Computing inferences for large-scale continuous-time Markov chains by combining lumping with imprecision

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SMPS 2018

Ghent University, ELIS, SYSTeMS

Yet another talk about imprecise Markov chains?

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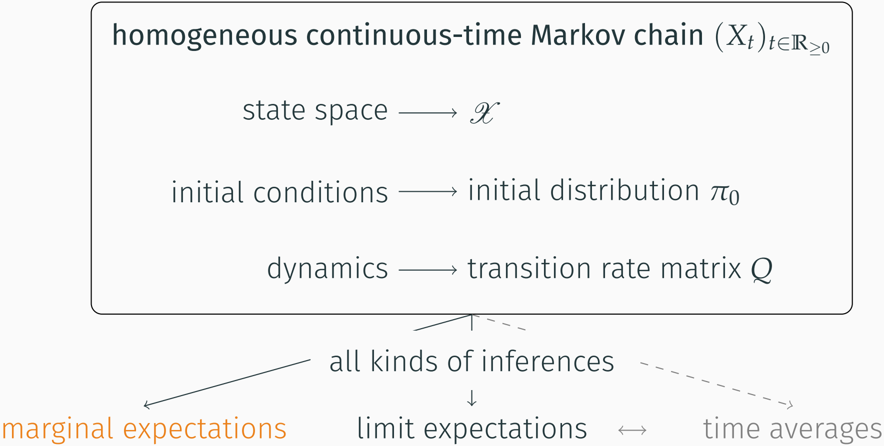
Teaser

Allowing for more imprecision results in faster computations!

Continuous-time Markov chain refresher



homogeneous continuous-time Markov chain $(X_t)_{t \in \mathbb{R}_{\geq 0}}$

state space $\longrightarrow \mathscr{X}$

initial conditions $\longrightarrow$ initial distribution π_0

dynamics $\longrightarrow$ transition rate matrix Q

all kinds of inferences

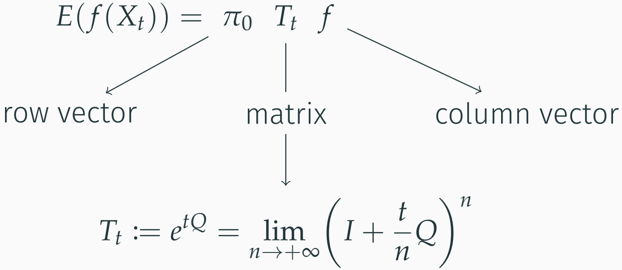
marginal expectations

limit expectations ↔ time averages

Marginal expectation

Fix some $f \colon \mathscr{X} \to \mathbb{R}$ and $t \in \mathbb{R}_{\geq 0}$.

It is well-known that



$$E(f(X_t)) = \pi_0 \ T_t \ f$$

row vector

matrix

column vector

$$T_t := e^{tQ} = \lim_{n \to +\infty} \left(I + \frac{t}{n} Q \right)^n$$

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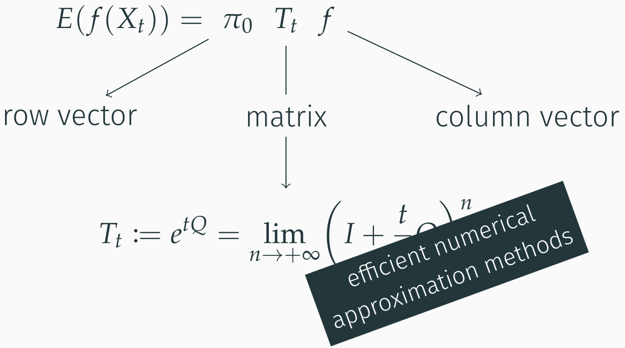

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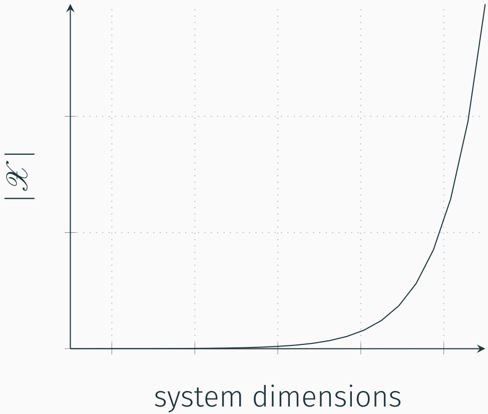
efficient numerical approximation methods

State space explosion



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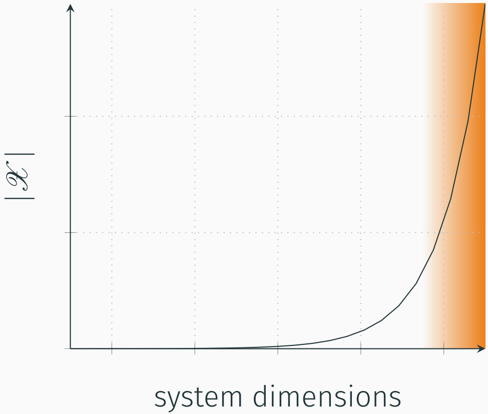


problem

computing $E(f(X_t))$ becomes intractable

solution

reduce the number of states *somehow*

Lumping

Informally

Taking together (lumping, aggregating) states yields the lumped stochastic process, which has a (significantly) smaller state space.

Formally

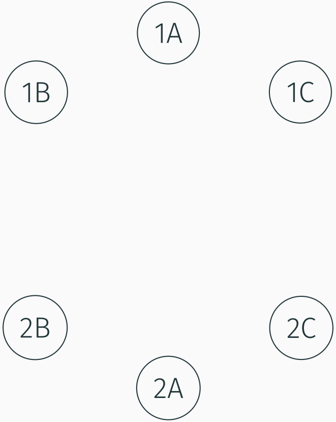
The lumped state space $\hat{\mathscr{X}}$ is a partition of $\mathscr{X}$. $[1 < |\hat{\mathscr{X}}| \ll |\mathscr{X}|]$

The lumping map $\Lambda \colon \mathscr{X} \to \hat{\mathscr{X}}$ maps states to corresponding lumps. [Λ is a surjection]

The *lumped stochastic process* is defined as

$$\hat{X}_t := \Lambda(X_t) \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

original model

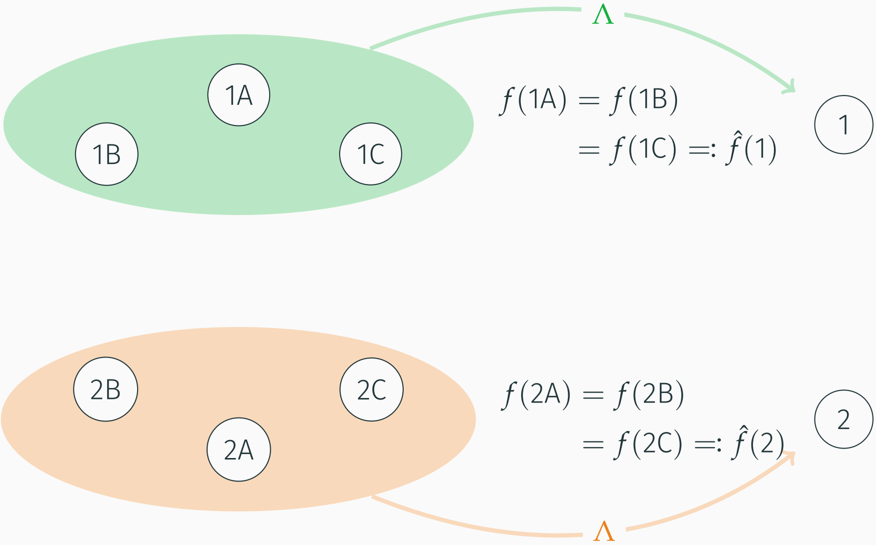


$$\begin{aligned} f(1\mathrm{A}) &= f(1\mathrm{B}) \\ &= f(1\mathrm{C}) \end{aligned}$$

$$\begin{aligned} f(2\mathrm{A}) &= f(2\mathrm{B}) \\ &= f(2\mathrm{C}) \end{aligned}$$

original model

lumped model



$$f(1A) = f(1B) = f(1C) =: \hat{f}(1)$$

$$f(2A) = f(2B) = f(2C) =: \hat{f}(2)$$

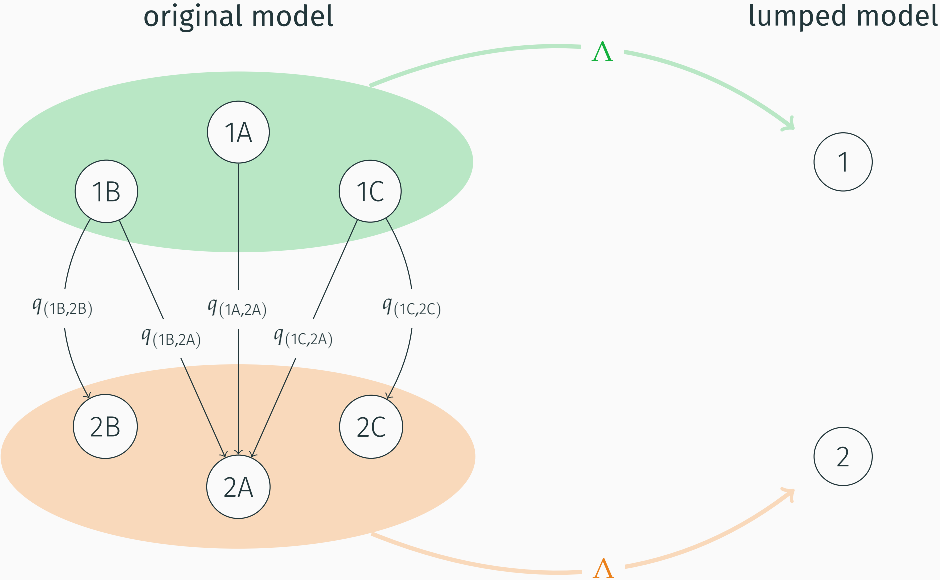
Marginal expectation » A trick with lumping

Fix some $f\colon \mathscr{X} \to \mathbb{R}$ and $\hat{f}\colon \hat{\mathscr{X}} \to \mathbb{R}$ such that $\hat{f} \circ \Lambda = f$.

Then since $\hat{X}_t = \Lambda(X_t)$,

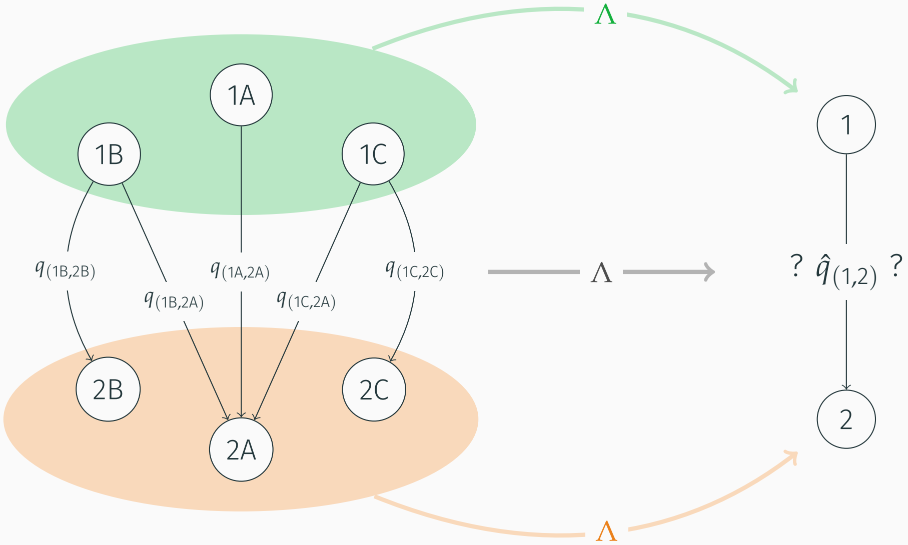
$$E(f(X_t)) = E(\hat{f}(\hat{X}_t)).$$

If we cannot compute $E(f(X_t))$ tractably,
can we compute $E(\hat{f}(\hat{X}_t))$ tractably?


original model
lumped model
Λ
1A
1B
1C
$q_{(1B,2B)}$
$q_{(1B,2A)}$
$q_{(1A,2A)}$
$q_{(1C,2A)}$
$q_{(1C,2C)}$
2B
2A
2C
1
2
Λ

original model

lumped model



The lumped stochastic process

In general, the lumped stochastic process $(\hat{X}_t)_{t\in\mathbb{R}_{\geq 0}}$

- has dynamics that cannot be determined “immediately” from Q and Λ, and
- is not homogeneous nor Markov.

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We therefore consider the *set* $\mathbb{P}_{\pi_0,Q,\Lambda}$ of *consistent stochastic processes*, that

- · contains the lumped stochastic process,
- \+ takes the form of an imprecise continuous-time Markov chain and
- \+ is fully parameterised by

  $\hat{\pi}_0$ the lumped initial distribution, and

  $\hat{\mathscr{Q}}$ the set of *possible* lumped transition rate matrices,

  which in turn are fully defined by π_0, Q and Λ.

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which in turn are fully defined by π_0, Q and Λ.

We can tractably compute lower and upper bounds for $\mathrm{E}(\hat{f}(\hat{X}_t))$.

The lumped stochastic process » Building blocks

Essential to our computations are the lumped initial distribution $\hat{\pi}_0$, given by

$$\hat{\pi}_0(\hat{x}) = \sum_{x \in \Lambda^{-1}(\hat{x})} \pi_0(x) \text{ for all } \hat{x} \in \hat{\mathscr{X}},$$

and the lower transition rate operator $\underline{\hat{Q}}$ that, for all $g \in \mathscr{L}(\hat{\mathscr{X}})$, is given by

$$[\underline{\hat{Q}}g](\hat{x}) = \min\left\{ \sum_{\hat{y} \in \hat{\mathscr{X}}} g(\hat{y}) \sum_{y \in \Lambda^{-1}(\hat{y})} Q(x, y) : x \in \Lambda^{-1}(\hat{x}) \right\} \text{ for all } \hat{x} \in \hat{\mathscr{X}}.$$

[Λ^{-1} is the set-valued inverse of Λ]

Note: in practice, this optimisation usually simplifies considerably

Marginal expectation » Lower and upper bound

Fix some $f\colon \mathscr{X} \to \mathbb{R}$ and $\hat{f}\colon \hat{\mathscr{X}} \to \mathbb{R}$ such that $\hat{f} \circ \Lambda = f$.

Then

$$\hat{\pi}_0 \;\; \underline{\hat{T}}_t \;\; \hat{f} \quad \leq \quad E(\hat{f}(\hat{X}_t)) = E(f(X_t)) = \pi_0 \; T_t \; f$$

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numerical approximation methods are intractable

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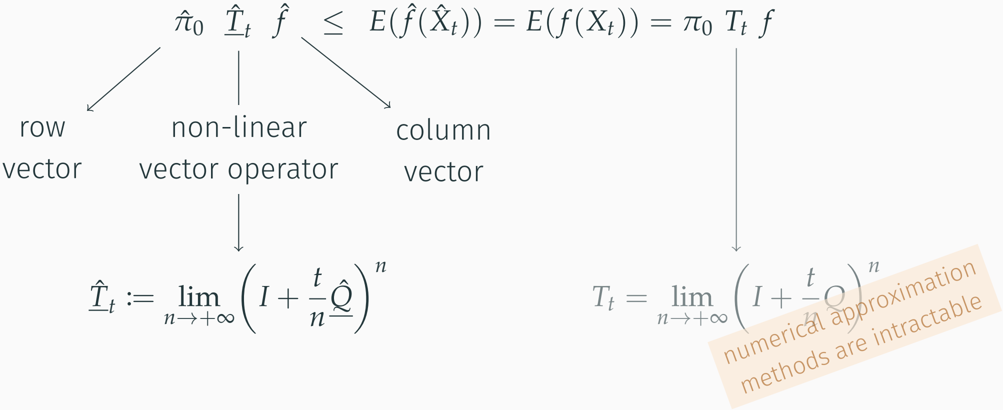

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row vector

non-linear vector operator

column vector

$$\underline{\hat{T}}_t := \lim_{n \to +\infty} \left(I + \frac{t}{n} \underline{\hat{Q}} \right)^n$$

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efficient numerical approximation methods

$$T_t = \lim_{n\to+\infty}\left(I + \frac{t}{n}Q\right)^n$$

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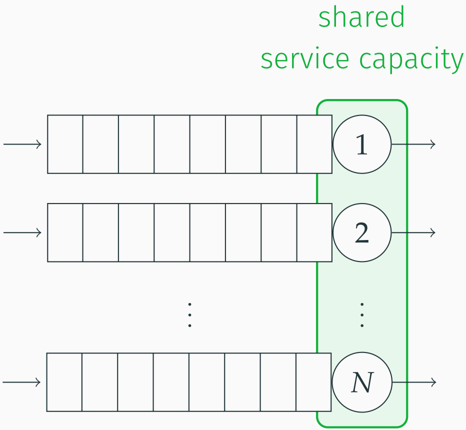
Then

$$\hat{\pi}_0 \; \underline{\hat{T}}_t \; \hat{f} \;\; \leq \;\; E(\hat{f}(\hat{X}_t)) = E(f(X_t)) = \pi_0 \; T_t \; f \;\; \leq \;\; -\hat{\pi}_0 \; \underline{\hat{T}}_t \; (-\hat{f}).$$

conjugacy

A queueing example » Set up

(Cardoen, 2018) studies a system of parallel queues with *processor sharing*.



shared
service capacity

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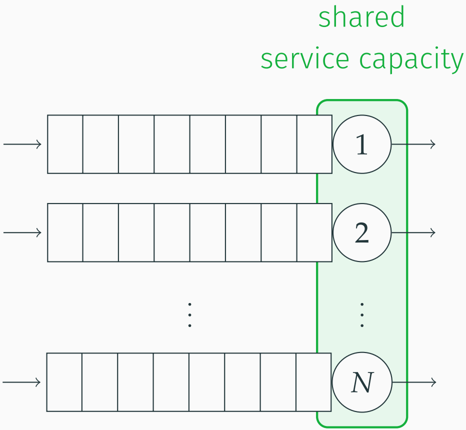

Full system description

- state is $(i_1, i_2, \ldots, i_N)$
- $|\mathscr{X}| = (K+1)^N$

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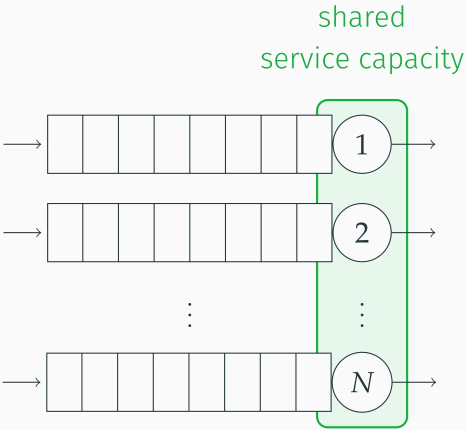

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Only interested in i_1

- $(i_1, i_2, \ldots, i_N)$ contains too much information
- ¿ lumped state space ?

A queueing example » Trade-off between imprecision and computation time

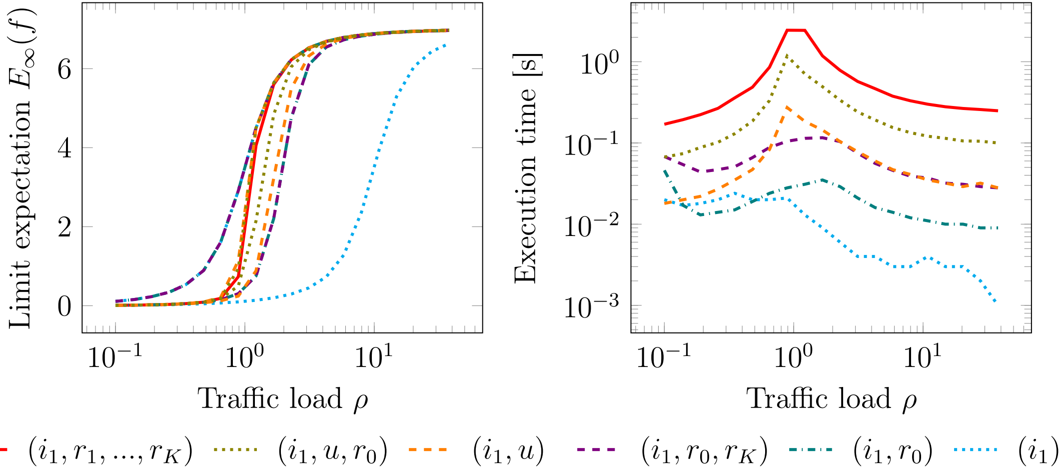

Figure taken from (Cardoen, 2018)

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