First Steps Towards an Imprecise Poisson Process

Alexander Erreygers ALEXANDER.ERREYGERS@UGENT.BE
Jasper De Bock JASPER.DEBOCK@UGENT.BE
ELIS – FLip, Ghent University, Belgium

Abstract

The Poisson process is the most elementary continuous-time stochastic process that models a stream of repeating events. It is uniquely characterised by a single parameter called the rate. Instead of a single value for this rate, we here consider a rate interval and let it characterise two nested sets of stochastic processes. We call these two sets of stochastic process imprecise Poisson processes, explain why this is justified, and study the corresponding lower and upper (conditional) expectations. Besides a general theoretical framework, we also provide practical methods to compute lower and upper (conditional) expectations of functions that depend on the number of events at a single point in time.

Keywords: Poisson process, counting process, continuous-time Markov chain, imprecision

1. Introduction

The *Poisson process* is arguably one of the most basic stochastic processes. At the core of this model is our subject, who is interested in something specific that occurs repeatedly over time, where time is assumed to be continuous. For instance, our subject could be interested in the arrival of a customer to a queue, to give an example from queueing theory. For the sake of brevity, we will call such a specific occurrence a *Poisson-event*,[1] whence our subject is interested in a stream of Poisson-events. The time instants at which subsequent Poisson-events occur are uncertain to our subject, hence the need for a probabilistic model. This set-up is not exclusive to queueing theory; it is also used in renewal theory and reliability theory, to name but a few applications.

There is a plethora of alternative but essentially equivalent characterisations of this Poisson process. Some of the more well-known and basic characterisations are as the limit of the Bernoulli process [5, Chapter VI, Sections 5 and 6] or as a sequence of mutually independent and exponentially distributed inter-Poisson-event times [6, Chapter 5, Section 3.A]. An alternative way to look at the Poisson process is as a random dispersion of points in some general space—that need not be the real number line—see for instance [1, Sections 2.1 and 2.2] or [8, Chapter 2]. More theoretically involved characterisations that are relevant to our set-up are as a counting process or as a continuous-time Markov chain, see for example [5, Chapter XVII, Section 2], [7, Section 1], [10, Section 2.4], [12, Section 2.1] or [13, Section 3].

Broadly speaking, these characterisations all make the same three assumptions: (i) *orderliness*, in the sense that the probability that two or more Poisson-events occur at the same time is zero; (ii) independence, more specifically the absence of after-effects or *Markovianity*; and (iii) *homogeneity*. It is essentially well-known that these three assumptions imply the existence of a parameter called the rate, and that this rate uniquely characterises the Poisson process. We here weaken the three aforementioned assumptions. First and foremost, we get rid of the implicit assumption that our subject's beliefs can be accurately modelled by a single stochastic process; instead, we assume that her beliefs only allow us to consider a *set* of stochastic processes. Specifically, we consider a rate interval instead of a precise value for the rate, and examine two distinct sets: (i) the set of all Poisson processes whose rate belongs to this rate interval; and (ii) the set of all processes that are orderly and "consistent" with the rate interval. We then define lower and upper conditional expectations as the infimum and supremum of the conditional expectations with respect to the stochastic processes in these respective sets. Aside from this general theoretical framework, we focus on computing the lower and upper expectation of functions that depend on the number of occurred Poisson-events at a single future time point. For the set of Poisson processes, we show that this requires the solution of a one-parameter optimisation problem; for the second set, we show that this can be computed using backwards recursion. Furthermore, we argue that both sets can be justifiably called imprecise Poisson processes: imprecise because their lower and upper expectations are not equal, and Poisson because their lower and upper expectations satisfy imprecise versions of the defining properties of the (precise) Poisson process. The interested reader can find proofs for all our results in the Appendix of the extended pre-print of this contribution [4], which is available on arXiv.

1. We use the term "Poisson-event" rather than just "event" to avoid confusion with the standard usage of event in probability theory, where event refers to a subset of the sample space; we are indebted to an anonymous reviewer for pointing out this potential confusion, and to Gert de Cooman for suggesting the adopted terminology.

Our approach is heavily inspired by the theory of imprecise continuous-time Markov chains [9]. For instance, we define the imprecise Poisson process via consistency with a rate interval, whereas Krak et al. [9] use consistency with a set of transition rate matrices. In the bigger picture, our contribution can therefore be seen as the first steps towards generalising the theory of imprecise continuous-time Markov chains from finite to countably infinite state spaces.

2. Counting Processes in General

Recall from the Introduction that our subject is interested in the occurrences of a Poisson-event. In this setting, it makes sense to consider the number of Poisson-events that have occurred from the initial time point $t_{\text{ini}} = 0$ up to a time point t, where t is a non-negative real number.

2.1. Counting Paths and the Sample Space

The temporal evolution of the number of occurred Poisson-events is given by a counting path $\omega\colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$; at any time point t in $\mathbb{R}_{\geq 0}$, $\omega(t)$ is the number of Poisson-events that have occurred from $t_{\text{ini}} = 0$ up to t.[2] Since the actual temporal evolution of the number of occurred Poisson-events is unknown to the subject, we need a probabilistic model, more specifically a continuous-time stochastic process. The sample space—the space of all possible outcomes—of this process is a set of counting paths, denoted by Ω. One popular choice for Ω is the set of all càdlàg—right-continuous with left limits—counting paths, in this set-up usually also assumed to be non-decreasing. However, our results do not require such a strong assumption. Before we state our assumptions on Ω, we first introduce some notation.

In the remainder, we frequently use increasing sequences $t_1, \ldots, t_n$ of time points, that is, sequences $t_1, \ldots, t_n$ in $\mathbb{R}_{\geq 0}$ of arbitrary length—that is, with n in $\mathbb{N}$—such that $t_i < t_{i+1}$ for all i in $\{1, \ldots, n-1\}$. For the sake of brevity, we follow [9, Section 2.1] in denoting such a sequence by u. We collect all increasing—but possibly empty—sequences of time points in $\mathscr{U}$, and let $\mathscr{U}_\emptyset := \mathscr{U} \setminus \{\emptyset\}$. Observe that as a sequence of time points u in $\mathscr{U}$ is just a finite and ordered set of non-negative real numbers, we can perform common set-theoretic operations on them like unions. In order to lighten our notation, we identify the single time point t with a sequence; as such, we can use $u \cup t$ as a notational shorthand for $u \cup \{t\}$. Also, a statement of the form $\max u < t$ is taken to be true if $u = \emptyset$; see for instance Lemma 3. With this convention, for any t in $\mathbb{R}_{\geq 0}$, we let $\mathscr{U}_{<t} := \{u \in \mathscr{U} : \max u < t\}$ be the set of all sequences of time points of which the last time point precedes t. Note that if $t = 0$, then there is no such non-empty sequence and so $\mathscr{U}_{<t} = \{\emptyset\}$.

In order to better distinguish between general non-negative integers and counts, we let $\mathscr{X} := \mathbb{Z}_{\geq 0}$. For any $u = t_1, \ldots, t_n$ in $\mathscr{U}_\emptyset$, we let $\mathscr{X}_u$ be the set of all n-tuples $x_u = (x_{t_1}, \ldots, x_{t_n})$ of non-negative integers that are non-decreasing:

$$\mathscr{X}_u := \{(x_{t_1}, \ldots, x_{t_n}) \in \mathscr{X}^n : x_{t_1} \leq \cdots \leq x_{t_n}\}.$$

If u is the empty sequence $\emptyset$, then we let $\mathscr{X}_u = \mathscr{X}_\emptyset$ denote the singleton containing the empty tuple, denoted by $x_\emptyset$.

With all this notation in place, we can now formally state our requirements on Ω:

A1. $(\forall \omega \in \Omega)(\forall t, \Delta \in \mathbb{R}_{\geq 0})\ \omega(t) \leq \omega(t + \Delta)$;

A2. $(\forall u \in \mathscr{U}_\emptyset)(\forall x_u \in \mathscr{X}_u)(\exists \omega \in \Omega)(\forall t \in u)\ \omega(t) = x_t$.

Assumption (A1) ensures that all paths are non-decreasing, which is essential if we interpret $\omega(t)$ as the number of Poisson-events that have occurred up to time t. Assumption (A2) ensures that the set Ω is sufficiently large, essentially ensuring that the finitary events of Equation (1) further on are non-empty.

2.2. Coherent Conditional Probabilities

We follow Krak et al. [9] in using the framework of coherent conditional probabilities to model the beliefs of our subject. What follows is a brief introduction to coherent conditional probabilities; we refer to [11] and [9, Section 4.1] for a more detailed exposition. For any sample space—that is, a non-empty set—S, we let $\mathscr{E}(S)$ denote the set all events—that is, subsets of S—and let $\mathscr{E}_\emptyset(S) := \mathscr{E}(S) \setminus \{\emptyset\}$ denote the set of all non-empty events. Before we introduce coherent conditional probabilities, we first look at full conditional probabilities.

Definition 1 *Let S be a sample space. A* full conditional probability *P is a real-valued map on $\mathscr{E}(S) \times \mathscr{E}_\emptyset(S)$ such that, for all A, B in $\mathscr{E}(S)$ and C, D in $\mathscr{E}_\emptyset(S)$,*

P1. $P(A \mid C) \geq 0$;

P2. $P(A \mid C) = 1$ *if* $C \subseteq A$;

P3. $P(A \cup B \mid C) = P(A \mid C) + P(B \mid C)$ *if* $A \cap B = \emptyset$;

P4. $P(A \cap D \mid C) = P(A \mid D \cap C) P(D \mid C)$ *if* $D \cap C \neq \emptyset$.

Note that (P1)–(P3) just state that $P(\cdot \mid C)$ is a finitely-additive probability measure, and that (P4) is a multiplicative version of Bayes' rule. We use the adjective *full* because the domain of P is $\mathscr{E}(S) \times \mathscr{E}_\emptyset(S)$. Next, we move to domains that are a subset of $\mathscr{E}(S) \times \mathscr{E}_\emptyset(S)$.

Definition 2 *Let S be a sample space. A coherent conditional probability P is a real-valued map on $\mathscr{D} \subseteq \mathscr{E}(S) \times \mathscr{E}_\emptyset(S)$ that can be extended to a full conditional probability.*

2. We use $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ to denote the non-negative integers and natural numbers (or positive integers), respectively. Furthermore, the real numbers, non-negative real numbers and positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$, respectively.

Important to emphasise here is that simply demanding that (P1)–(P4) hold on the domain $\mathscr{D}$ is in general *not* sufficient to guarantee that P can be extended to a full conditional probability. A necessary and sufficient condition for the existence of such an extension can be found in [11, Theorem 3] or [9, Corollary 4.3], but we refrain from stating it here because of its technical nature. We here only mention that this so-called *coherence* condition—hence explaining the use of the adjective coherent—has an intuitive betting interpretation, and that checking this condition is usually feasible while explicitly constructing the full conditional extension is typically not; this is extremely useful when constructing proofs. Another strong argument for using coherent conditional probabilities is that they can always be extended to a coherent conditional probability on a larger domain [11, Theorem 4]. This too is an essential tool in the proof of many of our main results, including Theorems 6, 15 and 19.

2.3. Events and Fields

For any $v = t_1, \ldots, t_n$ in $\mathscr{U}_\emptyset$ and $B \subseteq \mathscr{X}_v$, we define the *finitary event*

$$(X_v \in B) := \{\omega \in \Omega \colon (\omega(t_1), \ldots, \omega(t_n)) \in B\}. \quad (1)$$

Furthermore, we also let $(X_\emptyset = x_\emptyset) := \Omega =: (X_\emptyset \in \mathscr{X}_\emptyset)$. Then for any u in $\mathscr{U}$, we let $\mathscr{F}_u$ be the field of events—or algebra of sets—generated by the finitary events for all sequences with time points in or succeeding u:

$$\mathscr{F}_u := \langle \{(X_v \in B) \colon v \in \mathscr{U}, B \subseteq \mathscr{X}_v, \\ (\forall t \in v)\ t \in u \cup [\max u, +\infty)\} \rangle. \quad (2)$$

Lemma 3 *Consider some u in $\mathscr{U}$ and A in $\mathscr{F}_u$. Then there is some v in $\mathscr{U}$ with $\min v > \max u$ and some $B \subseteq \mathscr{X}_w$ with $w := u \cup v$ such that $A = (X_w \in B)$.*

2.4. Counting Processes as Coherent Conditional Probabilities

From here on, we focus on coherent conditional probabilities with the domain

$$\mathscr{D}_{\mathrm{CP}} := \{(A, X_u = x_u) \colon u \in \mathscr{U}, A \in \mathscr{F}_u, x_u \in \mathscr{X}_u\},$$

which essentially consists of future events conditional on the number of occurred Poisson-events at specified past time-points. The rationale behind this domain is twofold. First and foremost, it is sufficiently large to make most inferences that one is usually interested in. For example, this domain allows us to compute—tight lower and upper bounds on—the expectation of a real-valued function on the number of occurred Poisson-events at a single future time point, as we will see in Section 6. Second, it guarantees that every rate corresponds to a unique Poisson process, as we will see in Section 3.

Definition 4 *A* counting process *P is a coherent conditional probability on $\mathscr{D}_{\mathrm{CP}}$ such that*

CP1. $P(X_0 = 0) = 1$;

CP2. *for all t in $\mathbb{R}_{\geq 0}$, u in $\mathscr{U}_{<t}$ and (x_u, x) in $\mathscr{X}_{u \cup t}$,*

$$\lim_{\Delta \to 0^+} \frac{P(X_{t+\Delta} \geq x + 2 \,|\, X_u = x_u, X_t = x)}{\Delta} = 0$$

and, if $t > 0$,

$$\lim_{\Delta \to 0^+} \frac{P(X_t \geq x + 2 \,|\, X_u = x_u, X_{t-\Delta} = x)}{\Delta} = 0.$$

The second requirement (CP2) is—our version of—the *orderliness* property that we previously mentioned in the Introduction. In essence, it ensures that the probability that two or more Poisson-events occur at the same time is zero. We collect all counting processes in the set $\mathbb{P}$.

2.5. Conditional Expectation with Respect to a Counting Process

For any counting process P, we let E_P denote the associated (conditional) expectation, defined in the usual sense as an integral with respect to the measure P—see for instance [11, Theorem 6] or [14, Section 15.10.1].

Let $\mathscr{K}_{\mathrm{b}}(\Omega)$ denote the set of all real-valued functions on Ω that are bounded below.[3] Fix some u in $\mathscr{U}$. Then f in $\mathscr{K}_{\mathrm{b}}(\Omega)$ is *$\mathscr{F}_u$-measurable* if for all α in $[\inf f, +\infty)$, the level set $\{f > \alpha\} := \{\omega \in \Omega \colon f(\omega) > \alpha\}$ is an element of $\mathscr{F}_u$. We collect all such $\mathscr{F}_u$-measurable functions in $\mathscr{G}_u$.

The (conditional) expectation E_P has domain

$$\mathscr{G} := \{(f, X_u = x_u) \in \mathscr{K}_{\mathrm{b}}(\Omega) \times \mathscr{E}_\emptyset(\Omega) \colon \\ u \in \mathscr{U}, x_u \in \mathscr{X}_u, f \in \mathscr{G}_u\}.$$

For any $(f, X_u = x_u)$ in $\mathscr{G}$, we have

$$E_P(f \,|\, X_u = x_u) \\ := \inf f + \int_{\inf f}^{\sup f} P(\{f > \alpha\} \,|\, X_u = x_u)\, \mathrm{d}\alpha,$$

where the integral is a—possibly improper—Riemann integral. Note that this integral always exists because $P(\{f > \alpha\} \,|\, X_u = x_u)$ is a non-increasing function of α. This expression simplifies if f is an *$\mathscr{F}_u$-simple function*. To

3. Note that we could just as well consider arbitrary real-valued functions instead of restricting ourselves to bounded-below functions. Our main reason for doing so is that this facilitates a more elegant treatment. Furthermore, many functions of practical interest are bounded-below.

define these, we let $\mathbb{I}_A\colon \Omega \to \mathbb{R}$ denote the indicator of an event $A \subseteq \Omega$, defined for all ω in Ω as $\mathbb{I}_A(\omega) := 1$ if $\omega \in A$ and 0 otherwise. We then say that f is $\mathscr{F}_u$-simple if it can be written as $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$, with n in $\mathbb{N}$ and, for all i in $\{1,\ldots,n\}$, a_i in $\mathbb{R}$ and A_i in $\mathscr{F}_u$. In this case, the integral expression reduces to

$$E_P(f \,|\, X_u = x_u) = \sum_{i=1}^{n} a_i P(A_i \,|\, X_u = x_u). \tag{3}$$

For unconditional expectations, we have that

$$E(\cdot) := E_P(\cdot \,|\, \Omega) = E_P(\cdot \,|\, X_\emptyset = x_\emptyset) = E_P(\cdot \,|\, X_0 = 0),$$

where the final equality holds due to (CP1). Therefore, in the remainder, we can restrict ourselves to expectations of the form $E_P(\cdot \,|\, X_u = x_u, X_t = x)$, as $E(\cdot)$ corresponds to the case $u = \emptyset$, $t = 0$ and $x = 0$.

3. The Poisson Process in Particular

We now turn to the most well-known counting process, namely the Poisson process. As explained in the Introduction, there are plenty of alternative characterisations of the Poisson process. The following definition turns out to capture all its essential properties in our framework.

Definition 5 *A* Poisson process *P is a counting process such that, for all t,Δ in $\mathbb{R}_{\geq 0}$, u in $\mathscr{U}_{<t}$, (x_u, x) in $\mathscr{X}_{u\cup t}$ and y in $\mathscr{X}$ with $y \geq x$,*

PP1. $P(X_{t+\Delta} = y \,|\, X_u = x_u, X_t = x) = P(X_{t+\Delta} = y \,|\, X_t = x)$;

PP2. $P(X_{t+\Delta} = y \,|\, X_t = x) = P(X_{t+\Delta} = y - x \,|\, X_t = 0)$;

PP3. $P(X_{t+\Delta} = y \,|\, X_t = x) = P(X_\Delta = y \,|\, X_0 = x)$.

The first condition (PP1) states that the Poisson process is Markovian, while conditions (PP2) and (PP3) state that the Poisson process is homogeneous. Note that—unlike many of the characterisations mentioned in the Introduction—we do *not* impose that the transition probabilities are Poisson distributed, nor do we impose some value for the "rate". It was already observed by Feller [5, Chapter XVII, Section 2, Footnote 4] and Khintchine [7, Sections 1 and 2] that assuming—their version of—(PP1)–(PP3) is sufficient to obtain the Poisson process. Our results basically extend these characterisations to our framework for counting processes using coherent conditional probabilities.

First and foremost, we obtain that the transition probabilities are Poisson distributed, hence explaining the name of the process.

Theorem 6 *Consider a Poisson process P. Then there is a rate λ in $\mathbb{R}_{\geq 0}$ such that, for all t,Δ in $\mathbb{R}_{\geq 0}$, u in $\mathscr{U}_{<t}$, (x_u, x) in $\mathscr{X}_{u\cup t}$ and y in $\mathscr{X}$,*

$$P(X_{t+\Delta} = y \,|\, X_u = x_u, X_t = x) = \begin{cases} \psi_{\lambda\Delta}(y-x) & \text{if } y \geq x, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $\psi_{\lambda\Delta}$ is the Poisson distribution with parameter $\lambda\Delta$, defined for all k in $\mathbb{Z}_{\geq 0}$ as

$$\psi_{\lambda\Delta}(k) := e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!}.$$

Conversely, for every λ in $\mathbb{R}_{\geq 0}$, there is a unique coherent conditional probability P on $\mathscr{D}_{\mathrm{CP}}$ that satisfies (CP1) and Equation (4), and this P is a Poisson process.

Theorem 6 might seem somewhat trivial, but its proof is surprisingly lengthy. Note that it also establishes that any rate λ gives rise to a unique Poisson process, so in the remainder we can talk of *the* Poisson process with rate λ. Finally, it has the following obvious corollary.

Corollary 7 *Consider a Poisson process P. Then there is a rate λ in $\mathbb{R}_{\geq 0}$ such that, for all t in $\mathbb{R}_{\geq 0}$, u in $\mathscr{U}_{<t}$ and (x_u, x) in $\mathscr{X}_{u\cup t}$,*

$$\lim_{\Delta \to 0^+} \frac{P(X_{t+\Delta} = x+1 \,|\, X_u = x_u, X_t = x)}{\Delta} = \lambda \tag{5}$$

and, if $t > 0$,

$$\lim_{\Delta \to 0^+} \frac{P(X_t = x+1 \,|\, X_u = x_u, X_{t-\Delta} = x)}{\Delta} = \lambda. \tag{6}$$

We end our discussion of Poisson processes with the following result, which actually is a—not entirely immediate—consequence of Theorem 15 further on.

Theorem 8 *Consider a counting process P. If there is a rate λ in $\mathbb{R}_{\geq 0}$ such that P satisfies Equations (5) and (6), then P is the Poisson process with rate λ.*

4. Sets of Counting Processes

Instead of considering a single counting process, we now study *sets* of counting processes. With any subset $\mathscr{P}$ of $\mathbb{P}$, we associate a *lower expectation*

$$\underline{E}_{\mathscr{P}}(\cdot \,|\, \cdot) := \inf\{E_P(\cdot \,|\, \cdot) \colon P \in \mathscr{P}\} \tag{7}$$

and, similarly, an *upper expectation*

$$\overline{E}_{\mathscr{P}}(\cdot \,|\, \cdot) := \sup\{E_P(\cdot \,|\, \cdot) \colon P \in \mathscr{P}\}. \tag{8}$$

Since the expectation E_P associated with any counting process P in $\mathscr{P}$ has domain $\mathscr{G}$, $\underline{E}_{\mathscr{P}}$ and $\overline{E}_{\mathscr{P}}$ are well-defined on the same domain $\mathscr{G}$. Observe that for any $(f, X_u = x_u)$ in $\mathscr{G}$ such that f is bounded, the lower and upper expectations are conjugate in the sense that $\overline{E}_{\mathscr{P}}(f \,|\, X_u = x_u) = -\underline{E}_{\mathscr{P}}(-f \,|\, X_u = x_u)$. Therefore, it suffices to study one of the two if only considering bounded functions; we will focus on lower expectations in the remainder.

4.1. The Obvious Imprecise Poisson Process

From here on, we consider a closed interval $\Lambda := [\underline{\lambda}, \overline{\lambda}] \subset \mathbb{R}_{\geq 0}$ of rates instead of a single value for the rate λ. In order not to unnecessarily repeat ourselves, we fix one rate interval Λ that we use throughout the remainder. Due to Theorem 6, there is one obvious set of counting processes that is entirely characterised by this rate interval Λ: the set

$$\mathbb{P}^{\star}_{\Lambda} := \{P_{\lambda} : \lambda \in \Lambda\}$$

that consists of all Poisson processes with rate in this interval, where P_{λ} denotes the Poisson process with rate λ.

The lower and upper expectation associated with this set $\mathbb{P}^{\star}_{\Lambda}$ according to Equations (7) and (8) are denoted by $\underline{E}^{\star}_{\Lambda}$ and $\overline{E}^{\star}_{\Lambda}$, respectively. It is clear that by construction, determining $\underline{E}^{\star}_{\Lambda}(f \,|\, X_u = x_u)$ and/or $\overline{E}^{\star}_{\Lambda}(f \,|\, X_u = x_u)$ boils down to solving a one-parameter optimisation problem: one has to minimise and/or maximise $E_{P_{\lambda}}(f \,|\, X_u = x_u)$—the conditional expectation of f with respect to the Poisson process with rate λ—with respect to all values of λ in the rate interval Λ. For some specific functions f, see for example Proposition 16 further on, this one-parameter optimisation problem can be solved analytically. For more involved functions, the optimisation problem has to be solved numerically, for instance by evaluating $E_{P_{\lambda}}(f \,|\, X_u = x_u)$ over a (sufficiently fine) grid of values of λ in the rate interval Λ, where $E_{P_{\lambda}}(f \,|\, X_u = x_u)$ might also have to be numerically approximated.

4.2. A More Involved Imprecise Poisson Process

A second set of counting processes characterised by the rate interval Λ is inspired by Theorem 8. This theorem suggests that the dynamics of a counting process are captured by the rate—that is, the limit expressions in Equations (5) and (6) of Corollary 7. Essential to our second characterisation is the notion of consistency.

Definition 9 *A counting process P is* consistent *with the rate interval Λ, denoted by $P \sim \Lambda$, if for all t in $\mathbb{R}_{\geq 0}$, u in $\mathscr{U}_{<t}$ and (x_u, x) in $\mathscr{X}_{u \cup t}$,*

$$\underline{\lambda} \leq \liminf_{\Delta \to 0^+} \frac{P(X_{t+\Delta} = x+1 \,|\, X_u = x_u, X_t = x)}{\Delta} \leq \limsup_{\Delta \to 0^+} \frac{P(X_{t+\Delta} = x+1 \,|\, X_u = x_u, X_t = x)}{\Delta} \leq \overline{\lambda} \quad (9)$$

and, if $t > 0$,

$$\underline{\lambda} \leq \liminf_{\Delta \to 0^+} \frac{P(X_t = x+1 \,|\, X_u = x_u, X_{t-\Delta} = x)}{\Delta} \leq \limsup_{\Delta \to 0^+} \frac{P(X_t = x+1 \,|\, X_u = x_u, X_{t-\Delta} = x)}{\Delta} \leq \overline{\lambda}. \quad (10)$$

We let

$$\mathbb{P}_{\Lambda} := \{P \in \mathbb{P} \colon P \sim \Lambda\}$$

denote the set of all *counting* processes that are consistent with the rate interval Λ. Observe that, as every Poisson process is a counting process,

$$\mathbb{P}^{\star}_{\Lambda} \subseteq \mathbb{P}_{\Lambda}. \quad (11)$$

It is essential to realise that $\mathbb{P}^{\star}_{\Lambda}$ is *not* equal to $\mathbb{P}_{\Lambda}$, at least not in general. Indeed, the set $\mathbb{P}_{\Lambda}$ will contain counting processes that have much more exotic dynamics than Poisson processes, in the sense that they need not be Markovian nor homogeneous. However, if Λ is equal to the degenerate interval $[\lambda, \lambda]$, then it follows from Theorem 8 that

$$\mathbb{P}^{\star}_{\Lambda} = \mathbb{P}_{\Lambda} = \{P_{\lambda}\}, \quad (12)$$

where P_{λ} is the Poisson process with rate λ, as before. Therefore, both $\mathbb{P}_{\Lambda}$ and $\mathbb{P}^{\star}_{\Lambda}$ are proper generalisations of the Poisson process.

We let $\underline{E}_{\Lambda}$ and $\overline{E}_{\Lambda}$ denote the lower and upper expectations associated with the set $\mathbb{P}_{\Lambda}$ according to Equations (7) and (8). It is an immediate consequence of Equations (7), (8) and (11) that

$$\underline{E}_{\Lambda}(\cdot \,|\, \cdot) \leq \underline{E}^{\star}_{\Lambda}(\cdot \,|\, \cdot) \leq \overline{E}^{\star}_{\Lambda}(\cdot \,|\, \cdot) \leq \overline{E}_{\Lambda}(\cdot \,|\, \cdot). \quad (13)$$

The remainder of this contribution is concerned with computing these lower and upper expectations for a specific type of functions, with a particular focus on the outer ones.

We end this section by mentioning that $\mathbb{P}^{\star}_{\Lambda}$ and $\mathbb{P}_{\Lambda}$ are not the only two sets of counting processes that are of potential interest, but they are—to some extent—the two most extreme sets. One set of counting process that lies in between the two is that of the time-inhomogeneous Poisson processes—see for instance [7, Section 5] or [12, Section 2.4]—that are consistent with the rate interval Λ. In order not to unnecessarily complicate our exposition, we have chosen to limit ourselves to the two extreme cases.

5. The Poisson Generator and Its Corresponding Semi-Group

Our method for computing lower expectations is based on the method used in the theory of imprecise continuous-time Markov chains [9]. Essential to this method of Krak et al. [9] is a semi-group of "lower transition operators" that is generated by a "lower transition rate operator". In Section 5.2, we extend their method for generating this semi-group to a countably infinite state space, be it only for one specific type of lower transition rate operator. First, however, we introduce some necessary concepts and terminology.

5.1. Functions, Operators and Norms

Consider some non-empty ordered set $\mathscr{Y}$ that is at most countably infinite, and let $\mathscr{L}(\mathscr{Y})$ be the set of all bounded real-valued functions on $\mathscr{Y}$. Observe that $\mathscr{L}(\mathscr{Y})$ is clearly a vector space. Even more, it is well-known that this vector space is complete under the supremum norm

$$\|f\| := \sup\{|f(x)| : x \in \mathscr{Y}\} \quad \text{for all } f \in \mathscr{L}(\mathscr{Y}).$$

A *transformation* is any operator $A\colon \mathscr{L}(\mathscr{Y}) \to \mathscr{L}(\mathscr{Y})$. Such an operator A is *non-negatively homogeneous* if, for all f in $\mathscr{L}(\mathscr{Y})$ and γ in $\mathbb{R}_{\geq 0}$, $A(\gamma f) = \gamma A f$. The supremum norm induces an operator norm for non-negatively homogeneous transformations A:

$$\|A\| := \sup\{\|Af\| : f \in \mathscr{L}(\mathscr{Y}), \|f\| = 1\};$$

see [2] for a proof that this is indeed a norm. An important non-negatively homogeneous transformation is the *identity map* I that maps any f in $\mathscr{L}(\mathscr{Y})$ to itself.

5.2. The Poisson Generator

A non-negatively homogeneous transformation that will be essential in the remainder is the *Poisson generator* $\underline{Q}\colon \mathscr{L}(\mathscr{X}) \to \mathscr{L}(\mathscr{X})$ associated with the rate interval Λ, defined for all f in $\mathscr{L}(\mathscr{X})$ and x in $\mathscr{X}$ as

$$[\underline{Q}f](x) := \min\{\lambda f(x+1) - \lambda f(x) : \lambda \in [\underline{\lambda}, \overline{\lambda}]\}.$$

Fix any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$. If $t < s$, then we let $\mathscr{U}_{[t,s]}$ denote the set of all non-empty and increasing sequences of time points $t_0, \ldots, t_n$ that start with $t_0 = t$ and end with $t_n = s$. For any sequence u in this set $\mathscr{U}_{[t,s]}$, we let

$$\Phi_u := \prod_{i=1}^{n} (I + \Delta_i \underline{Q}), \tag{14}$$

where for any i in $\{1, \ldots, n\}$, we denote the difference between the consecutive time points t_i and t_{i-1} by $\Delta_i := t_i - t_{i-1}$. In the remainder, we let $\sigma(u) := \max\{\Delta_i : i \in \{1, \ldots, n\}\}$ be the largest of these time differences. If $t = s$, then we let $\mathscr{U}_{[t,s]} := \{t\}$, $\sigma(t) := 0$ and $\Phi_t := I$.

The Poisson generator $\underline{Q}$ generates a family of transformations, as is evident from the following result. This result is very similar to [9, Corollary 7.11], which establishes an analoguous result for imprecise Markov chains with finite state spaces; it should therefore not come as a surprise that their proofs are largely similar as well.

Theorem 10 *Fix any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$. For any sequence $\{u_i\}_{i \in \mathbb{N}}$ in $\mathscr{U}_{[t,s]}$ such that $\lim_{i \to +\infty} \sigma(u_i) = 0$, the corresponding sequence $\{\Phi_{u_i}\}_{i \in \mathbb{N}}$ converges to a unique non-negatively homogeneous transformation that does not depend on the chosen sequence $\{u_i\}_{i \in \mathbb{N}}$.*

For any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$, Theorem 10 allows us to define the non-negatively homogeneous transformation

$$\underline{T}_t^s := \lim_{\sigma(u) \to 0} \{\Phi_u : u \in \mathscr{U}_{[t,s]}\}, \tag{15}$$

where this unconventional notation for the limit denotes the unique limit mentioned in Theorem 10. The family of transformations thus defined has some very interesting properties: in the extended pre-print [4], we prove that for any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$, f,g in $\mathscr{L}(\mathscr{X})$ and γ in $\mathbb{R}_{\geq 0}$,

SG1. $\underline{T}_t^s(\gamma f) = \gamma \underline{T}_t^s f$;

SG2. $\underline{T}_t^s(f+g) \geq \underline{T}_t^s f + \underline{T}_t^s g$;

SG3. $\underline{T}_t^s f \geq \inf f$.

We furthermore prove that this family forms a time-homogeneous semi-group, in the sense that

SG4. $\underline{T}_t^t = I$;

SG5. $\underline{T}_t^s = \underline{T}_t^r \underline{T}_r^s$ for all r in $\mathbb{R}_{\geq 0}$ with $t \leq r \leq s$;

SG6. $\underline{T}_t^s = \underline{T}_0^{s-t}$.

While the induced transformation $\underline{T}_t^s$ is interesting in its own right, we will be mainly interested in (a single component of) the image $\underline{T}_t^s f$ of some bounded function f. Therefore, for any x in $\mathscr{X}$ and t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$, we define the operator $\underline{P}_t^s(\cdot \,|\, x)\colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$ as

$$\underline{P}_t^s(f \,|\, x) := [\underline{T}_t^s f](x) \qquad \text{for all } f \in \mathscr{L}(\mathscr{X}).$$

The following follows immediately from (SG1)–(SG3).

Corollary 11 *For any x in $\mathscr{X}$ and t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$, $\underline{P}_t^s(\cdot \,|\, x)$ is a coherent lower prevision in the sense of [14, Definition 4.10].*

In the remainder, we let $\overline{P}_t^s(\cdot \,|\, x) := -\underline{P}_t^s(-\cdot \,|\, x)$ be the conjugate coherent upper prevision of the coherent lower prevision $\underline{P}_t^s(\cdot \,|\, x)$.

5.3. The Reduced Poisson Generator

Fix any $\underline{x}, \overline{x}$ in $\mathscr{X}$ such that $\underline{x} \leq \overline{x}$, and let

$$\chi := \{x \in \mathscr{X} : \underline{x} \leq x \leq \overline{x}\}.$$

We define the *reduced Poisson generator* $\underline{Q}^{\chi}\colon \mathscr{L}(\chi) \to \mathscr{L}(\chi)$ for all g in $\mathscr{L}(\chi)$ and x in χ as

$$[\underline{Q}^{\chi} g](x) := \begin{cases} \min\limits_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} (\lambda g(x+1) - \lambda g(x)) & \text{if } \underline{x} \leq x < \overline{x}, \\ 0 & \text{if } x = \overline{x}. \end{cases}$$

In the extended pre-print, we verify that this reduced Poisson generator $\underline{Q}^{\chi}$ is a *lower transition rate operator* in the sense of [9, Definition 7.2]. As outlined in [9, Section 7], this lower transition rate operator generates a family of transformations as well. For any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$ and any u in $\mathscr{U}_{[t,s]}$, we let

$$\Phi_u^{\chi} := \prod_{i=1}^{n} (I + \Delta_i \underline{Q}^{\chi}).$$

Note the similarity between the equation above and Equation (14). Because $\underline{Q}^{\chi}$ is a lower transition rate operator, it follows from [9, Corollary 7.11]—a result similar to Theorem 10—that the transformation

$$\underline{T}_{t,s}^{\chi} := \lim_{\sigma(u)\to 0}\{\Phi_u^{\chi} : u \in \mathscr{U}_{[t,s]}\} \tag{16}$$

is non-negatively homogeneous. The limit in this definition is to be interpreted as the limit in Equation (15): it does not depend on the actual sequence $\{u_i\}_{i\in\mathbb{N}}$ as long as $\lim_{i\to+\infty}\sigma(u_i) = 0$. Unsurprisingly, Krak et al. [9] show that this family of transformations $\underline{T}_{t,s}^{\chi}$ also satisfies (SG1)–(SG6). Observe that Equation (16) suggests a method to evaluate $\underline{T}_{t,s}^{\chi}$ for some g in $\mathscr{L}(\chi)$: choose a sufficiently fine grid u, and compute $\Phi_u^{\chi} g$ via backwards recursion. There is much more to this approximation method than we can cover here; the interested reader is referred to [9, Section 8.2] and [3].

5.4. The Essential Case of Eventually Constant Functions

Our reason for introducing the restricted Poisson generator $\underline{Q}^{\chi}$ and its induced transformation $\underline{T}_{t,s}^{\chi}$ is because the latter can be used to compute $\underline{P}_t^s(f\,|\,x)$. Essential to our method are those functions f in $\mathscr{L}(\mathscr{X})$ that are *eventually constant*, in the sense that

$$(\exists \overline{x} \in \mathscr{X})(\forall x \in \mathscr{X}, x \geq \overline{x})\ f(x) = f(\overline{x}).$$

In this case, we say that f is *constant starting from* $\overline{x}$. We collect all real-valued bounded functions f on $\mathscr{X}$ that are eventually constant in $\mathscr{L}^{c}(\mathscr{X})$.

Our next result establishes a link between $\underline{P}_t^s(\cdot\,|\,x)$ and $\underline{T}_{t,s}^{\chi}$ for eventually constant functions.

Proposition 12 *Fix some t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$ and some f in $\mathscr{L}^{c}(\mathscr{X})$ that is constant starting from $\overline{x}$. Choose some $\underline{x}$ in $\mathscr{X}$ with $\underline{x} \leq \overline{x}$, and let $\chi := \{x \in \mathscr{X} : \underline{x} \leq x \leq \overline{x}\}$. Then for any x in $\mathscr{X}$ with $x \geq \underline{x}$,*

$$\underline{P}_t^s(f\,|\,x) = [\underline{T}_t^s f](x) = \begin{cases} [\underline{T}_{t,s}^{\chi} f^{\chi}](x) & \text{if } x \leq \overline{x}, \\ f(\overline{x}) & \text{if } x \geq \overline{x}, \end{cases}$$

where f^{χ} is the restriction of f to χ.

Note that we are free to choose $\underline{x}$. If we are interested in $\underline{P}_t^s(f\,|\,x)$ for a specific value of x, then choosing $\underline{x} = \min\{\overline{x}, x\}$ is the optimal choice. However, if we are interested in $\underline{P}_t^s(f\,|\,x)$ for a finite range $R \subset \mathscr{X}$ of different x values, the obvious choice is $\underline{x} = \min(R \cup \{\overline{x}\})$ because we then only have to determine $\underline{T}_{t,s}^{\chi} f^{\chi}$ once!

A method to compute $\underline{P}_t^s(\cdot\,|\,x)$ for all bounded functions f follows from combining Proposition 12 with the following result.

Proposition 13 *For any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$, f in $\mathscr{L}(\mathscr{X})$ and x in $\mathscr{X}$,*

$$\underline{P}_t^s(f\,|\,x) = \lim_{\overline{x}\to+\infty} \underline{P}_t^s(\mathbb{I}_{\leq\overline{x}} f + f(\overline{x})\mathbb{I}_{>\overline{x}}\,|\,x),$$

where $\mathbb{I}_{\leq\overline{x}}$ and $\mathbb{I}_{>\overline{x}}$ are the indicators of $\{z \in \mathscr{X} : z \leq \overline{x}\}$ and $\{z \in \mathscr{X} : z > \overline{x}\}$, respectively.

Observe that $\mathbb{I}_{\leq\overline{x}} f + f(\overline{x})\mathbb{I}_{>\overline{x}}$—with $\mathbb{I}_{\leq\overline{x}} f$ the point-wise multiplication of $\mathbb{I}_{\leq\overline{x}}$ and f—is constant starting from $\overline{x}$. Therefore, it follows from Proposition 12 that $\underline{P}_t^s(\mathbb{I}_{\leq\overline{x}} f + f(\overline{x})\mathbb{I}_{>\overline{x}}\,|\,x) = [\underline{T}_{t,s}^{\chi} f^{\chi}](x)$, where f^{χ} is the restriction of f to χ. We can combine this observation and Proposition 13 to obtain a method to compute $\underline{P}_t^s(f\,|\,x)$ for any bounded function f: (i) choose some sufficiently large $\overline{x}$ and let $\chi := \{y \in \mathscr{X} : x \leq y \leq \overline{x}\}$; (ii) compute $\underline{P}_t^s(\mathbb{I}_{\leq\overline{x}} f + f(\overline{x})\mathbb{I}_{>\overline{x}}\,|\,x) = [\underline{T}_{t,s}^{\chi} f^{\chi}](x)$, using one of the existing approximation methods mentioned at the end of Section 5.3; (iii) repeat (i)–(ii) for increasingly larger $\overline{x}$ until convergence is empirically observed.

6. Computing Lower Expectations of Functions on X_s

Let $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$ denote the set of all real-valued bounded-below functions on $\mathscr{X}$. With any f in $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$ and s in $\mathbb{R}_{\geq 0}$, we associate the real-valued bounded-below function

$$f(X_s) \colon \Omega \to \mathbb{R} \colon \omega \mapsto [f(X_s)](\omega) := f(\omega(s)).$$

In other words, and as suggested by our notation, $f(X_s)$ is the functional composition of f with the projector

$$X_s \colon \Omega \to \mathscr{X} \colon \omega \mapsto X_s(\omega) := \omega(s).$$

The (conditional) expectation of $f(X_s)$ exists for any counting process P, as is established by the following rather obvious result.

Lemma 14 *Consider some s in $\mathbb{R}_{\geq 0}$ and u in $\mathscr{U}$ with $\max u \leq s$. Then for any f in $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$, $f(X_s)$ is an $\mathscr{F}_u$-measurable function.*

In the remainder, we provide several methods for computing lower and upper expectations; first for those with respect to the consistent Poisson processes and second for those with respect to all consistent counting processes. For the latter, we first limit ourselves to bounded functions and subsequently move on to functions that are bounded-below.

6.1. With Respect to the Consistent Poisson Processes

Fix some rate λ in $\mathbb{R}_{\geq 0}$, and let P be the Poisson process with rate λ. It is essentially well-known—and a consequence of Theorem 6—that for any t,s in $\mathbb{R}_{\geq 0}$ with $t \leq s$, u in $\mathscr{U}_{<t}$, (x_u, x) in $\mathscr{X}_{u\cup t}$ and f in $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$,

$$E_P(f(X_s)\,|\,X_u = x_u, X_t = x) = \sum_{y=x}^{+\infty} f(y)\psi_{\lambda(s-t)}(y-x). \tag{17}$$

Because of this expression, $\underline{E}^{\star}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)$ can be computed using the straightforward method that we already discussed in Section 4.1: (i) fix a finite grid of λ's in $\Lambda=[\underline{\lambda},\overline{\lambda}]$, (ii) (numerically) evaluate the infinite sum in Equation (17) for each λ in this grid, and (iii) compute the minimum. In some specific cases, it is even possible to know beforehand for which λ this minimum will be achieved. For example, if f is monotone and bounded, or bounded below and non-decreasing, then as we will see in Propositions 16 and 17, it suffices to consider $\lambda=\underline{\lambda}$ or $\lambda=\overline{\lambda}$.

6.2. With Respect to the Consistent Counting Processes

Computing $\underline{E}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)$ is less straightforward, as in general this does not reduce to a one-parameter optimisation problem. Nevertheless, as we are about to show, the semi-group of Section 5 allows us to circumvent this issue. Our first result establishes a method to compute the lower—and hence also the upper—expectation of bounded functions.

Theorem 15 *For any t,s in $\mathbb{R}_{\geq 0}$ with $t\leq s$, u in $\mathscr{U}_{<t}$, f in $\mathscr{L}(\mathscr{X})$ and (x_u,x) in $\mathscr{X}_{u\cup t}$,*

$$\underline{E}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)=\underline{P}_t^s(f\,|\,x).$$

Indeed, because of this result, we can use the method that was introduced at the end of Section 5.4 to compute the lower expectation of f.

For the special case of monotone bounded functions, we obtain an even stronger result.

Proposition 16 *Fix any t,s in $\mathbb{R}_{\geq 0}$ with $t\leq s$, u in $\mathscr{U}_{<t}$, (x_u,x) in $\mathscr{X}_{u\cup t}$ and f in $\mathscr{L}(\mathscr{X})$. If f is monotone, then*

$$\begin{aligned}\underline{E}_{\Lambda}&(f(X_s)\,|\,X_u=x_u,X_t=x)\\&=\underline{E}^{\star}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)\\&\qquad=E_{P_\lambda}(f(X_s)\,|\,X_u=x_u,X_t=x),\end{aligned}$$

where P_λ is the Poisson process with rate $\lambda=\underline{\lambda}$ if f is non-decreasing and rate $\lambda=\overline{\lambda}$ if f is non-increasing.

Almost everything has now been set up to consider a general real-valued bounded below function of X_s. An essential intermediary step is an extension of Proposition 16.

Proposition 17 *Fix any t,s in $\mathbb{R}_{\geq 0}$ with $t\leq s$, u in $\mathscr{U}_{<t}$ and (x_u,x) in $\mathscr{X}_{u\cup t}$. Then for any f in $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$ that is non-decreasing,*

$$\begin{aligned}\underline{E}_{\Lambda}&(f(X_s)\,|\,X_u=x_u,X_t=x)\\&=\underline{E}^{\star}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)\\&\qquad=E_{P_{\underline{\lambda}}}(f(X_s)\,|\,X_u=x_u,X_t=x)\end{aligned}$$

and

$$\begin{aligned}\overline{E}_{\Lambda}&(f(X_s)\,|\,X_u=x_u,X_t=x)\\&=\overline{E}^{\star}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)\\&\qquad=E_{P_{\overline{\lambda}}}(f(X_s)\,|\,X_u=x_u,X_t=x),\end{aligned}$$

where $P_{\underline{\lambda}}$ and $P_{\overline{\lambda}}$ are the Poisson processes with rates $\underline{\lambda}$ and $\overline{\lambda}$, respectively.

As an immediate corollary of Proposition 17, we obtain an interpretation for the rate interval Λ: its bounds provide tight lower and upper bounds on the expected number of Poisson-events in any time period.

Corollary 18 *Fix any t,s in $\mathbb{R}_{\geq 0}$ with $t\leq s$, u in $\mathscr{U}_{<t}$ and (x_u,x) in $\mathscr{X}_{u\cup t}$. Then*

$$\underline{E}_{\Lambda}(X_s\,|\,X_u=x_u,X_t=x)=x+\underline{\lambda}(s-t)$$

and

$$\overline{E}_{\Lambda}(X_s\,|\,X_u=x_u,X_t=x)=x+\overline{\lambda}(s-t),$$

and similarly for $\underline{E}^{\star}_{\Lambda}$ and $\overline{E}^{\star}_{\Lambda}$.

A more important consequence of Proposition 17 is the following result, which can be regarded as an extension of (the combination of) Proposition 13 and Theorem 15.

Theorem 19 *Fix any t,s in $\mathbb{R}_{\geq 0}$ with $t\leq s$, u in $\mathscr{U}_{<t}$, (x_u,x) in $\mathscr{X}_{u\cup t}$ and f in $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$. If*

$$\sum_{y=x}^{+\infty}f_{\max}(y)\psi_{\overline{\lambda}(s-t)}(y-x)<+\infty,$$

where $f_{\max}$ in $\mathscr{K}_{\mathrm{b}}(\mathscr{X})$ is defined for all y in $\mathscr{X}$ as

$$f_{\max}(y):=\max\{f(z)\colon z\in\mathscr{X},z\leq y\},$$

then

$$\underline{E}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)=\lim_{\overline{x}\to+\infty}\underline{P}_t^s(\mathbb{I}_{\leq\overline{x}}f+f(\overline{x})\mathbb{I}_{>\overline{x}}\,|\,x),$$

$$\overline{E}_{\Lambda}(f(X_s)\,|\,X_u=x_u,X_t=x)=\lim_{\overline{x}\to+\infty}\overline{P}_t^s(\mathbb{I}_{\leq\overline{x}}f+f(\overline{x})\mathbb{I}_{>\overline{x}}\,|\,x),$$

where the two limits are finite.

Because of this result, we can compute the lower and upper expectation using the same method as before. Note that it makes no difference that f is no longer bounded; the method still works because $\mathbb{I}_{\leq\overline{x}}f+f(\overline{x})\mathbb{I}_{>\overline{x}}$ is bounded.

6.3. A Numerical Example

We end this section with a basic numerical example. We determine tight lower and upper bounds on

$$P(X_t=x\,|\,X_0=0)=E_P(\mathbb{I}_x(X_t)\,|\,X_0=0),$$

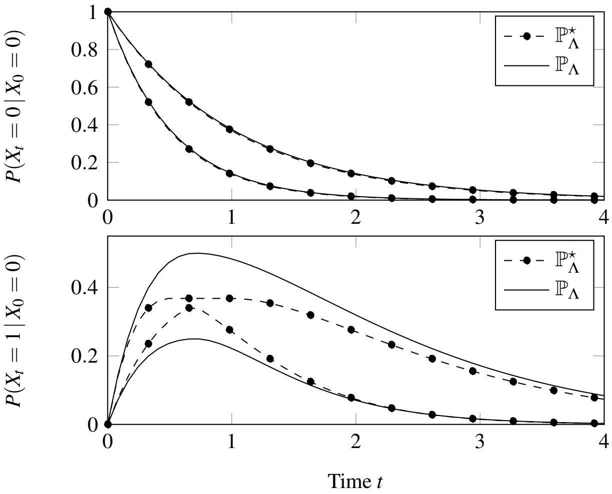

Figure 1: Bounds on transition probabilities as a function of t for the rate interval $\Lambda = [1,2]$.

with x equal to 0 or 1. We use the methods outlined in Sections 6.1 and 6.2 to compute lower and upper bounds with respect to the sets $\mathbb{P}^\star_\Lambda$ and $\mathbb{P}_\Lambda$ for $\Lambda = [1,2]$. The resulting bounds are depicted in Figure 1. Observe that for $x = 0$, the bounds with respect to $\mathbb{P}^\star_\Lambda$ and $\mathbb{P}_\Lambda$ are equal, as is to be expected due to Proposition 16 because $\mathbb{I}_x$ is monotone for $x = 0$. For $x = 1$, $\mathbb{I}_x$ is not monotone and the bounds with respect to $\mathbb{P}^\star_\Lambda$ are clearly *not* equal to those with respect to $\mathbb{P}_\Lambda$.

7. Justification for the Term Imprecise Poisson Process

Until now, we have provided little justification for why we call both $\mathbb{P}^\star_\Lambda$ and $\mathbb{P}_\Lambda$ imprecise Poisson processes. In Section 4.2, we already briefly mentioned that the two sets are proper generalisations of the Poisson process: if the rate interval Λ is degenerate, meaning that $\underline{\lambda} = \lambda = \overline{\lambda}$, then both sets reduce to the singleton containing the Poisson process with rate λ. Another argument for referring to $\mathbb{P}^\star_\Lambda$ and $\mathbb{P}_\Lambda$ as imprecise Poisson processes concerns the (tight lower and upper bounds on the) expected number of Poisson events in a time period of length Δ. For a Poisson process, it is well-known that this expectation is equal to $\Delta\lambda$, and we know from Corollary 18 that the corresponding lower and upper expectations are equal to $\Delta\underline{\lambda}$ and $\Delta\overline{\lambda}$, respectively.

We end this section with our strongest argument for using the term imprecise Poisson process to refer to both $\mathbb{P}^\star_\Lambda$ and $\mathbb{P}^\star_\Lambda$. The following result establishes that the corresponding lower expectations $\underline{E}^\star_\Lambda$ and $\underline{E}_\Lambda$—and, due to conjugacy, also the corresponding upper expectations $\overline{E}^\star_\Lambda$ and $\overline{E}_\Lambda$—satisfy imprecise generalisations of (CP1), (CP2) and (PP1)–(PP3), which are the defining properties of a Poisson process.

Proposition 20 *For all t, Δ in $\mathbb{R}_{\geq 0}$, u in $\mathscr{U}_{<t}$, (x_u, x) in $\mathscr{X}_{u\cup t}$ and f in $\mathscr{L}(\mathscr{X})$,*

(i) $\underline{E}_\Lambda(f(X_0)) = f(0)$;

(ii)
$$\lim_{\Delta \to 0^+} \frac{\underline{E}_\Lambda(\mathbb{I}_{(X_{t+\Delta} \geq x+2)} \mid X_u = x_u, X_t = x)}{\Delta} = 0$$
and, if $t > 0$,
$$\lim_{\Delta \to 0^+} \frac{\underline{E}_\Lambda(\mathbb{I}_{(X_t \geq x+2)} \mid X_u = x_u, X_{t-\Delta} = x)}{\Delta} = 0;$$

(iii) $\underline{E}_\Lambda(f(X_{t+\Delta}) \mid X_u = x_u, X_t = x) = \underline{E}_\Lambda(f(X_{t+\Delta}) \mid X_t = x)$;

(iv) $\underline{E}_\Lambda(f(X_{t+\Delta}) \mid X_t = x) = \underline{E}_\Lambda(f'_x(X_{t+\Delta}) \mid X_t = 0)$;

(v) $\underline{E}_\Lambda(f(X_{t+\Delta}) \mid X_t = x) = \underline{E}_\Lambda(f(X_\Delta) \mid X_0 = x)$;

with $f'_x \colon \mathscr{X} \to \mathbb{R} \colon z \mapsto f'_x(z) := f(x+z)$. The same equalities also hold for $\underline{E}^\star_\Lambda$.

8. Conclusion

In this contribution, we proposed two generalisations of the Poisson process in the form of two sets of counting processes: the set $\mathbb{P}^\star_\Lambda$ of all Poisson processes with rate λ in the rate interval Λ, and the set $\mathbb{P}_\Lambda$ of all counting process that are consistent with the rate interval Λ. We argued why both of these sets can be seen as proper generalisations of the Poisson process. First and foremost, for a degenerate rate interval they both reduce to the singleton containing the Poisson process with this rate. Second, the lower and upper expectations with respect to both sets satisfy imprecise generalisations of (CP1), (CP2) and (PP1)–(PP3), the defining properties of a Poisson process. We also presented several methods for computing lower and upper expectations for functions that depend on the number of occurred Poisson-events at a single time point.

We end with two suggestions for future research. An obvious open question is whether we can efficiently compute lower and upper expectations for functions that depend on the number of occurred Poisson-events at multiple points in time. Based on similar results of Krak et al. [9] for imprecise continuous-time Markov chains with a finite state space, we strongly believe that this will be the case for $\mathbb{P}_\Lambda$ but not for $\mathbb{P}^\star_\Lambda$, whence providing a practical argument in favour of the former. A perhaps slightly less obvious open question is whether Theorem 6 and Corollary 7 can be generalised to sets of counting processes, in the sense that we can infer the existence of a rate interval rather than specify one, by imposing appropriate conditions on the set of counting processes, including the imprecise generalisations of (CP1), (CP2) and (PP1)–(PP3).

Acknowledgments

We would like to express our gratitude to the three anonymous reviewers for their valuable time, constructive feedback and sound suggestions for improvements. Jasper De Bock's research was partially funded by the European Commission's H2020 programme, through the UTOPIAE Marie Curie Innovative Training Network, H2020-MSCA-ITN-2016, Grant Agreement number 722734.

References

[1] Daryl J. Daley and David Vere-Jones. *An Introduction to the Theory of Point Processes*, volume I: Elementary Theory and Methods. Springer, 2003.

[2] Jasper De Bock. The limit behaviour of imprecise continuous-time Markov chains. *Journal of Nonlinear Science*, 27(1):159–196, 2017. doi: 10.1007/s00332-016-9328-3.

[3] Alexander Erreygers and Jasper De Bock. Imprecise continuous-time Markov chains: Efficient computational methods with guaranteed error bounds. In *Proceedings of ISIPTA'17*, pages 145–156. PMLR, 2017.

[4] Alexander Erreygers and Jasper De Bock. First steps towards an imprecise poisson process. Extended preprint available on arXiv:1905.05734, 2019.

[5] William Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. Wiley, 1968.

[6] Samuel Karlin and Howard M. Taylor. *A first course in stochastic processes*. Academic Press, 1975.

[7] Aleksandr Yakovlevich Khintchine. *Mathematical methods in the theory of queueing*. Charles Griffin & Company, 1960.

[8] John F.C. Kingman. *Poisson Processes*. Clarendon Press, 1993.

[9] Thomas Krak, Jasper De Bock, and Arno Siebes. Imprecise continuous-time Markov chains. *International Journal of Approximate Reasoning*, 88:452–528, 2017. doi: 10.1016/j.ijar.2017.06.012.

[10] James R. Norris. *Markov chains*. Cambridge University Press, 1997.

[11] Eugenio Regazzini. Finitely additive conditional probabilities. *Rendiconti del Seminario Matematico e Fisico di Milano*, 55(1):69–89, 1985. doi: 10.1007/BF02924866.

[12] Sheldon M. Ross. *Stochastic Processes*. Wiley, 1996.

[13] Ke-iti Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, 1999.

[14] Matthias C.M. Troffaes and Gert de Cooman. *Lower Previsions*. Wiley, 2014.