RUSSELL'S SET VERSUS THE UNIVERSAL SET IN PARACONSISTENT SET THEORY*

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1. *Introduction*

Russell's set lead to the well-known troubles in Cantor's set theory. It was ruled out from classical set theory. Later, the paraconsistent logicians thought it possible and interesting to have it in paraconsistent set theory. The objective of this paper is to show that Russell's set causes other troubles in paraconsistent set theory than the ones discussed in [1] and [2].

In [2] one of us proved that in some of da Costa's paraconsistent set theories with Russell's set, $R = \hat{x} \neg(x \in x)$, $\bigcup \bigcup R$ is the universal set. In this paper we prove that this result is valid in all 'strong' paraconsistent set theories (particularly in all da Costa's paraconsistent set theories). Answering a question in Section 6 of [2] we also prove that in all strong paraconsistent set theories with Russell's set $\bigcup R$ is the universal set. Finally, we also prove that in some 'weak' paraconsistent set theories the existence of Russell's set implies the existence of the universal set.

A *paraconsistent set theory* is a set theory which is inconsistent but non-trivial; in other words, some formula is a theorem together with its negation, but nevertheless not all formulas are theorems. The underlying logic of a paraconsistent set theory should clearly be a paraconsistent logic, i.e., a logic in which there is a symbol of negation, say $\neg$, such that it is not possible in general to obtain any formula B from some formula A and its negation $\neg$A.

One of the prime motivations for constructing paraconsistent set theories resides in the attempt to articulate nontrivial set theories in which Russell's class is a set. It is well-known that set theories without universal set are richer than the ones with universal set in that the former allow for a larger number of distinct sets. Moreover, the former are more interesting for independent reasons. In view of this situation it is important to find out whether it is possible to devise paraconsistent set theories *with* Russell's set but *without* universal set, a problem already considered in [2]. The present paper contributes to the solution of this problem.

In view of the multitude of logics already developed, we shall distinguish between *strong* and *weak* paraconsistent set theories. To set a boundary we stipulate that *minimal strong paraconsistent set theories* are those that have as underlying logic the positive intuitionistic first-order logic with equality to which excluded middle, formulated as $A \vee \neg A$, is added. Notice that the propositional fragment of this logic is even weaker than the basic logic PI of [4]. Some weak paraconsistent set theories will be characterized as they are needed in the sequel of this paper.

In all paraconsistent set theories considered in this paper we suppose that Russell's class is a set.

The results presented in this paper might seem to lead to the conclusion that the paraconsistent programme is bound to fail in the context of set theory. In the last section we shall argue why we do not subscribe to this conclusion.

2. *The universal set in strong paraconsistent set theory*

In this section we first prove that $\bigcup \bigcup R$ is the universal set, i.e., that $(x) \,.\, x \in \bigcup \bigcup R$ in minimal strong paraconsistent set theories. This result may be obtained as a corollary to the second theorem in this section, $(x) \,.\, x \in \bigcup R$, but we prove it as a theorem because it was this result that gave rise to all other results presented in this paper. Moreover, Theorem 2.2 may be proved as Theorem 3.1, but as the proof of the latter is very long we give here a shorter proof of the former.

For the following proofs we need some properties of equality and some results of set theory. The properties of equality are the usual ones. The results of set theory we need concern the unitary set, $\{x\}$, and the union, $\bigcup x$ and $x \cup y$. All of them are easily proved in our minimal strong paraconsistent set theories.

LEMMA 2.1. $\vdash (x, y, z) : x, y \in \{z\} . \supset . x = y$.

LEMMA 2.2. I. $\vdash R \in R$.
II. $\vdash \neg(R \in R)$.

THEOREM 2.1. $\vdash (x) . x \in \bigcup \bigcup R$.
PROOF.

(1)	$\{R \cup \{x\}\} \in \{R \cup \{x\}\}$	hyp.
(2)	$\{R \cup \{x\}\} = R \cup \{x\}$	from (1) and Lemma 2.1.
(3)	$R \in R \cup \{x\}$	from Lemma 2.2 part I.
(4)	$R \in \{R \cup \{x\}\}$	from (2) and (3).
(5)	$R = R \cup \{x\}$	from (4) and Lemma 2.1.
(6)	$\{R \cup \{x\}\} \in R$	from (2), (5) and Lemma 2.2 part I.
(7)	$\neg(\{R \cup \{x\}\} \in \{R \cup \{x\}\})$	hyp.
(8)	$\{R \cup \{x\}\} \in R$	from (7) and the definition of R.
(9)	$\{R \cup \{x\}\} \in R$	from (1)-(6) and (7)-(8)
(10)	$\{R \cup \{x\}\} \subseteq \bigcup R$	from (9).
(11)	$R \cup \{x\} \in \bigcup R$	from (10).
(12)	$R \cup \{x\} \subseteq \bigcup \bigcup R$	from (11).
(13)	$x \in \bigcup \bigcup R$	from (12). ■

Now we shall prove that $\bigcup R$ is the universal set. To do this we need two lemmas.

LEMMA 2.3. I. $\vdash x \in R \supset \{x\} \in R$.
II. $\vdash x, y \in R \supset \{x, y\} \in R$.

PROOF.

I. If $\{x\} \in \{x\}$ then $\{x\} = x$. Thus, by the hypothesis that $x \in R$, we obtain $\{x\} \in R$. If $\neg(\{x\} \in \{x\})$ then $\{x\} \in R$. ■

II. If $\{x, y\} \in \{x, y\}$ then $\{x, y\} = x \vee \{x, y\} = y$. Thus, by the hypothesis that $x, y \in R$, we obtain $\{x, y\} \in R$. If $\neg(\{x, y\} \in \{x, y\})$, then $\{x, y\} \in R$. ■

LEMMA 2.4. $\vdash (x) . \{\{x, R\}\} \in R$.
PROOF.

(1)	$\{\{x, R\}\} \in \{\{x, R\}\}$	hyp.
(2)	$\{\{x, R\}\} = \{x, R\}$	from (1) and Lemma 2.1.

(3)	$x = R$	from (2) and Lemma 2.1.
(4)	$x, R \in R$	from (3) and Lemma 2.2, part I.
(5)	$\{x, R\} \in R$	from (4) and Lemma 2.3, part II.
(6)	$\{\{x, R\}\} \in R$	from (6) and Lemma 2.3, part I.
(7)	$\neg(\{\{x, R\}\} \in \{\{x, R\}\})$	hyp.
(8)	$\{\{x, R\}\} \in R$	from (7) and the definition of R.
(9)	$\{\{x, R\}\} \in R$	from (1)-(6) and (7)-(8). ■

THEOREM 2.2. $\vdash (x) \,.\, x \in \cup R$.
PROOF. First suppose that $\neg(\{x, R\} \in \{x, R\})$. In this case $\{x, R\} \in R$ by the definition of R and consequently $x \in \cup R$. Suppose next that $\{x, R\} \in \{x, R\}$. Hence either $\{x, R\} = R$ or $\{x, R\} = x$. If $\{x, R\} = R$ then $\{x, R\} \in R$ by Lemma 2.2, part I. If, on the other hand, $\{x, R\} = x$ then $\{\{x, R\}\} = \{x\}$ and consequently $\{x\} \in R$ by Lemma 2.4. In both cases, $x \in \cup R$. ■

Originally we found a more complicated proof of Theorem 2.2, proceeding along the lines of the proof of Theorem 4.1 in [2], but with the empty set defined as $\hat{x}\,(y) \,.\, x \in y$. The present proof has the advantage of showing very clearly why Theorem 2.2 holds, viz. that any set x is a member of some sets, viz. $\{x, R\}$ and $\{x\}$, of which at least one is itself a member of R for 'obvious' reasons. As fas as propositional logic is concerned, all we need for the present proof is Modus Ponens ($A, A \supset B \,/\, B$), Excluded Middle ($A \vee \neg A$), and Proof by Cases (From α, $\alpha \vdash A \vee B$, $\alpha \cup \{A\} \vdash C$, and $\alpha \cup \{B\} \vdash C$ to derive C).

3. *The universal set in the set theories based on* P

The weakest paraconsistent logic already developed is the Arruda and da Costa system P (see [3]). In this section we prove that even in the weak paraconsistent set theories based on P, the existence of Russell's set implies that $\cup R$ is the universal set.

The postulates of P are the following:

A1. $A \to A$	R1. $A, A \to B \,/\, B$
A2. $A \& B \to A$	R2. $A \to B, B \to C \,/\, A \to C$

A3. $A \& B \to B$

A4. $A \to A \vee B$

A5. $B \to A \vee B$

A6. $A \vee \neg A$

A7. $\neg\neg A \to A$

R3. $A, B \,/\, A \& B$

R4. $A \to B,\ A \to C \,/\, A \to B \& C$

R5. $A \& B \to D,\ A \& C \to D \,/\, A \& (B \vee C) \to D$

D≡. $A \equiv B =_{df} (A \to B) \& (B \to A)$

To obtain the corresponding predicate calculus P*, it was proposed in [3] that the following postulates be added, where the restrictions are the usual ones (as in Kleene [5]):

Q1. $(x)\,A(x) \to A(y)$

Q2. $A(y) \to (Ex)\,A(x)$

Q3. $C \to A(x) \,/\, C \to (x)\,A(x)$

Q4. $A(x) \to C \,/\, (Ex)\,A(x) \to C$

Q5. $(x)\,.\,C \vee A(x)\ :\to\ :C \vee (x)\,A(x)$

Q6. $C \& (Ex)\,A(x)\ :\to\ :(Ex)\,.\,C \& A(x)$

To obtain the corresponding predicate calculus with equality, $P^=$, we furthermore add the following postulates:

I1. $x = x$

I2. $x = y \to y = x$

I3. $x = y \ \&\ A(x) \to A(y)$

LEMMA 3.1. *The following statements hold true in* $P^=$:

T1. $\vdash A \& B \equiv B \& A$

T2. $\vdash (A \& B) \& C \equiv A \& (B \& C)$

T3. $\vdash x = y \equiv y = x$

DR1. $A \to B \vdash A \& C \to B$

DR2. $A \to B \vdash A \& C \to B \& C$

DR3. $A \to B \vdash A \& C \& D \to B \& C$

DR4. $A \& C \to B \vdash A \& C \to B \& C$

DR5. $A \to C,\ B \to C \vdash A \vee B \to C$

DR6. $A \equiv B \vdash A \& C \equiv B \& C$

DR7. $A \equiv B \vdash C \& A \equiv C \& B$

DR8. $A \equiv B \vdash A \vee C \equiv B \vee C$

DR9. $A \equiv B \vdash C \vee A \equiv C \vee B$

DR10. $A \equiv B,\ A \to C \vdash B \to C$

DR11. $A \equiv B,\ C \to A \vdash C \to B$

DR12. $A \equiv B,\ (x)A \vdash (x)B$

DR13. $A \equiv B,\ (Ex)A \vdash (Ex)B$

PROOF. Left to the reader (cf. also [3]). ■

LEMMA 3.2. *The following rule is derivable in* $P^=$:
EQ+ *If* $\vdash A \equiv B$ *and there is an occurrence of* A *in* C *outside the scope of a symbol for negation* $\neg$, *and* D *is the result of replacing that occurrence of* A *in* C *by* B, *then* $C \vdash D$.
PROOF. By induction on the depth of A in C and by application of DR6-DR13. ■

Let us consider now the paraconsistent set theory $SP^=$ based on $P^=$. For our needs the following specific postulates will do:

S1. $(Ex)(x) \,.\, x \in y \equiv \neg(x \in x)$	(Existence of Russell's set, R.)
S2. $(z)(Ey)(x) \,.\, x \in y \equiv x = z$	(Existence of the unitary set, $\{z\}$.)
S3. $(u, v)(Ey)(x) : x \in y \,.\equiv.\, x = u \vee x = v$	(Existence of the pair $\{u, v\}$.)
S4. $(u)(Ey)(x) : x \in y \equiv (Ez) \,.\, x \in z \;\&\; z \in u$	(Existence of the union of u, $\cup$ u.)

S2 is obviously $P^=$-derivable from S3. We list it separately for future reference.

LEMMA 3.3. I. $\vdash R \in R$
II. $\vdash x \in \{x\}$
III. $\vdash x \in \{x, R\}$
IV. $\vdash R \in \{x, R\}$

PROOF.
I. We obtain $\neg(R \in R) \rightarrow R \in R$ from S1, and $R \in R \rightarrow R \in R$ from A1.
Hence, $R \in R \vee \neg(R \in R) \,.\rightarrow.\, R \in R$ by DR5. But $R \in R \vee \neg(R \in R)$ derives from A6. Consequently, we obtain $R \in R$ by R1.
II. From S2 and I1.
III and IV. From S3 and I1. ■

LEMMA 3.4. $\vdash x \in y \;\&\; y \in z \,.\rightarrow.\, x \in \bigcup z$.
PROOF. From S4 and Q2 by R2. ■

In order to simplify the following proofs we shall denote formulas of the forms $x \in x$ and $(x \in x) \vee \neg(x \in x)$ by the expressions of the corresponding forms Kx and Tx respectively. We also shall denote $x \in z \,\&\, y \in z$ by $x, y \in z$.

LEMMA 3.5. $\vdash x \in R \,\&\, T\{x\} \,.\!\rightarrow.\, \{x\} \in R$.

PROOF.

(1) $K\{x\} \rightarrow x = \{x\}$ — S2, EQ+.

(2) $K\{x\} \,\&\, x \in R \,.\!\rightarrow.\, x = \{x\} \,\&\, x \in R$ — (1), DR2.

(3) $x = \{x\} \,\&\, x \in R \,.\!\rightarrow.\, \{x\} \in R$ — I3.

(4) $K\{x\} \,\&\, x \in R \,.\!\rightarrow.\, \{x\} \in R$ — (2), (3), R2.

(5) $\neg K\{x\} \rightarrow \{x \in R$ — S1.

(6) $\neg K\{x\} \,\&\, x \in R \,.\!\rightarrow.\, \{x\} \in R$ — (5), DR1.

(7) $x \in R \,\&\, T\{x\} \,.\!\rightarrow.\, \{x\} \in R$ — (4), (6), R5, EQ+. ■

LEMMA 3.6. $\vdash x, y \in R \,\&\, T\{x, y\} \,.\!\rightarrow.\, \{x, y\} \in R$.

PROOF.

(1) $K\{x, y\} \,.\!\rightarrow.\, \{x, y\} = x \vee \{x, y\} = y$ — S3.

(2) $K\{x, y\} \,\&\, x, y \in R \,.\!\rightarrow.\, (\{x, y\} = x \vee \{x, y\} = y) \,\&\, x, y \in R$ — (1), DR2.

(3) $\{x, y\} = x \,\&\, x, y \in R \,.\!\rightarrow.\, \{x, y\} \in R$ — A2, I3, R2, EQ+

(4) $\{x, y\} = y \,\&\, x, y \in R \,.\!\rightarrow.\, \{x, y\} \in R$ — A3, I3, R2, EQ+

(5) $(\{x, y\} = x \vee \{x, y\} = y) \,\&\, x, y \in R \,.\!\rightarrow.\, \{x, y\} \in R$ — (3), (4), R5, EQ-

(6) $K\{x, y\} \,\&\, x, y \in R \,.\!\rightarrow.\, \{x, y\} \in R$ — (2), (5), R2.

(7) $\neg K\{x, y\} \,.\!\rightarrow.\, \{x, y\} 9 R$ — S1.

(8) $\neg K\{x, y\} \,\&\, x, y \in R \,.\!\rightarrow.\, \{x, y\} \in R$ — (7), DR1.

(9) $x, y \in R \,\&\, T\{x, y\} \,.\!\rightarrow.\, \{x, y\} \in R$ — (6), (8), R5, EQ-

LEMMA 3.7. $\vdash (x) \,.\, \{\{x, R\}\} \in R$.

PROOF. To simplify the proof we shall denote $x, R \in \{x, R\} \,\&\, R \in R \,\&\, T\{x, R\} \,\&\, T\{\{x, R\}\}$ by H.

(1) $K\{\{x, R\}\} \rightarrow \{\{x, R\}\} = \{x, R\}$ — S2.

(2) $K\{\{x, R\}\} \,\&\, H \,.\!\rightarrow.\, \{\{x, R\}\} = \{x, R\} \,\&\, x \in \{x, R\}$ — (1), DR3.

(3) $\{\{x, R\}\} = \{x, R\} \,\&\, x \in \{x, R\} \,.\!\rightarrow.\, x \in \{\{x, R\}\}$ — I3, EQ+.

(4) $x \in \{\{x, R\}\} \rightarrow x = \{x, R\}$ — S2.

(5) $K\{\{x, R\}\} \,\&\, H \,.\!\rightarrow.\, x = \{x, R\}$ — (2), (3), (4), R2.

(6) $K\{\{x, R\}\} \,\&\, H \,.\!\rightarrow.\, \{\{x, R\}\} = \{x, R\} \,\&\, R \in \{x, R\}$ — (1), DR3, EQ+.

(7) $\{\{x, R\}\} = \{x, R\} \,\&\, R \in \{x, R\} \,.\!\rightarrow.\, R \in \{\{x, R\}\}$ — I3, EQ+.

(8) $R \in \{\{x, R\}\} \rightarrow R = \{x, R\}$ — S2.

(9) $K\{\{x, R\}\} \,\&\, H \,.\!\rightarrow.\, R = \{x, R\}$ — (6), (7), (8), R2.

(10) $K\{\{x, R\}\}$ & $H . \to . x = \{x, R\}$ & $R = \{x, R\}$ (5), (9), R4.
(11) $x = \{x, R\}$ & $R = \{x, R\} . \to . x = R$ I3, EQ+.
(12) $K\{\{x, R\}\}$ & $H . \to . x = R$ (10), (11), R2.
(13) $K\{\{x, R\}\}$ & $H . \to . x = R \& R \in R$ (12), DR4.
(14) $x = R \& R \in R . \to . x \in R$ I3, EQ+.
(15) $x = R \& R \in R . \to . x, R \in R$ (14), DR4, EQ+.
(16) $K\{\{x, R\}\}$ & $H . \to . x, R \in R$ (13), (15), R2.
(17) $K\{\{x, R\}\}$ & $H . \to . x, R \in R \& T\{x, R\}$ (16), DR4, EQ+.
(18) $x, R \in R \& T\{x, R\} . \to . \{x, R\} \in R$ Lemma 3.6.
(19) $K\{\{x, R\}\}$ & $H . \to . \{x, R\} \in R$ (17), (18), R2.
(20) $K\{\{x, R\}\}$ & $H . \to . \{x, R\} \in R \& T\{\{x, R\}\}$ (19), DR4.
(21) $\{x, R\} \in R \& T\{\{x, R\}\} . \to . \{\{x, R\}\} \in R$ Lemma 3.5.
(22) $K\{\{x, R\}\}$ & $H . \to . \{\{x, R\}\} \in R$ (20), (21), R2.
(23) $\neg K\{\{x, R\}\}$ & $H . \to . \{\{x, R\}\} \in R$ S1, DR1.
(24) $T\{\{x, R\}\}$ & $H . \to . \{\{x, R\}\} \in R$ (22), (23), R5, EQ+.
(25) $T\{\{x, R\}\}$ & H A6, Lemma 3.3, R3.
(26) $\{\{x, R\}\} \in R$ (24), (25), R1. ■

THEOREM 3.1. $\vdash (x) . x \in \bigcup R$.
PROOF. We shall denote $\{\{x, R\}\} \in R \& x \in \{x\} \& R \in R \& x \in \{x, R\}$ by G.

(1) $K\{x, R\} . \to . \{x, R\} = x \vee \{x, R\} = R$ S3.
(2) $K\{x, R\} \& G . \to . (\{x, R\} = x \vee \{x, R\} = R) \& G$ (1), DR2.
(3) $\{x, R\} = x \to x \in \{\{x, R\}\}$ S2, A3, EQ+.
(4) $\{x, R\} = x \& G . \to . x \in \{\{x, R\}\} \& \{\{x, R\}\} \in R$ (3), DR3.
(5) $x \in \{\{x, R\}\} \& \{\{x, R\}\} \in R . \to . x \in \bigcup R$ Lemma 3.4.
(6) $\{x, R\} = x \& G . \to . x \in \bigcup R$ (4), (5), R2.
(7) $\{x, R\} = R \& G . \to . \{x, R\} = R \& R \in R$ A3, EQ+.
(8) $\{x, R\} = R \& R \in R . \to . \{x, R\} \in R$ I3, EQ+.
(9) $\{x, R\} = R \& G . \to . \{x, R\} \in R$ (7), (8), R2.
(10) $\{x, R\} = R \& G . \to . \{x, R\} \in R \& x \in \{x, R\}$ (9), DR4.
(11) $\{x, R\} \in R \& x \in \{x, R\} . \to . x \in \bigcup R$ Lemma 3.4, EQ+.
(12) $\{x, R\} = R \& G . \to . x \in \bigcup R$ (10), (11), R2.
(13) $(\{x, R\} = x \vee \{x, R\} = R) \& G . \to . x \in \bigcup R$ (6), (12), R5, EQ+.
(14) $K\{x, R\} \& G . \to . x \in \bigcup R$ (2), (13), R2.

(15) $\neg K\{x, R\} \rightarrow \{x, R\} \in R$	S1.
(16) $\neg K\{x, R\}$ & $G . \rightarrow . \{x, R\} \in R$ & $x \in \{x, R\}$	(15), DR3, EQ+.
(17) $\{x, R\} \in R$ & $x \in \{x, R\} . \rightarrow . x \in \bigcup R$	Lemma 3.4.
(18) $\neg K\{x, R\}$ & $G . \rightarrow . x \in \bigcup R$	(16), (17), R2.
(19) $T\{x, R\}$ & $G . \rightarrow . x \in \bigcup R$	(14), (18), R5, EQ+.
(20) $T\{x, R\}$ & G	A6, lemmas 3.3 and 3.7, R3
(21) $x \in \bigcup R$	(19), (20), R1. ■

The system $P^=$ is rather weak. Yet, we didn't even need A4, A5, A7, Q3, Q4, Q5 and Q6. Anderson and Belnap's systems *E* and *R* and Routley's systems DMO and DK are extensions of P, and consequently all these systems lead to the existence of the universal set in view of S1-S4. We do, of course, consider only set theories in which all theorems of the underlying logic hold.

4. *Some remarks on a very weak paraconsistent set theory*

In this section we show that the universal set cannot be avoided even in very weak paraconsistent set theories with Russell's set, unless one avoids that either $\neg(x \in R)$ or $x \in x$ determine a set. To prove this we consider an extremely weak logic and, consequently, an extremely weak set theory, in which we take as postulates exactly what we need to prove Theorem 4.1 below. The absence of excluded middle should not be taken too serious: if both $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$ were added, excluded middle would be derivable.

Let L $\vee$ be a first order logic characterized by:

R1. $A \rightarrow B, B \rightarrow C \;/\; A \rightarrow C$
R2. $A \rightarrow B, \neg A \rightarrow B \;/\; B$
R3. $A(x) \;/\; (x)A(x)$.

In order to define Russell's set we need equivalence, which is not explicitly definable in the present logic. However, to show our point we do not even need Russell's set properly. We consider a very poor set theory SL $\vee$, based on L $\vee$, in which there is a set (still called R) that contains at least all members of Russell's set, and in which, for any two sets x and y, there is a set (still denoted by '$x \cup y$') that contains at least all members of x and of y. The postulates are the following (where x, y and z are sets):

S1. $\neg(x \in x) \rightarrow x \in R$
S2. $z \in x \rightarrow z \in x \cup y$
S3. $z \in y \rightarrow z \in x \cup y$

Let us now suppose that either $\neg(x \in R)$ or $x \in x$ 'determine' a set in SL $\vee$, X and Y respectively, such that:

F. $\neg(x \in R) \rightarrow x \in X$

or

F′. $x \in x \rightarrow x \in Y$.

THEOREM 4.1. I. $\vdash (x) . x \in R \cup X$ *in* SL $\vee$ *plus* F.
II. $\vdash (x) . x \in Y \cup R$ *in* SL $\vee$ *plus* F′.

PROOF.

I.	(1) $x \in R \rightarrow x \in R \cup X$	S2.
	(2) $\neg(x \in R) \rightarrow x \in X$	F.
	(3) $x \in X \rightarrow x \in R \cup X$	S3.
	(4) $\neg(x \in R) \rightarrow x \in R \cup X$	(2), (3), R1.
	(5) $x \in R \cup X$	(1), (4), R2.
	(6) $(x) . x \in R \cup X$	(5), R3.

II. The proof is similar to that of part I. ■

In a sense the present results are not surprising. Although the negation in SL $\vee$ is very weak, its occurrence in F guarantees that X contains all sets which are non-members of R, i.e. that the former is 'a complement' of the latter; and its occurrence in S1 guarantees that R is 'a complement' of Y. It is obvious that the universal set cannot be avoided in the presence of 'union' and 'complement'.

5. *Concluding remarks*

Any strong paraconsistent set theory with Russell's set is an extension of the minimal strong paraconsistent set theory considered in Section 2. Thus, $\cup R$ is the universal set in any strong paraconsistent set theory. In Section 3 we have seen furthermore that even in weak paraconsistent set theories which contain Russell's set and are based on a paraconsistent extension of P, $\bigcup R$ is the universal set.

It should be pointed out that P is very weak, and that paraconsistent set theories based on a further weakening of P will be extremely poor as far as theorems are concerned. Their mathematical usefulness is dubious. Yet, we have shown that even in paraconsistent set theories which are weaker than $SP^=$ (presented in Section 3) troubles arise. Although we were unable to show that the presence of Russell's set leads by itself to the existence of the universal set, we have demonstrated that Russell's set together with either the set $\hat{x}\ \neg(x \in R)$ or the set $\hat{x}\ (x \in x)$ leads to the existence of the universal set in a very weak fragment of $SP^=$. The same obviously holds in all richer paraconsistent set theories.

All the logics considered in this paper have the law of excluded middle as an axiom, and in the corresponding set theories Russell's class is a set. The conclusion we may obtain from the above results is that the law of excluded middle together with the existence of Russell's set lead to the existence of the universal set. Thus, if we want to have Russell's set in a set theory without universal set, we have either to eliminate the law of excluded middle from the underlying logic or to introduce Russell's set in an *ad hoc* way. Another consequence of the above results is that, in order to avoid the existence of the universal set in paraconsistent set theories with the law of excluded middle, we must prevent operating with union on Russell's set. Perhaps, a similar problem will arise with other non-classical sets.

Paraconsistent set theories without universal set may be very usefull for a deep analysis of the behavior of Russell's set and of other non-classical sets. Nonetheless, many difficulties arise in the construction of such paraconsistent set theories, e.g., (i) to decide which non-classical sets will be introduced apart from Russell's set, (ii) to avoid paradoxes like those developed in [1] and [2], and (iii) to decide which operations will be defined for the non-classical sets in order to avoid the aforementioned problems. Needless to say that it does not make much sense to introduce non-classical sets but to prevent at the same time to operate with them. If one were to do so, the non-classical sets would play the same role as the classes in classical set theories and hence paraconsistent set theory would not present any new mathematical interest. Finally, it should be pointed out that it is still an open question in which paraconsistent set theories Russell's set is different from the universal set; a very unpleasant situation indeed.

There can be no doubt that some expectations of paraconsistent logicians with respect to set theory did not came true. It does not follow from this that the development of paraconsistent logics has been useless. We even are convinced that one cannot conclude that the paraconsistent research tradition has failed or is bound to fail completely with respect to set theory. We will conclude this paper by offering two arguments in this connection.

It is a specific problem of paraconsistent logic to clarify the process of deduction within a theory in which an unexpected inconsistency has been derived and before a consistent alternative theory has been developed. Even if there are no interesting paraconsistent set theories, it may turn out necessary to develop further a set theory which originally was intended to be consistent but was proved to be inconsistent later. A contribution to the solution of this problem is offered in [5] by the development of *dynamic dialectical logics*, viz. logics which adapt themselves to the specific inconsistencies that arise within some theory. This part of the paraconsistent programme is not in any way affected by our present results.

For the sake of our second argument we first want to clarify a point concerning the paraconsistent programme. None of us has ever claimed or even believed – and the same holds for the vast majority of people working in the paraconsistent research traditon – that any set of postulates, or even a set of postulates which relies on 'clear mathematical intuitions', may be supplied with an underlying paraconsistent logic to the effect of arriving at an interesting theory in which specific unwanted results are avoided. Paraconsistent logic is not a wonderful remedy. None of da Costa's set theories and no set theory considered in this paper contains, e.g., the axiom of abstraction in its unrestricted form. It is not excluded that there be some restriction on S2-S4, perhaps even a restriction which is not *ad hoc*, that results in an interesting paraconsistent set theory with Russell's set but without universal set. Also, it has not been demonstrated that an interesting set theory cannot be based on a logic in which excluded middle fails.

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* This paper is the consequence of the seminars developed at the University of Gent, Belgium, when the first author was there as Visiting Professor with a partial grant (81/0522-4) of the 'Fundação de Amparo à Pesquisa do Estado de São Paulo' (FAPESP), Brazil.