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Inconsistency-adaptive logics

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Abstract

After a general description of adaptive logics and their intended applications, I study the proof theory and semantics of two closely related predicative inconsistency-adaptive logics, **APIL1** and **APIL2**. To this end, I first describe their monotonic basis: the paraconsistent logic **PIL** obtained by dropping the consistency requirement from classical logic. The propositional fragments of these inconsistency-adaptive logics have been studied elsewhere. The predicative versions involve several interesting difficulties that lead to new results.

1 Aim of this paper

After a general description of adaptive logics and their intended applications, I study two closely related inconsistency-adaptive predicative logics. To this end, I first describe the paraconsistent logic **PIL**, a basic paraconsistent logic obtained by dropping the consistency requirement from classical logic. Next, I investigate the proof theory as well as the semantics of two inconsistency-adaptive logics based on **PIL**, **APIL1** and **APIL2**. The proof theory of the propositional fragment of **APIL1** was described in [4], the semantics of the propositional fragment of **APIL2** in [3]. In those papers, I used the term "dynamic dialectical logic". I moved to "inconsistency-adaptive" because the general notion of an adaptive logic is not captured by the term "dialectical".

The fact that I present two (sensible) variants, **APIL1** and **APIL2**, is to a large extend the result of a mistake. Both the semantics presented in [3] and the proof theory presented in [4] are very appealing intuitively, but they define slightly different systems. The difference relates to an ambiguity in "maximally normal interpretation". Suppose that a theory, that was intended as consistent, contains $(p\& \sim p) \vee (q\& \sim q)$, but contains neither $p\& \sim p$, nor $q\& \sim q$, nor any other inconsistency. (This notion of containment becomes clear in the sequel). There are two different strategies to proceed in view of a maximally normal interpretation of the theory. According to the first, *reliability*, both p and q are unreliable; the theory does not allow us to rely on the consistent behaviour of either of them, even if we may rely on the consistent behaviour of all other formulas – the idea behind the proof theory of [4]. The second strategy *minimizes abnormality*: only one of p and q behaves inconsistently – the idea behind the semantics in [3]. I have been unable to find further (natural) strategies to obtain maximally normal interpretations.

The central importance of (especially inconsistency-)adaptive logics lies in their application to (reconstructions of) discovery processes and related creative episodes from the history of the sciences – see section 2.

2 Adaptive logics

A logic specifies a (type of) language by fixing the meaning of a set of logical terms. As a result, a logic comprises a number of presuppositions on the structure of the domain described – or rather, on the structure of the domain as it is pictured in the language. Those presuppositions are usually revealed by the semantics of the logic[1]. Classical logic (henceforth **CL**), for example, presupposes that A and B are not both true if $A\&B$ is false; it presupposes that an adequate description of the domain may be arrived at by describing actual properties of the domain, without recurring, for example, to possibilities; and it does not presuppose that $A(\alpha)$ is true for some individual constant α if $(\exists x)A(x)$ is true.

By a theory $T = \langle \Gamma, \mathbf{L} \rangle$ I shall mean $Cn_{\mathbf{L}}(\Gamma)$: the deductive closure, defined by a logic **L**, of a set of (non-logical) axioms Γ. 'Theory' is used here in a broad sense, as Γ may be any set of sentences of the language. But it is useful to keep in mind that Γ may be determined by criteria that link the language to the domain. For example, it may comprise an empirical theory (in the usual sense) together with a set of descriptions of the results of applications of observational criteria. In view of the process by which we arrive at Γ, and of the fact that we have to fix **L** at an early stage of that process, it is possible that Γ violates some of the presuppositions of **L**, in which case we shall say that Γ (and T) has *abnormal* properties (with respect to **L**).

Sometimes **L** turns out to be inadequate with respect to the domain. Remember intuitionism, that was forced to give up **CL** in view of its criteria for mathematical truth and falsehood. In other cases, one may have good reasons to modify Γ in such a way that the abnormal properties are removed (while certain non-logical criteria are still being fulfilled). In [2] I hinted to the process by which Russell removed the Russell paradox from Frege's set theory; in [6], Joke Meheus studies the case of Clausius who forged a consistent theory from the in-

[1] Some care is required here, as a logic may have characteristic semantics of different types. The difficulty is too complex to be dealt with here, and fortunately there is no need to do so for the purposes of the present paper.

consistent set comprising Carnot's thermodynamics as well as Joule's principle on the conversion of work to heat (and back) and a set of experimental results (mainly obtained by Joule). In such cases, the logic of the resulting theory – **CL** in the above cases – is the same as the originally intended logic. Yet, there is a period during which the original 'theory' displays abnormal properties with respect to this logic. It is typical for **CL** (and several other logics) that, if T displays abnormal properties with respect to it, these always surface in the form of inconsistencies and hence of triviality. It is this transitory stage that requires us to move to adaptive logics. In this sense, adaptive logics are typical for certain creative stages in the development of mathematical and empirical theories.

In the transitory stage, $Cn_{\mathbf{L}}(\Gamma)$ is trivial. In typical cases it is possible to show, as in [6], each of the following. (i) It is necessary to continue reasoning from Γ in order to localize and remove the abnormal properties. (ii) **L** is clearly inadequate for this purpose. (iii) **L** may still be applied sensibly to certain subsets of Γ, but the definition of these subsets requires reasoning from Γ itself. (iv) The logic **Lf**, defined by dropping the violated presuppositions from **L**, is too poor to serve as a device for reasoning from Γ. (v) Such reasoning proceeds (or may be reconstructed) in terms of a (particular) adaptive logic.

Before turning to a specific adaptive logic, I briefly list some general properties in a somewhat metaphorical way. An adaptive logic **La** localizes the abnormal properties of Γ, safeguards the theory for triviality by preventing specific rules of **L** from being applied to abnormal consequences of Γ, but behaves exactly like **L** for all other consequences of Γ. In a sense **La** oscillates between **L** and **Lf** (as defined in the previous paragraph), depending on the abnormal properties of the set of premises. The (dynamic) proof theory of adaptive logics is based on the idea that a formula is considered to behave normally 'unless and until proved otherwise'[2]. The semantics is better understood by another metaphor: **La** interprets Γ as maximally normal. We have seen that there are (at least) two strategies to do so. For **APIL2**, e.g., the **La**-semantic consequences of Γ are the formulas true in the **Lf**-models of Γ that are minimally inconsistent (not more inconsistent than required to make Γ true). By displaying these properties, adaptive logics enable us to reason from theories that are abnormal with respect to their logic, and to localize[3] the abnormal properties and thus to prepare for their elimination (in view of extra-logical considerations)[4].

[2] This formulation is somewhat inaccurate; two formulas may be connected with respect to their consistent behaviour, as in the example from section 1.

[3] This phrase is often used for paraconsistent logics in general, meaning that they do not lead from inconsistency to triviality. Adaptive logics, however, localize inconsistencies in a much deeper sense: each inference rule of classical logic applies except at points where a specific inconsistent consequence of the premises prevent this - this becomes fully transparent later.

[4] In [5] I show that some mixed non-monotonic logics (the logics intended for handling rules with exceptions) may be reconstructed as inconsistency- adaptive logics that are supplied with a preferential procedure to resolve inconsistencies that result from applying the rules

My view on adaptive logics is top-down. Given a theory that displays abnormal properties with respect to its intended logic **L**, **Lf** and next **La** are introduced in order to stay as close as possibly to the originally intended theory; extra-logical preferences should enable us to move from $Cn_{\mathbf{La}}(\Gamma)$ to a Γ' that is normal with respect to **L**. In [8] Graham Priest accommodates adaptive logics to a bottom-up view. As a dialetheist Priest believes in the existence of a true logic, which he contends to be paraconsistent. But he admits that in many situations we are justified in presupposing consistency. He then shows that, if his preferred paraconsistent logic **LP** (from his [7]) is extended into an adaptive logic $\mathbf{LP^m}$ by assuming consistency until and unless shown otherwise, then $\mathbf{LP^m}$ recaptures all classical reasoning where it is sensible (according to his dialetheist view). In the top-down view, adaptive logics are *provisional* instruments, to be used during the transitory stage in which the full *intended* logic **L** is inadequate. In the bottom-up view, one starts from a rather poor (in Priest's case paraconsistent) logic, taken to be the *correct* logic. By adopting an (inconsistency-)adaptive extension of this logic, its power is extended to that of **CL** whenever reasoning does not take place 'in the neighbourhood' of a true inconsistency.

I mentioned already that abnormal properties with respect to **CL** (and some other logics) always result in inconsistency (and hence triviality). This does not entail that all adaptive logics should be defined from a paraconsistent logic. Quite to the contrary. If, for example, empirical criteria lead to abnormal behaviour with respect to conjunction in that some criterion results in $\sim (A \& B)$ whereas other criteria result in A as well as in B, the violated presupposition is not consistency (that $\sim A$ is false whenever A is true) but the Conjunction rule (that $A \& B$ is true whenever A and B are true – remember Kyburg's attack on *conjunctivitis*). Another argument derives from the paradox of Curry and Moh Shaw-Kwei, that is independent of negation properties. So, many other adaptive logics deserve to be studied as well.

Some of the intended applications of adaptive logics are frequently approached by inference relations that apply to subsets or subbases of a set (or multiset, or base) of (possibly inconsistent) premises. It all started with [9] (and other work by Nicholas Rescher) but became popular recently in connection with AI applications. Such approaches are extremely dependent on the formulation of the premises. Many applications require that one logically analyzes the inconsistent set of premises, derives 'deeper' inconsistencies, attaches preferences to formulas that are not premises themselves, but consequences of the premises, etc. In all such cases, inference relations defined with respect to subsets (etc.) of the premises are inadequate. Adaptive logics differ from these approaches in that they proceed, syntactically, by dynamic proofs and, semantically, in terms of (specific inconsistent) models of the full set of premises.

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with exceptions. [10] studies indexed inconsistency-adaptive logics, with reference to some applications to knowledge bases.

3 The paraconsistent logic PIL

The consistency presupposition is rendered in the standard **CL**-semantics as

($*$) If $v_M(A) = 1$, then $v_M(\sim A) = 0$.

The system obtained by dropping ($*$) from the standard **CL**-semantics will be called **PIL**. It is not an adaptive logic, but a (paraconsistent) fragment of **CL** on which inconsistency-adaptive logics will be built in subsequent sections.

Let the language-schema (including the definition of terms, formulas, and wffs) be as for **CL**; including functions is straightforward, but will be disregarded. Let S be the set of sentential letters, P^r the set of letters for predicates of rank r, C and V the set of letters for individual constants and variables respectively, F the set of (open and closed) formulas, and N the set of formulas of the form $\sim A$. Let the members of C as well as the members of V be given in a certain order, denoted by "$<$". As usual, $A(x)$ is a formula in which x occurs free, and $A(a)$ is obtained from $A(x)$ by replacing every free occurrence of x in A by a.

PIL-SEMANTICS[5]

A model is a couple $M = \langle D, v\rangle$ in which D is a set and v is an assignment function defined by:

C1.1 $v : \mathsf{S} \to \{0, 1\}$

C1.2 $v : \mathsf{C} \cup \mathsf{V} \to D$ is such that $D = \{v(\alpha)\alpha \in \mathsf{C} \cup \mathsf{V}\}$[6]

C1.3 $v : \mathsf{P}^r \to \mathcal{P}(D^r)$ (the power set of the r-th Cartesian product of D)

C1.4 $v : \mathsf{N} \to \{0, 1\}$

The valuation function v_M determined by the model M is defined as follows:

C2.1 $v_M : \mathsf{F} \to \{0, 1\}$

C2.2 where $A \in \mathsf{S}, v_M(A) = v(A)$

C2.3 $v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$

C2.4 $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$

C2.5 $v_M(\sim A) = 1$ iff $v_M(A) = 0$ or $v(\sim A) = 1$

C2.6 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$

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[5] Throughout this paper, the metalanguage is classical. For example, in the true statement about the **PIL**-semantics "if $v(p) = 1$, then $v(p)$ is not 0", the "not" is classical.

[6] The requirement, which is obviously much weaker than ω-completeness, restricts the semantics to models with countable domain D, but greatly facilitates both some other clauses and the proofs.

C2.7 $v_M(A\&B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$

C2.8 $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$

C2.9 $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$

C2.10 $v_M((\forall\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for all $\beta \in \mathsf{C} \cup \mathsf{V}$.

C2.11 $v_M((\exists\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for at least one $\beta \in \mathsf{C} \cup \mathsf{V}$.

Truth in a model, semantic consequence and validity are defined as usual. **PIL** is the basic paraconsistent logic obtained by weakening **CL**.

PIL is the predicative version of **PI** from [1]; most properties are retained in the predicative version. It has a number of paraconsistent extensions that are closer to **CL**. **PIL** may indeed be extended by basically two kinds of properties. Vasil'ev properties are specific forms of $(*)$, e.g., "if $v_M({\sim} A) = 1$, then $v_M({\sim}{\sim} A) = 0$"; they make negation behave classically in front of specific complex formulas. Schütte properties reduce the meaning of negation in front of complex formulas to (negations of) subformulas or related formulas, as in "$v_M({\sim} (A\&B)) = v_M({\sim} A \vee {\sim} B)$". If only Schütte properties are added, the resulting systems are *strictly* paraconsistent in that all forms of contradictions have models. With some Vasil'ev properties, e.g., "$v_M({\sim} (A\&B)) = 1$ iff $v_M(A\&B) = 0$", classical negation is definable, e.g., by "$\neg A =_{df} {\sim} (A\&A)$" which results in "$v_M(\neg A) = 1$ iff $v_M(A) = 0$". Some **PIL**-extensions are *maximally* paraconsistent in that their only (proper) extensions are **CL** and the trivial logic (in which all wffs are theorems).

In some **PIL**-extensions, all negations in front of complex wffs are either governed by Vasil'ev or by Schütte properties. This entails that all inconsistencies are reduced to inconsistencies at the level of primitive wffs (propositional letters, primitive predicative wffs, and identities). This will allow for a more elegant formulation of the semantics. In **PIL**, however, $A\& \sim A$ may be true in a model M, even if no subformula of A behaves inconsistently in M; if A is true in M, the truth of $\sim A$ is fully independent of the truth of any subformula of A. The advantage of the present formulation is that it is more general than any other formulation known to me. In [8] for example, inconsistencies are handled by assigning a positive as well as a negative extension to predicates. That is only possible because all inconsistencies true in a model derive from true inconsistencies at the level of primitive predicative wffs. This may be expressed in the above semantics by (i) requiring that $v({\sim} A) = 1$ only if A is a primitive wff, and (ii) adding the suitable clauses on the valuation function.

For positive formulas, the above **PIL**-semantics is identical to the **CL**-semantics. Also, C2.5 warrants that $v_M(A) = 1$ or $v_M({\sim} A) = 1$. It is useful to look at the properties of inconsistencies in **PIL**. Unlike most of its extensions, **PIL** *does not spread inconsistencies*: no inconsistency entails another. As a consequence, the usual extensionality properties do not obtain within the scope of a negation. For example, it is possible that $v_M(A) = v_M(B)$, and even that

$v_{M'}(A) = v_{M'}(B)$ for all M', while $v_M(\sim (A\&C)) \neq v_M(\sim (B\&C))$. Similarly, it is possible that $v(a) = v(b)$, whereas $v_M(\sim a = c) \neq v_M(\sim b = c)$[7]. Of course, **PIL** may be extended in such a way that $a = b$, $A(a)| = A(b)$ holds, but it is certainly advisable not to have this in a basic logic in which the replacement rule does not obtain for wffs and predicates either. That **PIL** does not spread inconsistencies will prove valuable for the adaptive logics based on it.

PIL-models have an important property which is absent in many paraconsistent logics, e.g., in Priest's **LP** and in relevant logics.

Theorem 1 *For all non-trivial models* $M, \{Av_M(A) = 1\}$ *is maximally non-trivial.*

Proof. If $v_M(B) = 0$, then $v_M(B \supset C) = 1$ for all C, and hence $\{Av_M(A) = 1\} \cup \{B\}$ is trivial. ■

This is important for the relation between the logic and the language, as it warrants that some theories are complete (in the sense of Gödel's first theorem). For logics lacking this property, no theory $T = \langle \Gamma, \mathbf{L} \rangle$ is complete (maximally non-trivial) if there are at least two wffs A and B such that $A \notin Cn_{\mathbf{L}}(\Gamma)$ and $B \notin Cn_{\mathbf{L}}(\Gamma \cup A)$ – roughly speaking: if two partially independent wffs behave consistently in T. So, in practice, no interesting theory based on such logic is complete.

Clause C1.4 is independent of C1.1-3, as it should be. It follows that $v(\sim A) = 1$ does not necessarily cause an inconsistency in the model: if $v_M(A) = 0$, then $v_M(\sim A) = 1$ anyway. But some models have the property that, for any formula $A, v(\sim A) = 0$ whenever $v_M(A) = 0$; let us call these N-minimal models. Let us call M and M' *formula-equivalent* (respectively wff-equivalent) models iff $v_M(A) = v_{M'}(A)$ for all formulas (respectively wffs) A.

Theorem 2 *For any* **PIL**-*model* M*, there is a formula-equivalent* N-*minimal* **PIL**-*model* M'*.*

Proof. Define M' from M, by modifying v as follows: if $v_M(A) = 0$, then $v(\sim A) = 0$. Obviously, M' is N-minimal. Still, $v_{M'}(A) = v_M(A)$ for all A. Indeed, the change in C1.4 has no effect at all on C2.5 (and cannot possibly have any effect on any other $v_{M'}$-value). ■

Excursion. Truth in an ω-complete **CL**-model is characterized by the set of propositional letters, primitive predicative wffs, and identities. In view of C1.4, this property fails in **PIL**. Still, similar characterizations of ω-complete **PIL**-models are possible. The simplest (redundant) characterization of an ω-complete **PIL**-model M is a quartuple $\langle \mathsf{S}^\circ, I^\circ, E^\circ, \mathsf{N}^\circ \rangle$ in which $\mathsf{S}^\circ, I^\circ, E^\circ$, and

[7]Here is another instructive example. $(\exists x)(\forall y)(x = y \supseteq (Py\& \sim Py))$ is true in a model iff there is an $\alpha \in \mathsf{C} \cup \mathsf{V}$ such that $v(\alpha) \in v(P)$, and, for all $\beta \in \mathsf{C} \cup \mathsf{V}$ such that $v(\beta) = v(\alpha)$, $v(\sim P\beta) = 1$.

N° are respectively the members of S, the identities, the primitive predicative wffs, and the formulas of the form $\sim A$ that are true in M. To obtain a non-redundant characterization, define C^m, the minimization of C with respect to M, as the set of those $\alpha \in \mathsf{C}$ for which there is no $\beta \in \mathsf{C}$ such that $\beta < \alpha$ and $v_M(\alpha = \beta) = 1$; next define a non- redundant characterization of an ω-complete **PIL**-model M, by requiring that $\alpha = \beta \in I^\circ$ only if $\alpha \in \mathsf{C}^m$ and $\beta \notin \mathsf{C}^m$, and that no elements of $\mathsf{C} - \mathsf{C}^m$ occur in E°. Finally, one may recursively define a N-minimal characterization of a **PIL**-model M by *moreover* restricting N° as follows. Where the complexity of a wff is the number of occurrences of connectives and quantifiers, and the complexity of $\sim A$ is $n, \sim A \in \mathsf{N}^\circ$ only if $\sim A$ is not a semantic consequence of $\mathsf{S}^\circ \cup I^\circ \cup E^\circ$ together with the members of N° the complexity of which is smaller than n. A N-minimal characterization characterizes a N-minimal **PIL**-model and all models wff-equivalent to it.

I now list an axiomatization of **PIL**. As usual, the α and β should be interpreted in such a way that all (main) formulas are wffs.

SYNTAX

MP From A and $A \supset B$ to derive B

$A\supset 1$ $A \supset (B \supset A)$

$A\supset 2$ $((A \supset B) \supset A) \supset A$

$A\supset 3$ $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

$A\&1$ $(A\&B) \supset A$

$A\&2$ $(A\&B) \supset B$

$A\&3$ $A \supset (B \supset (A\&B))$

$A\vee 1$ $A \supset (A \vee B)$

$A\vee 2$ $B \supset (A \vee B)$

$A\vee 3$ $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

$A\equiv 1$ $(A \equiv B) \supset (A \supset B)$

$A\equiv 2$ $(A \equiv B) \supset (B \supset A)$

$A\equiv 3$ $(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$

$A\sim 1$ $(A \supset\sim A) \supset\sim A$ (alternatively: $A\vee \sim A$)

$R\forall$ To derive $\vdash A \supset (\forall\alpha)B(\alpha)$ from $\vdash A \supset B(\beta)$, provided β does not occur in either A or $B(\alpha)$.

$A\forall$ $(\forall\alpha)A(\alpha) \supset A(\beta)$

$R\exists$ To derive $\vdash (\exists\alpha)A(\alpha) \supset B$ from $\vdash A(\beta) \supset B$, provided β does not occur in either $A(\alpha)$ or B.

$A\exists$ $A(\beta) \supset (\exists\alpha)A(\alpha)$

$A{=}1$ $\alpha = \alpha$

$A{=}2$ $\alpha = \beta \supset (A \supset B)$ where B is obtained by replacing in A an occurrence of α that occurs outside the scope of a negation by β

I leave it to the reader to show that

Theorem 3 *If* $\Gamma_{\mathbf{PIL}} A$, *then* $\Gamma \models_{\mathbf{PIL}} A$.

To prove the completeness of the axiomatization with respect to the semantics, one proceeds very much as for **CL**. First, we prove (as for **CL**) that $\Gamma_{\mathbf{PIL}} A$ iff $\Gamma'_{\mathbf{PIL}} A'$, where Γ' and A' are obtained from Γ and A by a systematic reletering of the individual constants. This enables us to suppose that an infinite subset of C does not occur in either Γ or A in the following theorem.

Theorem 4 *If* $\Gamma \models_{\mathbf{PIL}} A$, *then* $\Gamma_{\mathbf{PIL}} A$.

Proof. Suppose that $\Gamma \not\vdash_{\mathbf{PIL}} A$. Consider, as for the proof in **CL**, a sequence B_1, B_2, ... that contains all wffs and in which each wff of the form $(\exists\alpha)A$ is followed immediately by an instance with a constant that does not occur in Γ, in A, or in any previous member of the sequence. We then define

$$\begin{array}{rcl} \Delta_0 & = & Cn_{\mathbf{PIL}}(\Gamma) \\ \Delta_{i+1} & = & Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\}) \text{ if } A \notin Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\}), \text{ and} \\ \Delta_{i+1} & = & \Delta_i \text{ otherwise} \\ \Delta & = & \Delta_0 \cup \Delta_1 \cup \ldots \end{array}$$

Each of the following is provable:

(i) $\Gamma \supseteq \Delta$ (by the definition of Δ).

(ii) $A \notin \Delta$ (by the definition of Δ).

(iii) Δ is deductively closed (by the definition of Δ).

(iv) Δ is maximally non-trivial. To see this, remark first that $A \supset C \in \Delta$ for all C. Indeed, if $A \supset C \notin \Delta$, there is a Δ_i such that $\Delta_i \cup \{A \supset C\} \vdash A$; hence $\Delta_i \vdash (A \supset C) \supset A$ by the deduction theorem; hence, in view of $A \supset 2, \Delta_i \vdash A$, which is impossible. If $E \notin \Delta$, then there is a Δ_i such that $\Delta_i \cup \{E\} \vdash A$ and hence $\Delta \cup \{E\} \vdash A$; as $A \supset C \in \Delta$ for all $C, \Delta \cup \{E\}$ is trivial.

(v) Δ is prime, i.e.: if $C \vee E \in \Delta$, then $C \in \Delta$ or $E \in \Delta$; obvious in view of the proof of (iv).

(vi) Δ is ω-complete. As for **CL**, the order of the sequence B_1, B_2, ... and $R\exists$ warrant that, whenever $(\exists\alpha)C(\alpha) \in \Delta$, then $C(\beta) \in \Delta$ for some $\beta \in \mathsf{C}$.

Define a **PIL**-model M as follows:

1. $D = \{\alpha\alpha \in \mathsf{C}$ and (there is no $\beta \in \mathsf{C}$ such that $\beta < \alpha$ and $\alpha = \beta \in \Delta)\}$[8];

2. for all $C \in \mathsf{S}$, $v(C) = 1$ iff $C \in \Delta$;

3. for all $\alpha \in \mathsf{C}$, if $\alpha \in D$, $v(\alpha) = \alpha$; if $\alpha \in \mathsf{C} - D$, $v(\alpha)$ is the $\beta \in D$ such that $\alpha = \beta \in \Delta$ (there is a unique such β by the definition of D);

4. for all $\pi \in \mathsf{P}^r$, $v(\pi) = \{\langle\alpha_1 \ldots \alpha_r\rangle\pi\alpha_1 \ldots \alpha_r \in \Delta\}$;

5. for all $\sim C \in \mathsf{N}$, $v(\sim C) = 1$ iff $C, \sim C \in \Delta$;

6. each $\alpha \in \mathsf{V}$ is associated (arbitrarily) with a constant β (which need not be a member of D) to the effect that $v(\alpha) = v(\beta)$ and, for any $A, v(\sim A(\alpha)) = v(\sim A(\beta))$.

I now show by an induction on the length of formulas that, for all C, if $C \in \Delta$, then $v_M(C) = 1$. First consider $\sim C \in \Delta$. If $C \notin \Delta$, then $v_M(C) = 0$ and hence $v_M(\sim C) = 1$. If $C \in D$, then $v(\sim C) = 1$ by (5); hence $v_M(\sim C) = 1$. The proof of the other cases proceeds as for **CL**. I give only two examples. For identity, suppose that $\alpha = \beta \in \Delta$; then, by the definition of D, there is a $\gamma \in D$ such that $\alpha = \gamma$, $\beta = \gamma \in \Delta$; hence $v(\alpha) = v(\beta) = \gamma$; hence $v_M(\alpha = \beta) = 1$. For the universal quantifier, suppose that $(\forall\alpha)C(\alpha) \in \Delta$; then $C(\beta) \in \Delta$ for all $\beta \in \mathsf{C}$, and hence $v_M(C(\beta)) = 1$ for all $\beta \in \mathsf{C}$; in view of (6), $v_M(C(\beta)) = 1$ for all $\beta \in \mathsf{V}$ as well; whence, by C2.10, $v_M((\forall\alpha)C(\alpha)) = 1$.

As Δ is maximally non-trivial and $v_M(C) = 1$ for all $C \in \Delta$, it follows immediately that $\Delta = \{Cv_M(C) = 1\}$ and hence that $v_M(A) = 0$. But $v_M(B) = 1$ for all $B \in \Gamma$. Hence $\Gamma \not\models_{\mathbf{PIL}} A$. ■

Corollary 1 $\Gamma_{\mathbf{PIL}}A$ *iff* $\Gamma \models_{\mathbf{PIL}} A$.

In the following two sections, I shall implicitly rely on Corollary 1 to freely move from the **PIL**-syntax to the **PIL**-semantics in metatheoretic proofs. Consistent **PIL**-models – see also the next section – are (formula-)equivalent to **CL**-models[9]. I shall, however, conventionally identify **CL**-models with consistent **PIL**-models, whenever this does not cause any troubles.

[8] If $a < b$ and $a = b \in \Delta$, then $a \in D$, $b \notin D$, even if $\sim Pa \in \Delta$ and $\sim Pb \notin \Delta$.

[9] If the **PIL**-model is N-minimal, then $v(\sim A) = 0$ for all A.

4 Proof theory of the inconsistency-adaptive logic APIL1

The proof theory for the propositional case is explained in detail in [4]. A basic idea was the following theorem, in which **PI** is the propositional fragment of **PIL**:

(†) $\vdash_{\mathbf{CL}} A$ iff, for some $C_1, \ldots, C_n (n \geq 0)$, $\vdash_{\mathbf{PI}} ((C_1 \& \sim C_1) \vee \ldots \vee (C_n \& \sim C_n)) \vee A$

Where A has the form $(B_1 \& \ldots \& B_m) \supset C$, (†) suggests that we derive C from $B_1, \ldots, B_m$, provided all of $C_1, \ldots C_n$ 'behave consistently'. A specific interpretation of this suggestion leads to an inconsistency-adaptive logic, as we shall see.

The equivalent of (†) is not provable for **PIL**. Consider, for example, the following **CL**-theorem:

$$((\forall x)(Px \supset Qx) \& (\exists x) \sim Qx) \supset (\exists x) \sim Px$$

The antecedent is true and the consequent is false in, e.g., **PIL**-models in which $v(P) = v(Q) = D$, $v(\sim Qx) = 1$ and $v(\sim A) = 0$ whenever $A \neq Qx$. But $Qx \& \sim Qx$ is not well-formed.

Where $(A \& \sim A)$ is a formula in which the variables $\alpha_1, \ldots, \alpha_k$ $(k \geq 0)$ occur free (in that order), let $\exists(A \& \sim A)$ be $(\exists \alpha_1) \ldots (\exists \alpha_k)(A \& \sim A)$. Let $DEK(A_1, \ldots, A_n)$ refer to $\exists(A_1 \& \sim A_1) \vee \ldots \vee \exists(A_n \& \sim A_n)$ – a disjunction of (where necessary) existentially quantified contradictions. I shall say that $A_1, \ldots, A_n$ are the *factors* of $DEK(A_1, \ldots, A_n)$. As permutations of the factors and of the quantifiers in "$\exists$" result in equivalent formulas, I shall from now on use sets to refer to any of those permutations. Remark that $DEK(\Sigma \cup \{Px\})$ is **PIL**-equivalent to $DEK(\Sigma \cup \{Py\})$ and is **PIL**-derivable from $DEK(\Sigma \cup \{Pa\})$, but that neither Pa nor Py is a factor of $DEK(\Sigma \cup \{Px\})$. For the sake of generality, $DEK(\emptyset) \vee A$ will be A.

Incidentally, a **PIL**-model M is consistent iff $v_M(\exists(A \& \sim A)) = 0$ for all formulas A.

Theorem 5 *If* $\vdash_{\mathbf{PIL}} DEK\{C_1, \ldots, C_n\} \vee A$*, then* $\vdash_{\mathbf{CL}} A$*.*

Proof. As all **CL**-models are **PIL**-models, $DEK\{C_1, \ldots, C_n\} \vee A$ is true in all **CL**-models. But $DEK\{C_1, \ldots, C_n\}$ is false in all of them. Hence, A is true in all **CL**-models. ■

Define the set $bsf(A)$ of (open and closed) basic subformulas of A as follows:

(i) if A is a propositional letter, a primitive predicative formula, or an identity, then $bsf(A) = \{A\}$,

(ii) $bsf(\sim B) = \sim B \cup bsf(B)$,

(iii) $bsf(B \supset C) = bsf(B \& C) = bsf(B \vee C) = bsf(B \equiv C) = bsf(B) \cup bsf(C)$,

(iv) $bsf((\forall\alpha)A(\alpha)) = bsf((\exists\alpha)A(\alpha)) = bsf(A(\alpha))$.

Obviously, $bsf(A)$ is finite for any A. Moreover, inspection of the **PIL**-semantics teaches that an induction on the length of formulas gives us:

Lemma 1 *For any* **PIL***-model* M*, if there is no* B *such that* $\sim B \in bsf(A)$ *and* $v_M(\exists(B\& \sim B)) = 1$*, then there is a consistent* **PIL***-model* M' *such that* $v_M(A) = v_{M'}(A)$.

Proof. Suppose that the antecedent is true for some M. Let M' be obtained from M by putting $v(\sim B) = 0$ for all B. We proceed by an induction on the complexity of A (the number of connectives and quantifiers that occur in A).

If the complexity of A is 0, then $v_{M'}(A) = v_M(A)$. Supposing that $v_{M'}(A) = v_M(A)$ for all A with complexity less than n, I show that $v_{M'}(A) = v_M(A)$ for all A with complexity n. Of the seven cases to be considered, four are obvious, viz. the ones where A is either $B \supset C$ or $B\&C$ or $B \vee C$ or $B \equiv C$.

Case 5: A is $\sim B$. If $v_{M'}(B) = v_M(B) = 0$, then $v_{M'}(\sim B) = v_M(\sim B) = 1$ by C2.5. Suppose that $v_{M'}(B) = v_M(B) = 1$. Then $v_{M'}(\sim B) = 0$ (as M' is a consistent model). But also $v_M(\sim B) = 0$, for otherwise (in view of C2.5) $v(\sim B) = v_M(B\& \sim B) = v_M(\exists(B\& \sim B)) = 1$, which contradicts the main supposition.

Case 6: A is $(\forall\alpha)B(\alpha)$. Suppose first that $v_M(A) = 1$. Then $v_M(B(\beta)) = 1$ for all $\beta \in \mathsf{C} \cup \mathsf{V}$. Hence, by the induction hypothesis, $v_{M'}(B(\beta)) = 1$ for all $\beta \in \mathsf{C} \cup \mathsf{V}$. But then $v_{M'}((\forall\alpha)B(\alpha)) = 1$. Suppose next that $v_M(A) = 0$. Then there is a $\beta \in \mathsf{C} \cup \mathsf{V}$ such that $v_M(B(\beta)) = 0$. Hence, by the induction hypothesis, $v_{M'}(B(\beta)) = 0$ for some (the same) $\beta \in \mathsf{C} \cup \mathsf{V}$. But then $v_{M'}((\forall\alpha)B(\alpha)) = 0$.

Case 7: A is $(\exists\alpha)B(\alpha)$. The proof is wholly analogous to that of case 6. ■

Remark that, if $v_M((\exists x)(Px\& \sim Px)) = 0$, then $v_M(P\alpha\& \sim P\alpha) = 0$ for all $\alpha \in \mathsf{C} \cup \mathsf{V}$. So, the requirement (from the antecedent of the Lemma) on the finite $bsf(A)$ has implications for an infinite number of formulas.

Theorem 6 *If* $\vdash_{\mathbf{CL}} A$*, then, for some* $C_1, \ldots, C_n (n \geq 0)$*,* $\vdash_{\mathbf{PIL}} DEK\{C_1, \ldots, C_n\} \vee A$.

Proof. Suppose that $\vdash_{\mathbf{CL}} A$. Hence $v_M(A) = 1$ for all consistent **PIL**-models M. As $bsf(A)$ is finite, $DEK\{B \sim B \in bsf(A)\} \vee A$ is a wff, which is easily shown to be **PIL**-valid. Consider indeed a **PIL**-model M. If, for some $\sim B \in bsf(A)$, $v_M(\exists(B\& \sim B)) = 1$, then $v_M(DEK\{B \sim B \in bsf(A)\}) = 1$. If, for no $\sim B \in bsf(A)$, $v_M(\exists(B\& \sim B)) = 1$, then $v_M(A) = 1$ by Lemma 1. ■

The following **PIL**-theorems illustrate Theorem 6:

1. $((p \supset q) \& p) \supset q$
2. $((\forall x)(Px \supset Qx) \& Pa) \supset Qa$
3. $(p \& \sim p) \vee (((p \vee q) \& \sim p) \supset q)$
4. $((p \& \sim p) \vee (q \& \sim q)) \vee (((r \supset (p \vee q)) \& (\sim p \& \sim q)) \supset \sim r)$
5. $(\exists x)(Qx \& \sim Qx) \vee (((\forall x)(Px \supset Qx) \& (\exists x) \sim Qx) \supset (\exists x) \sim Px)$
6. $((\forall x)Px \& \sim (\forall x)Px) \vee (\sim (\forall x)Px \supset (\exists x) \sim Px)$

Remark that (6) is equivalent to each of the following:

(6.1) $\sim (\forall x)Px \supset ((\exists x) \sim Px \vee ((\forall x)Px \& \sim (\forall x)Px))$

(6.2) $(((\forall x)Px \& \sim (\forall x)Px) \supset \perp) \supset (\sim (\forall x)Px \supset (\exists x) \sim Px)$

(6.3) $(\forall x)Px \vee (\sim (\forall x)Px \supset (\exists x) \sim Px)$

In (6.3), $\perp$ is the 'falsum' (intuitively, the conjunction of all formulas). Similarly for all **PIL**-theorems of the form $DEK\{\ldots\} \vee A$. Each of these formulations is useful to grasp the idea of the next paragraph.

The first inconsistency-adaptive logic based on **PIL** will be called **APIL1**. I now define its proof theory. The idea is that we apply all rules of (or derivable in) **PIL** unconditionally, whereas *other* rules of (or derivable in) **CL** are applied on a provisional basis and on the condition that certain formulas are *reliable* (with respect to their consistent behaviour). To keep the matter algorithmic, the consistent behaviour of a formula will be determined by the stage of the proof, not by the (abstract) notion of derivability. As a result, proofs will be dynamic in that wffs written at some stage may be deleted at a later stage. Yet, I shall show that each set of premises has a unique set of (final) **APIL1**-consequences.

As I explained in [4] for the propositional version, it is handy to write **APIL1**-proofs in a specific format according to which each line of a proof consists of five elements:

(i) a line number,

(ii) the formula derived,

(iii) the line numbers of the wffs from which (ii) is derived,

(iv) the rule of inference that justifies the derivation[10], and

(v) the formulas on the consistent behaviour of which we rely in order for (ii) to be derivable *by* (iv) *from* the formulas of the lines enumerated in (iii).

[10] This is an application of RU or RC, rather than these (meta-)rules themselves. See below in the text.

The extension to the predicative level does not require any changes, except in that (v) will sometimes contain open formulas.

Definition 1 *A occurs unconditionally at some line of a proof iff the fifth element of that line is empty.*

Definition 2 *A behaves consistently at a stage of a proof iff* $\exists(A\& \sim A)$ *does not occur unconditionally in the proof at that stage.*

Definition 3 *The consistent behaviour of* A_1 *is connected to the consistent behaviour of* $A_2, \ldots, A_n$ *at a stage of a proof if* $DEK\{A_1, \ldots, A_n\}$ *occurs unconditionally in the proof at that stage whereas* $DEK\{A_2, \ldots, A_n\}$ *does not occur unconditionally*[11] *in it.*

Definition 4 *A is reliable at a stage of a proof iff A behaves consistently at that stage and its consistent behaviour is not connected to the consistent behaviour of other formulas.*

Given these definitions, proofs in **APIL1** are governed by an unconditional rule, a conditional rule and a deletion rule. The application of a rule to a proof at a stage produces the next stage.

RU If $\vdash_{\mathbf{PIL}} (A_1\&\ldots\&A_n) \supset B$, and $A_1, \ldots, A_n$ occur in the proof, then add B to it. The fifth element of the new line is the union of the fifth elements of the lines mentioned in its third element.

RC If $\vdash_{\mathbf{PIL}} DEK\{C_1, \ldots, C_m\} \vee ((A_1\&\ldots\&A_n) \supset B)$, and $A_1, \ldots, A_n$ occur in the proof, then add B to it *provided* that each factor of $DEK\{C_1, \ldots, C_m\}$ is reliable (at that stage). The fifth element of the new line is the union of $\{C_1, \ldots, C_m\}$ and of the fifth elements of the lines mentioned in its third element.

RD If C is not (any more) reliable, then delete from the proof all lines the fifth element of which contains C[12].

At any stage of a proof, it is obligatory to apply RD and permitted to apply RU and RC. If the fifth element of a line is empty, the formula (its second element) is **PIL**-derivable from the premises (and cannot possibly be deleted later); if the fifth element is not empty, its formula is provisionally derived and, unless it can also be derived at a line the fifth element of which is empty, not a **PIL**-consequence of the premises.

Please remark that the (unconditional) occurrence of (plain) DEK-formulas in the proof determines whether some formulas are reliable, and hence which applications of RC are permitted in view of **PIL**-theorems of the form $DEK\{C_1,$

[11] See sections 4 and 5 of [4] for the rationale of this requirement; see also Lemma 2 below.

[12] Instead of deleting wffs, one might require that a rule of inference should only be applied to formulas that are *still derivable* at the stage of the proof; see [4].

$1\ldots, C_m\} \vee ((A_1\&\ldots\&A_n) \supset B)$. As usual, proofs may be sped up by derived rules – e.g., RC may be restricted in such a way that the occurrence of $DEK\{Py, \ldots\}$ makes the consistent behaviour of Px unreliable.

As RU and RC are in a sense meta-rules, it may be useful to have a look at some object rules. Of course, all positive rules of PC are valid unconditionally in **APIL1**: $A, A \supset B/B$; $A, B/A\&B$; $A\&B/A$; $A/A \vee B$; $A \vee B, A \supset C, B \supset C/C$; $(\forall\alpha)A(\alpha)/A(\beta)$; etc. Here are some rules that may be applied at a stage of a proof provided A is reliable: $A \vee B, \sim A/B$; $B \supset A, \sim A/ \sim B$; $A/ \sim\sim A$; $\sim A/ \sim (A\&B)$. For $\sim (\exists\alpha)A/(\forall\alpha) \sim A, (\exists\alpha)A$ should be reliable; for $(\forall\alpha) \sim A/ \sim (\exists\alpha)A$, (the possibly open formula) A should be reliable – i.e., $(\exists\alpha)(A\& \sim A)$ is not derived unconditionally[13].

With respect to natural deduction proofs, it is useful to remark that Conditional Proof (to derive $A \supset B$ from a proof of B on the hypothesis A) holds unconditionally and that Reductio (to derive $\sim A$ from a proof of B and $\sim B$ on the hypothesis A) may only be applied provided (that) B is reliable. It is useful to remark that *only the premises* determine whether some formula is reliable. With respect to Reductio, for example, it is quite all right that $B\& \sim B$ is derivable from the premises together with hypotheses that are not yet eliminated – but this is not the place to discuss these matters any further.

As explained in section 7 of [4], "theorem" may be defined in two different ways in **APIL1**, resulting in different sets of theorems, viz. those of **CL** and those of **PIL**. As **APIL1** is non-monotonic, proofs from premises and axioms should be carried out in such a way that some axioms (of **CL**) are written conditionally (and that the conditions are transformed in applications of Uniform Substitution). But let me return to the formulation in terms of RU, RC, and RD.

As the proofs are dynamic, a wff may be derivable (and derived) at some stage of a proof, and may be deleted (and not derivable any more) at a later stage. So, we need to distinguish between provisional consequences and *final* consequences.

Definition 5 *An extension of an* **APIL1***-proof from* Γ *is intelligent iff it has the following property: if both* $DEK(\Sigma)$ *and* $DEK(\Sigma\cup\Pi)$ *occur unconditionally in the extension, then the former precedes the latter.*

Definition 6 *A is finally derived at some line in an* **APIL1***-proof iff it is the second element of that line and the line will not be deleted in any intelligent extension of the proof.*

Definition 7 $\Gamma \vdash_{\mathbf{APIL1}} A$ *(A is an* **APIL1***-consequence of* Γ *or is finally derivable from* Γ*) iff A is finally derived at some line in an* **APIL1***-proof from* Γ.

[13] A (structurally) simple way to implement the proof procedure is to apply only **PIL**-rules and to restrict RC as follows: If $DEK\{C_1, \ldots, C_m\} \vee A$, occurs at line j of the proof at a stage, then add A to the proof *provided* that, at that stage, each factor of $DEK\{C_1, \ldots, C_m\}$ is reliable; the fifth element of the new line is the union of $\{C_1, \ldots, C_m\}$ and of the fifth element of line j.

PIL is not decidable. Hence, unlike for its propositional fragment, there is no algorithm for writing intelligent proofs. A dramatic consequence is that, while we have a positive test for **CL**-derivability, we lack even this for **APIL1**. Of course, some fragments of **APIL1** are decidable, and we may have reasons to assume consistency or local consistency – most people applying **APIL1** to elementary Peano Arithmetic will assume that it is consistent.

Yet, it is possible to prove that $Cn_{\mathbf{APIL1}}(\Gamma)$, the set of **APIL1**-consequences of Γ, may be characterized without referring to the dynamics of the proofs. This is important in itself as well as with respect to the semantics introduced in section 5. The characterization refers to **PIL** only. The central point is that it depends only on **PIL**-derivability (which is monotonic) whether a wff is reliable in an intelligent extension of a proof.

Lemma 2 *If, in an* **APIL1***-proof from* Γ*,* A *occurs as the second element and* $\{C_1, \ldots, C_m\}$ $(0 \leq m)$ *occurs as the fifth element of a line, then* $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$*.*

The proof proceeds exactly as for Lemma 1 of [4], by induction on the number of the line at which A occurs. If the antecedent of Lemma 2 is true, then it is possible to add to the proof a line which has $A \vee DEK\{C_1, \ldots, C_m\}$ as its second element and the fifth element of which is empty.

Where a *DEK-formula* is a formula of the form $DEK\{A_1, \ldots, A_m\}$, we define:

Definition 8 *A DEK-consequence of* Γ *is a* DEK*-formula which is* **PIL***-derivable from* Γ*.*

Definition 9 *A minimal* DEK*-consequence of* Γ *is a* DEK*-consequence of* Γ *such that no result of dropping a disjunct from it is a* DEK*-consequence of* Γ*.*

Theorem 7 $\Gamma \vdash_{\mathbf{APIL1}} A$*, iff there are* $C_1, \ldots, C_m$ $(m \geq 0)$ *such that* $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$*, and none of* $C_1, \ldots, C_m$ *is a factor of a minimal* DEK*-consequence of* Γ*.*

Proof. For the first direction, suppose that $\Gamma \vdash_{\mathbf{APIL1}} A$. Hence, A is finally derived at some line j of a proof from Γ. Let the fifth element of this line be $\{C_1, \ldots, C_m\}$. Hence, $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$ $(m \geq 0)$ by Lemma 2. Suppose now that C_i is a factor of a minimal DEK-consequence of Γ; then there is an intelligent extension of the proof in which that DEK-consequence of Γ occurs unconditionally; but then line j is deleted by RD, which contradicts the fact that A is finally derived at line j.

For the second direction, suppose that there are $C_1, \ldots, C_m$ $(m \geq 0)$ such that (i) $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$ and (ii) no C_i is a factor of a minimal DEK-consequence of Γ. Then, there is an **APIL1**-proof from Γ in which A occurs as the second element of a line the fifth element of which is $\{C_1, \ldots, C_m\}$.

Figure 1:

Moreover, this line will not be deleted in any intelligent extension of the proof. It follows that A is finally derived at that line. Whence $\Gamma \vdash_{\mathbf{APIL1}} A$. ■

An important feature of **APIL1** is expressed by the following theorem:

Theorem 8 *If* $\Gamma \vdash_{\mathbf{APIL1}} A$*, then it is possible to extend any proof from* Γ *into a proof in which* A *is finally derived from* Γ*.*

Proof. If $\Gamma \vdash_{\mathbf{APIL1}} A$, there is a proof from Γ in which A is finally derived. Call this Proof 1. Consider an arbitrary proof from Γ, call it Proof 2, that consists of j lines. For any DEK-formula in proof 2, it is possible to add to Proof 2 a minimal DEK-formula from which it is derivable. This will extend Proof 2 with at most j lines. In view of RU, RC, and RD, it is obvious that the sequence consisting of the lines of Proof 1 followed by the lines of the extended Proof 2 form a proof in which A is finally derived from Γ. ■

5 Semantics of the inconsistency-adaptive logic APIL1

The dynamic proof-procedure of **APIL1** displays an interesting similarity to real thought processes. Still, the semantics offers a different and insightful perspective.

All **CL**-models are **PIL**-models. If Γ is consistent, its consistent **PIL**- models are its **CL**-models; but it has inconsistent models as well (except for the border case where Γ is maximally non-trivial, i.e. the set of wffs true in some ω-complete **CL**-model). If Γ is inconsistent, it has inconsistent **PIL**-models only (and no **CL**-models).

The **APIL1**-semantics is obtained from the **PIL**-semantics by defining, for each Γ, a subset of the **PIL**-models of Γ. The idea is that any Γ defines a set of (semantically) unreliable formulas, and that the **APIL1**-models of Γ are those **PIL**-models of Γ in which only unreliable formulas behave inconsistently. I summarize the situation in Figure 1: the two larger ellipses represent all **PIL**-models and all **CL**- models respectively; the circle represents the **PIL**-models of Γ; the smallest field marked Γ represents the **APIL1**-models of Γ.

Definition 10 *Where* M *is a* **PIL**-*model,* $EK(M) = \{A v_M(\exists(A \& \sim A)) = 1\}$.

Remark that, if $A(x) \in EK(M)$, then $A(\alpha) \in EK(M)$ for all $\alpha \in \mathsf{V}$. The set of formulas that are (semantically) *unreliable* with respect to Γ is defined as follows:

Definition 11 *$U(\Gamma)$ is the set of the factors of minimal DEK-consequences of Γ*[14].

Definition 12 *A **PIL**-model M is maximally **APIL1**-normal with respect to Γ iff (i) M is a **PIL**-model of Γ, and (ii) $EK(M) \supseteq U(\Gamma)$.*

Definition 13 *M is an **APIL1**-model of Γ iff M is maximally **APIL1**-normal with respect to Γ.*

Definition 14 *$\Gamma \models_{\mathbf{APIL1}} A$ iff A is true in all **APIL1**-models of Γ.*

Theorem 9 *If $\Gamma \vdash_{\mathbf{APIL1}} A$, then $\Gamma \models_{\mathbf{APIL1}} A$.*

Proof. Suppose that $\Gamma \vdash_{\mathbf{APIL1}} A$. By Corollary 1 and Theorem 7, there are $C_1, \ldots, C_m$ ($m \geq 0$) such that $\Gamma \models_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$ and $C_1, \ldots, C_m \notin U(\Gamma)$. If $\Gamma \models_{\mathbf{PIL}} A$ then $\Gamma \models_{\mathbf{APIL1}} A$. If there are $C_1, \ldots, C_m$ ($m \geq 1$) such that $\Gamma \models_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$ and $C_1, \ldots, C_m \notin U(\Gamma)$, then, for all **APIL1**-models M of Γ, $v_M(A \vee DEK\{C_1, \ldots, C_m\}) = 1$ and $v_M(DEK\{C_1, \ldots, C_m\}) = 0$, and hence $v_M(A) = 1$. ■

Theorem 10 *If $\Gamma \models_{\mathbf{APIL1}} A$ then $\Gamma \vdash_{\mathbf{APIL1}} A$.*

Proof. Suppose that $\Gamma \nvdash_{\mathbf{APIL1}} A$. Let B_1, B_2, ... the sequence from the proof of Theorem 4 and define:

$$
\begin{aligned}
\Delta_0 &= Cn_{\mathbf{PIL}}(\Gamma \cup \{\exists(B\& \sim B) \supset AB \in \mathsf{F} - U(\Gamma)\}) \\
\Delta_{i+1} &= Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\}) \text{ if } A \notin Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\}), \text{ and} \\
\Delta_{i+1} &= \Delta_i \text{ otherwise} \\
\Delta &= \Delta_0 \cup \Delta_1 \cup \ldots
\end{aligned}
$$

Each of the following is provable:

(i) $\Gamma \supseteq \Delta$ (by the definition of Δ).

(ii) $A \notin \Delta$. By the definition of Δ, if $A \in \Delta$, then $A \in \Delta_0$. The latter, however, is impossible. Indeed, if $A \in \Delta_0$, then there are $C_1, \ldots, C_m \in \mathsf{F} - U(\Gamma)$ ($m \geq 1$) such that $\Gamma \cup \{DEK\{C_1, \ldots, C_n\} \supset A\}_{\mathbf{PIL}} A$[15]; hence, by the deduction theorem, $\Gamma \vdash_{\mathbf{PIL}} (DEK\{C_1, \ldots, C_n\} \supset A) \supset A$; hence $\Gamma \vdash_{\mathbf{PIL}} DEK\{C_1, \ldots, C_n\} \vee A$; but as $C_1, \ldots, C_m \in \mathsf{F} - U(\Gamma)$, it follows that $\Gamma \vdash_{\mathbf{APIL1}} A$, which contradicts the main supposition.

(iii) If $C \notin U(\Gamma)$, then $\exists(C\& \sim C) \notin \Delta$. Indeed, if $C \notin U(\Gamma)$, then $\exists(C\& \sim C) \supset A \in \Delta_0$; so, if $\exists(C\& \sim C) \in \Delta$, then $A \in \Delta$, which contradicts (ii).

[14] In view of Corollary 1, the minimal DEK-consequences of Γ are determined by the **PIL**-models of Γ.

[15] In view of the fact that any proof is finite, and of $A \supset B, C \supset B_{\mathbf{PIL}}(A \vee C) \supset B$.

(iv) Δ is deductively closed (by the definition of Δ).

(v) Δ is maximally non-trivial (as in the proof of Theorem 4).

(vi) Δ is prime (as in the proof of Theorem 4).

(vii) Δ is ω-complete (as in the proof of Theorem 4).

As in the proof of Theorem 4, a **PIL**-model M is defined from Δ. In view of (i) and (ii), all members of Γ are true in M and A is false in M. In view of (iii), M is an **APIL1**-model of Γ. Hence $\Gamma \not\models_{\mathbf{APIL1}} A$. ■

Theorems 7, 9 and 10 give us:

Corollary 2 *$\Gamma \vdash_{\mathbf{APIL1}} A$ iff $\Gamma \models_{\mathbf{APIL1}} A$.*

Theorem 11 *If Γ is consistent, then $Cn_{\mathbf{APIL1}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$.*

Proof. If Γ is consistent, the **APIL1**-models of Γ are the consistent **PIL**- models of Γ, viz. the **CL**-models of Γ. ■

Theorem 12 *If there is an A such that (i) $A \notin U(\Gamma)$, and (ii) and $Cn_{\mathbf{PIL}}(\Gamma \cup \{\exists(A \& \sim A)\})$ is not trivial, then $Cn_{\mathbf{PIL}}(\Gamma)$ is a proper subset of $Cn_{\mathbf{APIL1}}(\Gamma)$.*

Proof. Suppose that the antecedent is true. In view of (ii), there is a B such that $\Gamma \nvdash_{\mathbf{PIL}} \exists(A \& \sim A) \supset B$. But $\vdash_{\mathbf{PIL}} \exists(A \& \sim A) \vee (\exists(A \& \sim A) \supset B)$. Hence, in view of (i), $\Gamma \vdash_{\mathbf{APIL1}} \exists(A \& \sim A) \supset B$. ■

In [8] Priest discusses an interesting property which he calls reassurance. For **APIL1** reassurance is provable:

Theorem 13 *$Cn_{\mathbf{APIL1}}(\Gamma)$ is not trivial unless $Cn_{\mathbf{PIL}}(\Gamma)$ is trivial. (Reassurance)*

Proof. Suppose that $Cn_{\mathbf{APIL1}}(\Gamma)$ is trivial whereas $Cn_{\mathbf{PIL}}(\Gamma)$ is not, and let $A \notin Cn_{\mathbf{PIL}}(\Gamma)$. Then $A \& \sim A \notin Cn_{\mathbf{PIL}}(\Gamma)$ whereas $A \& \sim A \in Cn_{\mathbf{APIL1}}(\Gamma)$. In view of Theorem 7, this is only possible if there are $C_1, \ldots, C_n$ $(n > 0)$ such that $DEK\{C_1, \ldots, C_n\} \vee (A \& \sim A) \in Cn_{\mathbf{PIL}}(\Gamma)$ and $C_1, \ldots, C_n \notin U(\Gamma)$. But then $DEK\{C_1, \ldots, C_n, A\} \in Cn_{\mathbf{PIL}}(\Gamma)$. As no member of $\{C_1, \ldots, C_n\}$ is a factor of a minimal DEK-consequence of Γ, $(DEK\{A\} =)A \& \sim A \in Cn_{\mathbf{PIL}}(\Gamma)$, which is impossible. ■

6 Semantics of the inconsistency-adaptive logic APIL2

For **APIL2**, our second inconsistency-adaptive logic, it appears advisable to start from the semantics (which is an extension of the semantics in [3] to the predicative level). The central difference with the **APIL1**-semantics lies in the different strategy to obtain maximally normal models. Here the strategy will be *minimizing abnormality*. This will result in a stronger selection of models: it will appear that all **APIL2**-models of some set of premises are **APIL1**-models of it, but that the converse does not always hold. If, e.g., $DEK(p, q)$ is the only minimal DEK-consequence of Γ, then, unlike for **APIL1**-models of Γ, either $p\& \sim p$ or $q\& \sim q$ is false in any **APIL2**-model of Γ.

Definition 15 *A* **PIL***-model M is maximally* **APIL2***-normal with respect to Γ iff (i) M is a* **PIL***-model of Γ, and (ii) there is no* **PIL***-model M' of Γ such that $EK(M') \subset EK(M)$.*

To avoid misunderstanding, it is useful to repeat that, if $A(x) \in EK(M)$, then $A(\alpha) \in EK(M)$ for all $\alpha \in \mathsf{V}$[16].

Definition 16 *M is an* **APIL2***-model of Γ iff M is maximally* **APIL2***-normal with respect to Γ.*

Definition 17 $\Gamma \models_{\mathbf{APIL2}} A$ *iff A is true in all* **APIL2***-models of Γ.*

In general, the relation between **PIL**-models, **APIL2**-models, and **CL**-models is still as in Figure 1. If Γ is consistent and not maximally non-trivial, it has consistent as well as inconsistent **PIL**-models, but its **APIL2**-models are its **CL**-models. So, if Γ is consistent, **APIL2** and **CL** define the same set of semantic-consequences for it. If Γ is inconsistent, it has no consistent **PIL**-models (it has no **CL**- models). Still, some **PIL**-models of Γ will (except for border cases) be more inconsistent than is required to make Γ true. In other words, some **PIL**-models of Γ will not be maximally **APIL2**-normal with respect to Γ; these are not **APIL2**-models of Γ. As the **APIL2**-models of Γ are, in general, a subset of its **PIL**-models, Γ has, in general, more **APIL2**-semantic-consequences than **PIL**-semantic-consequences.

It is not, however, just a matter of having less models and more semantic consequences. More importantly, the definitions show that **APIL2** interprets a set of premises *as consistent as possible* in the sense that no more formulas of the form $\exists(A\& \sim A)$ are true than is required by the set of premises.

It should be mentioned that there are other senses of interpreting premises as consistent as possible. If $Px\& \sim Px$ and $Py\& \sim Py$ are the only inconsistent formulas true in M and only $Px\& \sim Px$ is true in M', then $EK(M') = EK(M)$. Indeed, as far as wffs are concerned, M is not more inconsistent than M'. This situation is largely due to the fact that I choose to consider inconsistencies as linguistic matters and not as ontological ones. In the **PIL**-semantics, inconsistencies are provoked by the values assigned to formulas of the form $\sim A$. I gave

[16] In view of this (and of the **PIL**-semantics), making the domain as small as possible does not result in general in models that are less inconsistent. In this sense **APIL1** and **APIL2** differ drastically from $\mathbf{LP}^m$ – see p. 325 of [8].

a reason for this decision in section 3, viz. symmetry with respect to the absence of other replacements of extensionally identicals within the scope of negation. Once the decision was taken, it became rather artificial (as an inspection of the semantics shows) to try to minimize inconsistencies in another sense[17]. Let me add a final comment on the matter. People that consider the way in which **APIL2** minimizes inconsistencies unsatisfactory should consider its functioning in ω-complete models before settling their minds. In the semantics of $\mathbf{LP^m}$, for example, Graham Priest locates inconsistencies at the level of the elements of the domain, but considers ω-complete models only – see [8].

7 Proof theory of the inconsistency-adaptive logic APIL2

Let the format of proofs be as for **APIL1**, except that no lines are deleted in **APIL2**- proofs, but that there may be tentative lines, indicated with a mark after the line number. Marked lines are not considered as occurring in the proof and may not be relied upon for adding further lines. After each step, the marks are updated, viz. removed or added.

The updating of the marks will be governed by an *integrity criterion*. The intuitive idea is as follows. Suppose that A is derived on one or more lines the fifth element of which is not empty. A is considered as derived (at a stage of the proof) and the lines become a full part of the proof if A comes out true under any maximally normal 'interpretation' of the least DEK- formulas (at that stage). Since we are in the proof theory, "interpretation" should refer to formal properties of the formulas that occur in the proof.

Let us first try to get some grasp on DEK-formulas. Clearly, *at least* one disjunct of each DEK-formula should be true. As we shall have to look at combinations of factors of DEK- formulas, it is useful to remark that some DEK-formulas that occur unconditionally in a proof may be disregarded. Suppose that a Gödel-numbering (or some other ordering) of formulas is given. Where A and B are DEK-formulas, let us stipulate that

Definition 18 *$A \prec B$ iff either (i) $A \vdash_{\mathbf{PIL}} B$ and $B \nvdash_{\mathbf{PIL}} A$, or (ii) A and B are* **PIL***-equivalent and the Gödel-number of A is smaller than the Gödel number of A.*

Definition 19 *A is a least DEK-formula (at a stage of the proof) if it occurs unconditionally in the proof and no DEK-formula B such that $B \prec A$ occurs unconditionally in the proof.*

[17]Suppose that $v(x) = v(y) \in v(P)$ and that $v(\sim Px) = 1$. The model seems to become more inconsistent if also $v(\sim Py) = 1$; if, however, $v(\sim Py) = 0$, then $(\exists x)(\exists y)(x = y \&(\sim Px \&(\sim Py \supset \bot)))$ is true in the model, which also seems a kind of inconsistency (as $\sim Py \supset \bot$ is a strong negation of $\sim Py$).

If $DEK(\Gamma \cup \{Px\})$ and $DEK(\Gamma \cup \{Py\})$ occur unconditionally in the proof and the Gödel-number of the former is smaller than that of the latter, then at best the former will be a least DEK-formula. Neither of them is a least DEK-formula if $DEK(\Gamma \cup \{Pa\})$ also occurs unconditionally in the proof. Clearly, if one disjunct of each least DEK- formula is true, then all DEK-formulas are true (at that stage).

Let $*\Phi_s$ be the set of all sets that contain one factor out of each least DEK-formula (at stage s of the proof). $*\Phi_s$ may contain redundant elements for two different reasons. The first is related to the individual variables. Where neither x nor y occurs free in $A(z), \exists(A(x)\& \sim A(x))$ is **PIL**-equivalent to $\exists(A(y)\& \sim A(y))$. But $A(x)$ may be a factor of some least DEK-formula and $A(y)$ of another. Hence, $*\Phi_s$ may contain $\{Px, Py\}$, or may contain both $\{Px, p\}$ and $\{Py, p\}$. To reduce these, we define $^\circ\Phi_s$ from $*\Phi_s$ by relettering all open formulas in the members of $*\Phi_s$ in such a way that the free variables occur always in the same order (for all formulas, the first occurring free variable is always x_1, the second always x_2, etc.). The second reason for redundant elements is that the same factor may occur in different least DEK-formulas. If $DEK\{p, q\}$ and $DEK\{p, r\}$ are the least DEK-formulas, $*\Phi_s =^\circ \Phi_s = \{\{p\}, \{p, r\}, \{p, q\}, \{q, r\}\}$. Of these $\{p, r\}$ and $\{p, q\}$ are redundant: both $DEK\{p, q\}$ and $DEK\{p, r\}$ are true if $p\& \sim p$ is true; there is no need that also $r\& \sim r$ or $q\& \sim q$ be true. So, let Φ_s be obtained from $^\circ\Phi_s$ by eliminating elements from it that are proper supersets of other elements. The members of Φ_s are sets of formulas, such that, if $\exists(A\& \sim A)$ is true for all members A of such a set, then all DEK-formulas that occur unconditionally in the proof are true. To see this, it is sufficient to realize that, if A and B are different formulas (and not reletterings of each other with respect to the individual variables), then $\exists(A\& \sim A)$ and $\exists(B\& \sim B)$ are **PIL**-independent formulas – remember that **PIL** does not spread inconsistencies.

Where Φ_s is as defined above and A is the second element of line j, line j *fulfils the integrity criterion* (at stage s) iff (i) the intersection of some member of Φ_s and of the fifth element of line j is empty, and (ii) for each $\varphi \in \Phi_s$ there is a line k such that the intersection of φ and of the fifth element of line k is empty and A is the second element of line k. As a (very) simple illustration, consider:

(j)	$DEK\{p, q, r\}$	...	...	$\emptyset$
(j+1)	A	...	...	$\{p, q\}$
(j+2)	A	...	...	$\{q, r\}$
(j+3)	A	...	...	$\{p, r\}$

If (j) is the only least DEK-formula in the proof, $\Phi_s = \{\{p\}, \{q\}, \{r\}\}$ and lines (j+1)-(j+3) fulfil the integrity criterion. They also fulfil the integrity criterion if the second element of line (j) is $DEK\{p, q, r, s\}$[18].

[18] The integrity criterion has a helpful geometrical interpretation. A least DEK-formula corresponds to a (finite) dimension of a many- dimensional space; its factors correspond to

Let us now turn to the **APIL2**-rules.

RU As for **APIL1**.

RC If $\vdash_{\mathbf{PIL}} DEK\{C_1, \ldots, C_m\} \vee ((A_1 \& \ldots \& An) \supset B)$, and $A_1, \ldots, A_n$ occur in the proof at a stage, then add B to it *provided*[19] that, at that stage, each factor of $DEK\{C_1, \ldots, C_m\}$ *behaves consistently.* The fifth elements of the new line is the union of $\{C_1, \ldots, C_m\}$ and of the fifth elements of the lines mentioned in its third element.

RQ+ A mark is added to a line that does not fulfil the integrity criterion, and to all lines derived from it.

RQ- If a line fulfils the integrity criterion and is marked, the mark is removed.

This format requires many duplicated lines and lots of updating of the marks. More convenient formats involve complications that are expendable with respect to the aim of the present paper.

Definition 20 *An extension of an* **APIL2***-proof from* Γ *is intelligent iff it has the following two properties: (i) if both* $DEK(\Sigma)$ *and* $DEK(\Sigma \cup \Pi)$ *occur unconditionally in the extension, then the former precedes the latter, and (ii) if* $DEK(\Sigma)$ *may be added unconditionally to the proof, and if some line becomes marked when* $DEK(\Sigma)$ *is added at once but does not become marked when some other lines are added first, then these other lines are added before* $DEK(\Sigma)$*.*

Remark that Φ_s is always a finite set of finite sets of formulas. Suppose that A occurs as the second element of some unmarked line, and that it becomes marked if $DEK(\Sigma)$ is added unconditionally. Whether it is possible to prevent this line from being marked depends on the possibility to add lines that have A as their second element and some out of a finite number of sets as fifth elements.

Definition 21 *A is finally derived at some line of an* **APIL2***-proof iff it is the second element of that line, and the line is not marked and will not be marked in any intelligent extension of the proof.*

Definition 22 $\Gamma \vdash_{\mathbf{APIL2}} A$ *(A is an* **APIL2***-consequence of* Γ *or is finally derivable from* Γ*) iff A is finally derived at some line of an* **APIL2***-proof from* Γ*.*

As for **APIL1**, we have:

the elements of the dimension. A point of the space corresponds to a way to make all least DEK-formulas true. A fifth element of a line corresponds to a union of hyperplanes; the intersection of all these unions is empty if the integrity criterion is fulfilled.

[19]If the line were added while the proviso is not fulfilled, it would be marked at all stages of the proof.

Lemma 3 *If, in an* **APIL2***-proof from* Γ*, A occurs as the second element and* $\{C_1, \ldots, C_m\}$ $(0 \leq m)$ *occurs as the fifth element of a line, then* $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK\{C_1, \ldots, C_m\}$.

Definition 23 *A strongest minimal DEK-consequence of* Γ *is one that is not* **PIL***-derivable from any non-equivalent minimal DEK-consequence of* Γ.

If $DEK\{Pb, Qa\}$ is a minimal DEK-consequence of Γ, then $DEK\{Px, Qa\}$ is a minimal DEK-consequence of Γ, but not a strongest one. Any DEK-consequence of Γ, A, is derivable from a strongest minimal DEK-consequence of Γ, possibly A itself.

Definition 24 *A is a least DEK-consequence of* Γ *iff (i) it is a strongest minimal DEK- consequence of* Γ *and (ii) no* $B \prec A$ *is a strongest minimal DEK-consequence of* Γ.

Definition 25 Φ_Γ *is the set of all sets that contain exactly one factor of each least DEK- consequence of* Γ *and that are not proper supersets of such a set*[20].

Theorem 14 $\Gamma \vdash_{\mathbf{APIL2}} A$ *iff there are one or more (possibly empty) finite sets* $\Sigma_1, \Sigma_2, \ldots$ *such that* $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_1), \Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_2), \ldots$*, and, for any* $\varphi \in \Phi_\Gamma$*, one of the* Σ_i *is such that* $\Sigma_i \cap \varphi = \emptyset$.

Proof. For the first direction, suppose that $\Gamma \vdash_{\mathbf{APIL2}} A$. Hence, A is finally derived at some line j of a proof from Γ. Let the fifth element of this line be (the finite set) Σ. Hence, $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma)$ by Lemma 3. If Σ is empty or, for any $\varphi \in \Phi_\Gamma, \Sigma \cap \varphi = \emptyset$, then the consequent of the first direction is true. So, suppose that, for some $\varphi \in \Phi_\Gamma$, there is no Σ such that $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma)$ and $\Sigma \cap \varphi \neq \emptyset$. It follows that, for any set $\{\Sigma_1, \ldots, \Sigma_n\}$ such that $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_1), \ldots, \Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_n)$, there is a finite $\varphi' \supseteq \phi$ such that, $\Sigma_1 \cap \varphi' \neq \emptyset, \ldots, \Sigma_n \cap \varphi' \neq \emptyset$. Hence, there is an intelligent extension of the proof in which a finite set of least DEK-consequences of Γ occur unconditionally and in which line j does not fulfil the integrity criterion (some member of Φ_s overlaps with the fifth element of each old or new line of which A is the second element). But then A is not finally derived at line j, which contradicts the supposition.

For the second direction, suppose that there are one or more (possibly empty) finite sets $\Sigma_1, \Sigma_2, \ldots$ such that $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_1), \Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_2), \ldots$, and, for any $\varphi \in \Phi_\Gamma$, one of the Σ_i is such that $\Sigma_i \cap \varphi = \emptyset$. I now show that A is finally derivable from Γ. Consider a proof from Γ in which (i) some unmarked line j has A as its second element and has one of the aforementioned Σ_i, viz. Σ_j, as its fifth element, and in which (ii) each least DEK-formula is a minimal DEK-consequence of Γ[21]. Consider also any formula $DEK(\Pi)$ that is

[20] I suppose that the Gödel numbering warrants that free individual variables always occur in the same order in factors of least DEK-consequences of Γ.

[21] There is bound to be such a proof, at the worst one in which no DEK-formula occurs.

a minimal DEK-consequence of Γ. Let the addition of $DEK(\Pi)$ to the proof result in Φ_s. Whether we first add non-DEK- formulas to the proof has no effect on Φ_s. Remark that Φ_s has a finite number of members, each of which is a finite set. Remark also that each $\varphi \in \Phi_s$ is a subset of some $\varphi' \in \Phi_\Gamma$. As, for some $\varphi' \in \Phi_\Gamma, \Sigma_j \cap \varphi' = \emptyset$, there is a $\varphi \in \Phi_s$ such that $\Sigma_j \cap \varphi = \emptyset$. Moreover, for any (of the finitely many) $\varphi \in \Phi_s$, there is a $\varphi' \in \Phi_\Gamma$ and a finite Σ_k such that $\Gamma \vdash_{\mathbf{PIL}} A \vee DEK(\Sigma_k), \Sigma_k \cap \varphi' = \emptyset$ and $\varphi \supseteq \varphi'$; but then also $\Sigma_k \cap \varphi = \emptyset$. Hence there is an intelligent extension of the proof in which $DEK(\Pi)$ occurs unconditionally and in which line j is not marked. ■

As for **APIL1** it is easy (and hence left to the reader) to prove:

Theorem 15 *If* $\Gamma \vdash_{\mathbf{APIL2}} A$*, then it is possible to extend any proof from* Γ *into a proof in which* A *is finally derived from* Γ*.*

Let us now move on to the relation between the syntax and the semantics. As before, I shall rely on Corollary 1 without referring to it.

The following lemma is very central and deserves some explanation. By an *equivalence set* of formulas I shall mean a set of formulas that are **PIL**-equivalent. Even if **PIL** only sparsely relates inconsistencies to each other, some DEK-formulas are **PIL**-equivalent, e.g., $DEK\{Px, Qa\}$ and $DEK\{Py, Qa\}$, and some DEK-formulas are stronger than others, as we have seen. It is interesting to have a closer look at the relation between DEK-consequences of Γ and the inconsistencies true in **APIL2**-models of Γ. First, the number of equivalence sets of strongest minimal DEK-consequences of Γ does not uniquely determine the number of inconsistencies true in a model M of Γ. For example, if $DEK\{p, q\}$ and $DEK\{p, r\}$ are the minimal DEK-consequences of Γ, then in some models $p \& \sim p$ will be the only true inconsistency, whereas both $q \& \sim q$ and $r \& \sim r$ will be true in other models. Next, more than one factor of some strongest minimal DEK- consequence of Γ may behave inconsistently in an **APIL2**-model of Γ. For example, if $DEK\{p, q\}, DEK\{q, r\}$, and $DEK\{p, r\}$ are the minimal DEK-consequences of Γ, then *two* out of $p \& \sim p$, $q \& \sim q$ and $r \& \sim r$ are bound to be true in any **APIL2**-model of Γ. Finally, if $DEK\{Pxy, \ldots\}$ is and $DEK\{Pxx, \ldots\}$ is not a minimal DEK-consequence of Γ, then $v_M(Pxx \& \sim Pxx) = 0$ for any **APIL2**-model of Γ; indeed, if it were true, then not only $v_M((\exists x)(\exists y)(Pxy \& \sim Pxy)) = 1$, but also $v_M((\exists x)(Pxx \& \sim Pxx)) = 1$, whence M would not be maximally normal with respect to Γ.

In any **APIL2**-model M of Γ, there is a set of (open and closed) formulas A such that $v_M(A \& \sim A) = 1$. As a result, there are wffs of the form $\exists(A \& \sim A)$ such that $v_M(\exists(A \& \sim A)) = 1$. For any set of equivalent such wffs, we may select a *paradigmatic* member on the basis of the occurrence of free variables in $A \& \sim A$, which I supposed before to agree with the order introduced by the Gödel numbering of formulas. The degree of inconsistency of a model is determined by these paradigmatic inconsistencies, as appears from the definition of an **APIL2**-model.

For any **APIL2**-model M of Γ, let ψ_M be the set of A such that $A\& \sim A$ is a paradigmatic inconsistency true in M. Let Ψ_Γ be the set of all these ψ_M.

Lemma 4 *For any $\varphi \in \Phi_\Gamma$, Γ has an* **APIL2***-model M such that $\psi_M = \varphi$.*

Proof. Let Γ be prepared as for Theorem 4, and let $B_1, B_2, \ldots$ be the sequence from the proof of that theorem. Select some $\varphi \in \Phi_\Gamma$, and let $\Pi = \{\exists(A\& \sim A)A \in \varphi\}$. Define:

$$\Delta_0 = Cn_{\mathbf{PIL}}(\Gamma \cup \Pi)$$

If there is no Σ such that $DEK(\Sigma) \in Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\})$ and $\Pi \nvdash_{\mathbf{PIL}} DEK(\Sigma)$, then

$$\Delta_{i+1} = Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\}),$$

otherwise

$$\Delta_{i+1} = Cn_{\mathbf{PIL}}(\Delta_i \cup \sim B_{i+1}).$$

$$\Delta = \Delta_1 \cup \Delta 2 \cup \ldots$$

It is possible to prove that there is an ω-complete **PIL**-model M such that (i) $v_M(A) = 1$ iff $A \in \Delta$, and (ii) $\psi_M = \varphi$. The proof is completely analogous to that of Theorem 4, except for the following bit.

If there is a Σ such that $DEK(\Sigma) \in Cn_{\mathbf{PIL}}(\Delta_i \cup \{B_{i+1}\})$ and $\Pi \nvdash_{\mathbf{PIL}} DEK(\Sigma)$, then there is no Σ' such that $DEK(\Sigma') \in Cn_{\mathbf{PIL}}(\Delta_i \cup \sim B_{i+1})$ and $\Pi \nvdash_{\mathbf{PIL}} DEK(\Sigma')$. To see this, suppose that $\Pi \nvdash_{\mathbf{PIL}} DEK(\Sigma), \Pi \nvdash_{\mathbf{PIL}} DEK(\Sigma'), \Delta_i \cup \{B_{i+1}\} \vdash_{\mathbf{PIL}} DEK(\Sigma)$, and $\Delta_i \cup \{\sim B_{i+1}\} \vdash_{\mathbf{PIL}} DEK(\Sigma')$. Then, by the Deduction Theorem, the fact that $B_{i+1} \vee \sim B_{i+1}$ is a **PIL**-theorem, Dilemma, and the fact that $DEK(\Sigma) \vee DEK(\Sigma') = DEK(\Sigma \cup \Sigma')$, it follows that $\Delta_i \vdash_{\mathbf{PIL}} DEK(\Sigma \cup \Sigma')$. But then $\Pi \vdash_{\mathbf{PIL}} DEK(\Sigma \cup \Sigma')$ by the definition of Δ. But as Π is a set of formulas of the form $\exists(A\& \sim A)$, it follows that either $\Pi \vdash_{\mathbf{PIL}} DEK(\Sigma)$ or $\Pi \vdash_{\mathbf{PIL}} DEK(\Sigma')$, which contradicts the supposition. It follows that $\psi_M = \varphi$.

M is maximally normal with respect to Γ. Indeed, M is a **PIL**-model of Γ and, for all $\Pi' \subset \Pi$, there is a Σ such that $\Gamma \cup \Pi' \vdash_{\mathbf{PIL}} DEK(\Sigma)$ and $\Pi' \nvdash_{\mathbf{PIL}} DEK(\Sigma)$ (by the definition of Φ_Γ). ■

Remark that the least DEK-consequences of Γ are true in all **PIL**-models of Γ, and that all DEK-consequences of Γ are true in all **PIL**-models in which all least DEK-consequences of Γ are true (remember Corollary 1).

Lemma 5 $\Phi_\Gamma = \Psi_\Gamma$

Proof. $\Phi_\Gamma \supseteq \Psi_\Gamma$ follows from Lemma 4. For the opposite direction, let $\psi_M \in \Psi_\Gamma$. Hence M is an **APIL2**-model of Γ. Any least DEK- consequence of Γ has a factor B such that $B \in \psi_M$ and $v_M(\exists(B\& \sim B)) = 1$. Hence, there is a $\varphi \in \Phi_\Gamma$

such that $\varphi \supseteq \psi_M$. If $\psi_M \neq \varphi$, then, again by Lemma 4, there is a **APIL2**-model M' of Γ such that $\psi_{M'} = \varphi$. But then $EK(M') \subset EK(M)$, which is impossible. ■

Theorem 16 *$\Gamma \models_{\mathbf{APIL2}} A$ iff $\Gamma \vdash_{\mathbf{APIL2}} A$.*

Proof. Immediate in view of Lemma 5, Theorem 14, and Corollary 1. ■

As for the propositional fragment of **APIL1** – see Theorems 7 and 8 of [4] – we have (i) $Cn_{\mathbf{PIL}}(\Gamma) \supseteq Cn_{\mathbf{APIL2}}(\Gamma) \supseteq Cn_{\mathbf{CL}}(\Gamma)$, (ii) if Γ is consistent then $Cn_{\mathbf{APIL2}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$, and (iii) if Γ is inconsistent then, except for border cases, $Cn_{\mathbf{PIL}}(\Gamma) \subset Cn_{\mathbf{APIL2}}(\Gamma) \subset Cn_{\mathbf{CL}}(\Gamma)$. The proof of Theorem 17 is identical to that of Theorem 11.

Theorem 17 *If Γ is consistent, then $Cn_{\mathbf{APIL2}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$.*

Theorem 18 *If there are wffs $A_1, \ldots, A_n$ $(n > 0)$ such that (i) $Cn_{\mathbf{PIL}}(\Gamma \cup \{A_1 \& \sim A_1, \ldots, A_n \& \sim A_n\})$ is not trivial, and (ii) $\{A_1, \ldots, A_n\}$ is not a subset of any $\varphi \in \Phi_\Gamma$, then $Cn_{\mathbf{PIL}}(\Gamma)$ is a proper subset of $Cn_{\mathbf{APIL2}}(\Gamma)$.*

Proof. Suppose that the antecedent is true. In view of (i), $\Gamma \cup \{A_1 \& \sim A_1, \ldots, A_n \& \sim A_n\}$ has a non-trivial **PIL**-model M, which also is a **PIL**-model of Γ. Hence there is a B such that $\Gamma \nvdash_{\mathbf{PIL}} ((A_1 \& \sim A_1) \& \ldots \& (A_n \& \sim A_n)) \supset B$. In view of (ii), it follows from Lemma 5 that, for all **APIL2**-models M of $\Gamma, v_M((A_1 \& \sim A_1) \& \ldots \& (A_n \& \sim A_n)) = 0$ and hence $v_M(((A_1 \& \sim A_1) \& \ldots \& (A_n \& \sim A_n)) \supset B) = 1$. Hence $\Gamma \vdash_{\mathbf{APIL2}} ((A_1 \& \sim A_1) \& \ldots \& (A_n \& \sim A_n)) \supset B$. ■

The result is very general. I cannot imagine any interesting Γ such that *any* finite set of wffs is a subset of some $\varphi \in \Phi_\Gamma$.

Theorem 19 *$Cn_{\mathbf{APIL2}}(\Gamma)$ is trivial iff $Cn_{\mathbf{PIL}}(\Gamma)$ is trivial. (Reassurance)*

Proof. If $Cn_{\mathbf{PIL}}(\Gamma)$ is not trivial, then Γ has non-trivial **PIL**-models. These determine the set of least DEK-consequences of Γ, and hence Φ_Γ. In view of Lemmas 4 and 5, the **APIL2**-models of Γ are those **PIL**-models M for which $\psi_M \in \Phi_\Gamma$. And there are such models: the proof of Lemma 4 shows how to construct them. ■

8 Concluding remarks

As promised, I studied the syntax and semantics of two inconsistency-adaptive logics that are based on a poor and simple paraconsistent logic. With respect to the top-down view on adaptive logics, viz. where their role is restricted to transitory stages and the aim is to reach a consistent theory, richer paraconsistent bases are less suitable, as I explained in section 9 of [4].

A point that still deserves to be stressed is the need for a dynamic proof procedure: lines are deleted or marked; formulas derivable at some stage of a proof may become underivable at a later stage, or vice versa. This is related to the fact that proofs are concrete entities. Although the meta-theoretic characterizations of final derivability and semantic consequence are static, it is not possible to devise static proof procedures. This should not be considered a disadvantage of adaptive logics. The systematic way in which this dynamics is captured and its static meta- theoretic characterization should rather be considered a positive step towards an understanding of the many forms of dynamics that occur in human (creative and other) thought processes.

Both **APIL1** and **APIL2** have nice and interesting tableau methods. The related results are important from a computational point of view, but have to be postponed to another paper. So has the study of logics that are adaptive with respect to other abnormal properties.

There are many open problems about adaptive logics, even about inconsistency-adaptive ones. A rather urgent and easy one concerns those based on the **PIL**-variant which locates inconsistencies at the level of elements of the domain, rather than at the linguistic level. Even if I argued for my present choice, such inconsistency-adaptive logics may be useful both from a dialetheist point of view and, in general, with respect to theories that are not meant to be replaced by consistent ones. The most suitable approach to the **PIL**-variant is as follows. Consider all formulas of the form $\sim A$ in which all constants and free variables are replaced by the metavariables $\alpha_1, \alpha_2, \ldots$ (in that order). Any such formula is a propositional function of (some) arity n. Replace C1.4 of the **PIL**-semantics by a clause that assigns, to each propositional function of arity n, a set of n-tuples of members of D, and adapt C2.5 accordingly. This will immediately result in $(\forall x)(\forall y)(x = y \equiv (A(x) \equiv A(y)))$ being a valid formula, as desired by this approach.

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This is a very old paper. Some hints for translating to recent terminology:

	old terminology	recent terminology
conjunction	&	$\wedge$
standard negation	$\sim$	$\neg$
classical negation	$\neg$	$\sim$
disjunction of abnormalities	$DEK(\ldots)$	Dab(…)
Tarski logic	**PIL**	**CLuN**
adaptive logics	**APIL1**	$\mathbf{CLuN}^r$
adaptive logics	**APIL2**	$\mathbf{CLuN}^m$
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