes the consequence relations by means of abstract definitions, the dynamic proof theory enables one to explicate the actual reasoning processes. Moreover, the metatheory enables one to show that, notwithstanding their dynamic character, these proofs (i) lead to the correct conclusions in the long run, and (ii) lead to conclusions that provide a basis for decision and action even in the short run.

2 The Problem

The most common mechanism that, in specific situations, leads to general knowledge is inductive generalization. To be more precise, I mean the 'derivation' from data of formulas of the form $\forall A$ (the universal closure of A) and of **CL**-

[2]Logic also had a curative function: to prevent people from talking nonsense. I shall not discuss this here, as the topic has long been settled by now.

[3]See, for example, the work of Hintikka and associates, who were among the few philosophers of science that tried to keep applying logic.

consequences of the data and the generalizations.[4] It has often been argued that there is no logic of induction. The only argument that has ever been adduced for this claim is that the resulting consequence relation is not monotonic: a generalization that is derivable from a set of data need not be derivable after the set of data is extended. Today, however, many non-monotonic consequence relations have been decently defined and well-studied.

Non-monotonic consequence relations display an *external dynamics*. Suppose that $\Gamma \vdash_{\mathbf{LI}} A$ states that A is an inductive consequence (defined by the logic of induction **LI**) of Γ.[5] If Γ is the set of data available at some point in time, then $\Gamma \vdash_{\mathbf{LI}} A$ enables one to accept A. At a later point in time, the set of available data might be $\Gamma \cup \Delta$ and if $\Gamma \cup \Delta \nvdash_{\mathbf{LI}} A$, one has to give up the conclusion A. This dynamics is external in that it does not derive from the reasoning process. Both $\Gamma \vdash_{\mathbf{LI}} A$ and $\Gamma \cup \Delta \nvdash_{\mathbf{LI}} A$ always were true and always will be true. If one knows them to be true, then one justly accepted A at the point in time where Γ was the set of available data, and one justly rejected A after Γ was extended with Δ.

Suppose that one is only interested in generalizations $\forall A$ in which A does not contain quantifiers or individual constants. The basic mechanism behind (thus restricted) inductive reasoning is joint compatibility. A generalization G is inductively derivable from a set of data iff it holds for all sets Δ of generalizations that $\Delta \cup \{G\}$ is compatible with the data whenever Δ is compatible with the data. Given that the data are singular formulas and given the form of the generalizations, the matter is (effectively) decidable. Now, let us make the picture slightly more realistic and suppose that background theories are available. Suppose moreover, to keep things simple, that the data do not contradict the background theories. The available knowledge now consists of the data, the background theories, and their (**CL**-)consequences. Which set of inductive generalizations is compatible with this knowledge is not in general a decidable matter. Worse, there is no positive test for it.[6]

Given the absence of a positive test, how is it possible that people ever arrive at inductive generalizations? The answer is quite obvious: by reasoning. In specific cases, the reasoning may enable one to arrive at a final judgement. However, the reasoning cannot in general lead to a final judgement, even if the premises (data and background generalizations) remain stable during the reasoning process. It can, however, lead to a good estimate. Some people will arrive at a better estimate than others, and the efficiency of such reasoning processes may be studied.

Even if the reasoning does not result in a final judgement, one may consider its outcome sufficiently reliable for making a decision—one may know that a final judgement is impossible, one may consider it too expensive or time consuming to obtain a better judgement, etc.

The absence of a positive test makes the reasoning process necessarily dynamic. Even when reasoning from a stable set of premises, one will have to

[4]Nearly always, background knowledge plays a role—see below in the text.

[5]Actually, the adaptive logic **LI** is described in [15] and several further results are forthcoming in papers by Lieven Haesaert and myself, for example [19]. However, one needs not to know those systems in order to follow the argument in the text.

[6]A positive test for a property is a systematic procedure that leads, after finitely many steps, to a "yes" if the property applies, but may go on forever if it does not. See [27] for such matters.

consider certain formulas as derived provisionally. In other words, it cannot be avoided that, at some point in the reasoning process, one considers as derived certain formulas that later have to be considered as not derived. This I shall call the *internal dynamics*. It is not caused by the introduction of new premises, but is a property of the reasoning process itself, even if it proceeds from a stable premise set. For example, if background knowledge is present, one cannot avoid deriving certain generalizations that later turn out incompatible with the (stable) available knowledge.

In the subsequent paragraphs, I give some more examples of reasoning processes that display an internal dynamics. However, let me stress at once that the discussed logic of inductive generalization is only a special case of a broader phenomenon. Whenever a new theory is adduced, it is supposed to be compatible[7] with available knowledge (the data and formerly accepted theories, or at least part of them). As there is no positive test for compatibility, any reasoning that leads to accepting the new theory necessarily displays the internal dynamics and, except in the specific cases in which a final judgement can be reached, the decision taken as a result of this reasoning is necessarily defeasible and hence provisional.

A recent version of the theory of the process of explanation is presented by Ilpo Halonen and Jaakko Hintikka in [32]. In Section 6, they discuss the conditions on (nonstatistical) explanations (with a number of restrictions). The conditions (I slightly change their notation) concern an explanandum Pb, a background theory T (in which the predicate P occurs) and an initial condition (antecedent condition) I (in which b occurs). Among the six conditions are the following:

(iii) I is not inconsistent ($\nvdash_{\mathbf{CL}} \sim I$).
(iv) The explanandum is not implied by T alone ($T \nvdash_{\mathbf{CL}} Pb$).
(vi) I is compatible with T, i.e. the initial condition does not falsify the background theory ($T \nvdash_{\mathbf{CL}} \sim I$).

Obviously, there is no positive test for any of the three conditions. In other words, no finite reasoning process can (in general) lead to the conclusion that Pb is explained by I and T.

So although the 'logic' appears to be **CL** (see the formal conditions above), it is quite obvious that the reasoning process that leads to the conclusion that I and T together explain Pb cannot possibly be explicated in terms of **CL** (at the object level). The reasoning is *about* **CL**-derivability, and necessarily displays the internal dynamics. This is why it cannot be explicated by a **CL**-proof but only by a dynamic proof.[8]

The logic of questions forms a further example. According to [54] and [53], where this problem is studied and solved, a question Q is evoked by a set of declarative statements Γ iff the (prospective) presupposition of Q is derivable from Γ but no direct answer to Q is derivable from Γ. Note that there is no positive test for non-derivability[9] from Γ. Hence, although the definition is itself

[7] See [20] for the adaptive logic of compatibility (in the framework of **CL**).

[8] See [22] for adaptive logics that explicate several forms of reasoning that underly the search for explanations.

[9] I mean non-**CL**-derivability, in agreement with the cited papers, but the matter is the same for any other sensible logic.

unobjectionable, only a dynamic proof may (in general) lead to the conclusion that Q is evoked by Γ.

Another example concerns handling inconsistency. Consider the case in which a scientific (empirical or mathematical) theory T was meant to be consistent and was formulated with **CL** as its underlying logic, but turned out to be inconsistent. As we know from the literature,[10] scientists do not just throw away such a theory. They *reason* from T in search for a consistent replacement. Of course, they do not reason in terms of **CL**, because this is known to lead to triviality. However, they also do not reason in terms of some monotonic paraconsistent logic **PL**. In their reasoning, they want to interpret T *as consistently as possible*. After all, T was meant as a consistent theory.

Let us consider an utterly simplistic but instructive example. Let the 'theory' consist of $\sim p$, $p \vee r$, $\sim q$, $q \vee s$, and p. One obviously should not derive r from $\sim p$ and $p \vee r$ by Disjunctive Syllogism. To do so would lead to triviality: $p \vee r$ is itself a consequence of p, and so is any formula of the form $p \vee A$. However, as the theory was meant to be consistent, one will apply Disjunctive Syllogism to derive s from $\sim q$ and $q \vee s$. Indeed, it is quite obvious that q behaves consistently on this theory. To be more precise, q is consistently false on the theory, for $\sim q$ is obviously derivable whereas q is not, except of course by explicitly or implicitly applying Ex Falso Quodlibet to p and $\sim p$.[11]

So the reasoning from T should proceed in such a way that one obtains T in its full richness, except for the pernicious consequence of its inconsistency. Precisely for this reason, the reasoning cannot proceed in terms of some monotonic paraconsistent logic **PL**. Indeed, **PL** will invalidate certain **PL**-rules, for example Disjunctive Syllogism.[12] However, as we saw from the previous example, the requested reasoning should not invalidate certain rules of inference of **CL**, but only certain *applications* of these rules. Let me express this more precisely. For certain rules, an application should be valid if specific involved formulas behave consistently on the theory, and invalid otherwise. Precisely this proviso causes the reasoning to be internally dynamic: there is no positive test for the consistent behaviour of some formula on a set of premises.[13]

Up to now we have considered forms of reasoning that display an internal dynamics. All of them concerned a single unstructured set of premises. In many cases, however, the premises are structured, usually as a n-tuple of sets. I shall now consider some examples of this type.

Let us return for a moment to inductive generalization. Usually background knowledge is available in addition to the data. Let us restrict the discussion to the case where the background knowledge consists of generalizations in the sense meant before. An obvious complication is that the data may falsify some of the background generalizations. So two forms of dynamics have to be combined. First, we retain the background generalizations in as far as they are not falsified by the data. Next, from the data and the retained background generalizations we obtain new generalizations by the aforementioned logic **LI**. Remark that it

[10]See for example [43], [44], [49], [28], [40], [37] and [39]. Not all these authors side with me on the required approach, but that is immaterial.

[11]One way to implicitly apply Ex Falso Quodlibet proceeds by first applying Addition to obtain $p \vee q$ from p and next applying Disjunctive Syllogism to obtain q from $\sim p$ and $p \vee q$.

[12]See [38] for an exception: the paraconsistent logic **AN** validates Disjunctive Syllogism (and all 'analysing' rules of **CL**) but invalidates Addition (and Irrelevance and similar rules).

[13]Explicating this kind of reasoning was at the origin of the adaptive logic programme—see [5], [8], and many other papers.

is in general impossible to perform the first selection (of background generalizations) before proceeding to the second selection (of new inductive generalizations). This means that both forms of dynamics are necessarily combined in the reasoning process. After deriving some new inductive generalizations, one may be forced to change one's judgement on the compatibility of some background generalization with the data, and this will affect the derivability of new generalizations.[14]

Often not all background generalizations will be considered equally trustworthy. So instead of a set of background generalizations, one confronts a sequence of such sets, each having a different priority. In this case one has to combine a multiplicity of dynamics concerning the background generalizations with the dynamics that pertains to the new generalizations. Moreover, even certain falsified background generalizations may be considered as applying 'normally'. This means that an *instance* of the generalization is considered to hold unless and until proven incompatible with the data. Such 'pragmatic generalizations' may also be ordered by some priority relation. All this leads to more forms of dynamics (which, however, are all of three kinds).

Let us now consider a very different example. A participant in a discussion may change his or her position in view of arguments adduced by other participants. As a result, the interventions of the participant will be mutually incompatible, even if the participant's position is consistent during each intervention. However, the participant will not state his or her full new position whenever there is a change. So, after an intervention, the participant's position has to be reconstructed from the sequence of his or her interventions. In order to do so, one starts with (the consistent part of) the last intervention, to this one adds that part of the previous intervention that is compatible with it, and so on. Remark that, while doing so, one does not select statements that are made during an intervention, but rather their consequences.[15]

Diagnostic reasoning forms a further example in which the premises are prioritized and hence require a multiplicity of dynamics. Reasoning proceeds from data on the one hand and expectancies (that may have varying degrees of trustworthiness) on the other hand. The expectancies, or rather their consequences, are retained (in their order of priority) until and unless proven inconsistent with the data. (See [52], [47] and [24] for the adaptive logics.)

In all examples mentioned before, the flat ones as well as the prioritized ones, the reasoning displays both the internal dynamics and the external dynamics. It is worth mentioning that, whenever the external dynamics (non-monotonicity) is present, the reasoning necessarily displays the internal dynamics (even if the premises are stable). The converse, however, does not hold. The *Weak* consequence relation, of Rescher and Manor—see, for example, [48] and [25]—is monotonic. Nevertheless, it may be shown that the reasoning from premises to weak consequences requires an internal dynamics.[16] Some consequence relations that are monotonic as well as decidable may even be characterized (in an enlightening and attractive way) by a form of reasoning that displays an internal dynamics—see [13] for an example.

[14]The complication of falsifiable background generalizations is dealt with in [15]. Most complications discussed in the subsequent paragraph of the text are handled in [19].

[15]Adaptive logics for this reconstruction are spelled out in [50] and [16].

[16]A is a Weak consequence of Γ iff it is a **CL**-consequence of some consistent subset of Γ—remember that there is no positive test for consistency.

The preceding paragraphs do by no means contain an exhaustive list of the reasoning mechanisms (or even of the types of reasoning mechanisms) that display an internal dynamics. Nevertheless, the problem should be clear by now. Essential forms of human reasoning, that are common and that are important for understanding the way in which humans arrive at knowledge and revise it, display an internal dynamics.

In order to arrive at a precise theory of knowledge, one needs to explicate such forms of reasoning. To do so requires specific logics: adaptive logics. These unavoidably have some non-standard properties. More important, however, is that they are characterized in a formally stringent way, and that their properties are studied in agreement to the professional standards.

3 What are Adaptive Logics?

In the loose sense of the term, a logic is adaptive iff it adapts itself to the specific premises to which it is applied. This should not be misunderstood. First, I do not mean to say that the consequence set, determined by the logic, depends on the set of premises. This obviously holds for nearly all[17] logics. I mean that the *logic* adapts to the premises in that it depends on properties of the premise set whether some formula is derivable from some of the premises. Next, I really mean that the logic adapts *itself* to the premises. The reasoner does not interfere in this. The logic is defined by a set of rules as well as by a semantics. Both lead to the adaptive effect, independently of any decision of the human or machine that applies the rules.

The previous paragraph describes the underlying idea. I shall also present a more technical characterization. This should not be understood as a definition, but rather as a hypothesis on the properties of all adaptive logics. It relies on present best insights. These may change as more logics are studied or new insights in them are gained. I have a good reason to insert this remark: during the last twenty years the dynamics of the adaptive logics programme forced the Ghent logic group several times to revise the technical characterization.

Flat (non-prioritized) adaptive logics. I start with these because the prioritized ones may be seen as (systematic) combinations of them. An adaptive logic **AL** may be characterized by a triple: the lower limit logic, the set of abnormalities, and the strategy. The *lower limit logic* **LLL** is a monotonic logic for which it holds that $Cn_{\mathbf{LLL}}(\Gamma) = \bigcap\{A \mid \Gamma \cup \Delta \vdash_{\mathbf{AL}} A; \emptyset \subseteq \Delta \subseteq \mathcal{W}\}$, in which $\mathcal{W}$ is the set of all closed formulas of the language.[18] Intuitively, the lower limit logic is the stable part of the adaptive logic, the part that is not subject to any adaptation. From a proof theoretic point of view, the lower limit logic delineates the rules of inference that hold unexceptionally. From a semantic point of view,

[17]The two obvious exceptions are zero logic, according to which nothing is derivable from any premise set (not even the premises themselves) and trivial logic according to which everything is derivable from any premise set. These logics may seem completely uninteresting, but actually zero logic is not. From it, an adaptive logic may be defined, thus making all logical reasoning contingent on specific properties of the premises (put differently: on 'the world'). For example, adaptive zero logic assigns to consistent sets of premises exactly the same consequence set as **CL**. See [10] for a study of zero logic and the (most straightforward) adaptive logic definable from it.

[18]$Cn_{\mathbf{L}}(\Gamma)$ abbreviates $\{A \in \mathcal{W} \mid \Gamma \vdash_{\mathbf{L}} A\}$ as usual.

all adaptive models of Γ are lower limit models of Γ (but not conversely). It follows that $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma)$.

Suppose that we are dealing with a context in which **CL** is taken as the standard of deduction. If the lower limit logic of **AL** is **CL** (or, for example, a modal extension of **CL**), it is said that **AL** is *ampliative*. This is the case for inductive generalization (without background knowledge), for compatibility, etc. If the lower limit logic is weaker than **CL**, as in the case of inconsistency-adaptive logics, the adaptive logic is called *corrective*—the theory was intended to be interpreted in terms of **CL**, but turned out to be inconsistent and hence is interpreted as consistently as possible.[19]

If the lower limit logic is a fragment of **CL**, it is wise to extend it with the missing classical logical symbols (by means of explicit definitions or, where this is impossible, by a straightforward extension of the language—see [10]). This changes nothing to the way in which the premises are handled, but greatly simplifies the metatheory.

The second component of an adaptive logic is the set of *abnormalities* Ω. These are the formulas that are presupposed to be false, unless and until proven otherwise. In the standard format, Ω is characterized by a metalinguistic formula that may be restricted. In handling inconsistency, the set of abnormalities comprises the formulas of the form $\exists(A \wedge \sim A)$, in which $\exists A$ abbreviates the existential closure of A. The set may be restricted. For some lower limit logics the set is restricted by the requirement that A be a primitive formula (a formula containing no logical symbols except for identity). In the case of an inductive logic, the set of abnormalities may consist of all formulas of the form $\exists A \wedge \exists{\sim}A$ with the restriction that no individual constants or quantifiers occur in A.

Extending the lower limit logic with the requirement that no abnormality is logically possible results in a monotonic logic, which is called the *upper limit logic*. The effect is presumably most easily seen by considering the semantics. The upper limit logic is characterized by the lower limit logic models that verify no abnormality. If the adaptive logic is corrective, the lower limit logic is weaker than **CL**, and the upper limit logic will usually be (and in all cases studied up to now is) **CL**. If the adaptive logic is ampliative, the lower limit is (in all cases studied so far) **CL** or a modal extension of **CL**, and the upper limit logic is an extension of this.

Some examples are useful to clarify the matter. If the lower limit logic is **CL** and the set of abnormalities comprises all formulas of the form $\exists A \wedge \exists{\sim}A$ (see two paragraphs ago), then the upper limit logic is **CLU**, obtained by adding to **CL** the axiom $\exists A \supset \forall A$.[20] If, as in the case of an inconsistency-adaptive logic, the lower limit logic is a paraconsistent logic **PL** that is a fragment of **CL**, and the set of abnormalities comprises all formulas of the form $\exists(A \wedge \sim A)$, then the upper limit logic is **CL**. The importance of the set of abnormalities will be obvious once a strategy is chosen—see below. If the premise set does not require any abnormality to obtain, the adaptive logic will deliver the same consequences as the upper limit logic. If the premise set requires some abnormalities to obtain, the adaptive logic will still deliver more consequences than the lower limit logic,

[19]Remark that the lower limit logic may be zero logic—see footnote 17.

[20]Semantically, this logic is characterized by those **CL**-models in which, for every predicate π of adicity i, $v(\pi) \in \{D^i, \emptyset\}$ in which D^i is the i-th Cartesian product of the domain. The name **CLU** refers to the fact that this logic is characterized by **CL**-models that are (completely) uniform: all elements of the domain have the same properties.

viz. all upper limit consequences that are not 'blocked' by those abnormalities.

It became only clear during the last years that the combination of a lower limit logic with different sets of abnormalities may result in the same upper limit logic but in a different adaptive logic. This may be easily exemplified in terms of inconsistency-adaptive logics. Consider a lower limit logic that validates all reductions of negations (double negation, de Morgan laws, the standard negation laws for the quantifiers, etc.). Whether the set of abnormalities Ω comprises all formulas of the form $\exists(A \wedge {\sim}A)$, or is restricted to the case in which A is primitive, the upper logic is **CL**. However, on both the Reliability strategy and the Minimal Abnormality Strategy, the resulting adaptive logic is very different. The latter set of abnormalities defines an inconsistency-adaptive logic of the usual kind, whereas the former set of abnormalities defines an adaptive logic of a somewhat weird kind, viz. a flip–flop—see below.

A very important matter has to be brought up at this point. For all that was said before, an adaptive logic is obtained by presupposing that all formulas behave normally, except for those that need to behave abnormally in view of the premises. This formulation suggests that there is a well-defined set of formulas that behave abnormally in view of the premises, but this need not be the case. The complication derives from the fact that a set of premises may entail a disjunction of abnormalities (members of Ω) without entailing any of its disjuncts.[21] Let us again consider the adaptive logic of induction and let the premise set be $\{Pa, Qa, Rb, {\sim}Qb\}$. Even with so small a data set, $(\forall x)(Px \supset Qx)$ and $(\forall x)(Rx \supset {\sim}Qx)$ are derivable. Suppose next that the premise set is extended to $\{Pa, Qa, Rb, {\sim}Qb, Pc, Rc\}$. Neither $(\exists x)(Px \wedge Qx) \wedge (\exists x){\sim}(Px \wedge Qx)$ nor $(\exists x)(Rx \wedge Qx) \wedge (\exists x){\sim}(Rx \wedge Qx)$ is **CL**-derivable from these premises. However, their (classical) disjunction *is* **CL**-derivable from the premises.

Classical disjunctions[22] of abnormalities will be called *Dab-formulas* and will be written as $Dab(\Delta)$, in which $\Delta \subset \Omega$ is finite.[23] The *Dab*-formulas that are derivable by the lower limit logic from the premise set Γ will be called *Dab-consequences* of Γ. If $Dab(\Delta)$ is a *Dab*-consequence of Γ, then so is $Dab(\Delta \cup \Theta)$ for any (finite) Θ. For this reason, the following definition is important. Let **LLL** be the lower limit logic as before.

Definition 1 *$Dab(\Delta)$ is a* minimal *Dab-consequence of Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and there is no $\Theta \subset \Delta$ such that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta)$.*

That $Dab(\Delta)$ is a minimal *Dab*-consequence of Γ means that it is derivable (by the lower limit logic) from Γ that some member of Δ behaves abnormally, whereas it is not derivable which member of Δ behaves abnormally. Adaptive logics are obtained by interpreting a set of premises 'as normally as possible'. But clearly, this phrase is not unambiguous. This is why we need to disambiguate it by choosing a specific adaptive strategy.

The oldest known *strategy* is *Reliability* from [5], where it is discussed at the propositional level. Let $U(\Gamma) = \{A \mid A \in \Delta$ for some minimal *Dab*-consequence $Dab(\Delta)$ of $\Gamma\}$ (the set of formulas that are unreliable on Γ). The Reliability

[21] This holds for nearly all combinations of lower limit logics and sets of abnormalities. I shall mention exceptions when I introduce the Simple strategy.

[22] Remember that, whenever a logical symbol of the original language is not classical, then the language is extended with the corresponding classical symbol.

[23] Note that $Dab(\Delta)$ is the classical disjunction of the members of Δ. In some previous papers on specific adaptive logics, $Dab(\Delta)$ has a slightly different function.

strategy considers a formula as behaving abnormally iff it is a member of $U(\Gamma)$. As for the other strategies, the effect of this on the semantics and proof theory will be discussed in subsequent sections.

The *Minimal Abnormality* strategy (presented in [4] but first fully studied in [8]) delivers some more consequences than the Reliability strategy. If, for example, $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal *Dab*-consequences of Γ, the Minimal Abnormality strategy takes one member of each Δ_i to behave abnormally, while all other formulas behave normally.[24] Obviously, the Minimal Abnormality strategy does not pick out a specific such combination, but considers all of them. Here is a simple propositional example for an inconsistency-adaptive logic: $\Gamma = \{\sim p, \sim q, p \vee q, p \vee r, q \vee r\}$. If the lower limit logic validates all of full positive logic, $(p \wedge \sim p) \vee (q \wedge \sim q)$ is a minimal *Dab*-consequence of Γ. On the Reliability strategy, both $p \wedge \sim p$ and $q \wedge \sim q$ are unreliable with respect to Γ, and hence r is not an adaptive consequence of Γ. However, if the Minimal Abnormality strategy is chosen, then r is an adaptive consequence of Γ. Indeed, if $p \wedge \sim p$ behaves abnormally, then $q \wedge \sim q$ behaves normally and hence r is true in view of $\sim q$ and $q \vee r$; if $q \wedge \sim q$ behaves abnormally, then $p \wedge \sim p$ behaves normally and hence r is true in view of $\sim p$ and $p \vee r$. In subsequent sections, we shall see that both strategies are simple and perspicuous from a semantic point of view, and that the Reliability strategy leads to simple dynamic proofs, but that the dynamic proofs determined by the Minimal Abnormality strategy are rather complicated. Which strategy is adequate in a specific context of application is obviously a very different matter.

For some specific lower limit logics and sets of abnormalities, any minimal *Dab*-consequence $Dab(\Delta)$ of any premise set is such that Δ is a singleton. In such cases, the Reliability and Minimal Abnormality strategies lead to the same result and coincide with what is called the *Simple* strategy: a formula behaves abnormally just in case the abnormality is derivable from the premise set. Examples may be found in [38] and [20].

Several other strategies have been studied, but seem to have a less general import. Most of them were the result of characterizing an existing consequence relation by an adaptive logic. Examples may be found in [12], [18], [30] and [51].

A different way to characterize most flat adaptive logics is by seeing them as formula-preferential systems. The idea was first presented in [36] (see also [2]). This means that Ω is taken to be an arbitrary set of formulas. We shall see that the idea is sensible in general for so-called direct formulations of prioritized adaptive logics. I am not sure that it will work for all adaptive logics. Moreover, many sets of formulas do not define a sensible upper limit logic. The idea to interpret a set of premises 'as much as possible' in agreement with the upper limit logic is an attractive feature of the adaptive logic enterprise and is lost if they are rephrased as formula-preferential systems.

Ongoing work concerns an adaptive logic of inductive prediction. If ω-incomplete models are considered, one needs to define the 'abnormal part' of a model in terms of the model itself rather than in terms of the formulas it verifies. The first example of an approach in terms of models was Graham Priest's system $\mathbf{LP}^m$ from [46]—see [9] for a discussion of some mistakes in this construction.

[24]For some sets of minimal *Dab*-consequences, at least two members of some Δ_i behave abnormally—see [8, p. 468] for an example.

Prioritized adaptive logics and combining adaptive logics. Consider $\Sigma = \langle \Gamma_0, \ldots, \Gamma_n \rangle$, in which Γ_0 is a set of data (that are taken to be certain) and $\Gamma_1, \ldots, \Gamma_n$ are sets of expectancies—formulas that are supposed to obtain but may be overruled. The members of Γ_i ($1 \leq i \leq n$) carry a higher degree of certainty as i is smaller.

One prioritized adaptive logic to handle such n-tuples is obtained as follows. Where $\lozenge^i$ abbreviates a sequence of i diamonds, $\Sigma^{\lozenge} = \{\lozenge^i A \mid A \in \Gamma_i\}$. The lower limit logic is (for example) the modal logic[25] **T** and the set of abnormalities comprises all formulas of the form $\lozenge^i A \wedge \sim A$, in which A is either a primitive formula or the negation of a primitive formula and $1 \leq i \leq n$. The upper limit logic is **Triv**, which is obtained by extending **T** with (for example) the axiom $\lozenge A \supset A$. Finally, one combines the above with either the Reliability or Minimal Abnormality strategy with the following proviso: an abnormality $\lozenge^i A \wedge \sim A$ is considered as worse according as i is smaller. This means that, if either an abnormality of level i or an abnormality of level j is unavoidable in view of the premises, then the abnormality of level i is avoided if $i < j$. The results are the nice adaptive logics $\mathbf{T}^{sr}$ and $\mathbf{T}^{sm}$ from [24]. For more examples see [19], [50] and [51], the latter containing adaptive logics that characterize all prioritized Rescher–Manor consequence relations from [26].

A different way to characterize prioritized adaptive logics is by seeing them as the result of applying a sequence of flat adaptive logics, each of these logics having a (nearly) similar structure. This characterization of prioritized adaptive logics is a special case of a more general mechanism, viz. that adaptive logics, which may have a very different structure, are combined with each other. Consider the search for an explanation of some singular fact, given a theory. If one relies on Hintikka's conditions, one needs an adaptive logic to deal with the conditions for which there is no positive test—see Section 2. Suppose that moreover the involved theory is inconsistent (or that the data are inconsistent)—see [17] for a discussion of such inconsistencies. In such a case one needs to combine the adaptive logic for the process of explanation with an inconsistency-adaptive logic that interprets the theory as consistently as possible. The result is a sequence of (two) adaptive logics.

The term 'sequence' deserves a comment. The definition of the logic refers to such a sequence, and intuitively one may understand the logic as resulting from applying one adaptive mechanism after the other. However, as there is no positive test for any of the two, it is essential that the dynamic proof theory is able to handle all the adaptive steps in any order.

Applying the above logics $\mathbf{T}^{sr}$ and $\mathbf{T}^{sm}$ requires the transition from the n-tuple Σ to the set $\Sigma^{\lozenge}$.[26] Given such transition, the characterization of the prioritized consequence relation comes to a combination of (similar) flat adaptive logics.

It might be objected that the prioritized adaptive logic does not define the consequence relation itself, but defines it only under some translation. The matter is not as simple as it looks. If A is not a premise but only an expectancy, it seems desirable that this is expressed in the object language. Typically, in

[25] Given the variety of predicative extensions of propositional modal logics, one needs to pick a specific predicative version of **T**. I shall only discuss the propositional case here and refer to [24] for a predicative version that is adequate for diagnosis logic.

[26] Actually, the situation is similar for several flat adaptive logics that characterize formerly known consequence relations.

[52], which started the work on adaptive logics for diagnosis, $E(A)$ is used to express that A is an expectancy. So, from a philosophical viewpoint, rendering an expectancy of degree i as $\Diamond^i A$ is superior to an approach in terms of a sequence of sets of premises.

Still, if the adaptive logic is intended to characterize an existing prioritized consequence relation, one might be interested in a faithfully covering. In all cases studied up to now, we were able to also articulate direct formulations. These are formulations in terms of the original language. The same applies for the dynamic proofs. It not completely clear whether these formulations fit within the technical characterization of an adaptive logic. Apparently, more work is required before the matter can be settled. However, it is quite obvious that the direct formulations may be seen as adaptive logics in the sense of formula-preferential logics: the logic selects the models that verify 'as much as possible' the members of each $\Gamma_i \in \Sigma$ (in its order of priority). If adaptive logics are seen as formula preferential, and the set of abnormalities is arbitrary (and not defined by some logical form), the lower limit logic and the requirement that no abnormalities obtain might not together define an upper limit logic. If the priorities are expressed in the object language (for example as $\Diamond^i A$), abnormalities have a specific logical form (in the example $\Diamond^i A \wedge {\sim}A$ in which A is either a primitive formula or its negation). This matter too requires more study.

Some further comments. In the subsequent paragraphs, I mention some miscellaneous findings. Some of them created created confusion at a time, others have not yet been settled.

It is not absolutely clear whether all adaptive logics are either flat or prioritized. Several times our research group discovered items that did not seem to fit in either category. Sooner or later, the exception turned out to be apparent only. Often such apparent exceptions led to a better understanding of adaptive logics, and sometimes to broadening the notion.

A very different matter concerns the introduction of new premises. From a traditional point of view, extensions of the premise set are logically uninteresting. An extension of the premise set leads to a distinct premise set, and distinct premise sets may define distinct consequence sets. If the logic is non-monotonic, extending the premise set may result in a consequence set that displays gain as well as loss with respect to the original consequence set. In the context of adaptive logics the matter is slightly more interesting. Often the very reasoning that is performed in terms of an adaptive logic, or in terms of the consequence relation that is explicated by the adaptive logic, leads to the introduction of new premises. I shall mention two simple examples, a straightforward one first, and a somewhat unexpected one next—both concern forthcoming work.

Suppose that one is reasoning in terms of an adaptive logic—say a flat one— and that $Dab\{A_1, \ldots, A_n\}$ appears to be a minimal *Dab*-consequence of the premise set. Often this very fact causes one to search for good reasons to narrow down the minimal *Dab*-consequence and sometimes good reasons are found. So reasoning from the premises may cause one to extend the premises. Of course the new premises do not provide from the reasoning; the aforementioned good reasons are not provided by the reasoning. But the search for specific new premises is.

Let us consider an example. Suppose that one applies an inconsistency-adaptive logic and hence that the abnormalities have the form $\exists(B \wedge \sim B)$. Finding out that $Dab\{A_1, \ldots, A_n\}$ is apparently a minimal disjunction of abnormalities may trigger the question as well as the resulting insight that there are good reasons not to blame specific A_i. For example, these A_i may pertain to well-entrenched theories, or to observational criteria that are considered as unproblematic. In such cases, one may want to posit that those A_i are not abnormal. There are several ways to handle such situation. Each of them involves some complications that fall beyond the scope of the present paper—not because they are difficult, but because they require too much space.[27] This feature is not typical for corrective adaptive logics. For example, in the context of inductive adaptive logics it leads to introducing generalizations that are not justifiable in terms of the data, but possibly in terms of a worldview or some other bias. Typically, the new premises are conjectures, expressing claims that transcend theories and observations, and hence should be handled in such a way that they are revokable in view of the internal dynamics of the reasoning—that is in view of later gained insights in the meaning of the original premises.

I promised a second, somewhat unexpected example. Consider a problem-solving process, even one of a very basic kind, in which empirical data are relevant. Empirical data may be gathered, by observation or experiment, but gathering them may be expensive or time consuming. So some observations and experiments will be postponed until they turn out relevant for the problem-solving process. And indeed, the reasoning in terms of the suitable adaptive logic will indicate that some information is relevant or is possibly relevant.[28] Note that such considerations are quite remote from the traditional logical viewpoint. The problem one is facing is not whether A is derivable from a premise set Γ; the problem is not what is the consequence set of Γ. The problem is, in the simplest case, to settle whether A is true or not in view of the premises *and* of the available empirical means.

A very different type of adaptive logic is most easily illustrated in terms of inconsistency. Suppose that one confronts an inconsistent theory, that the inconsistency is taken to render the theory inadequate, and that one is interested in the 'consistent part' of the theory, which one deems unproblematic. To locate this consistent part (in other words, to obtain consistency by brute force) is a task for logic. Given the absence of a positive test for consistency, it is a task for adaptive logic. The task seems to be a difficult one; so far no adequate adaptive logic has been characterized.

The last comment concerns an amusing phenomenon. After first reading or hearing about adaptive logics, some people think that adaptive logics are (what I like to call) flip–flops. Thus some people think that inconsistency-adaptive logics (i) deliver the classical consequences of consistent premise sets, and (ii) deliver the paraconsistent consequences (defined by the lower limit logic) of inconsistent premise sets. While (i) is correct, (ii) is false for most inconsistency-adaptive logics. Even if the premise set is abnormal, most adaptive logics still interpret the set as normally as possible, and hence deliver more consequences than the lower limit logic. Nevertheless, it was amusing to discover that it is very easy

[27] The central point is that, in the absence of a positive test for the minimality of the *Dab*-formula, the new premises have to be defeasible—see also Section 5.

[28] Obviously interesting problem-solving processes are not algorithmic. A road that seems attractive at one point may later turn out to be a dead end.

to define flip–flops. These are indeed adaptive logics: they adapt themselves to the premise set, even if only in the crudest possible way.

4 Semantics

The dynamic proof theory of adaptive logics is certainly their most fascinating feature. It was this proof theory that led to the discovery of adaptive logics—see [5]. I nevertheless start by discussing the semantics because this will be easier for most people.

Let us consider an arbitrary flat adaptive logics **AL**, defined from a lower limit logic **LLL**, a set of abnormalities Ω and a strategy. I shall moreover suppose that the following conditions are fulfilled:

C1 **LLL** is a monotonic logic.
C2 **LLL** is left and right compact.[29]
C3 Ω is a set of formulas characterized by a (possibly restricted) logical form.
C4 **AL** is a flat adaptive logic defined from **LLL** and Ω by either the Reliability strategy or the Minimal Abnormality strategy.[30]
C5 All classical logical symbols are present in **LLL**—see Section 3.

Where it matters, I refer to the strategy by the name of the adaptive logic thus: $\mathbf{AL}^r$ and $\mathbf{AL}^m$.

The use of C5 was explained in Section 3.[31] Even for corrective adaptive logics the presence of classical negation, which I shall write as $\neg$, is often not required for extending the axiomatic characterization of the lower limit logic **LLL** into an axiomatic characterization of the upper limit logic **ULL**.[32]

Definition 2 *The upper limit logic* **ULL** *is semantically characterized by the* **LLL***-models that verify no member of* Ω*.*

I shall suppose that an adequate semantics for **LLL** is present, whence Definition 2 provides an adequate semantics for **ULL**. In view of this I shall pass freely from the proof theory to the semantics.

The **AL**-models of a premise set Γ are a subset of the **LLL**-models of Γ. How the selection is made depends on the strategy. For both strategies, we need the set of abnormalities verified by the model M: $Ab(M) = \{A \in \Omega \mid M \models A\}$. For the Reliability strategy, we moreover need the minimal *Dab*-consequences of Γ (defined by the semantic counterpart of Definition 1), and $U(\Gamma)$ is defined as in Section 3, viz. as $\{A \mid A \in \Delta$ for some minimal *Dab*-consequence $Dab(\Delta)$ of $\Gamma\}$.

Definition 3 *A* **LLL***-model* M *of* Γ *is* reliable *iff* $Ab(M) \subseteq U(\Gamma)$*.*

[29] Here are semantic versions. Left compactness: Γ has a model iff every finite $\Gamma' \subseteq \Gamma$ has a model; right compactness: every model of Γ verifies a member of Δ iff every model of Γ verifies a member of some finite $\Delta' \subseteq \Delta$. In the presence of classical negation, left compactness warrants right compactness.

[30] Whenever the Simple strategy is sensible, the theorems below extend to it immediately because both Reliability and Minimal Abnormality come to the Simple strategy in such cases.

[31] In [8] the adaptive logics **ACLuN1** and **ACLuN2** are defined and studied in the presence of an object language that does not contain classical negation. In [11], Strong Reassurance is proved in the presence of a similar language.

[32] If A is the logical form characterizing Ω, then, in the presence of classical implication, extending **LLL** with the axiom schema $\mathsf{A} \supset B$ delivers a characterization of **ULL**.

Definition 4 $\Gamma \vDash_{\mathbf{AL}^r} A$ *iff* A *is verified by all reliable models of* Γ.

Definition 5 *A* **LLL**-*model* M *of* Γ *is* minimal abnormal *iff there is no* **LLL**-*model* M' *of* Γ *such that* $Ab(M') \subset Ab(M)$.

Definition 6 $\Gamma \vDash_{\mathbf{AL}^m} A$ *iff* A *is verified by all minimal abnormal models of* Γ.

I shall prove several theorems for all adaptive logics under consideration. To simplify the notation, let $\mathcal{M}_\Gamma^{\mathbf{L}}$ denote the set of **L**-models of Γ. Where the logic is $\mathbf{AL}^r$ or $\mathbf{AL}^m$, I shall write $\mathcal{M}_\Gamma^r$ and $\mathcal{M}_\Gamma^m$ respectively.

Given that the adaptive logics were defined by a selection of lower limit models of the premise set, it is important to prove that this selection has suitable properties. A very strong property is that, for any lower limit model M that is not selected ($M \notin \mathcal{M}_\Gamma^{\mathbf{AL}}$), there is a selected model M' that is less abnormal than M ($Ab(M') \subset Ab(M)$). I call this property Strong Reassurance, Avron calls it Stopperedness, and it is closely related to what is called Smoothness in [34]. That its absence leads to undesired results is shown, for example, in [11]. I first prove the property for the Minimal Abnormality strategy. An unqualified "model" will always refer to a **LLL**-model.

Theorem 1 *If* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$, *then there is a* $M' \in \mathcal{M}_\Gamma^m$ *such that* $Ab(M') \subset Ab(M)$. *(Strong Reassurance for Minimal Abnormality.)*

Proof. The theorem holds (vacuously) if Γ has no **LLL**-models or if $\mathcal{M}_\Gamma^m = \mathcal{M}_\Gamma^{\mathbf{LLL}}$. Consider a $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$, let $D_1, D_2, \ldots$ be a list of all members of Ω, and define[33]

$$\Delta_0 = \emptyset$$

$$\Delta_{i+1} = \Delta_i \cup \{\neg D_{i+1}\}$$

if there is a model M' of $\Gamma \cup \Delta_i \cup \{\neg D_{i+1}\}$ such that $Ab(M') \subseteq Ab(M)$, and

$$\Delta_{i+1} = \Delta_i$$

otherwise. Finally,

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$$

Given the left compactness of **LLL**, $\Gamma \cup \Delta$ has models in view of the construction.

Step 1. I first show that, if M' is a model of $\Gamma \cup \Delta$, then $Ab(M') \subset Ab(M)$. Suppose that there is a $D_j \in \Omega$ such that $D_j \in Ab(M') - Ab(M)$. Let M'' be a model of $\Gamma \cup \Delta_{j-1}$ for which $Ab(M'') \subseteq Ab(M)$. As $D_j \notin Ab(M)$, $D_j \notin Ab(M'')$. Hence M'' is a model of $\Gamma \cup \Delta_{j-1} \cup \{\neg D_j\}$ and $Ab(M'') \subseteq Ab(M)$. So $\neg D_j \in \Delta_j \subseteq \Delta$. As M' is a model of $\Gamma \cup \Delta$, $D_j \notin Ab(M')$. But this contradicts the supposition.

Step 2. I now show that every model of $\Gamma \cup \Delta$ is a minimal abnormal model of Γ. Suppose that M' is a model of $\Gamma \cup \Delta$, but is not a minimal abnormal model of Γ. Hence, by Definition 5, there is a model M'' of Γ for which $Ab(M'') \subset Ab(M')$.

It follows that M'' is a model of $\Gamma \cup \Delta$. If it were not, then, as M'' is a model of Γ, there is a $\neg D_j \in \Delta$ such that M' verifies $\neg D_j$ and M'' falsifies

[33] Recall that $\neg$ is classical negation.

$\neg D_j$. But then M' falsifies D_j and M'' verifies D_j, which is impossible in view of $Ab(M'') \subset Ab(M')$.

Consider any $D_j \in Ab(M') - Ab(M'') \neq \emptyset$. As M'' is a model of $\Gamma \cup \Delta_{j-1}$ that falsifies D_j, it is a model of $\Gamma \cup \Delta_{j-1} \cup \{\neg D_j\}$. As $Ab(M'') \subset Ab(M')$ and $Ab(M') \subseteq Ab(M)$, $Ab(M'') \subset Ab(M)$. It follows that $\Delta_j = \Delta_{j-1} \cup \{\neg D_j\}$ and hence that $\neg D_j \in \Delta$. But then $D_j \notin Ab(M')$. Hence, $Ab(M'') = Ab(M')$. So the supposition leads to a contradiction. ■

As $\mathcal{M}^m_\Gamma \subseteq \mathcal{M}^r_\Gamma$ (by property 1 of Theorem 3 below), it follows that:

Theorem 2 *If $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^r_\Gamma$, then there is a $M' \in \mathcal{M}^r_\Gamma$ such that $Ab(M') \subset Ab(M)$. (Strong Reassurance for Reliability.)*

Corollary 1 *If Γ has lower limit models, then it has minimal abnormal models as well as reliable models. (Reassurance.)*

At this point, I need some lemmas that require a specific characterization of the minimal abnormal models. In [8], such a characterization is offered in terms of a set Φ_Γ, which is a set of sets of abnormalities.[34] It is shown there that, where M is a minimal abnormal model of some Γ, $Ab(M)$ is characterized by some $\phi \in \Phi_\Gamma$—all members of $Ab(M)$ are **LLL**-consequences of some $\phi \in \Phi_\Gamma$.

Recently, I found a drastically simpler such characterization, which only has the disadvantage to be less 'finitistic'. I shall now redefine Φ_Γ. The proofs in [8] may be easily modified in view of this change and may be generalized to all adaptive logics considered, but I cannot, in the present paper, spell out the required modifications to the proofs. Let Φ°_Γ comprise all sets that contain a disjunct out of each minimal *Dab*-consequence of Γ and that are **LLL**-closed with respect to Ω.[35] Let Φ_Γ contain all members of Φ°_Γ that are not supersets of other members of Φ°_Γ. Suitably modifying and generalizing the proof of Lemmas 7.2 and 7.3 of [8] gives us:

Lemma 1 *M is a minimal abnormal model of Γ iff $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ and $Ab(M) \in \Phi_\Gamma$.*

The strength of this lemma may be seen from the fact that each of the following properties are immediate or nearly immediate consequences of it:[36]

Theorem 3 *Each of the following holds:*

1. *Every minimal abnormal model of Γ is a reliable model of Γ ($\mathcal{M}^m_\Gamma \subseteq \mathcal{M}^r_\Gamma$). Hence $Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma)$.*

2. *If $A \in \Omega - U(\Gamma)$, then $\neg A \in Cn_{\mathbf{AL}^r}(\Gamma)$.*

3. *If $Dab(\Delta)$ is a minimal Dab-consequence of Γ and $A \in \Delta$, then there is a minimal abnormal model M of Γ that verifies A and falsifies all members (if any) of $\Delta - \{A\}$.*

[34] In [8] Φ_Γ is a set of sets of factors of abnormalities, but this modification is inconsequential and factors of abnormalities are a nuisance in the present setup.

[35] By the second half of the requirement I mean that $\varphi = Cn_{\mathbf{LLL}}(\varphi) \cap \Omega$.

[36] As the proof theory is only defined in the following section, expressions such as $Cn_{\mathbf{AL}^r}(\Gamma)$ refer to $\{A \in \mathcal{W} \mid \Gamma \vDash_{\mathbf{AL}^r} A\}$.

4. *All minimal abnormal models of* Γ *are minimal abnormal models of* $Cn_{\mathbf{AL}^m}(\Gamma)$ *and vice versa* ($\mathcal{M}^m_\Gamma = \mathcal{M}^m_{Cn_{\mathbf{AL}^m}(\Gamma)}$) *and hence* $Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{AL}^m}(Cn_{\mathbf{AL}^m}(\Gamma))$. *(Fixed Point.*[37]*)*

5. *All reliable models of* Γ *are reliable models of* $Cn_{\mathbf{AL}^r}(\Gamma)$ *and vice versa* ($\mathcal{M}^r_\Gamma = \mathcal{M}^r_{Cn_{\mathbf{AL}^r}(\Gamma)}$) *and hence* $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^r}(Cn_{\mathbf{AL}^r}(\Gamma))$. *(Fixed Point.)*

6. *For all* $\Delta \subseteq \Omega$, $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ *iff* $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. *(Immunity.)*

7. *If* $\Gamma \vDash_{\mathbf{AL}} A$ *for every* $A \in \Gamma'$, *and* $\Gamma \cup \Gamma' \vDash_{\mathbf{AL}} B$, *then* $\Gamma \vDash_{\mathbf{AL}} B$. *(Cautious Cut.)*

8. *If* $\Gamma \vDash_{\mathbf{AL}} A$ *for every* $A \in \Gamma'$, *and* $\Gamma \vDash_{\mathbf{AL}} B$, *then* $\Gamma \cup \Gamma' \vDash_{\mathbf{AL}} B$. *(Cautious Monotonicity.)*

These properties are well-known from the study of non-monotonic logics. They ensure that, although adaptive logics are non-monotonic, they fulfil a set of desirable properties. Thus Immunity warrants that the transition from the lower limit logic to the adaptive logic does not lead to the derivability of new disjunctions of abnormalities (or to the non-derivability of old disjunctions of abnormalities). Cautious Cut warrants that extending a premise set with some of its adaptive consequences does not lead to any gain in consequences, and Cautious Monotonicity warrants that extending a premise set with some of its adaptive consequences does not lead to any loss of consequences. So, if the adaptive logic is applied to a belief set, its consequences may be considered as beliefs themselves.

A premise set Γ will be called normal if $\mathcal{M}^{\mathbf{ULL}}_\Gamma \neq \emptyset$; it is called abnormal otherwise. Note that Γ is normal iff $\Omega \cap Cn_{\mathbf{LLL}}(\Gamma) = \emptyset$.

Theorem 4 *Each of the following obtains:*

1. $\mathcal{M}^{\mathbf{ULL}}_\Gamma \subseteq \mathcal{M}^m_\Gamma \subseteq \mathcal{M}^r_\Gamma \subseteq \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *and hence* $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

2. *If* Γ *is normal, then* $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \mathcal{M}^m_\Gamma = \mathcal{M}^r_\Gamma$ *and hence* $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

3. *If* Γ *is abnormal and* $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$, *then* $\mathcal{M}^{\mathbf{ULL}}_\Gamma \subset \mathcal{M}^m_\Gamma$ *and hence* $Cn_{\mathbf{AL}^m}(\Gamma) \subset Cn_{\mathbf{ULL}}(\Gamma)$.[38]

4. $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *iff* $\Gamma \cup \{A\}$ *is* **LLL***-satisfiable for some* $A \in \Omega - U(\Gamma)$.

5. $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^r}(\Gamma)$ *iff* $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$.

6. $\mathcal{M}^m_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *iff there is a (possibly infinite)* $\Delta \subseteq \Omega$ *such that* $\Gamma \cup \Delta$ *is* **LLL***-satisfiable and there is no* $\varphi \in \Phi_\Gamma$ *for which* $\Delta \subseteq \varphi$.

7. *If there are* $A_1, \ldots, A_n \in \Omega$ $(n \geq 1)$ *such that* $\Gamma \cup \{A_1, \ldots, A_n\}$ *is* **LLL***-satisfiable and* $\{A_1, \ldots, A_n\} \nsubseteq \varphi$ *for every* $\varphi \in \Phi_\Gamma$, *then* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^m}(\Gamma)$.

[37]The label might suggest that recurrent applications of some closure operation ultimately lead to a fixed point. This, however, is not the case: a single application of the closure operation leads to a fixed point ($Cn_{\mathbf{AL}^m}(\Gamma)$ is a fixed point with respect to $\mathbf{AL}^m$-closure).

[38]If Γ is abnormal, it has no **ULL**-models and $Cn_{\mathbf{ULL}}(\Gamma)$ is trivial.

8. $Cn_{\mathbf{AL}^m}(\Gamma)$ and $Cn_{\mathbf{AL}^r}(\Gamma)$ are non-trivial iff $\mathcal{M}_\Gamma^{\mathbf{LLL}} \neq \emptyset$.

Proof. Ad 2. If Γ is normal, then $U(\Gamma) = \emptyset$ and only **ULL**-models of Γ are minimal abnormal.

Ad 3. If Γ is abnormal, then $\mathcal{M}_\Gamma^{\mathbf{ULL}} = \emptyset$.

Ad 1. $\mathcal{M}_\Gamma^{\mathbf{ULL}} \subseteq \mathcal{M}_\Gamma^m$ follows from 2 and 3. $\mathcal{M}_\Gamma^r \subseteq \mathcal{M}_\Gamma^{\mathbf{LLL}}$ is immediate in view of the definition of reliable model of Γ. $\mathcal{M}_\Gamma^m \subseteq \mathcal{M}_\Gamma^r$ is item 1 of Theorem 3.

Ad 4. Immediate in view of Definitions 3 and 4.

Ad 5. By 4, there is an $A \in \Omega - U(\Gamma)$ such that all $M \in \mathcal{M}_\Gamma^r$ verify $\neg A$ whereas some $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^r$ does not.

Ad 6. Immediate in view of Definitions 5 and 6.

Ad 7. Suppose that the antecedent is true. All $M \in \mathcal{M}_\Gamma^m$ verify $\neg A_1 \vee \ldots \vee \neg A_n$ whereas some $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ (viz. an $M \in \mathcal{M}_{\Gamma \cup \{A_1,\ldots,A_n\}}^{\mathbf{LLL}}$) does not.

Ad 8. Immediate in view of Reassurance (Theorem 1) and the fact that no **LLL**-model is trivial. ■

This theorem states that adaptive logics are well-behaved. They deliver at least the lower limit consequences and at most the upper limit consequences, as desired (property 1). It moreover specifies the cases in which there is a gain with respect to the lower limit (properties 4–5 and 7 respectively). Property 2 states that all upper limit consequences are delivered if the premise set is normal.

Other known adaptive logics are obtained by combining adaptive logics of the type described above. Usually, it is easy to check that all aforementioned properties extend to them.[39]

5 Dynamic Proof Theory

Just like any other proof, a dynamic proof consists of a sequence of formulas. Annotated proofs consist of a sequence of lines that have five elements: (i) a line number, (ii) the derived formula A, (iii) the line numbers of the formulas from which A is derived, (iv) the rule by which A is derived, and (v) a (possibly empty) 'condition'. The condition specifies which formulas have to behave normally in order for A to be so derivable.

Apart from the fifth element of the lines, the only unusual thing is that lines of a dynamic proof may be marked. The marks may change from one stage of the proof to the next—adding a line to the proof brings the proof to its next stage.[40] The formula (second element) of a line that is marked at stage s is considered as not derived at stage s. Marking is governed by a definition, which depends on the strategy.

What does all this mean? The conditions and marks enable us to control the internal dynamics of the proofs in a formally precise way. The Derivability Adjustment Theorem (Theorem 5 below) is central for this. It states that a formula A is derivable from the premises by the upper limit logic—this is the ideal that we want to approach 'as much as possible' by the adaptive logic—just

[39]More properties (see for example [1] and [2]) may be established for the adaptive logics under consideration. For example, Right Cautious Cut (and hence Plausibility) holds for $\mathbf{AL}^m$ (but not for $\mathbf{AL}^r$).

[40]A stage of a proof is obviously a sequence of lines and not a line. The stage that results from adding line n (and applying the marking definitions) comprises line 1 up to line n.

in case $A \vee Dab(\Delta)$ (the classical disjunction of A and of a classical disjunction of abnormalities) is derivable from the premises by the lower limit logic (for some finite $\Delta \subseteq \Omega$). In other words, whenever $A \vee Dab(\Delta)$ is derivable from the premises (for some Δ) by the lower limit logic, we want A to be derivable by the adaptive logic unless the premises require that the members of Δ behave abnormally.

There is no negative test for "the members of Δ behave abnormally". If $A \vee Dab(\Delta)$ is derivable from the premises by the lower limit logic, we cannot in general find out whether the members of Δ behave abnormally. This is why we derive A *on the condition* Δ. We presume that the members of Δ do not behave abnormally, and hence that A is derivable from the premises, *unless and until* shown otherwise. Technically this is indicated by the fact that the line in which A is derived on the condition Δ is unmarked at a stage of the proof unless it has been shown at that stage that the members of Δ behave abnormally.

Of course one needs to distinguish between derivability at a stage and final derivability—the latter provides the 'final judgement' from Section 2. The matter is discussed in the next to last paragraph of this section.

Let us now move to technical matters. As before, I shall suppose that $\vee$ denotes classical disjunction.

Theorem 5 $\Gamma \vdash_{\mathbf{ULL}} A$ *iff there is a finite* $\Delta \subset \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$. *(Derivability Adjustment Theorem.)*

Proof. For the left–right direction suppose that $\Gamma \vdash_{\mathbf{ULL}} A$. It follows that all **ULL**-models of Γ verify A. All *other* **LLL**-models of Γ verify some member of Ω. Hence, there is a $\Delta' \subseteq \Omega$ such all **LLL**-models of Γ verify a member of $\Delta' \cup \{A\}$. By the right compactness of **LLL**, there is a finite $\Delta \subseteq \Delta'$ such that all **LLL**-models of Γ verify a member of $\Delta \cup \{A\}$. In other words, all **LLL**-models of Γ' verify $A \vee Dab(\Delta)$.

For the right–left direction suppose that $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$. As no **ULL**-model verifies any member of Ω, all **ULL**-models of Γ (if any) verify A. ■

This theorem provides the motor for the dynamic proof theory. I shall list the rules for a proof from Γ in the form of generic rules.[41] Apart from a premise rule, there is an unconditional rule and a conditional rule.

PREM If $A \in \Gamma$, then one may add a line consisting of

(i) the appropriate line number,
(ii) A,
(iii) "–",
(iv) "Prem", and
(v) $\emptyset$.

RU If $B_1, \ldots, B_m \vdash_{\mathbf{LLL}} A$ and $B_1, \ldots, B_m$ occur in the proof with the conditions $\Delta_1, \ldots, \Delta_m$ respectively, then one may add a line consisting of

(i) the appropriate line number,
(ii) A,

[41] While generic rules are unavoidable in the present setup, they are in general most convenient and transparent for characterizing the proof theory of specific adaptive logics.

(iii) the line numbers of the B_i,
(iv) "RU", and
(v) $\Delta_1 \cup \ldots \cup \Delta_m$.

RC If $B_1, \ldots, B_m \vdash_{\mathbf{LLL}} A \vee Dab(\Theta)$ and $B_1, \ldots, B_m$ occur in the proof with the conditions $\Delta_1, \ldots, \Delta_m$ respectively, then one may add a line consisting of

(i) the appropriate line number,
(ii) A,
(iii) the line numbers of the B_i,
(iv) "RC", and
(v) $\Theta \cup \Delta_1 \cup \ldots \cup \Delta_m$.

To wind up the characterization of the dynamic proofs, I now present the marking definitions for the two discussed strategies. At any stage of the proof, zero or more *Dab*-formulas will be derived. Some of them are minimal (at that stage). Let $U_s(\Gamma)$ be the union of all Δ for which $Dab(\Delta)$ is a minimal *Dab*-formula at stage s. Let $\Phi_s^\circ(\Gamma)$ be the set of all sets that contain one disjunct out of each minimal *Dab*-formula at stage s, and let $\Phi_s(\Gamma)$ contain those members of $\Phi_s^\circ(\Gamma)$ that are not proper supersets of other members of $\Phi_s^\circ(\Gamma)$.[42]

Definition 7 *Marking for* $\mathbf{AL}^r$*: Line i is marked at stage s iff, where* Δ *is its fifth element,* $\Delta \cap U_s(\Gamma) \neq \emptyset$.

Definition 8 *Marking for* $\mathbf{AL}^m$*: Line i is marked at stage s iff, where A is the second element and* Δ *the fifth element of line i, (i) there is no* $\varphi \in \Phi_s(\Gamma)$ *such that* $\varphi \cap \Delta = \emptyset$*, or (ii) for some* $\varphi \in \Phi_s(\Gamma)$*, there is no line k that has A as its second element and has as its fifth element some* Θ *such that* $\varphi \cap \Theta = \emptyset$.

At this point I can define **AL**-derivability:

Definition 9 *A is* derived at stage *s in an* **AL***-proof from* Γ *iff A is the second element of a line that is not marked in the proof (at stage s).*

Definition 10 *A is* finally derived *on line i of an* **AL***-proof (at a stage) from* Γ *iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) any extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 11 $\Gamma \vdash_{\mathbf{AL}} A$ *(A is* finally derivable *from* Γ*) iff A is finally derived on some line of an* **AL***-proof from* Γ.

The marking rules deserve some comments. For both strategies, the minimal *Dab*-consequences of the premises are estimated in terms of the *Dab*-formulas that have been derived on the empty condition at the stage. To see the relation with the semantics, suppose that the estimate is correct. $U_s(\Gamma)$ is then identical to $U(\Gamma)$ and $\Phi_s(\Gamma)$ to $\Phi(\Gamma)$. Consider a line in which A has been derived on a condition Δ, which corresponds to $A \vee Dab(\Delta)$ being verified by all **LLL**-models

[42]The proofs may made be somewhat more efficient by introducing some closing operations in the definitions of $U_s(\Gamma)$ and $\Phi_s(\Gamma)$. In doing so one should take computational matters into account: it should be decidable whether a line is marked or unmarked.

of the premises. This line is unmarked on the Reliability strategy iff no element of Δ is a member of $U_s(\Gamma)$. The latter corresponds to A being verified by all reliable models of the premises.

On the Minimal Abnormality strategy, a line in which A has been derived on a condition Δ warrants that A is verified by every model that does not verify any element of Δ. So the line is *un*marked on the Minimal Abnormality strategy just in the following case: for every minimal abnormal model of the premises—any such model is characterized by some $\varphi \in \Phi(\Gamma)$—A has been derived on a condition that is falsified by that model and if Δ is falsified by some minimal abnormal model of the premises. In other words the line witnesses, together with other lines, that every minimal abnormal model of the premises verifies A.

For the specific logics that were studied, the Soundness and Completeness of the dynamic proof theory with respect to the semantics was proved. Apparently, these proofs may be generalized for all adaptive logics under consideration. For examples of dynamic proofs, I refer to the many papers on specific logics.[43]

While Definition 10 may be taken at face value for Reliability, some weird premise sets require that, in the case of Minimal Abnormality, infinite extensions of proofs are considered—see [8, p. 466] for an example.[44]

The following theorem is central for the dynamic proof theory.

Theorem 6 *If $\Gamma \vdash_{\mathbf{AL}} A$, then any proof from Γ can be extended into a proof in which A is finally derived from Γ. (Proof Invariance.)*

Proof. Consider any proof from Γ—call it P1. If $\Gamma \vdash_{\mathbf{AL}} A$, there is a proof from Γ—call it P2—in which A has been finally derived at some line i and that, if extending it with P1 results in line i being marked, may be further extended in such a way that line i is unmarked. Call the last extension E. Definitions 7 and 8 warrant that,[45] if P1 is first extended with P2 and then with E, then the line that had number i in P2 is unmarked. ∎

What about decidability? The propositional fragments (and some other fragments) of most adaptive logics are decidable. This means that the dynamics of the proofs can in principle be avoided by deriving formulas in a suitable order and by not deriving any formulas that are marked in view of formulas that were derived earlier. The full predicative versions of adaptive logics are obviously undecidable and have no positive test for final derivability.

Even in undecidable waters there may be certain criteria that enable one to decide that a specific formula has been finally derived in some line of a dynamic proof from Γ. Some such criteria provide from work on the block approach (see for example [7]) and from work on tableau methods for adaptive logics (see [21] and [23]). Much more efficient criteria derive from goal directed dynamic proofs ([14] and work in progress, partly with Dagmar Provijn).

But what if no such criterion applies? It was shown in [7]—the result may be easily generalized to all considered adaptive logics—that as dynamic proofs proceed, the sets of formulas derived at subsequent stages offer increasingly better estimates of the set of finally derivable formulas. This estimate is not merely a computational approximation, but there is an idea behind it: as the

[43] A list is available: `http://logica.UGent.be/adlog/albib.html`.

[44] Extensions of infinite proofs are obtained by inserting formulas in the proof.

[45] All we need is that the order of the lines of a stage is immaterial for the marking definitions.

proof proceeds, it provides an increasingly better insight in the premises, and hence in the minimal *Dab*-formulas that are derivable from them. Moreover, the goal directed dynamic proofs provide means to speed up the gain of insight in the premises. The upshot is that dynamic proofs form a sensible basis for decision and action. In this sense, they not only enable one to explicate actual forms of dynamic reasoning, but also justify such forms of reasoning.

I shall be brief on prioritized and combined adaptive logics. The essential point was already mentioned: where different adaptive mechanisms are combined, one obtains dynamic proofs in which the dynamic mechanisms do not operate consecutively but at the same time. As a result, the dynamic proofs obtain their full explicatory and justificatory function.

6 In Conclusion

Several open problems were mentioned in the previous sections and I shall not repeat them here. I shall rather add a final comment concerning the epistemological function of adaptive logics.

Adaptive logics explicate consequence relations for which there is no positive test and that may be fit into the scheme: a lower limit logic, a set of abnormalities and a strategy. We have seen in Section 2 that those consequence relations abound in epistemological contexts and play a central role in them. The dynamic proofs not only provide the logics with a proof theory. With their conditions and marking definitions, they explicate the actual reasoning in terms of such consequence relations. This is extremely important because they thus provide a clear and transparent conceptual analysis for forms of reasoning that were often qualified as mere tinkering or even as logically flawed.

In my own work on epistemology (see for example [3] and [6]), I have stressed that the dynamics of human knowledge depends essentially on the fact that humans (as individuals or as groups) shift from one context to the other in solving problems. Adaptive logics do not offer an explication for this inter-contextual dynamics. They apply within contexts and explicate the intra-contextual dynamics in a formally precise way. It may be hoped that one will be able to move on to understand the inter-contextual dynamics, to explicate it, and to find means to increase its computational as well as its problem-solving efficiency.[46]

References

[1] Ofer Arieli and Arnon Avron. General patterns for nonmonotonic reasoning: From basic entailments to plausible relations. *Logic Journal of the IGPL*, 8:119–148, 2000.

[2] Arnon Avron and Iddo Lev. Formula-preferential systems for paraconsistent non-monotonic reasoning (an extended abstract). To appear.

[3] Diderik Batens. Meaning, acceptance, and dialectics. In Joseph C. Pitt, editor, *Change and Progress in Modern Science*, pages 333–360. Reidel, Dordrecht, 1985.

[46]Unpublished papers in the reference section (and many others) are available from the internet address `http://logica.UGent.be/centrum/writings/`.

[4] Diderik Batens. Dialectical dynamics within formal logics. *Logique et Analyse*, 114:161–173, 1986.

[5] Diderik Batens. Dynamic dialectical logics. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.

[6] Diderik Batens. Do we need a hierarchical model of science? In John Earman, editor, *Inference, Explanation, and Other Frustrations. Essays in the Philosophy of Science*, pages 199–215. University of California Press, 1992.

[7] Diderik Batens. Blocks. The clue to dynamic aspects of logic. *Logique et Analyse*, 150–152:285–328, 1995. Appeared 1997.

[8] Diderik Batens. Inconsistency-adaptive logics. In Orłowska [45], pages 445–472.

[9] Diderik Batens. Linguistic and ontological measures for comparing the inconsistent parts of models. *Logique et Analyse*, 165–166:5–33, 1999. Appeared 2002.

[10] Diderik Batens. Zero logic adding up to classical logic. *Logical Studies*, 2:15, 1999. (Electronic Journal: `http://www.logic.ru/LogStud/02/LS2.html`).

[11] Diderik Batens. Minimally abnormal models in some adaptive logics. *Synthese*, 125:5–18, 2000.

[12] Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.

[13] Diderik Batens. A dynamic characterization of the pure logic of relevant implication. *Journal of Philosophical Logic*, 30:267–280, 2001.

[14] Diderik Batens. On a partial decision method for dynamic proofs. In Hendrik Decker, Jørgen Villadsen, and Toshiharu Waragai, editors, *PCL 2002. Paraconsistent Computational Logic*, pages 91–108. (= *Datalogiske Skrifter* vol. 95), 2002. Also available as cs.LO/0207090 at `http://arxiv.org/archive/cs/intro.html`.

[15] Diderik Batens. On a logic of induction. In Roberto Festa, Atocha Aliseda, and Jeanne Peijnenburg, editors, *Confirmation, Empirical Progress, and Truth Approximation. Essays in Debate with Theo Kuipers*, volume 1. Poznan Studies in the Philosophy of the Sciences and the Humanities, Rodopi, Amsterdam, 2003. In print.

[16] Diderik Batens. Aspects of the dynamics of discussions and logics handling them. *Logical Studies*, in print.

[17] Diderik Batens. The theory of the process of explanation generalized to include the inconsistent case. *Synthese*, in print.

[18] Diderik Batens. A strengthening of the Rescher–Manor consequence relations. *Logic and logical Philosophy*, To appear.

[19] Diderik Batens and Lieven Haesaert. On classical adaptive logics of induction. *Logique et Analyse*, 173–175:255–290, 2001.

[20] Diderik Batens and Joke Meheus. The adaptive logic of compatibility. *Studia Logica*, 66:327–348, 2000.

[21] Diderik Batens and Joke Meheus. A tableau method for inconsistency-adaptive logics. In Roy Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Artificial Intelligence Vol. 1847, pages 127–142. Springer, 2000.

[22] Diderik Batens and Joke Meheus. On the logic and pragmatics of the process of explanation. In Mika Kiikeri and Petri Ylikoski, editors, *Explanatory Connections. Electronic Essays Dedicated to Matti Sintonen.* `http://www.valt.helsinki.fi/kfil/matti/`, 2001. 22 pp.

[23] Diderik Batens and Joke Meheus. Shortcuts and dynamic marking in the tableau method for adaptive logics. *Studia Logica*, 69:221–248, 2001.

[24] Diderik Batens, Joke Meheus, Dagmar Provijn, and Liza Verhoeven. Some adaptive logics for diagnosis. *Logic and Logical Philosophy*, 11/12:39–65, 2003.

[25] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 1: The flat case. *Studia Logica*, 58:17–45, 1997.

[26] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case. In Orłowska [45], pages 473–511.

[27] George S. Boolos and Richard J. Jeffrey. *Computability and Logic*. Cambridge University Press, 1989. (Third edition).

[28] Bryson Brown. How to be realistic about inconsistency in science. *Studies in History and Philosophy of Science*, 21:281–294, 1990.

[29] Newton C.A. da Costa. Calculs propositionnels pour les systèmes formels inconsistants. *Comptes rendus de l'Académie des sciences de Paris*, 259:3790–3792, 1963.

[30] Kristof De Clercq. Two new strategies for inconsistency-adaptive logics. *Logic and Logical Philosophy*, 8:65–80, 2000. Appeared 2002.

[31] Paul K. Feyerabend. Against method: outline of an anarchistic theory of knowledge. In M. Radner and S. Winokur, editors, *Analyses of Theories and Methods of Physics and Psychology*, volume 4 of *Minnesota Studies in the Philosophy of Science*, pages 17–30. University of Minnesota Press, Minneapolis, 1970.

[32] Ilpo Halonen and Jaakko Hintikka. Toward a theory of the process of explanation. *Synthese*, to appear.

[33] Stanisław Jaśkowski. Propositional calculus for contradictory deductive systems. *Studia Logica*, 24:243–257, 1969.

[34] Sarit Kraus, Daniel Lehman, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.

[35] Larry Laudan. *Progress and its Problems.* University of California Press, Berkeley, 1977.

[36] Iddo Lev. Preferential systems for plausible non-classical reasoning. Master's thesis, Department of Computer Science, Tel-Aviv University, 2000.

[37] Joke Meheus. Adaptive logic in scientific discovery: the case of Clausius. *Logique et Analyse*, 143–144:359–389, 1993. Appeared 1996.

[38] Joke Meheus. An extremely rich paraconsistent logic and the adaptive logic based on it. In Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, editors, *Frontiers of Paraconsistent Logic*, pages 189–201. Research Studies Press, Baldock, UK, 2000.

[39] Joke Meheus. Inconsistencies in scientific discovery. Clausius's remarkable derivation of Carnot's theorem. In Helge Krach, Geert Vanpaemel, and Pierre Marage, editors, *History of Modern Physics*, pages 143–154. Brepols, Brepols, 2002.

[40] Nancy Nersessian. Inconsistency, generic modeling, and conceptual change in science. In Joke Meheus, editor, *Inconsistency in Science*, pages 197–211. Kluwer, Dordrecht, 2002.

[41] Thomas Nickles, editor. *Scientific Discovery, Logic, and Rationality.* Reidel, Dordrecht, 1980.

[42] Thomas Nickles, editor. *Scientific Discovery: Case Studies.* Reidel, Dordrecht, 1980.

[43] John Norton. The logical inconsistency of the old quantum theory of black body radiation. *Philosophy of Science*, 54:327–350, 1987.

[44] John Norton. A paradox in Newtonian gravitation theory. *PSA 1992*, 2:421–420, 1993.

[45] Ewa Orłowska, editor. *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa.* Physica Verlag (Springer), Heidelberg, New York, 1999.

[46] Graham Priest. Minimally inconsistent **LP**. *Studia Logica*, 50:321–331, 1991.

[47] Dagmar Provijn and Erik Weber. The adaptive logics $\mathbf{D}^{nexp}$ and $\mathbf{D}^{exp}$ for non-explanatory and explanatory diagnostic reasoning. In Lorenzo Magnani, Nancy J. Nersessian, and Claudio Pizzi, editors, *Logical and Computational Aspects of Model-Based Reasoning*, pages 117–142. Kluwer, Dordrecht, 2002.

[48] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.

[49] Joel Smith. Inconsistency and scientific reasoning. *Studies in History and Philosophy of Science*, 19:429–445, 1988.

[50] Liza Verhoeven. Changing one's position in discussions. Some adaptive approaches. *Logic and Logical Philosophy*, 11–12:277–197, 2003.

[51] Liza Verhoeven. Proof theories for some prioritized consequence relations. To appear.

[52] Erik Weber and Dagmar Provijn. A formal analysis of diagnosis and diagnostic reasoning. *Logique et Analyse*, 165–166:161–180, 1999. Appeared 2002.

[53] Andrzej Wiśniewski. *The Posing of Questions. Logical Foundations of Erotetic Inferences*. Kluwer, Dordrecht, 1995.

[54] Andrzej Wiśniewski. The logic of questions as a theory of erotetic arguments. *Synthese*, 109:1–25, 1996.