Extending the domain of Markovian imprecise jump processes

Alexander Erreygers Jasper De Bock

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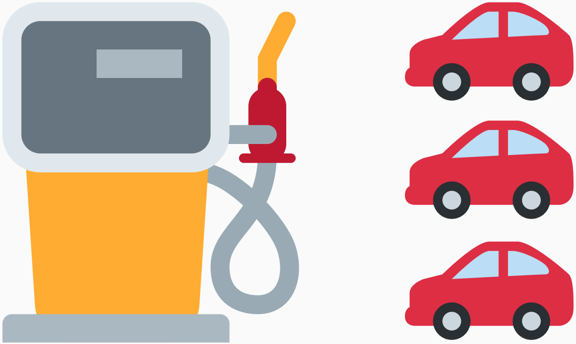
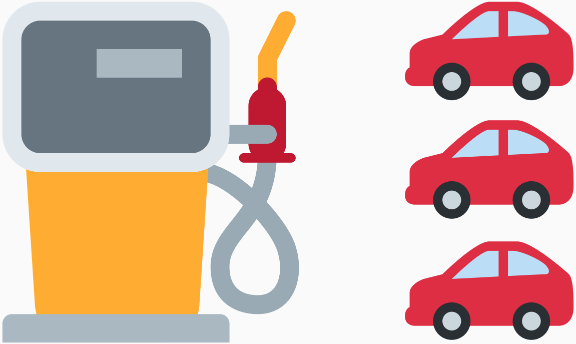
Ghent University, ELIS, Foundations Lab for imprecise probabilities





The manager  is interested in things like

- the expected average number of  over the following 24 hours;
- the expected duration of  in the following hour;
- the expected time until ;
- the probability of  in the following hour.

The abstract framework

We want to make inferences about the *state* of some *system* ⛽

which evolves over continuous time in a non-deterministic manner.

The *state* X_t at the time point t in $\mathbb{R}_{\geq 0}$ is an uncertain variable,

and we assume that it takes values in a finite *state space* $\mathscr{X}$ (⛽: $\{0, 1, 2, 3\}$) .

We are interested in the expectation/probability of idealised variables/events like

- temporal averages: $\frac{1}{T}\int_0^T f(X_t)\,\mathrm{d}t$; (⛽: avg. # of 🚗, dur. of ⛔)
- hitting times: $\inf\{t \in \mathbb{R}_{\geq 0} \colon X_t \in A\}$; (⛽: time until ⛔)
- hitting events: $\bigcup_{t\in\mathcal{T}}\{X_t \in A\}$. (⛽: ⛔ in the following hour)

The measure-theoretic model

$$(\Omega, \mathscr{F}, P) \quad \text{and} \quad (X_t)_{t \in \mathbb{R}_{\geq 0}}$$

The measure-theoretic model

σ-field over Ω

possibility space

probability measure on $\mathscr{F}$

$$(\Omega, \mathscr{F}, P) \quad \text{and} \quad (X_t)_{t \in \mathbb{R}_{\geq 0}}$$

The measure-theoretic model

$$(\Omega, \mathscr{F}, P) \quad \text{and} \quad (X_t)_{t \in \mathbb{R}_{\geq 0}}$$

$\downarrow$

X_t is an $\mathscr{F}$-measurable map from Ω to $\mathscr{X}$

Constructing a measure-theoretic model

Ω is the set of all maps from $\mathbb{R}_{\geq 0}$ to $\mathscr{X}$

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Ω is the set of all maps from $\mathbb{R}_{\geq 0}$ to $\mathscr{X}$

$\mathscr{F}$ is the field of all *cylinder events* of the form $\{\omega \in \Omega : (\omega(t_1), \ldots, \omega(t_n)) \in A\}$

↓

$\sigma(\mathscr{F})$ is the σ-field generated by $\mathscr{F}$

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P is a probability charge on $\mathscr{F}$

↓ P is trivially σ-additive on $\mathscr{F}$

P on $\mathscr{F}$ $\xrightarrow{\text{Carathéodory's Theorem}}$ P_σ on $\sigma(\mathscr{F})$ $\xrightarrow{\text{Lebesgue integration}}$ $E^{\mathrm{L}}_{P_\sigma}$ on $\sigma(\mathscr{F})$-measurables

Constructing a measure-theoretic model

$(\Omega, \sigma(\mathscr{F}), P_\sigma)$ is a probability space, and for all time points t in $\mathbb{R}_{\geq 0}$,

$$X_t : \Omega \to \mathscr{X} : \omega \mapsto \omega(t) \qquad \text{is trivially } \sigma(\mathscr{F})\text{-measurable.}$$

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Under some 🗝 continuity condition 🗝 on P, there is a modification $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$ of $(X_t)_{t \in \mathbb{R}_{\geq 0}}$
— meaning that $Y_t : \Omega \to \mathscr{X}$ is $\sigma(\mathscr{F})$-measurable and $P_\sigma(X_t = Y_t) = 1$ —
that has càdlàg sample paths.

Càdlàg paths

A path $\omega : \mathbb{R}_{\geq 0} \to \mathscr{X}$ is càdlàg if it is continuous from the right and has limits from the left.

$$(\forall t \in \mathbb{R}_{\geq 0}) \lim_{\Delta \searrow 0} \omega(t + \Delta) = \omega(t) \quad \text{and} \quad (\forall t \in \mathbb{R}_{>0}) \lim_{\Delta \searrow 0} \omega(t - \Delta) \text{ exists.}$$

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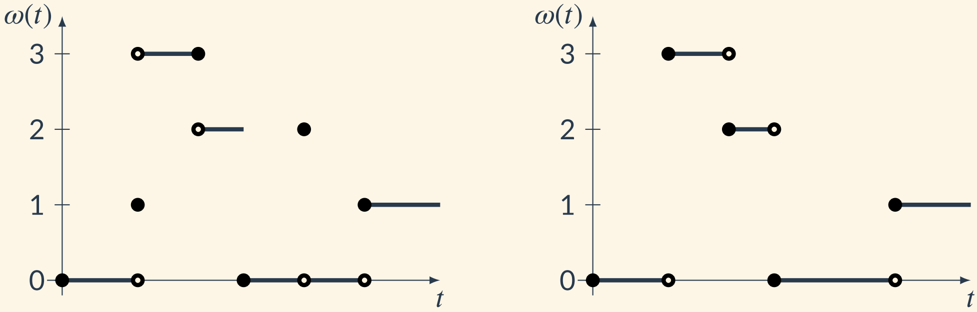

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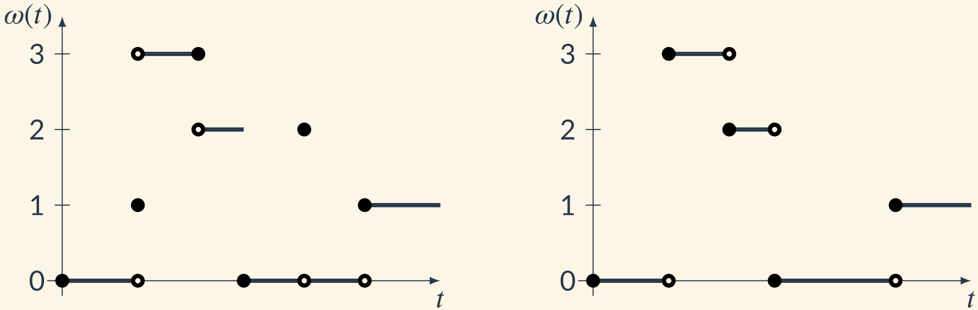

Every càdlàg path $\omega : \mathbb{R}_{\geq 0} \to \mathscr{X}$ is fully defined by its values on any countable dense subset of $\mathbb{R}_{\geq 0}$.

Constructing a measure-theoretic model

$(\Omega, \sigma(\mathscr{F}), P_\sigma)$ is a probability space, and for all time points t in $\mathbb{R}_{\geq 0}$,

$$X_t : \Omega \to \mathscr{X} : \omega \mapsto \omega(t) \qquad \text{is trivially } \sigma(\mathscr{F})\text{-measurable.}$$

🚨 Unfortunately, the idealised variables w.r.t. $(X_t)_{t \in \mathbb{R}_{\geq 0}}$ are not $\sigma(\mathscr{F})$-measurable! 🚨

Under some 🗝 continuity condition 🗝 on P, there is a modification $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$ of $(X_t)_{t \in \mathbb{R}_{\geq 0}}$ that has càdlàg sample paths.

🎉 Therefore, the idealised variables w.r.t. $(Y_t)_{t \in \mathbb{R}_{\geq 0}}$ are $\sigma(\mathscr{F})$-measurable! 🎉

Constructing a measure-theoretic model

$(\Omega, \sigma(\mathscr{F}), P_\sigma)$ is a probabilit… and fo… points …$_{\geq 0}$,

$$X_t : \Omega \to \ldots \mapsto \omega(t)$$ …ivially σ… …easurable.

Unfortunately, the ideal… variables … $(X_t)_{t\in\mathbb{R}}$… …not $\sigma(\mathscr{F})$-measurable!

Under some continuity condition on P, there is a modification $(Y_t)_{t\in\mathbb{R}_{\geq 0}}$ of $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ that has càdlàg sample paths.

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An alternative construction

Ω_{cl} is the set of all càdlàg maps from $\mathbb{R}_{\geq 0}$ to $\mathscr{X}$

$\mathscr{F}_{\text{cl}}$ is the field of all cylinder events of the form $\{\omega \in \Omega_{\text{cl}} : (\omega(t_1), \dots, \omega(t_n)) \in A\}$

idealised variables are $\sigma(\mathscr{F}_{\text{cl}})$-measurable

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P is a probability charge on $\mathscr{F}_{\mathrm{cl}}$

🗝 continuity condition 🗝 $\Rightarrow P$ is σ-additive on $\mathscr{F}_{\mathrm{cl}}$

P on $\mathscr{F}_{\mathrm{cl}}$

$\downarrow$ Carathéodory's Theorem

P_σ on $\sigma(\mathscr{F}_{\mathrm{cl}})$

$\downarrow$ Lebesgue integration

$E^{\mathrm{L}}_{P_\sigma}$ on $\sigma(\mathscr{F}_{\mathrm{cl}})$-measurables

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idealised variables are $\sigma(\mathscr{F}_{\text{cl}})$-measurable

idealised variables are the limit of
—monotone or uniformly bounded—
sequences of $\mathscr{F}_{\text{cl}}$-simples

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$E^{\text{L}}_{P_\sigma}$ on $\sigma(\mathscr{F}_{\text{cl}})$-measurables

E_P on $\mathscr{F}_{\text{cl}}$-simples

$\downarrow$ Daniell integration

E^{D}_P on 'limits of $\mathscr{F}_{\text{cl}}$-simples'

Markovian jump process

A jump process P on $\mathscr{F}_{\text{cl}}$ is completely defined by

the initial probabilities of the form

$$P(X_0 = x_0)$$

and the transition probabilities of the form

$$P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \dots, X_{t_n} = x_n, X_t = x).$$

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$$P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) = P(X_{t+\Delta} = y \mid X_t = x).$$

Markovian jump process

A homogeneous Markovian jump process P on $\mathscr{F}_{\mathrm{cl}}$ is completely defined by the initial probabilities of the form

$$P(X_0 = x_0)$$

and the transition probabilities of the form

$$P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \dots, X_{t_n} = x_n, X_t = x) = P(X_\Delta = y \mid X_0 = x).$$

Markovian jump process

A homogeneous Markovian jump process P on $\mathscr{F}_{\text{cl}}$ is completely defined by the initial probabilities of the form

$$P(X_0 = x_0)$$

and the transition probabilities of the form

$$P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \dots, X_{t_n} = x_n, X_t = x) = P(X_\Delta = y \mid X_0 = x).$$

Whenever it satisfies the 🗝 continuity condition 🗝, the homogeneous Markovian jump process P is completely defined by its

$$\text{initial distribution } p : \; p(x_0) = P(X_0 = x_0)$$

$$\text{(transition) rate matrix } Q : \; Q(x, y) = \frac{\mathrm{d}}{\mathrm{d}\Delta} P(X_\Delta = y \mid X_0 = x).$$

Imprecise jump process

- Damjan Škulj. "Efficient computation of the bounds of continuous time imprecise Markov chains". In: *AMC* 250 (2015), pp. 165–180
- Thomas Krak, Jasper De Bock, and Arno Siebes. "Imprecise continuous-time Markov chains". In: *IJAR* 88 (2017), pp. 452–528

- Max Nendel. *Markov chains under nonlinear expectation*. 2019. arXiv: 1803.03695 [math.PR]

Imprecise jump process

Thomas Krak, Jasper De Bock, and Arno Siebes. “Imprecise continuous-time Markov chains”. In: *IJAR* 88 (2017), pp. 452–528

a set $\mathscr{p}$ of initial distributions and a set $\mathscr{Q}$ of rate matrices

↓

$\mathscr{P}_{\mathscr{p},\mathscr{Q}}$ consists of all jump processes P that are consistent with $\mathscr{p}$ and $\mathscr{Q}$

↓

$\underline{E}_{\mathscr{p},\mathscr{Q}}$ is the lower envelope of $\{E_P : P \in \mathscr{P}_{\mathscr{p},\mathscr{Q}}\}$: for any $\mathscr{F}_{\mathrm{cl}}$-simple variable f,

$$\underline{E}_{\mathscr{p},\mathscr{Q}}(f) := \inf\{E_P(f) : P \in \mathscr{P}_{\mathscr{p},\mathscr{Q}}\}$$

Extending a Markovian imprecise jump process

a set $\mathscr{p}$ of initial distributions and a set $\mathscr{Q}$ of rate matrices

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$\mathscr{P}_{\mathscr{p},\mathscr{Q}}$ is the set of all jump processes P that are consistent with $\mathscr{p}$ and $\mathscr{Q}$

$\mathscr{Q}$ bounded $\Rightarrow$ every P in $\mathscr{P}_{\mathscr{p},\mathscr{Q}}$ satisfies the 🗝 continuity condition 🗝

↓

$(\forall P \in \mathscr{P}_{\mathscr{p},\mathscr{Q}})$ E_P on $\mathscr{F}_{\mathrm{cl}}$-simples $\xrightarrow{\text{Daniell integration}}$ E_P^{D} on 'limits of $\mathscr{F}_{\mathrm{cl}}$-simples'

↓

$\underline{E}_{\mathscr{p},\mathscr{Q}}$ is the lower envelope of $\{E_P^{\mathrm{D}} : P \in \mathscr{P}_{\mathscr{p},\mathscr{Q}}\}$

For a jump process P that satisfies the 🗝 continuity condition 🗝, E_P^{D}

- ⚙ satisfies monotone convergence; [~ Monotone Convergence Theorem]
- ⚙ satisfies uniformly bounded convergence. [~ Lebesgue's Dominated Convergence Theorem]

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If $\mathscr{Q}$ is bounded, the lower envelope $\underline{E}_{\mathscr{P},\mathscr{Q}}$

- 🎉 satisfies monotone convergence from above;
- 😐 is conservative for monotone convergence from below;
- 😐 is conservative for uniformly bounded point-wise convergence.

For a jump process P that satisfies the 🗝️ continuity condition 🗝️, E_P^{D}

- ⚙️ satisfies monotone convergence; [~ Monotone Convergence Theorem]
- ⚙️ satisfies uniformly bounded convergence. [~ Lebesgue's Dominated Convergence Theorem]

If $\mathcal{Q}$ is bounded, the lower envelope $\underline{E}_{\mathscr{P},\mathcal{Q}}$

- 🎉 satisfies monotone convergence from above;
- 😐 is conservative for monotone convergence from below;
- 😐 is conservative for uniformly bounded point-wise convergence;
- 🤓 is continuous for idealised inferences over $[0, T]$. (temporal averages, hitting times and hitting events)

Is $\underline{E}_{\mathscr{P},\mathscr{Q}}$ continuous for idealised inferences over $\mathbb{R}_{\geq 0}$?

Is $\underline{E}_{\mathscr{P},\mathscr{Q}}$ continuous for a larger class of idealised inferences?

Why not extend $\underline{E}_{\mathscr{P},\mathscr{Q}}$ directly?

How does this compare to the framework of (Nendel, 2018)?