Allowing for imprecision in the game-theoretic characterisation of the Poisson process

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Abstract. In their 1993 paper 'Forecasting point and continuous processes: Prequential analysis' in *Test*, Vovk put forward a game-theoretic definition of the Poisson process. A key assumption therein is that the rate of the Poisson process is known or specified exactly. In contrast, I replace this assumption with the less stringent—and arguably more realistic—one that the available information about the process takes the form of bounds the rate rather than a single, exact value. The resulting process has properties similar to the standard, 'precise' Poisson process, albeit with an imprecise flavour to them, thus justifying the moniker 'imprecise Poisson process'.

Keywords: Counting process · Capital process · Bid–ask spread

1 Introduction

More than twenty years ago, Vovk [9] put forward a 'prequential' definition of the Poisson and Wiener processes[1], with an (upper) expectation operator that is derived from the capital in a gambling game. Nowadays, this approach towards modelling uncertainty is known as the *game-theoretic* one, as laid out and popularised by Shafer & Vovk in their seminal monographs [6,7]. Discrete-time processes have been studied extensively in this game-theoretic framework, often allowing for imprecision in the local uncertainty models—or a bid–ask spread in the price of the gambles available to skeptic. Game-theoretic continuous-time processes have also received quite some attention in this framework; in contrast to discrete-time processes, I'm not aware of work in the setting of continuous time that allows for imprecision. This is why in this contribution, I set out to allow for imprecision, or a bid–ask spread, in Vovk's [9] game-theoretic definition of the Poisson process.

I lay down the foundation for doing so in Section 2, in the form of basic notation and terminology regarding (counting) paths, variables and processes. Section 3 introduces, in a relatively general manner, the basics of Shafer & Vovk's game-theoretic foundations for modelling uncertainty regarding a continuous-time process. Next, I specialise this to the imprecise Poisson process in Section 4,

[1] More generally, he considers the reduction of continuous martingales to these processes through a change of time.

and investigate the properties of the resulting conditional upper expectation operator in Section 5.

I've relegated the proofs for most of the results to Appendix A; the sole exceptions are Propositions 5 to 7 and Theorem 2. Unless mentioned otherwise, these relegated proofs are modifications of the proofs of related results in [9].

2 Counting paths, variables & processes

The set-up for this contribution is essentially standard, and follows Vovk's [9] rather closely; for the reader's sake, I feel it's nonetheless necessary to take some care in introducing it clearly. Let Ω be the set of all *counting paths*: those paths $\omega\colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ that start in 0, are increasing with unit jumps[2] and right-continuous; these paths also have limits from the left, whence they form a subset of the well-known càdlàg paths [3, Ch. 3, § 5]. A *variable* is a function on Ω, which need not be real. For example, for any time point $t \in \mathbb{R}_{\geq 0}$, the corresponding *coordinate variable*

$$N_t\colon \Omega \to \mathbb{Z}_{\geq 0}\colon \omega \mapsto \omega(t)$$

is an integer-valued variable. Similarly, a *partial variable* is a function defined on some subset of Ω; whenever necessary, we define binary operations on partial variables only on the intersection of their domains.

A (partial) variable f is t*-measurable* if it's completely determined by the value of the path $\omega \in \operatorname{dom} f$ on $[0, t]$: if for every two paths $\omega_1, \omega_2 \in \operatorname{dom} f$,

$$\omega_1|_{[0,t]} = \omega_2|_{[0,t]} \Rightarrow f(\omega_1) = f(\omega_2).$$

Obviously, the coordinate variable N_t is t-measurable for all $t \in \mathbb{R}_{\geq 0}$. A *process* is a family of extended real variables $(\mathcal{S}_t)_{t\in\mathbb{R}_{\geq 0}}$ such that $\mathcal{S}_t$ is t-measurable for all $t \in \mathbb{R}_{\geq 0}$; the canonical example is the *coordinate process* $(N_t)_{t\in\mathbb{R}_{\geq 0}}$.

While (partial) variables need not be (extended) real-valued, we will almost exclusively consider those that are; henceforth, we denote the set of these by $\overline{\mathbb{V}} := \overline{\mathbb{R}}^{\Omega}$. For example, for any process $(\mathcal{S}_t)_{t\in\mathbb{R}_{\geq 0}}$, we define the extended real variable

$$\liminf_{t\to+\infty} \mathcal{S}_t\colon \Omega \to \overline{\mathbb{R}}\colon \omega \mapsto \liminf_{t\to+\infty} \mathcal{S}_t(t).$$

Similarly, with $g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$ and $t \in \mathbb{R}_{\geq 0}$,

$$g(N_t)\colon \Omega \to \mathbb{R}\colon \omega \mapsto g(\omega(t))$$

is a real variable that is bounded if and only if g is bounded. Indicator variables are also real-valued and bounded: for any *event* $A \subseteq \Omega$, its corresponding *indicator* $\mathbb{I}_A$ is the $\{0, 1\}$-valued variable that is 1 on A and 0 elsewhere.

Finally, an essential class of extended real variables are the *stopping times*: this class $\mathfrak{T}$ consists of the positive extended real variables τ—so maps from Ω

[2] The assumption of unit jumps is only used in one specific place in the proof of Proposition 8; I leave getting rid of this assumption for future work.

to $\overline{\mathbb{R}}_{\geq 0} = [0, +\infty]$—such that for all $\omega_1, \omega_2 \in \Omega$, if $\omega_1|_{[0,\tau(\omega_1)]} = \omega_2|_{[0,\tau(\omega_1)]}$ then $\tau(\omega_2) = \tau(\omega_1)$. For any stopping time $\tau \in \mathfrak{T}$ and path ω, we let $\mathcal{I}_\tau(\omega)$ be the set of counting paths that agree with ω on $[0, \tau(\omega)] \cap \mathbb{R}_{\geq 0}$:

$$\mathcal{I}_\tau(\omega) \coloneqq \{\varpi \in \Omega \colon \varpi|_{[0,\tau(\omega)]\cap\mathbb{R}_{\geq 0}} = \omega|_{[0,\tau(\omega)]\cap\mathbb{R}_{\geq 0}}\}.$$

Note that the constant real variable $t \colon \omega \mapsto t$ is a stopping time, and that for all stopping times $\tau_1, \tau_2 \in \mathfrak{T}$, their pointwise minimum $\tau_1 \wedge \tau_2$ and maximum $\tau_1 \vee \tau_2$ are also stopping times. Given such a stopping time $\tau \in \mathfrak{T}$, a partial real variable f is *τ-measurable* if for all $\omega_1, \omega_2 \in \operatorname{dom} f$,

$$\omega_1|_{[0,\tau(\omega_1)]\cap\mathbb{R}_{\geq 0}} = \omega_2|_{[0,\tau(\omega_1)]\cap\mathbb{R}_{\geq 0}} \implies f(\omega_1) = f(\omega_2);$$

differently put, f is constant on $\mathcal{I}_\tau(\omega)$ for all $\omega \in \operatorname{dom} f$.

Although a stopping time τ can take the value $+\infty$, we are typically interested in the set of paths for which it is finite/real; we denote this set by

$$\{\tau < +\infty\} \coloneqq \{\omega \in \Omega \colon \tau(\omega) < +\infty\}.$$

For any stopping time $\tau \in \mathfrak{T}$, let $\overline{\mathbb{V}}_\tau$ be the set of extended real partial variables whose domain includes $\{\tau < +\infty\}$. Finally, then, for every stopping time $\tau \in \mathfrak{T}$ and path $\omega \in \{\tau < +\infty\}$, we shorten $\omega(\tau(\omega))$ to $\omega(\tau)$ and consider the 'stopped process'

$$\mathcal{S}_\tau \colon \{\tau < +\infty\} \to \overline{\mathbb{R}} \colon \omega \mapsto \mathcal{S}_{\tau(\omega)}(\omega(\tau)).$$

3 Game-theoretic upper expectations

In this contribution, we'll use Shafer & Vovk's game-theoretic approach to model the uncertain (future) evolution of the coordinate process $(N_t)_{t\in\mathbb{R}_{\geq 0}}$. A lot can be said about their powerful game-theoretic framework, but we'll stick to what is necessary for the remainder of this contribution; we refer the interested reader to their monographs [6,7] and references therein for more background. In the setting of continuous time, Shafer & Vovk consider a game between two players, called *trader* and *market*: first trader announces their strategy to trade in a number of securities, then market determines the price path of the securities. As in discrete time, one can also imagine a third player, called *forecaster*: they choose the securities that are available to trader. In the context of this contribution, the securities are derived from the coordinate process $(N_t)_{t\in\mathbb{R}_{\geq 0}}$ and market chooses one realisation ω from Ω.

Trader's strategy has to obey some restrictions for this game to make sense. For example, it makes sense to assume that at time $t \in \mathbb{R}_{\geq 0}$, they are uncertain about the future prices of the securities, or differently put, only have exact information about the prices of the securities in the past. We'll get back to these restrictions in Section 4 further on. For now, we'll assume that each allowed trading strategy gives rise to a process $(\mathcal{K}_t)_{t\in\mathbb{R}_{\geq 0}}$, which we'll call trader's *capital process* under this strategy; recall from Section 2 that for $(\mathcal{K}_t)_{t\in\mathbb{R}_{\geq 0}}$ to be a process, it must be that $\mathcal{K}_t$ is t-measurable for all $t \in \mathbb{R}_{\geq 0}$, meaning that it's completely defined by the value of the paths on $[0, t]$—or, in other words, doesn't depend on the future values of the prices of the securities.

The set $\mathfrak{K}$ collects the capital processes corresponding to all allowed trading strategies. Then the conditional upper expecation $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau] \colon \overline{\mathbb{V}}_\tau \times \{\tau < +\infty\} \to \overline{\mathbb{R}}$ is defined for all $f \in \overline{\mathbb{V}}_\tau$ and $\omega \in \{\tau < +\infty\}$ by

$$\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) := \inf\Big\{\mathcal{K}_\tau(\omega) \colon \mathcal{K}_\bullet \in \mathfrak{K}, (\forall \varpi \in \mathcal{I}_\tau(\omega)) \liminf_{r \to +\infty} \mathcal{K}_r(\varpi) \geq f(\varpi)\Big\},$$

where here and in the remainder, we write $\mathcal{K}_\bullet$ as a shorthand for $(\mathcal{K}_t)_{t \in \mathbb{R}_{\geq 0}}$. This conditional expectation $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ is the infimum capital the trader needs at time $\tau(\omega)$ to superhedge f in 'all possible futures' $\mathcal{I}_\tau(\omega)$—and at time $+\infty$. Differently put, if the trader has capital $c > \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ at $\tau(\omega)$ (and knowing the partial counting path $\omega|_{[0,\tau\omega]}$), they can trade such that they'll have $f(\varpi)$, whatever the actual realisation $\varpi \in \mathcal{I}_\tau(\omega)$ turns out to be. Consequently, we can interpret $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ as the trader's infimum selling price for the uncertain pay-off f, given the information they have at time $\tau(\omega)$. Since $\omega(0) = 0$ for all $\omega \in \Omega$, $\mathcal{K}_0$ is constant on Ω, and the same is true for $\overline{\mathrm{E}}_{\mathfrak{K}}[f|0]$, which for this reason we'll shorten to $\overline{\mathrm{E}}_{\mathfrak{K}}[f]$ from now on; this can be interpreted as the infimum initial capital the trader needs at time 0 to superhedge f 'in the long run', or alternatively, their infimum selling price for the uncertain pay-off f.

For every stopping time $\tau \in \mathfrak{T}$ and extended variable $f \in \overline{\mathbb{V}}_\tau$, our definition ensures that the partial variable $\omega \mapsto \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ is constant on $\mathcal{I}_\tau(\omega)$, and therefore τ-measurable. Furthermore, for every extended variable $f \in \overline{\mathbb{V}}$ and path $\omega \in \Omega$, the function $t \mapsto \overline{\mathrm{E}}_{\mathfrak{K}}[f|t](\omega)$ is increasing because the map $t \mapsto \mathcal{I}_t(\omega)$ is decreasing.

Our calling $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\bullet]$ a (conditional) upper expectation—after [12,8]—is justified under some mild assumptions on $\mathfrak{K}$.

Proposition 1. *Suppose $\mathfrak{K}$ contains all constant processes and is a cone—that is, closed under pointwise addition and multiplication with positive scalars. Then for all $\tau \in \mathfrak{T}$, $f, g \in \overline{\mathbb{V}}_\tau$ and $\mu \in \mathbb{R}$,*

E1. $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] \leq \sup f$*;*
E2. $\overline{\mathrm{E}}_{\mathfrak{K}}[f + g|\tau] \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] + \overline{\mathrm{E}}_{\mathfrak{K}}[g|\tau]$ *whenever f and g are real;*
E3. $\overline{\mathrm{E}}_{\mathfrak{K}}[\mu f|\tau] = \mu \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau]$ *whenever $\mu > 0$;*
E4. $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] \leq \overline{\mathrm{E}}_{\mathfrak{K}}[g|\tau]$ *whenever $f \leq g$;*
E5. $\overline{\mathrm{E}}_{\mathfrak{K}}[f + \mu|\tau] = \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] + \mu$.

Following Vovk [9, p. 196], we call the set $\mathfrak{K}$ of allowed capital processes *coherent* if

$$\inf\Big\{\liminf_{r \to +\infty} \mathcal{K}_r(\varpi) \colon \varpi \in \mathcal{I}_t(\omega)\Big\} \leq \mathcal{K}_t(\omega) \quad \text{for all } \mathcal{K}_\bullet \in \mathfrak{K}, t \in \mathbb{R}_{\geq 0}, \omega \in \Omega; \quad (1)$$

since this condition ensures that no trading strategy can result in a guaranteed profit, one might also think of it as *avoiding sure gain*. One consequence of coherence is that in the definition of the conditional upper expectation $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau](\omega)$, we can limit ourselves to capital processes that are always greater than the infimum of f on $\mathcal{I}_\tau(\omega)$.

Lemma 1. *Suppose $\mathfrak{K}$ is coherent, and fix some $\tau \in \mathfrak{T}$, $f \in \overline{\mathbb{V}}_\tau$, $\omega \in \{\tau < +\infty\}$ and $\mathcal{K}_\bullet \in \mathfrak{K}$ that superhedges f on $\mathcal{I}_\tau(\omega)$, meaning that*

$$\liminf_{r \to +\infty} \mathcal{K}_r(\varpi) \geq f(\varpi) \quad \text{for all } \varpi \in \mathcal{I}_\tau(\omega).$$

Then

$$\mathcal{K}_t(\varpi) \geq \inf\{f(\varpi') \colon \varpi' \in \mathcal{I}_t(\varpi)\} \quad \text{for all } t \in [\tau(\omega), +\infty[, \varpi \in \mathcal{I}_\tau(\omega).$$

Corollary 1. *Suppose $\mathfrak{K}$ is coherent, and fix some $\sigma, \tau \in \mathfrak{T}$ such that $\sigma \leq \tau$. If $\{\sigma < +\infty\} = \{\tau < +\infty\}$, then for all τ-measurable $f \in \overline{\mathbb{V}}_\sigma$ and $\omega \in \{\sigma < +\infty\}$,*

$$\overline{\mathrm{E}}_{\mathfrak{K}}[f|\sigma](\omega) = \inf\{\mathcal{K}_\sigma(\omega) \colon \mathcal{K}_\bullet \in \mathfrak{K}, (\forall \varpi \in \mathcal{I}_\sigma(\omega))\ \mathcal{K}_\tau(\varpi) \geq f(\varpi)\}.$$

Corollary 2. *If $\mathfrak{K}$ contains all constant processes and is coherent, then for all $\tau \in \mathfrak{T}$, $f \in \overline{\mathbb{V}}_\tau$ and $\omega \in \{\tau < +\infty\}$,*

E6. $\inf f|_{\mathcal{I}_\tau(\omega)} \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \leq \sup f|_{\mathcal{I}_\tau(\omega)}$.

That we demand something akin to the coherence condition should come to no surprise to the reader who is familiar with coherent (upper) expectations a la de Finetti [4], Walley [10] and Williams [12]. Following the latter two authors, we define the conjugate conditional lower expectation by

$$\underline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \coloneqq -\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \quad \text{for all } \tau \in \mathfrak{T}, f \in \overline{\mathbb{V}}_\tau, \omega \in \{\tau < +\infty\};$$

this can be interpreted as trader's supremum buying price for f given the information they have at $\tau(\omega)$. Vovk [9] essentially proves that, at least for real variables, trader's buying price is never higher than their seling price.

Proposition 2. *If $\mathfrak{K}$ satisfies the conditions in Proposition 1 and is coherent, then $\underline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau]$ for all $\tau \in \mathfrak{T}$ and all real $f \in \overline{\mathbb{V}}_\tau$.*

4 Betting strategies & capital processes for the Poisson process

As announced in the Introduction, this contribution aims to extend Vovk's [9] game-theoretic description of the Poisson process to allow for imprecision. Their game-theoretic characterisation draws inspiration from Watanabe's martingale characterisation of the Poisson process [11, Theorem 2.3 and following Remark]: the coordinate process $(N_t)_{t \in \mathbb{R}_{\geq 0}}$ is a Poisson process if and only if the compensated process $(N_t - \lambda t)_{t \in \mathbb{R}_{\geq 0}}$ is a martingale—that is, if conditional on the information up to the current time point $s \in \mathbb{R}_{\geq 0}$, for every future time point $t \in]s, +\infty[$ the increment $N_t - N_s - \lambda(t - s)$ has expectation zero. The allowed trading strategies are exactly those that bet on this increment.

If forecaster puts forth some *rate* $\lambda \in \mathbb{R}_{\geq 0}$, trader is allowed to bet on a series of increments of the form $N_{\tau_{k+1}} - N_{\tau_k} - \lambda(\tau_{k+1} - \tau_k)$, for stopping times $\tau_k \leq \tau_{k+1}$. More formally, a *two-sided elementary trading strategy* G is an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_{n+1}$ and a sequence $h_1, \ldots, h_n$ of variables such that for all $k \in \{1, \ldots, n\}$, the *stake* h_k is a (possibly partial) bounded real variable with $\operatorname{dom} h_k \supseteq \{\tau_k < +\infty\}$ that is τ_k-measurable. We interpret such an elementary trading strategy G as follows: at time τ_k, the trader puts stake h_k on the increment $N_{\tau_{k+1}} - N_{\tau_k} - \lambda(\tau_{k+1} - \tau_k)$. Consequently, if the trader starts with initial capitcal $c \in \mathbb{R}$ and follows the two-sided elementary trading strategy G, their capital at time $t \in \mathbb{R}_{\geq 0}$ is the real variable

$$\mathcal{K}_t^{G,c} := \sum_{k=1}^{n} h_k \cdot \Big(N_{\tau_{k+1} \wedge t} - N_{\tau_k \wedge t} - \lambda(\tau_{k+1} \wedge t - \tau_k \wedge t) \Big), \tag{2}$$

where $\cdot$ indicates pointwise multiplication of two real variables and where we ignore the zero terms in the sum that arise when $\tau_k(\omega) \wedge t = \tau_{k+1}(\omega) \wedge t$. Our assumptions on the stopping times τ_k and stakes h_k ensures that the family $(\mathcal{K}_t^{G,c})_{t \in \mathbb{R}_{\geq 0}}$ is a real process, which we call a *two-sided elementary capital process*. We collect these capital processes in the set $\mathfrak{K}_\lambda$, and denote the corresponding conditional upper and lower expecation by $\overline{\mathrm{E}}_\lambda[\bullet|\bullet]$ and $\underline{\mathrm{E}}_\lambda[\bullet|\bullet]$, respectively. It's easy to see that $\mathfrak{K}_\lambda$ includes the constant processes and is closed under scalar multiplication, but it takes a bit more work to verify that $\mathfrak{K}_\lambda$ is coherent and closed under pointwise addition; we don't prove this here separately, since it's a special case of Proposition 3 further on.

In the two-sided betting strategies, the trader bets on the counting increment $N_{\tau_{k+1}} - N_{\tau_k}$ being 'large'—greater than $\lambda(\tau_{k+1} - \tau_k)$—when $h_k \geq 0$, and on $N_{\tau_{k+1}} - N_{\tau_k}$ being 'small'—smaller than $\lambda(\tau_{k+1} - \tau_k)$—when $h_k \leq 0$. In the one-sided betting strategies that we're going introduce next, there's a spread between the prices for these two bets. This time around, forecaster puts forth *rate bounds* $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_{\geq 0}$ such that $\underline{\lambda} \leq \overline{\lambda}$, where $\overline{\lambda}(\tau_{k+1} - \tau_k)$ is forecaster's selling price for the gamble that pays out the increment $N_{\tau_{k+1}} - N_{\tau_k}$, and $\underline{\lambda}(\tau_{k+1} - \tau_k)$ is their buying price for this gamble. More formally, a *one-sided elementary trading strategy* G is an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_{n+1}$ and a sequence $\overline{h}_1, \underline{h}_1, \ldots, \overline{h}_n, \underline{h}_n$ of variables such that for all $k \in \{1, \ldots, n\}$, the *stakes* $\overline{h}_k$ and $\underline{h}_k$ are τ_k-measurable (possibly partial) bounded *positive* real variables whose domain includes $\{\tau_k < +\infty\}$. If the trader starts with initial capitcal $c \in \mathbb{R}$ and follows the one-sided elementary trading strategy G, their corresponding capital at time $t \in \mathbb{R}_{\geq 0}$ is the real variable

$$\begin{aligned} \mathcal{K}_t^{G,c} := \sum_{k=1}^{n} \overline{h}_k \cdot \Big(N_{\tau_{k+1} \wedge t} - N_{\tau_k \wedge t} - \overline{\lambda}(\tau_{k+1} \wedge t - \tau_k \wedge t) \Big) \\ + \underline{h}_k \cdot \Big(\underline{\lambda}(\tau_{k+1} \wedge t - \tau_k \wedge t) - N_{\tau_{k+1} \wedge t} + N_{\tau_k \wedge t} \Big). \end{aligned} \tag{3}$$

Our assumptions on the stopping times τ_k and stakes $\overline{h}_k$ and $\underline{h}_k$ ensure that the family $(\mathcal{K}_t^{G,c})_{t \in \mathbb{R}_{\geq 0}}$ is a real process, which we call a *one-sided elementary* *capital process*. We collect these capital processes in the set $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, and denote the corresponding conditional upper and lower expectations by $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\bullet]$ and $\underline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\bullet]$, respectively.

For every capital process $(\mathcal{K}_t^{G,c})_{t\in\mathbb{R}_{\geq 0}} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, index $k \in \{1,\dots,n\}$, counting path $\omega \in \Omega$ and time points $t, r \in [\tau_k(\omega), \tau_{k+1}(\omega)] \cap \mathbb{R}_{\geq 0}$ such that $t \leq r$, it follows from Eq. (3) that, with $\overline{\underline{h}}_k \coloneqq \overline{h}_k - \underline{h}_k$,

$$\mathcal{K}_r^{G,c}(\omega) - \mathcal{K}_t^{G,c}(\omega) = \overline{\underline{h}}_k(\omega)\big(\omega(r) - \omega(t) - \underline{\lambda}(r-t)\big) - \overline{h}_k(\omega)(\overline{\lambda} - \underline{\lambda})(r-t). \quad (4)$$

This expression makes it obvious that for all $\lambda \in \mathbb{R}_{\geq 0}$, the set $\mathfrak{K}_{\{\lambda\}}$ of one-sided elementary capital processes for $[\lambda, \lambda] = \{\lambda\}$ is equal to the set $\mathfrak{K}_\lambda$ of two-sided elementary capital processes for λ.

It's crucial for the remainder of this contribution that the set of capital processes $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ satisfies the conditions in Proposition 2.

Proposition 3. *For all $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_{\geq 0}$ with $\underline{\lambda} \leq \overline{\lambda}$, the set $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ of one-sided elementary capital processes includes the constants, is a cone and is coherent.*

5 Properties of the conditional upper expectation for the Poisson process

In the remainder of this contribution, I list some properties of the conditional upper expectation operator $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\tau]$ which I hope should convince the reader of the sensibility of the game-theoretic characterisation of the imprecise Poisson process.

5.1 Properties related to the rate bounds

The first property looks at nested rate bounds: the conditional upper expectation for $[\underline{\lambda}_\mathrm{i}, \overline{\lambda}_\mathrm{i}]$ is dominated by the one for $[\underline{\lambda}_\mathrm{o}, \overline{\lambda}_\mathrm{o}]$ whenever $[\underline{\lambda}_\mathrm{i}, \overline{\lambda}_\mathrm{i}] \subseteq [\underline{\lambda}_\mathrm{o}, \overline{\lambda}_\mathrm{o}]$.

Proposition 4. *For all $\underline{\lambda}_\mathrm{o}, \underline{\lambda}_\mathrm{i}, \overline{\lambda}_\mathrm{i}, \overline{\lambda}_\mathrm{o} \in \mathbb{R}_{\geq 0}$ with $\underline{\lambda}_\mathrm{o} \leq \underline{\lambda}_\mathrm{i} \leq \overline{\lambda}_\mathrm{i} \leq \overline{\lambda}_\mathrm{o}$, $\tau \in \mathfrak{T}$ and $f \in \overline{\mathbb{V}}_\tau$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda}_\mathrm{i},\overline{\lambda}_\mathrm{i}]}[f|\tau] \leq \overline{\mathrm{E}}_{[\underline{\lambda}_\mathrm{o},\overline{\lambda}_\mathrm{o}]}[f|\tau].$$

Corollary 3. *For all $\underline{\lambda}, \lambda, \overline{\lambda} \in \mathbb{R}_{\geq 0}$ such that $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$, $\tau \in \mathfrak{T}$ and $f \in \overline{\mathbb{V}}_\tau$,*

$$\overline{\mathrm{E}}_\lambda[f|\tau] \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau].$$

The second property involves the interpretation of the rate bounds $\overline{\lambda}$ and $\underline{\lambda}$ as proportionality constants for forecaster's selling and buying prices. Since, as explained in Section 3, $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_\tau - N_\sigma|\sigma](\omega)$ can be interpreted as the trader's infimum selling price for the uncertain increment $N_\tau - N_\sigma$ given the information they have at time $\sigma(\omega)$, one would expect that this is again proportional to $\overline{\lambda}$; the next result confirms that this is indeed the case, at least when these stopping times are *constant*.

Proposition 5. *For any two time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_t - N_s|s] = \overline{\lambda}(t-s) \quad \textit{and} \quad \underline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_t - N_s|s] = \underline{\lambda}(t-s).$$

Proof. We prove first that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_t - N_s|s] \leq \overline{\lambda}(t-s) \quad \text{and} \quad \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(N_t - N_s)|s] \leq -\underline{\lambda}(t-s). \tag{5}$$

For the first inequality, observe that for the trading strategy G with stopping times $\tau_1 := s$ and $\tau_2 := t$ and stakes $\overline{h}_1 := 1$ and $\underline{h}_1 := 0$,

$$\mathcal{K}_r^{G,\overline{\lambda}(t-s)} = \overline{\lambda}(t-s) + \overline{h}_1 \cdot \big(N_t - N_s - \overline{\lambda}(t-s)\big) = N_t - N_s \quad \text{for all } r \in [t, +\infty[.$$

For the second inequality, consider a similar trading strategy but with $\overline{h}_1 := 0$ and $\underline{h}_1 := 1$.

If $\underline{\lambda} = \lambda = \overline{\lambda}$, we infer from the two inequalities in (5) that

$$\lambda(t-s) \leq \underline{\mathrm{E}}_{\lambda}[N_t - N_s|s] \leq \overline{\mathrm{E}}_{\lambda}[N_t - N_s|s] \leq \lambda(t-s).$$

If $\underline{\lambda} < \overline{\lambda}$, the equality follows from (5) and the equality above for $\lambda = \overline{\lambda}$ and $\lambda = \underline{\lambda}$ due to Proposition 4 with $[\underline{\lambda}_{\mathrm{i}}, \overline{\lambda}_{\mathrm{i}}]$ equal to $\{\overline{\lambda}\}$ or $\{\underline{\lambda}\}$ and $[\underline{\lambda}_{\mathrm{o}}, \overline{\lambda}_{\mathrm{o}}] = [\underline{\lambda}, \overline{\lambda}]$. □

The next property we turn to is the alternative characterisation of the rate parameter λ of the 'classical' Poisson process as the inverse of the expected time until the next increment. The time until the next increment after the stopping time $\tau \in \mathfrak{T}$ is the partial variable

$$\rho_\tau \colon \{\tau < +\infty\} \to \overline{\mathbb{R}}_{\geq 0} \colon \omega \mapsto \inf\{r \in]\tau(\omega), +\infty[\colon \omega(r) > \omega(\tau)\};$$

note that we can extend the domain of ρ_τ to Ω by setting it equal to $+\infty$ on $\{\tau < +\infty\}^{\mathrm{c}}$, which makes this variable a stopping time. It's fairly easy to verify that the upper and lower expectation of $\rho_\tau - \tau$ conditional on τ are the inverse of the forecaster's rate bounds.

Proposition 6. *For every stopping time $\tau \in \mathfrak{T}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau] = \frac{1}{\underline{\lambda}} \quad \textit{and} \quad \underline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau] = \frac{1}{\overline{\lambda}}.$$

Proof. Due to a similar argument as the one at the end of our proof for Proposition 5, it suffices to prove that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau] \leq \frac{1}{\underline{\lambda}} \quad \text{and} \quad \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(\rho_\tau - \tau)|\tau] \leq -\frac{1}{\overline{\lambda}}. \tag{6}$$

For the first inequality, consider the trading strategy G with stopping times $\tau_1 := \tau$ and $\tau_2 := \rho_\tau$ and stakes $\overline{h}_1 := 0$ and $\underline{h}_1 := \frac{1}{\underline{\lambda}}$. Let $c := 1/\underline{\lambda}$ [the case $\underline{\lambda} = 0$ should be treated separately: then there can be no elementary capital process that superhedges ρ_τ]. For any counting path $\omega \in \{\tau < +\infty\}$, there's a unique counting path $\varpi \in \mathcal{I}_s(\omega)$ for which $\rho_\tau(\varpi) = +\infty$: the one that remains constant after τ; for this path $\varpi = \omega \oplus_\tau 0$ and all $r \in [\tau(\omega), +\infty[$,

$$\mathcal{K}_r^{G,c}(\varpi) = c - \overline{h}_1(\varpi)\overline{\lambda}(r - \tau(\varpi)) + \underline{h}_1(\varpi)\underline{\lambda}(r - \tau(\varpi)) = \frac{1}{\underline{\lambda}} + (r - \tau(\varpi)),$$

and therefore $\liminf_{r\to+\infty} \mathcal{K}_r^{G,c}(\varpi) = +\infty = \rho_\tau(\varpi)$. For every path $\varpi \in \mathcal{I}_\tau(\omega)$ for which $\rho_\tau(\varpi) < +\infty$,

$$\begin{aligned}\liminf_{r\to+\infty} \mathcal{K}_r^{G,c}(\varpi) &= \mathcal{K}_{\rho_\tau}^{G,c}(\varpi) \\ &= \frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}}\big(\underline{\lambda}(\rho_\tau(\varpi) - \tau(\varpi)) - 1\big) \\ &= \rho_\tau(\varpi) - \tau(\varpi).\end{aligned}$$

This implies the first inequality in (6). For the second inequality, use a similar trading strategy with the same stopping times, stakes $\overline{h}_1 := \frac{1}{\overline{\lambda}}$ and $\underline{h}_1 := 0$ and initial capital $c := -1/\overline{\lambda}$. □

5.2 The strong Markov property

The next property of the 'classical' Poisson process on the list is that it satisfies the (strong) Markov property. We'll actually show a stronger 'memorylessness'-like result first: that the conditional upper expectation $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\tau]$ can be 'shifted' from the stopping time τ to 0—intuitively, this is a straightforward consequence of the fact that trading strategies can be 'shifted'. In the formal statement of this result, we need to stitch together two counting paths $\omega, \varpi \in \Omega$ at the stopping time $\tau \in \mathfrak{T}$ as follows:

$$\omega \oplus_\tau \varpi \colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0} \colon t \mapsto \begin{cases} \omega(t) & \text{if } t < \tau(\omega), \\ \omega(\tau(\omega)) + \varpi(t - \tau(\omega)) & \text{if } t \geq \tau(\omega); \end{cases}$$

the reader will have no trouble verifying that $\omega \oplus_\tau \varpi$ is indeed a counting path. Similarly, for any time point $s \in \mathbb{R}_{\geq 0}$ and counting path $\omega \in \Omega$, we'll need to look at the derived counting path

$$\omega_{[s,+\infty[} \colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0} \colon t \mapsto \omega(t + s) - \omega(s)$$

which looks only at the increments of ω starting from s.

Proposition 7. *For all $\tau \in \mathfrak{T}$, $f \in \overline{\mathbb{V}}_\tau$ and $\omega \in \{\tau < +\infty\}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f(\omega \oplus_\tau \bullet)] \quad \textit{where } f(\omega \oplus_\tau \bullet) \colon \Omega \to \overline{\mathbb{R}} \colon \varpi \mapsto f(\omega \oplus_\tau \varpi).$$

Proof. We'll verify the equality in the statement by proving that the left-hand side is lower than or equal to the right-hand side and vice versa.

First, we show that the left-hand side is lower than or equal to the right-hand side. Consider any capital process $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\liminf_{r\to+\infty} \mathcal{K}_r^{G,c}(\varpi) \geq$

$f(\omega \oplus_\tau \varpi)$ for all $\varpi \in \Omega$. Let us enumerate the stopping times and stakes of G as $\tau_1, \ldots, \tau_{n+1}$ and $\overline{h}_1, \ldots, \underline{h}_{n+1}$, respectively. Consider the trading strategy G' with stopping times $\tau'_1, \ldots, \tau'_{n+1}$ defined for all $k \in \{1, \ldots, n+1\}$ as

$$\tau'_k \colon \Omega \to \overline{\mathbb{R}}_{\geq 0} \colon \varpi \mapsto \begin{cases} \tau(\omega) + \tau_k(\varpi_{[\tau(\omega),+\infty[}) & \text{if } \varpi \in \mathcal{I}_\tau(\omega) \\ \tau(\omega) & \text{if } \varpi \notin \mathcal{I}_\tau(\omega) \end{cases}$$

and, for all $k \in \{1, \ldots, n\}$, stakes defined by

$$\overline{h}'_k(\varpi) \coloneqq \overline{h}_k(\varpi_{[\tau(\omega),+\infty[}) \quad \text{and} \quad \underline{h}'_k(\varpi) \coloneqq \underline{h}_k(\varpi_{[\tau(\omega),+\infty[}) \quad \text{for all } \varpi \in \Omega.$$

Our construction ensures that $\mathcal{K}^{G',c}_\tau(\omega) = c$ and

$$\mathcal{K}^{G',c}_r(\varpi) = \mathcal{K}^{G,c}_{r-\tau(\omega)}(\varpi_{[\tau(\omega),+\infty[}) \quad \text{for all } \varpi \in \mathcal{I}_\tau(\omega), r \in [\tau(\omega), +\infty[.$$

For all $\varpi \in \mathcal{I}_\tau(\omega)$, we infer from all this that

$$\liminf_{r\to+\infty} \mathcal{K}^{G',c}_r(\varpi) = \liminf_{r\to+\infty} \mathcal{K}^{G,c}_r(\varpi_{[\tau(\omega),+\infty[}) \geq f(\omega \oplus_\tau \varpi_{[\tau(\omega),+\infty[}) = f(\varpi),$$

where for the last equality we used that $\varpi = \omega \oplus_\tau \varpi_{[\tau(\omega),+\infty[}$. So for any capital process $\mathcal{K}^{G,c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges $f(\omega \oplus_\tau \bullet)$ on Ω, there is some capital process $\mathcal{K}^{G',c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ with $\mathcal{K}^{G'c}_\tau(\omega) = c$ that superhedges f on $\mathcal{I}_\tau(\omega)$. From this, we infer that indeed

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau](\omega) \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f(\omega \oplus_\tau \bullet)].$$

The proof of the reverse inequality is similar. Consider any capital process $\mathcal{K}^{G,c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges f on $\mathcal{I}_\tau(\omega)$. Enumerating the stopping times and stakes of G as $\tau_1, \ldots, \tau_{n+1}$ and $\overline{h}_1, \ldots, \underline{h}_{n+1}$, we now consider the trading strategy G' with stopping times $\tau'_1, \ldots, \tau'_{n+1}$ defined for all $k \in \{1, \ldots, n+1\}$ by

$$\tau'_k \colon \Omega \to \overline{\mathbb{R}} \colon \varpi \mapsto \begin{cases} 0 & \text{if } \tau_k(\omega) \leq \tau(\omega) \\ \tau_k(\omega \oplus_\tau \varpi) - \tau(\omega) & \text{if } \tau_k(\omega) > \tau(\omega) \end{cases}$$

and, for all $k \in \{1, \ldots, n\}$, stakes defined by

$$\overline{h}'_k(\varpi) \coloneqq \overline{h}_k(\omega \oplus_\tau \varpi) \quad \text{and} \quad \underline{h}'_k(\varpi) \coloneqq \underline{h}_k(\omega \oplus_\tau \varpi) \quad \text{for all } \varpi \in \Omega.$$

With $c' \coloneqq \mathcal{K}^{G,c}_\tau(\omega)$, our construction ensures that

$$\mathcal{K}^{G',c'}_r(\varpi) = \mathcal{K}^{G,c}_{\tau(\omega)+r}(\omega \oplus_\tau \varpi) \quad \text{for all } \varpi \in \Omega, r \in \mathbb{R}_{\geq 0},$$

and therefore

$$\liminf_{r\to+\infty} \mathcal{K}^{G',c'}_r(\varpi) = \liminf_{r\to+\infty} \mathcal{K}^{G,c}_r(\omega \oplus_\tau \varpi) \geq f(\omega \oplus_\tau \varpi) \quad \text{for all } \varpi \in \Omega.$$

Since $\mathcal{I}_\tau(\omega) = \{\omega \oplus_\tau \varpi \colon \varpi \in \Omega\}$, we've shown that for any capital process $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges f on $\mathcal{I}_\tau(\omega)$, there is some capital process $\mathcal{K}_\bullet^{G',c'} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ with $c' = \mathcal{K}_\tau^{G,c}(\omega)$ that superhedges $f(\omega \oplus_\tau \bullet)$. From this, we infer that indeed

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau](\omega) \geq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f(\omega \oplus_\tau \bullet)].$$

□

The Markov property—see, for example, [3, Ch. 4, Eq. (1.2)]—follows almost immediately from this memoryless character of the conditional upper expectation: for all time points $t \in \mathbb{R}_{\geq 0}$ and time periods $\Delta \in \mathbb{R}_{\geq 0}$,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(X_{t+\Delta})|t](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(t) + X_\Delta)] \quad \text{for all } g \in \overline{\mathbb{R}}^{\mathbb{Z}_{\geq 0}}, \omega \in \Omega;$$

so does the strong Markov property [3, Ch. 4, Eq. (1.17)]: for all stopping times $\tau \in \mathfrak{T}$ and time periods $\Delta \in \mathbb{R}_{\geq 0}$,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(X_{\tau+\Delta})|\tau](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(\tau) + X_\Delta)] \quad \text{for all } g \in \overline{\mathbb{R}}^{\mathbb{Z}_{\geq 0}}, \omega \in \{\tau < +\infty\}.$$

5.3 Law of iterated upper expectations

Next up is the crucial *law of iterated (upper) expectations*, also known as the tower property. For the sake of simplicity, we'll only establish a version of this law for constant stopping times and bounded and so-called finitary variables: those that depend only on the value of the counting path at finitely many time points. Obviously, a variable $f \in \overline{\mathbb{V}}$ is bounded and finitary if and only if $f = g(X_{t_1}, \ldots, X_{t_k})$ for some $k \in \mathbb{N}$, $t_1 \leq \cdots \leq t_k \in \mathbb{R}_{\geq 0}$ and some $g \in \mathbb{G}_k$, where here and in the remainder we let $\mathbb{G}_k$ be the set of bounded real functions on $(\mathbb{Z}_{\geq 0})^k$—for $k = 1$, we'll simply write $\mathbb{G}$.

Theorem 1. *For all consecutive time points $t_1, \ldots, t_{k+1} \in \mathbb{R}_{\geq 0}$ such that $k \geq 1$ and $t_1 < \cdots < t_{k+1}$ and gambles $g \in \mathbb{G}_{k+1}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\big[g(N_{t_1}, \ldots, N_{t_{k+1}})|t_1\big] = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \ldots, N_{t_{k+1}})|t_k]|t_1\Big].$$

My proof is inspired by Shafer & Vovk's [6, Proposition 8.7] proof for a similar result in discrete time, but adds a move to ensure the 'cut' is finite. Because it's rather lengthy, I've relegated it to Appendix A.

One can turn the law of iterated upper expectations in Theorem 1 into a (theoretical) recursive computational scheme to compute $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \ldots, N_{t_k})|t_1]$, at least once one realises that the variable

$$\varpi \mapsto \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \ldots, N_{t_{k+1}})|t_k](\varpi)$$

in the conditional upper expecation on the right-hand side of the equality in Theorem 1 is only functionally dependent on the values ϖ takes in $t_1, \ldots, t_k$ rather than on the values it takes on the entire interval $[0, t_k]$—so only on a countable set rather than an uncountable one.

Corollary 4. *For all consecutive time points $t_1, \dots, t_{k+1} \in \mathbb{R}_{\geq 0}$ such that $t_1 < \dots < t_{k+1}$ and $k \geq 1$, bounded functions $g \in \mathbb{G}_{k+1}$ and paths $\omega \in \Omega$, and with $\Delta \coloneqq t_{k+1} - t_k$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \dots, N_{t_{k+1}})|t_k](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(t_1), \dots, \omega(t_k), \omega(t_k) + N_\Delta)].$$

5.4 Connection to the sublinear Poisson semigroup

Corollary 4 tells us that it's important to be able to compute conditional upper expectations of the form $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s]$ for bounded functions $g \in \mathbb{G}$ and consecutive time points $s, t \in \mathbb{R}_{\geq 0}$. For the classical Poisson process with rate λ, as well for Vovk's measure-theoretic Poisson process [9, Theorem 5], these conditional expectations are related to the Poisson distribution $\psi_{\lambda(t-s)}$ with parameter $\lambda(t-s)$:

$$\overline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = \underline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = \sum_{z \in \mathbb{Z}_{\geq 0}} g(\omega(s) + z)\psi_{\lambda(t-s)}(z).$$

To generalise this property to our imprecise setting, it will be elucidative to express the right-hand side of this equality using the so-called *Poisson semigroup* $(\mathrm{S}_\Delta)_{\Delta \in \mathbb{R}_{\geq 0}}$ for $\Delta = t - s$:

$$\overline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = \underline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = [\mathrm{S}_{(t-s)}g](\omega(s))$$

This family $\mathrm{S}_\bullet$ of linear operators—linear maps from $\mathbb{G}$ to $\mathbb{G}$—is generated by the Poisson generator $\mathrm{G}\colon \mathbb{G} \to \mathbb{G}$, which maps any $g \in \mathbb{G}$ to

$$\mathrm{G}g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}\colon n \mapsto \lambda(g(n+1) - g(n)),$$

as follows:

$$\mathrm{S}_\Delta = e^{\Delta \mathrm{G}} = \lim_{k \to +\infty} \left(\mathrm{I} + \frac{\Delta}{k}\mathrm{G}\right)^k \quad \text{for all } \Delta \in \mathbb{R}_{\geq 0}.$$

Crucially, a similar result holds for the imprecise game-theoretic Poisson process, at least if we replace the Poisson semigroup $\mathrm{S}_\bullet$ by the sublinear Poisson semigroup $\overline{\mathrm{S}}_\bullet$. As explained in [1], this family of sublinear operators is generated by the sublinear Poisson generator $\overline{\mathrm{G}}\colon \mathbb{G} \to \mathbb{G}$, which maps any $g \in \mathbb{G}$ to

$$\overline{\mathrm{G}}g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}\colon n \mapsto \max\{\lambda\big(g(n+1) - g(n)\big)\colon \lambda \in \{\underline{\lambda}, \overline{\lambda}\}\},$$

through the operator exponential:

$$\overline{\mathrm{S}}_\Delta \coloneqq e^{\Delta \overline{\mathrm{G}}} = \lim_{k \to +\infty} \left(\mathrm{I} + \frac{\Delta}{k}\overline{\mathrm{G}}\right)^k \quad \text{for all } \Delta \in \mathbb{R}_{\geq 0}.$$

Theorem 2. *For all time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$ and $g \in \mathbb{G}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s] = [\overline{\mathrm{S}}_{t-s}g](N_s).$$

As an intermediary step towards proving this result, we establish the following 'generalisation'[3] of Vovk's [9] Theorem 5; the proof is a rather straightforward modification of the original one.

Proposition 8. *For all $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$ and $g \in \mathbb{G}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s] \leq \left[\overline{\mathrm{S}}_{t-s}g\right](N_s).$$

Proof (Proof of Theorem 2). Recall from Proposition 7 that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(s) + N_{t-s})] \quad \text{for all } \omega \in \Omega.$$

Consequently, it suffices to show that

$$\overline{\mathrm{E}}[g(z + N_\Delta)] = [\overline{\mathrm{S}}_\Delta g](z) \quad \text{for all } g \in \mathbb{G}, z \in \mathbb{Z}_{\geq 0}, \Delta \in \mathbb{R}_{\geq 0}.$$

So for all $\Delta \in \mathbb{R}_{\geq 0}$, let $\overline{\mathrm{T}}_\Delta \colon \mathbb{G} \to \mathbb{G}$ map $g \in \mathbb{G}$ to

$$\overline{\mathrm{T}}_\Delta g \colon \mathbb{Z}_{\geq 0} \to \mathbb{R} \colon z \mapsto \overline{\mathrm{E}}[g(z + N_\Delta)].$$

It follows immediately from the properties of the upper expectation that $\overline{\mathrm{T}}_\Delta$ is a sublinear transition operator, from Proposition 8 and [2, Lemma 53] that $\overline{\mathrm{T}}_\Delta$ is dominated by $\overline{\mathrm{S}}_\Delta$, and from (E6) that $\overline{\mathrm{T}}_0 = \mathrm{I}$.

Furthermore, we can use the law of iterated upper expectation [Theorem 1] to show that $\overline{\mathrm{T}}_\bullet$ is a semigroup. To this end, fix some $\Delta_1, \Delta_2 \in \mathbb{R}_{>0}$, $g \in \mathbb{G}$ and $z \in \mathbb{Z}_{\geq 0}$, and observe that

$$[\overline{\mathrm{T}}_{\Delta_1+\Delta_2} g](z) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z + N_{\Delta_1+\Delta_2})] = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(x + N_{\Delta_1+\Delta_2})|s]]$$

Now for all $\omega \in \Omega$, it follows from Proposition 7 that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(x+N_{\Delta_1+\Delta_2})|\Delta_1](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(x+\omega(\Delta_1)+N_{\Delta_2})] = [\overline{\mathrm{T}}_{\Delta_2} g](x+\omega(\Delta_1)).$$

Substituting into the preceding equality, we find that

$$[\overline{\mathrm{T}}_{\Delta_1+\Delta_2} g](x) = \overline{\mathrm{E}}\big[[\overline{\mathrm{T}}_{\Delta_2} g](x + N_{\Delta_1})\big] = [\overline{\mathrm{T}}_{\Delta_1}\overline{\mathrm{T}}_{\Delta_2} g](x),$$

as required.

For any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and $\Delta \in \mathbb{R}_{\geq 0}$, let $\mathrm{T}^\lambda_\Delta$ map $g \in \mathbb{G}$ to

$$\mathrm{T}^\lambda_\Delta g \colon \mathbb{Z}_{\geq 0} \to \mathbb{R} \colon z \mapsto \overline{\mathrm{E}}_\lambda[g(z + N_\Delta)].$$

Now from Proposition 8, it follows that for all $\Delta \in \mathbb{R}_{\geq 0}$ and $g \in \mathbb{G}$,

$$\mathrm{T}^\lambda_\Delta g \leq \mathrm{S}^\lambda_\Delta g \quad \text{and} \quad -\mathrm{T}^\lambda_\Delta(-g) \geq -\mathrm{S}^\lambda_\Delta(-g) = \mathrm{S}^\lambda_\Delta g,$$

where $\mathrm{S}^\lambda_\bullet$ is the Poisson semigroup with rate λ. Consequently, $\mathrm{T}^\lambda_\Delta = \mathrm{S}^\lambda_\Delta$ for all $\Delta \in \mathbb{R}_{\geq 0}$ and $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. From this, Propositions 4 and 8, it follows that

$$\mathrm{S}^\lambda_\Delta g = \mathrm{T}^\lambda_\Delta g \leq \overline{\mathrm{T}}_\Delta g \leq \overline{\mathrm{S}}_\Delta g \quad \text{for all } \lambda \in [\underline{\lambda}, \overline{\lambda}], \Delta \in \mathbb{R}_{\geq 0}, g \in \mathbb{G}.$$

[3] We only generalise their result in the particular case $A_t = \lambda t$.

Since $\overline{\mathrm{S}}_\bullet$ is equal to Nendel's [5] so called *Nisio semigroup* induced by the family $\{\mathrm{S}^\lambda_\bullet : \lambda \in [\underline{\lambda}, \overline{\lambda}]\}$ [1, Proposition 5.2], and this Nisio semigroup is the point-wise smallest semigroup that dominates the family $\{\mathrm{S}^\lambda_\bullet : \lambda \in [\underline{\lambda}, \overline{\lambda}]\}$ [5, Remark 5.3], it follows from these inequalities that $\overline{\mathrm{T}}_\Delta = \overline{\mathrm{S}}_\Delta$ for all $\Delta \in \mathbb{R}_{\geq 0}$, which concludes our proof. □

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A Relegated proofs

A.1 Proofs for results in Section 3

Proof of Proposition 1. These properties follow immediately from the assumptions in the statement on $\mathfrak{K}$ and the definition of $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau]$. $\square$

Proof of Lemma 1. Follows almost immediately from the definition of coherence. $\square$

Proof of Corollary 1. Follows immediately from Lemma 1 and the τ-measurability of f. $\square$

Proof of Corollary 2. The upper found follows almost immediately from the definition of $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau](\omega)$, while the lower bound follows from Lemma 1 for $t = \tau(\omega)$ and $\varpi = \omega$. $\square$

Proof of Proposition 2. The outer inequalities follow from (E6). The middle equality holds because by (E6) and (E2),

$$0 = \overline{\mathrm{E}}_{\mathfrak{K}}[0|\tau](\omega) = \overline{\mathrm{E}}_{\mathfrak{K}}[f - f|\tau](\omega) \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) + \overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega).$$

$\square$

A.2 Proofs for results in Section 4

Proof of Proposition 3. That $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ contains the constants follows from choosing $\tau_1 = 0 = \tau_2$ in the definition of the trading strategy G, and that it's closed under multiplication with positive scalars clearly follows immediately from the definition of trading strategies. That $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is closed under pointwise addition takes a bit more work to prove formally, but should be intuitively clear. We'll only prove explicitly that $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is coherent. To this end, we fix some $\mathcal{K}_{\bullet}^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, $t \in \mathbb{R}_{\geq 0}$ and $\omega \in \Omega$, and set out to show that for any $\epsilon \in \mathbb{R}_{>0}$, there is some $\varpi \in \mathcal{I}_t(\omega)$ such that

$$\liminf_{r\to+\infty} \mathcal{K}_r^{G,c}(\varpi) < \mathcal{K}_t^{G,c}(\omega) + \epsilon. \tag{7}$$

So fix any such $\epsilon \in \mathbb{R}_{>0}$.

We'll show the existence of the path ϖ by constructing it recursively such that the capital doesn't increase. To this end, we let $\mathscr{K}$ be the set of indices $k \in \{1, \ldots, n\}$ such that $t < \tau_{k+1}(\omega)$. In case this index set $\mathscr{K}$ is empty, the path $\varpi = \omega$ satisfies the inequality (7), so henceforth we assume that $\mathscr{K} \neq \emptyset$ and let $k \coloneqq \min \mathscr{K}$.

If $\tau_k(\omega) > t$, let $\varpi_k \coloneqq \omega$. Otherwise, we set out to construct some counting path $\varpi_k \in \mathcal{I}_t(\omega)$ for which that the 'currently active' bets with stakes $\overline{h}_k(\omega)$ and $\underline{h}_k(\omega)$ don't increase the capital after t by more than ϵ/n. Choose some $\delta_k \in \mathbb{R}_{>0}$ such that $\omega(t + \delta_k) = \omega(t)$ [this is possible because ω is continuous from the right], $-\overline{h}_k(\omega)\underline{\lambda}\delta_k < \epsilon/n$, $\delta_k < \tau_{k+1}(\omega) - t$ and $\underline{\lambda}\delta_k \leq 1$. [4] It follows from Eq. (4) that the counting path ϖ_k given by

$$r \mapsto \begin{cases} \omega(r) & \text{if } r < t + \delta_k, \\ \omega(t) & \text{if } r \geq t + \delta_k \text{ and } \overline{h}_{k-1}(\omega) \geq 0 \\ \omega(t) + \lceil \underline{\lambda}(r-t) \rceil & \text{if } r \geq t + \delta_k \text{ and } \overline{h}_{k-1}(\omega) < 0 \end{cases}$$

does exactly that. Note that $\tau_{k+1}(\varpi_k) \geq t + \delta_k$—if this were not the case, then since $\varpi_k|_{[0,\tau_{k+1}(\varpi_k)]} = \omega|_{[0,\tau_{k+1}(\varpi_k)]}$ by definition and τ_{k+1} is a stopping time, it must be that $\tau_{k+1}(\omega) = \tau_{k+1}(\varpi_k) < t + \delta_k$, which contradicts our requirement that $\delta_k < \tau_{k+1}(\omega) - t$. Similarly, note that our construction also ensures that $\overline{h}_\ell(\varpi_k) = \overline{h}_\ell(\omega)$ and $\underline{h}_\ell(\varpi_k) = \underline{h}_\ell(\omega)$ for all $\ell \in \{1, \ldots, k\}$, as the stakes $\overline{h}_\ell$ and $\underline{h}_\ell$ are τ_ℓ-measurable and $\varpi_k|_{[0,\tau_k(\omega)]} = \omega|_{[0,\tau_k(\omega)]}$.

If $\tau_{k+1}(\varpi_k) = +\infty$ or $k = n$, our construction ensures that $\mathcal{K}_r^{G,c}(\varpi_k) < \mathcal{K}_t^{G,c}(\omega) + \epsilon/n$ for all $r \in [t, t+\infty[$, which implies the inequality in (7). If on the other hand $k < n$ and $t_k \coloneqq \tau_{k+1}(\varpi_k) < +\infty$, we let ℓ be the smallest index in $\{k+1, \ldots, n\}$ such that $\tau_\ell(\varpi_k) < \tau_{\ell+1}(\varpi_k)$. We repeat the same argument as before, but modifying ϖ_k from $\tau_\ell(\varpi_k)$ onwards rather than ω from t onwards. After at most $n - k$ repetitions of the same argument, we'll end up with a counting path in $\mathcal{I}_t(\omega)$ that satisfies the inequality in (7). □

A.3 Additional results and relegated proofs for results in Section 5

Proof of Proposition 4. It suffices to make the following observation. Fix any elementary capital process $\mathcal{K}_\bullet^{G_\mathrm{o},c_\mathrm{o}} \in \mathfrak{K}_{[\underline{\lambda}_\mathrm{o},\overline{\lambda}_\mathrm{o}]}$ and any path $\omega \in \{\tau < +\infty\}$. Then with $c_\mathrm{i} \coloneqq \mathcal{K}_\tau^{G_\mathrm{o},c_\mathrm{o}}(\omega)$ and G_i a copy of the trading strategy G_o but with stopping times $\tau_k^\mathrm{i} \coloneqq \tau_k^\mathrm{o}$ instead of τ_k^o, the capital process $\mathcal{K}_\bullet^{G_\mathrm{i},c_\mathrm{i}} \in \mathfrak{K}_{[\underline{\lambda}_\mathrm{i},\overline{\lambda}_\mathrm{i}]}$ has capital c_i in τ for ω and is constructed in such a way that for all $t \in \,]\tau(\omega), +\infty[$ and $\varpi \in \mathcal{I}_\tau(\omega)$,

$$\begin{aligned} &\mathcal{K}_t^{G_\mathrm{i},c_\mathrm{i}}(\varpi) - \mathcal{K}_t^{G_\mathrm{o},c_\mathrm{o}}(\varpi) \\ &\quad = \sum_{k=1}^{n} (\tau_{k+1}^\mathrm{i}(\varpi) \wedge t - \tau_k^\mathrm{i}(\varpi) \wedge t)\big(\overline{h}_k(\varpi)(\overline{\lambda}_\mathrm{o} - \overline{\lambda}_\mathrm{i}) + \underline{h}_k(\varpi)(\underline{\lambda}_\mathrm{i} - \underline{\lambda}_\mathrm{o})\big) \geq 0, \end{aligned}$$

where the inequality follows from the assumptions in the statement. □

The following corollary of Proposition 5 will be useful in the proof of Theorem 1 and Proposition 8—through Lemma 2—further on.

Corollary 5. *For any two time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$ and $m \in \mathbb{N}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{\{N_t - N_s \geq m\}}|s] \leq \frac{\overline{\lambda}(t-s)}{m}.$$

Proof. Since $\mathbb{I}_{\{N_t - N_s \geq m\}} \leq (N_t - N_s)/m$, this follows immediately from Proposition 5 and (E4) & (E3) in Proposition 1. □

[4] The final condition is only necessary because we need unit jumps.

Proof of Theorem 1. Fix any $\omega \in \Omega$ and $\epsilon \in \mathbb{R}_{>0}$, and let $f := g(N_{t_1}, \ldots, N_{t_{k+1}})$. Since $f = (f - \inf f) + \inf f$ and $\inf(f - \inf f) \geq 0$, thanks to (E5) we may assume without loss of generality that $\inf f \geq 0$. Furthermore, since f is positive and bounded, it follows from (E6) that so are $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1]$ and $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]$.

Since f is t_{k+1}-measurable, it follows from Corollary 1 that there is some capital process $\mathcal{K}_\bullet^{G_1,c_1} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) \leq \mathcal{K}_{t_1}^{G_1,c_1}(\omega) < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) + \epsilon$ and $\mathcal{K}_{t_{k+1}}^{G_1,c_1}(\varpi) \geq f(\varpi)$ for all $\varpi \in \mathcal{I}_{t_1}(\omega)$. Consequently, it must be that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k](\varpi) \leq \mathcal{K}_{t_k}^{G_1,c_1}(\varpi)$ for all $\varpi \in \mathcal{I}_{t_k}(\omega)$; thanks to (E4) and Corollary 1, we infer from this that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]|t_1\Big](\omega) \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\mathcal{K}_{t_k}^{G_1,c_1}|t_1\Big](\omega).$$

Clearly, the t_k-measurable varibale $\mathcal{K}_{t_k}^{G_1,c_1}$ is hedged on $\mathcal{I}_{t_1}(\omega)$ (and at t_k) by $\mathcal{K}_\bullet^{G_1,c_1}$; consequently,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]|t_1\Big](\omega) \leq \mathcal{K}_{t_1}^{G_1,c_1}(\omega) < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) + \epsilon. \tag{8}$$

For the converse inequality, recall from Section 2 that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]$ is t_k-measurable. Hence, it follows from Corollary 1 that there is some capital process $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]|t_1\Big](\omega) \leq \mathcal{K}_{t_1}^{G,c}(\omega) < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]|t_1\Big](\omega) + \frac{\epsilon}{3}$$

and $\mathcal{K}_{t_k}^{G,c}(\varpi) \geq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k](\varpi)$ for all $\varpi \in \mathcal{I}_{t_1}(\omega)$. Let $\Delta := t_{k+1} - t_k$, and fix some natural number m such that $3(\sup f)\overline{\lambda}\Delta < m\epsilon$. Then by Corollary 5, there is some capital process $\mathcal{K}_\bullet^{G_0,c_0} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ with $\mathcal{K}_{t_1}^{G_0,c_0}(\omega) \leq \overline{\lambda}\Delta/m + \epsilon/(3\sup f)$ such that $\mathcal{K}_{t_k}^{G_0,c_0}(\varpi) \geq \mathbb{I}_{\{N_{t_k} - N_{t_1} \geq m\}}(\varpi)$ for all $\varpi \in \mathcal{I}_{t_1}(\omega)$. On the other hand, for all $z = z_{1:k} \in (\mathbb{Z}_{\geq 0})^k$ with $\omega(t_1) = z_1 \leq z_2 \leq \cdots \leq z_k < \omega(t_1) + m$, there is some capital process $\mathcal{K}_\bullet^{G_z,c_z} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z_1, \ldots, z_k, z_k + N_\Delta)] \leq \mathcal{K}_0^{G_z,c_z} = c_z < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z_1, \ldots, z_k, z_k + N_\Delta)] + \frac{\epsilon}{3}$$

and, thanks to Corollary 1,

$$\mathcal{K}_\Delta^{G_z,c_z}(\varpi) \geq g(z_1, \ldots, z_k, z_k + \varpi(\Delta)) \quad \text{for all } \varpi \in \Omega.$$

Consider now the trading strategy G' that trades according to G until t_k and, for $\varpi \in \mathcal{I}_{t_1}(\omega)$ with $\varpi(t_k) < \omega(t_1) + m$, from t_k onwards trades according to $G_{(\varpi(t_1)\ldots,\varpi(t_k))}$ shifted by t_k—it shouldn't take too much effor from the reader to understand that this is still a valid trading strategy.

By construction, $\mathcal{K}_{t_1}^{G',c+\frac{\epsilon}{3}}(\omega) = \mathcal{K}_{t_1}^{G,c}(\omega) + \epsilon/3$, and $\mathcal{K}_{t_{k+1}}^{G',c+\frac{\epsilon}{3}}(\varpi) \geq f(\varpi)$ for all $\varpi \in \mathcal{I}_{t_1}(\omega)$ such that $\varpi(t_k) < \omega(t_1) + m$. Additionally, for all $\varpi \in \mathcal{I}_{t_1}(\omega)$—so in particular for those with $\varpi(t_k) \geq \omega(t_1) + m$—$(\sup f)\mathcal{K}_{t_k}^{G_0,c_0}(\varpi) \geq$

$(\sup f)\mathbb{I}_{\{N_{t_k}-N_{t_1}\geq m\}}(\varpi)$. Since furthermore $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is closed under positive linear combinations, it follows that

$$\begin{aligned}\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) &\leq \mathcal{K}_{t_1}^{G',c+\frac{\epsilon}{3}}(\omega)+(\sup f)\mathcal{K}_{t_1}^{G_0,c_0}(\omega)\\ &\leq \mathcal{K}_{t_1}^{G,c}(\omega)+\frac{\epsilon}{3}+(\sup f)\frac{\overline{\lambda}\Delta}{m}+\frac{\epsilon}{3}\\ &< \mathcal{K}_{t_1}^{G,c}(\omega)+\epsilon\\ &< \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]|t_1\Big](\omega)+\epsilon. \end{aligned} \tag{9}$$

Since ϵ can be made arbitrarily small, the equality in the statement follows from the inequalities (8) and (9). □

Proof of Corollary 4. This follows immediately from Proposition 7 once one realises that for all $\varpi\in\mathcal{I}_{t_k}(\omega)$,

$$\big[g(N_{t_1},\dots,N_{t_{k+1}})\big](\omega\oplus_{t_k}\varpi)=g(\omega(t_1),\dots,\omega(t_k),\omega(t_k)+\varpi(t_{k+1}-t_k)).$$

□

In our proof for Proposition 8, we'll need a bound on the upper probability of having more than one jump in one of a sequence of consecutive intervals of the same length.

Lemma 2. *Fix some $s,t\in\mathbb{R}_{\geq 0}$ such that $s<t$. For all $n\in\mathbb{N}$, we define $\Delta_n:=(t-s)/n$, $t_k^n:=s+(k-1)\Delta_n$ for all $k\in\{1,\dots,n+1\}$ and*

$$A_n:=\{\omega\in\Omega\colon(\forall k\in\{1,\dots,n\})\ \omega(t_{k+1}^n)\leq\omega(t_k^n)+1\}.$$

Then

$$(\forall\epsilon\in\mathbb{R}_{>0})(\exists n_\epsilon\in\mathbb{N})(\forall n\in\mathbb{N},n\geq n_\epsilon)\ \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A_n^c}|s]<\epsilon$$

Proof. Fix any $\omega\in\Omega$, $\epsilon\in\mathbb{R}_{>0}$ and $m\in\mathbb{N}$ such that $m>3\overline{\lambda}(t-s)/\epsilon$. Then by Corollary 5, there is some capital process $\mathcal{K}_\bullet^{G_m,c_m}\in\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\mathcal{K}_s^{G_m,c_m}(\omega)<\overline{\lambda}(t-s)/m+\epsilon/3$ and $\mathcal{K}_t^{G_m,c_m}(\varpi)\geq\mathbb{I}_{\{N_t-N_s\geq m\}}(\varpi)$ for all $\varpi\in\mathcal{I}_s(\omega)$.

Now consider the elementary betting strategy G that bets with unit stake from the moment there is some jump in the interval $[t_k^n,t_{k+1}^n]$ and stops at the end of this interval, and stops trading altogether once the path has increased by more than m; more formally, we consider the stopping times $\tau_1:=\tau_1'\wedge\sigma$, …, $\tau_{2m+1}:=\tau_{2m+1}'\wedge\sigma$ with

$$\sigma\colon\Omega\to\overline{\mathbb{R}}_{\geq 0}\colon\varpi\mapsto\inf\{r\in\mathbb{R}_{\geq 0}\colon\varpi(r)\geq\varpi(s)+m\}$$

and with τ_1', …, τ_{2m+1}' defined recursively by $\tau_1':=t_1^n=s$ and, for all $j\in\{1,\dots,m\}$, by

$$\tau_{2j}'\colon\Omega\to\overline{\mathbb{R}}_{\geq 0}\colon\varpi\mapsto\inf\left\{r\in\mathbb{R}_{\geq 0}\colon\tau_{2j-1}(\varpi)<r\leq t,\varpi(r)>\lim_{r_-\nearrow r}\varpi(r_-)\right\}$$

and

$$\tau'_{2j+1}\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}\colon \varpi \mapsto \inf\{t_k^n\colon k \in \{1,\dots,n+1\}, t_k^n \geq \tau_{2j}(\varpi)\},$$

and the stakes $\overline{h}_{2j} \coloneqq 1$, $\overline{h}_{2j-1} \coloneqq 0$ and $\underline{h}_{2j} \coloneqq 0 \eqqcolon \underline{h}_{2j-1}$. Then with $c \coloneqq \epsilon/3$, our construction ensures that $\mathcal{K}_t^{G,c}(\varpi) \geq c - m\overline{\lambda}\Delta_n$ for all $\varpi \in \Omega$, and in particular that

$$\mathcal{K}_t^{G,c}(\varpi) \geq c - m\overline{\lambda}\Delta_n + 1 \quad \text{for all } \varpi \in \{N_t - N_s < m\} \cap A_n^{\mathrm{c}}.$$

Consequently, whenever $n \geq n_\epsilon \coloneqq 2m\overline{\lambda}(t-s)/\epsilon$, $\mathbb{I}_{A_n^{\mathrm{c}}}$ is superhedged on $\mathcal{I}_s(\omega)$ by $\mathcal{K}_\bullet^{G,c} + \mathcal{K}_\bullet^{G_m,c_m} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, whence indeed

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A_n^{\mathrm{c}}}|s](\omega) \leq \mathcal{K}_s^{G,c}(\omega) + \mathcal{K}_s^{G_m,c_m}(\omega) < c + \frac{\overline{\lambda}(t-s)}{m} + \frac{\epsilon}{3} < \epsilon.$$

□

Proof of Proposition 8. In the degenerate case $s = t$, the equality in the statement is immediate because (i) $\overline{\mathrm{S}}_0 = \mathrm{I}$; and (ii) $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|s]$ maps the s-measurable variable $g(N_s)$ to itself due to (E6). Henceforth, we therefore assume that $s < t$.

We fix any $\omega \in \Omega$ and $g \in \mathbb{G}$, and set out to prove that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s](\omega) \leq \big[\overline{\mathrm{S}}_{t-s}g\big](\omega(s)).$$

For all $n \in \mathbb{N}$, let Δ_n, t_1^n, …, t_{n+1}^n and A_n be defined as in Lemma 2, and let $g_k^n \coloneqq (\mathrm{I} + \Delta_n\overline{\mathrm{G}})^{n+1-k}g$ for all $k \in \{1,\dots,n+1\}$. Note that $g_k^n = (\mathrm{I} + \Delta_n\overline{\mathrm{G}})g_{k+1}^n$ and that the stopping time

$$\sigma_n\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}\colon \varpi \mapsto \inf \bigcup_{k=1}^{n} \{r \in [t_k^n, t_{k+1}^n]\colon \varpi(r) \geq \varpi(t_k^n) + 2\}$$

is equal to $+\infty$ on the event A_n.

Fix any $\epsilon \in \mathbb{R}_{>0}$. Then by Lemma 2 and [1, Theorem 3.1], there is some $n \in \mathbb{N}$ such that $\delta \coloneqq (\sup g - \inf g)\underline{\lambda}\Delta_n < \epsilon$, $0 \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A_n^{\mathrm{c}}}|s](\omega) < \epsilon/2(\sup g - \inf g)$ and $|[\overline{\mathrm{S}}_{t-s}g](\omega(s)) - g_1^n(\omega(s))| < \epsilon$. For this n, we consider the initial capital $c \coloneqq g_1^n(\omega(s)) + \delta$ in combination with the elementary betting strategy G with stopping times $\tau_1 \coloneqq t_1^n \wedge \sigma_n, \dots, \tau_{n+1} \coloneqq t_{n+1}^n \wedge \sigma_n$ and stakes defined for all $k \in \{1,\dots,n\}$ and $\varpi \in \Omega$ by

$$\overline{h}_k(\varpi) \coloneqq \Big(+g_{k+1}^n\big(\varpi(\tau_k)+1\big) - g_{k+1}^n\big(\varpi(\tau_k)\big)\Big) \wedge 0$$

and

$$\underline{h}_k(\varpi) \coloneqq \Big(-g_{k+1}^n\big(\varpi(\tau_k)+1\big) + g_{k+1}^n\big(\varpi(\tau_k)\big)\Big) \wedge 0.$$

This way, for all $\varpi \in \mathcal{I}_s(\omega)$, $k \in \{1,\dots,n\}$ with $\tau_k(\varpi) < \sigma_n(\varpi)$ and $r \in [\tau_k(\varpi), \tau_{k+1}(\varpi)]$ such that $r < \sigma_n(\varpi)$,

$$\begin{aligned}\mathcal{K}_r^{G,c}(\varpi) &- \mathcal{K}_{\tau_k}^{G,c}(\varpi) \\ &= g_{k+1}^n(\varpi(r)) - g_k^n(\varpi(\tau_k)) + \big(\overline{h}_k(\varpi)\overline{\lambda} - \underline{h}_k(\varpi)\underline{\lambda}\big)(t_{k+1}^n - r). \quad (10)\end{aligned}$$

To verify this equality, observe that by construction of the capital process $\mathcal{K}^{G,c}_{\bullet}$,

$$\begin{aligned}
&\mathcal{K}^{G,c}_r(\varpi) - \mathcal{K}^{G,c}_{\tau_k}(\varpi) \\
&\quad = \overline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \overline{\lambda}(r - \tau_k(\varpi))\big) \\
&\qquad - \underline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \underline{\lambda}(r - \tau_k(\varpi))\big) \\
&\quad = \overline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \overline{\lambda}\Delta_n\big) + \overline{h}_k(\varpi)\overline{\lambda}(t^n_{k+1} - r) \\
&\qquad - \underline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \underline{\lambda}\Delta_n\big) - \underline{h}_k\underline{\lambda}(\varpi)(t^n_{k+1} - r).
\end{aligned}$$

Recall that $g^n_k = (\mathrm{I} + \Delta_n\overline{\mathrm{G}})g^n_{k+1} = g^n_{k+1} + \Delta_n\overline{\mathrm{G}}g^n_{k+1}$ by definition. Hence, if $\overline{h}_k(\varpi) > 0$ (and therefore $\underline{h}_k(\varpi) = 0$),

$$\begin{aligned}
&g^n_{k+1}(\varpi(r)) - g^n_k(\varpi(\tau_k)) \\
&\quad = g^n_{k+1}\big(\varpi(r)\big) - g^n_{k+1}\big(\varpi(\tau_k)\big) - \overline{\lambda}\Delta_n\Big(g^n_{k+1}\big(\varpi(\tau_k) + 1\big) - g^n_{k+1}\big(\varpi(\tau_k)\big)\Big);
\end{aligned}$$

a similar equality holds if $\underline{h}_k(\varpi) > 0$ (and therefore $\overline{h}_k(\varpi) = 0$) with $\underline{\lambda}$ in place of $\underline{\lambda}$. Because $r < \sigma_n(\varpi)$ by assumption, it's guaranteed that $0 \leq \omega(r) - \omega(\tau_k) \leq 1$; it's therefore straightforward to verify that if $\overline{h}_k(\omega) > 0$,

$$\overline{h}_k(\omega)\big(\omega(r) - \omega(\tau_k) - \overline{\lambda}\Delta_n\big) = g^n_{k+1}(\varpi(r)) - g^n_k(\varpi(\tau_k)),$$

and similarly for $\underline{h}_k(\omega) > 0$; this verifies Eq. (10). In the particular case that $\tau_{k+1}(\omega) < \sigma_n(\omega)$, Eq. (10) for $r = \tau_{k+1}(\omega)$ reduces to

$$\mathcal{K}^{G,c}_{\tau_{k+1}}(\omega) - \mathcal{K}^{G,c}_{\tau_k}(\omega) = g^n_{k+1}(\omega(\tau_{k+1})) - g^n_k(\omega(\tau_{k+1})). \tag{11}$$

Since $\mathcal{K}^{G,c}_{\tau_1}(\varpi) = g^n_1(\varpi(s)) + \delta$ for all $\varpi \in \mathcal{I}_s(\omega)$, it follows from Eq. (11) that for all $k \in \{1, \dots, n\}$ such that $\tau_k(\varpi) < \sigma_n(\varpi)$,

$$\mathcal{K}^{G,c}_{\tau_k}(\varpi) = g^n_k\big(\varpi(\tau_k)\big) + \delta. \tag{12}$$

Now recall that $\sigma_n(\omega) < +\infty$ for any $\varpi \in A_n \cap \mathcal{I}_s(\omega)$, so it follows from the preceding equality for $k = n + 1$ that

$$\mathcal{K}^{G,c}_t(\varpi) = \mathcal{K}^{G,c}_{\tau_{n+1}}(\varpi) = g^n_{n+1}\big(\omega(\tau_{n+1})\big) + \delta = g\big(\varpi(t)\big) + \delta.$$

Next, observe that for $\varpi \in A^c_n \cap \mathcal{I}_s(\omega)$, there is some largest $k \in \{1, \dots, n\}$ such that $\tau_k(\varpi) < \sigma_n(\varpi)$ (and of course $\tau_{k+1}(\varpi) = \sigma_n(\varpi)$); then since ϖ has unit jumps,

$$\lim_{r \nearrow \sigma_n(\varpi)} \mathcal{K}^{G,c}_{\sigma_n}(\varpi) - \mathcal{K}^{G,c}_r(\varpi) = \lim_{r \nearrow \sigma_n(\varpi)} \overline{\underline{h}}_k(\varpi)(\varpi(\sigma_n) - \varpi(r)) = \overline{\underline{h}}_k(\omega).$$

It follows from this, Eqs. (10) and (12) that

$$\begin{aligned}
\mathcal{K}^{G,c}_{\sigma_n}(\varpi) &= \lim_{r \nearrow \sigma_n(\varpi)} \delta + g^n_{k+1}(\varpi(r)) + \big(\overline{h}_k(\varpi)\overline{\lambda} - \underline{h}_k(\varpi)\underline{\lambda}\big)(t^n_{k+1} - r) + \overline{\underline{h}}_k(\varpi) \\
&\geq \delta + \inf g - (\sup g - \inf g)\underline{\lambda}\Delta_n - (\sup g - \inf g) \\
&= 2\inf g - \sup g.
\end{aligned}$$

Recall now that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A_n^{\mathrm{c}}}|\tau](\omega) < \epsilon$, so there is some capital process $\mathcal{K}_{\bullet}^{G',c'} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\mathcal{K}_s^{G',c'}(\omega) < \epsilon/2(\sup f - \inf g)$ and $\mathcal{K}_t^{G',c'}(\varpi) \geq \mathbb{I}_{A_n^{\mathrm{c}}}(\varpi)$ for all $\varpi \in \mathcal{I}_s(\omega)$. Since $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is a cone, we conclude from this that

$$\begin{aligned}\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s](\omega) &\leq \mathcal{K}_s^{G,c}(\omega) + 2(\sup g - \inf g)\mathcal{K}_s^{G',c'}(\omega) \\ &< g_1^n(\omega(s)) + \delta + \epsilon \\ &< [\overline{\mathrm{S}}_{t-s}g](\omega(s)) + 3\epsilon.\end{aligned}$$

Since $\omega \in \Omega$, $g \in \mathbb{G}$ and $\epsilon \in \mathbb{R}_{\geq 0}$ were arbitrary, this proves the inequality in the statement. □