Towards an *imprecise* **Poisson process**

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The Poisson process

The *Poisson process* ...

- ... is a well-known continuous-time stochastic process,
- ... was *not* introduced by Poisson,
- ... is often used in applications, but the underlying assumptions are not always justified,
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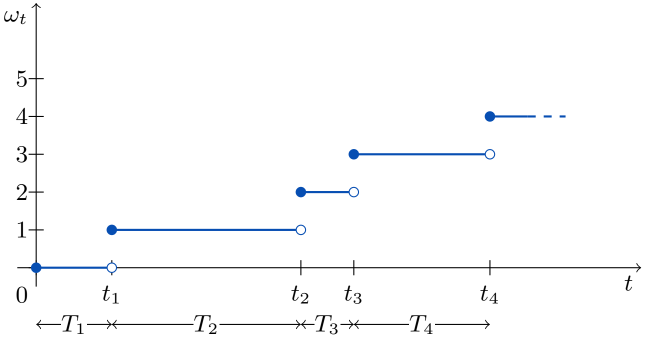
A Poisson process is a *counting* process: at all times $t \in \mathbb{R}_{\geq 0}$,

N_t takes values in $\mathbb{N}$

N_t is interpreted as the number of “events”, “occurences” or “*arrivals*” since the starting point $t_0 = 0$.



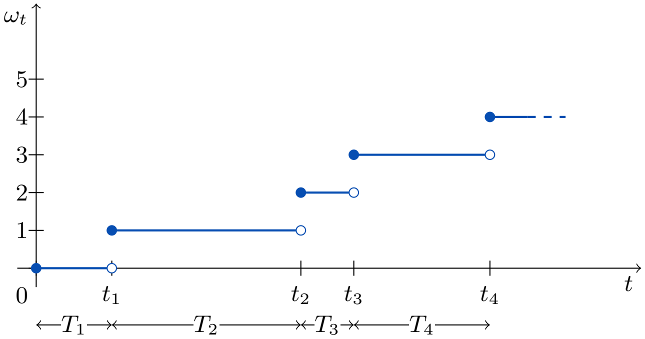
Note that a realisation (or sample path) ω_t is always monotonously increasing!

Often it is demanded that sample paths are càdlàg (right continuous with left limits).

We assume that having more than one arrival in a (very) small interval is highly unlikely (alt.: jumps in sample path have height one).

The Poisson process

A basic counting process

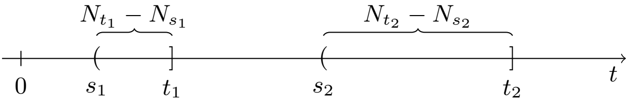


We typically want to determine the expected

- time $T' := t_{n+k} - t$ until the following k arrivals,
- number of arrivals $N_{t+\Delta} - N_t$ in some time period Δ.

The precise Poisson process

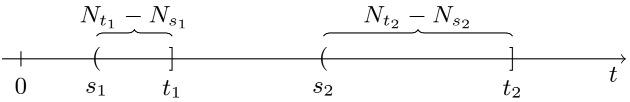
A straightforward definition



The number of arrivals $N_{t_1} - N_{s_1}$ in the (finite) interval $(s_1, t_1]$

- is independent of the number of arrivals $N_{t_2} - N_{s_2}$ in the disjoint interval $(s_1, t_1]$,
- follows a Poisson distribution with parameter $\lambda\Delta_i := \lambda(t_i - s_i) \in \mathbb{R}_{\geq 0}$:

$$\mathrm{P}(N_{t_i} - N_{s_i} = n) = \frac{(\lambda\Delta_i)^n}{n!}e^{-\lambda\Delta_i} = \frac{(\lambda(t_i - s_i))^n}{n!}e^{-\lambda(t_i - s_i)}.$$



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Consequently, the interarrival time T_j (as well as $T' = t_{n+1} - t$) is a random variable that is

- independent of the previous interarrival times $T_1, \ldots, T_{j-1}$,
- exponentially distributed with rate or intensity λ:

$$\mathrm{P}(T_j \leq s) = 1 - \mathrm{P}(N_{t_{j-1}+s} - N_{t_{j-1}} = 0) = 1 - e^{-\lambda s} \text{ for all } s \in \mathbb{R}_{\geq 0}.$$

The precise Poisson process

A more technical axiomatic definition

A counting process P (with state space $\mathbb{N}$) is a *Poisson process* if

(i) it is *Markov*, in the sense that

$$\mathrm{P}(N_t = n_t | N_{t_1} = n_1, \ldots, N_{t_m} = n_m, N_s = n_s) = \mathrm{P}(N_t = n_t | N_s = n_s)$$

for all $t_1 < \cdots < t_m < s \leq t$ and all $n_1 \leq \cdots \leq n_m \leq n_s \leq n_t$;

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for all $t_1 < \cdots < t_m < s \leq t$ and all $n_1 \leq \cdots \leq n_m \leq n_s \leq n_t$;

(ii) it is both time- and state-*homogeneous*, in the sense that

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(iii) it is *orderly*, in the sense that

$$\lim_{\delta \to 0+} \frac{\mathrm{P}(N_\delta \geq 2)}{\delta} = 0$$

or $\mathrm{P}(N_\delta \geq 2) = \mathcal{O}(\delta^2)$.

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(iv) Often, it is also required that there is some $\lambda \in \mathbb{R}_{\geq 0}$ such that

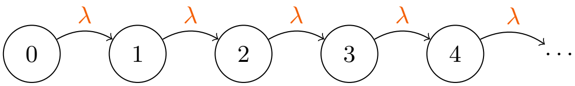
$$\lim_{\delta \to 0+} \frac{\mathrm{P}(N_\delta = 1)}{\delta} = \lambda \text{ and } \lim_{\delta \to 0+} \frac{1 - \mathrm{P}(N_\delta = 0)}{\delta} = \lambda.$$

or $\mathrm{P}(N_\delta = 1) = \lambda\delta + \mathcal{O}(\delta^2)$ and $\mathrm{P}(N_\delta = 0) = 1 - \lambda\delta + \mathcal{O}(\delta^2)$.
The existence of λ actually follows from the previous three requirements!

The precise Poisson process

As a continuous-time Markov chain

The Poisson process is a pure birth process: it is equal to a CTMC with state space $\mathscr{X} = \mathbb{N}$ and transition diagram



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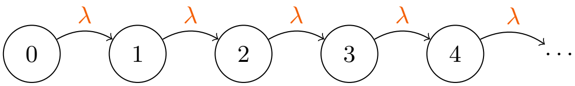
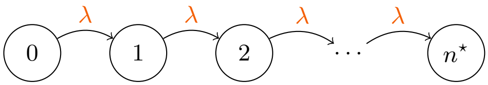


We now want to determine $\mathrm{E}(f(N_t)|N_0 = 0)$, where there is some $n^\star \in \mathbb{N}$ such that $f(n^\star + k) = 0$ for all $k \geq 0$. It then suffices to consider the CTMC with state space $\mathscr{X} = \{0, \dots, n^\star\}$ and transition rate diagram



Other alternative definitions of the Poisson process are

- as a martingale: any counting process N_t such that $\mathrm{E}(N_t - \lambda t) = 0$ is a Poisson process with rate λ;
- as the continuous-time limit of the Bernoulli process (with $\lambda\Delta/n$ the probability of having an arrival in the time period Δ/n).

The precise Poisson process

The rate λ

How do we interpret the parameter λ that "fully" characterises the Poisson process?

1. As the "rate" of having an arrival in a small interval with length δ

$$P(N_{t+\delta} = n + 1 | N_t = n) = P(N_\delta = 1) = \lambda\delta + \mathcal{O}(\delta^2)$$

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How do these alternative definitions relate?

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The imprecise Poisson process

As a collection of orderly counting processes

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- state-inhomogeneous Poisson processes with $\lambda_n \in [\underline{\lambda}, \overline{\lambda}]$ for all $n \in \mathbb{N}$,
- state- and time-inhomogeneous Poisson processes (or simply orderly but Markovian counting processes) such that $\lambda_{n,t} \in [\underline{\lambda}, \overline{\lambda}]$ for all $t \in \mathbb{R}_{\geq 0}$ and all $n \in \mathbb{N}$, or

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The imprecise Poisson process

As an imprecise continuous-time Markov chain

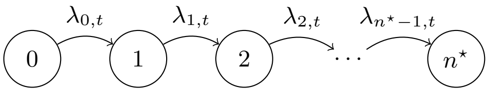
Consider the set of all state- and time-inhomogeneous Poisson processes such that $\lambda_{n,t} \in [\underline{\lambda}, \overline{\lambda}]$ for all $t \in \mathbb{R}_{\geq 0}$ and all $n \in \mathbb{N}$.

We now want to determine

$$\underline{\mathrm{E}}(f(N_t)|N_0 = 0),$$

where there is some $n^\star \in \mathbb{N}$ such that $f(n^\star + k) = 0$ for all $k \geq 0$.

It suffices to consider an imprecise CTMC, and more specifically the set of all inhomogeneous CTMCs with state space $\mathscr{X} = \{0, \ldots, n^\star\}$ and transition rate diagram



where $\underline{\lambda} \leq \lambda_{n,t} \leq \overline{\lambda}$ for all $0 \leq n < n^\star$ and all $t \in \mathbb{R}_{\geq 0}$.

What makes the Poisson process the Poisson process?

What properties should an imprecise Poisson process definitely have?

In the precise case, all the different definitions are equal.

Does the same hold for their imprecise versions?