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The dynamic proofs of defeasible logics

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Outline

Warnings and Conventions

My approach: describe defeasible reasoning in a way that resembles the description and metatheory of *Tarski logics*, including proof theory and semantics, and including soundness, completeness and other metatheoretic properties.

I see this not as opposed to a more application oriented approach, but as complementary with it (and not because both are needed, but because both clarify and justify each other).

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Let $\Delta \subseteq_{\mathrm{fin}} \Gamma$ abbreviate "a finite $\Delta \subseteq \Gamma$".

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example: Inductive generalization: Which set of 'generalisations' is *jointly compatible* with a given set of empirical data?
(generalisation: purely functional; only unary predicates)

$\forall x(A(x) \supset B(x))$ compatible with Γ
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Complication: If an instance of $(A(\alpha) \wedge \neg B(\alpha)) \vee (C(\beta) \wedge \neg D(\beta))$ derivable from Γ, then the members of $\{\forall x(A(x) \supset B(x)), \forall x(C(x) \supset D(x))\}$ are **jointly incompatible** with Γ.

Other examples

- Handling inconsistency: interpret inconsistent premise sets/theories as consistently as possible. (In preparation of forging a consistent replacement.)
- deciding on a person's position in a discussion.
- abduction: deriving from a theory / set of theories potential explanations of a given fact
- etc. etc.

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Rule-Based approaches may be integrated – see below.

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· an external dynamics: *non-monotonicity*: conclusions revised in view of the addition of new premises

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most crucial: no positive test for "consistent" (affects: handling inconsistency, classical compatibility, inductive generalisation, explanation (cf. Hintikka-Halonen), . . .)

Outline

Adaptive Logics

· oldest work (late 1970s): inconsistency-adaptive logics [Batens, 1989] [occasion, irony]

· soon semantic approach spelled out [Batens, 1986] [minimal abnormality]

· generalisation to predicative level [Batens, 1999] [$\leftrightarrow$ GP]

· students (especially J. Meheus) pushed to generalize the inconsistency-adaptive approach to other defeasible processes, ultimately the aim was to incorporate all defeasible reasoning forms, first and foremost all methods from PoS and daily life

· as more adaptive logics were studied, need for a general characterization: SF. This defined a AL as a triple, and offered generic definitions of the proof theory and the semantics [Batens, 2001]. Generic proofs were provided of the metatheory (including Soundness, Completeness and many metatheoretic properties) [Batens, 2007].

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$\Gamma \vdash_{\mathbf{CL}} A$ iff there are $B_1, \ldots, B_n$ such that
$\Gamma \vdash_{\mathbf{CLuN}} A \vee ((B_1 \wedge \neg B_1) \vee \ldots \vee B_n \wedge \neg B_n))$
similarly for other paraconsistent logics. Generic proof: [Batens, 2007]
example: $p \vee q, \neg p \vdash_{\mathbf{CL}} q$ and $p \vee q, \neg p \vdash_{\mathbf{CL}} q \vee (p \wedge \neg p)$

Where $\mathcal{L}$ is a language schema and $\mathcal{W}$ its set of closed formulas. An adaptive logic in standard format, **AL** : $\wp(\mathcal{W}) \to \wp(\mathcal{W})$ is defined by a triple:

(1) A *lower limit logic* **LLL**: a logic that is defined over $\mathcal{L}$, has *static proofs* in $\mathcal{L}$.
simplification relies on *trusty semantics* [Batens, 2021].

(2) A decidable *set of abnormalities* $\Omega \subseteq \mathcal{W}^*$: a set of formulas characterized by a (possibly restricted) logical form F; or a decidable union of such sets.

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For all adaptive logics in standard format, the format defines by generic means the semantics, the proof theory, and a very extensive meta-theory (soundness, completeness, stopperedness, etc., etc. — see [Batens, 2007] and several later results by others).

adaptive logics in SF have a complexity up to Π^1_1 [Verdée, 2009, Odintsov and Speranski, 2012, Odintsov and Speranski, 2013]

Some examples of proofs

simple propositional proof that illustrates handling inconsistency:

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8	$\neg r$	3, 7; RC	$\{p \wedge \neg p, q \wedge \neg q\}$

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Static proofs

Intuitively, **L** has static proofs iff, for every decidable Γ,
(i) every formula derived in a **L**-proof from Γ remains derived if the proof is extended
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I shall consider *annotated* proofs (number and a justification on each line)
- easier for handling dynamic proofs
- non-annotated proofs parasitic on annotated ones

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- N_l is a (possibly empty) set of lines (referred to by their numbers)
- R_l is a S-rule as introduced below

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A S-rule is finitary iff Υ is finite
the members of Υ called *local premises* (of the S-rule)

S-rules may have a restriction attached to them.
Essential that it can be decided whether the restriction is fulfilled by *inspecting* the list of lines to which the application of the rule belongs.

Examples of such restrictions, e.g., in the rule R$\forall$: “To derive $\vdash A \supset \forall\alpha B(\alpha)$ from $\vdash A \supset B(\beta)$, provided β does not occur in either A or $B(\alpha)$.”

- the restriction on β is established by inspection.
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Some S-rules may have the form $\emptyset/$A, possibly with a restriction attached to it.

If there is no restriction, A is usually called an axiom schema.
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Explicit definitions may also be seen as (couples of) rules. The definition A $=_{\mathrm{df}}$ B corresponds to the S-rule "from a formula C that contains an occurrence of A, to infer the formula obtained from C by replacing A by B, and *vice versa*".

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Numerous odd logics may be defined with otherwise restricted rules $\emptyset/A$, or with unrestricted such rules, or without Prem. examples: (i) the empty logic **Em** has $\mathrm{Cn}_{\mathbf{Em}}(\Gamma) = \emptyset$ for all Γ, (ii) the constant logic **Tr** has $\mathrm{Cn}_{\mathbf{Tr}}(\Gamma) = \mathcal{W}$ for all Γ

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Let $\mathcal{R}$ denote a set of S-rules that contains Prem. Given $\mathcal{R}$ and a list L of lines, a line l of L is $\mathcal{R}$-correct iff
(i) all members of N_l precede l in L
(ii) $R_l \in \mathcal{R}$
(iii) the formula of l is the result of applying R_l to the formulas of the lines in N_l.

Definition

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Definition 4 comes to: a static $\mathcal{R}$-*proof of A from* Γ is a static $\mathcal{R}$-proof from Γ in which A is the formula of a line of a stage.

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In the same way, the infinitary $A, C_1 \wedge D_1, C_2 \wedge D_2, \ldots /A \vee B$ generates the more specific finitary rule $A, C_1 \wedge D_1/A \vee B$.
In general, every infinitary S-rule R generates zero or more finitary rules. The set of these, say $\mathrm{fin}(R)$, is recursive.

Theorem

If $\mathcal{R}$ is a recursive set of S-rules, then there is a recursive set $\mathcal{R}'$ of finitary S-rules such that $\Gamma \vdash_{\mathcal{R}'} A$ iff $\Gamma \vdash_{\mathcal{R}} A$.

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*A logic **L** has static proofs iff there is a recursive set $\mathcal{R}$ of finitary S-rules such that $\Gamma \vdash_{\mathbf{L}} A$ iff $\Gamma \vdash_{\mathcal{R}} A$.*

Different sets of rules define the same logic. If **L** has static proofs, let $\mathcal{R}_{\mathbf{L}}$ be a recursive set of finitary +S-rules such that $\Gamma \vdash_{\mathbf{L}} A$ iff $\Gamma \vdash_{\mathcal{R}_{\mathbf{L}}} A$. The Corollary warrants that there is such a set. In view of the proof of last Theorem, a further corollary is available.

Corollary

Every line of every stage of a static proof has a finite path.

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Definition

A *standard* $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ is a $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ in which A is the formula of the *last* line of the *last* stage.

It is easily provable that all logics that have static proofs share many interesting properties.

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In such cases **L** cannot be characterized by any recursive set of S-rules in which the metavariables range over formulas from $\mathcal{L}$.

Not all known logics have static proofs (in a language schema). E.g., is second order **CL**, which has no dynamic proofs either.

In some logics **L** the meaning of some logical symbols cannot be fixed by S-rules, but only by more complex rules, sometimes called *metarules* [Routley, 1982, Brady, 2006]; the cause is often the weakness of the involved implication.
In view of this, S-rule is better redefined as follows:

$$\text{If } \Gamma_1 \vdash_{\mathbf{L}} A_1 \text{ and} \ldots \ \Gamma_n \vdash_{\mathbf{L}} A_n, \text{ then } \Upsilon/A \quad (n \in \{0, 1, \ldots\}).$$

example: If $A \vdash C$ and $B \vdash C$ then $A \vee B/C$. Where $n > 0$, the application of a S-rule in a static proof $\mathbb{P}$ refers to finitely many other finite and static proofs. Adjust “stage” etc.

Outline

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A rule is a metalinguistic expression of the form $\Upsilon/A{:}\Pi$ – read as "to derive A on the condition Π from Υ", in which A is a metalinguistic formula and Υ and Π are recursive sets of metalinguistic formulas.

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Rules specify that from formulas of a certain form, a formula of a corresponding form may be derived on a condition, which is a set of formulas of a further form.

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Example: in the 'direct proof theory' for Rescher–Manor relations a premise is introduced on the condition that it is consistent. The underlying dynamic logic is not adaptive, but can be characterized by an adaptive logic *under a translation.*

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A line of a dynamic annotated proof will be a quadruple comprising a line number, a formula, a justification, and a condition. The first three elements are as for static proofs, except that the justification now contains a rule instead of a S-rule. As before R_l will denote the rule applied to add the line, and N_l is the set of lines to which the rule is applied. The condition is a set of formulas.

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The application of a rule carries over conditions. If (formulas of the form of) all members of Υ occur on lines of a list, and Π' is the union of the conditions of those lines, then the application of the rule $\Upsilon/A{:}\Pi$ leads to adding a line that has A as its formula and $\Pi \cup \Pi'$ as its condition.

As for S-rules, a *restriction* may be attached to a rule, provided that it can be decided whether the restriction is fulfilled by inspecting the list of lines to which the application of the rule belongs.
Given a set $\mathcal{R}$ of rules and a list L of lines, a line l of L is $\mathcal{R}$-*correct* iff
(i) it is the result of applying a rule $R_l \in \mathcal{R}$ to the formulas and conditions of the members of N_l and
(ii) all members of N_l precede l in the list.

Definition

A *marking definition* determines, for every stage of a dynamic proof, and for every line i of the stage whether it is marked or unmarked. The definition proceeds in terms of a requirement on the condition of line i and on the formulas unconditionally derived at other lines of the stage.

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A static $\mathcal{R}$-proof from Γ is a chain of $\mathcal{R}$-stages from Γ, the first element of which is the empty list and all other elements of which are extensions of their predecessors.

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A static $\mathcal{R}$-*proof of* A:Δ *from* Γ is a static $\mathcal{R}$-proof from Γ in which, from a certain stage on, there is a line that has A as its formula and Δ as its condition.

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If $\mathcal{R}$ *is a recursive set of rules, then there is a recursive set* $\mathcal{R}'$ *of finitary rules such that* $\Gamma \vdash_{\mathcal{R}'} A$:$\Delta$ *iff* $\Gamma \vdash_{\mathcal{R}} A$:$\Delta$*.*

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$\Gamma \vdash^{M}_{\mathcal{R}} A$ (A is $\mathcal{R}$-M-derivable *from* Γ) iff A is the formula of an unmarked line i of a stage of an $\mathcal{R}$-M-proof from Γ and the proof is stable with respect to line i.

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Definition

Where $\mathbf{L}_1$ has dynamic proofs and $\mathbf{L}_2$ has static proofs, $\mathbf{L}_1$ *S-agrees with* $\mathbf{L}_2$ iff there is a function $f : \mathcal{W}_+ \times \wp(\mathcal{W}_+) \rightarrow \mathcal{W}_+$ such that $\Gamma \vdash_{\mathbf{L}_1} A{:}\Delta$ iff $\Gamma \vdash_{\mathbf{L}_2} f(A, \Delta)$.

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If **L** *is defined by a recursive set* $\mathcal{R}$ *of rules and a marking definition, and* $\Gamma \vdash_{\mathbf{L}} A{:}\Delta$*, then there is a static* $\mathcal{R}$*-proof of* $A{:}\Delta$ *from* Γ *in which* A *is the formula and* Δ *the condition of the last line of the last stage.*

Definition

Where $\mathbf{L}_1$ has dynamic proofs and $\mathbf{L}_2$ has static proofs, $\mathbf{L}_1$ *S-agrees with* $\mathbf{L}_2$ iff there is a function $f : \mathcal{W}_+ \times \wp(\mathcal{W}_+) \rightarrow \mathcal{W}_+$ such that $\Gamma \vdash_{\mathbf{L}_1} A{:}\Delta$ iff $\Gamma \vdash_{\mathbf{L}_2} f(A, \Delta)$.

Corollary

Every adaptive logic **AL** *S-agrees with its lower limit logic* **LLL**.

Outline

The essential specificity concerns final derivability (as opposed to derivability at a stage):

Definition

A is *finally derived* in line i of the finite stage s of a proof from Γ iff (i) A is derived in line i of stage s and (ii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.

cf. dialogues

Outline

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