First steps towards an imprecise Poisson process

Alexander Erreygers & Jasper De Bock // Foundations Lab for Imprecise Probabilities

GHENT UNIVERSITY

Poisson-events

We are interested in the repeated occurrences of a **Poisson-event** over time, but the exact time instants of these occurrences are uncertain to us; for example, the arrival of a customer to some queue.

For every time instant t, we let X_t be the number of Poisson-events that have occurred up to t; hence, X_t is non-decreasing with t.

In general, we model our beliefs by specifying the transition probabilities

$$P(X_{t+\Delta} = y \mid X_t = x, \underbrace{X_{t_n} = x_n, \dots, X_{t_1} = x_1}_{X_u = x_u}),$$

where $t_1, \dots, t_n, t$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ and $x_1, \dots, x_n, x$ is a non-decreasing sequence in $\mathbb{Z}_{\geq 0}$.

Counting processes in general

For a **counting process**, we assume that

CP1. we start at zero:

$$P(X_0 = 0) = 1;$$

CP2. two Poisson-events can not occur at the same time:

$$P(X_{t+\Delta} \geq x+2 \mid X_t = x, X_u = x_u) = o(\Delta).$$

The Poisson process in particular

For a **Poisson process**, one additionally assumes that the transition probabilities

PP1. are Markov:

$$P(X_{t+\Delta} = y \mid X_t = x, X_u = x_u) = P(X_{t+\Delta} = y \mid X_t = x);$$

PP2. only depend on the length of the time interval:

$$P(X_{t+\Delta} = y \mid X_t = x) = P(X_\Delta = y \mid X_0 = x);$$

PP3. only depend on the number of occurred events in the time interval:

$$P(X_\Delta = y \mid X_0 = x) = P(X_\Delta = y - x \mid X_0 = 0).$$

It is well-known that a Poisson process is uniquely characterised by a single parameter: the **rate** λ.

In particular, the transition probabilities are given by the Poisson distribution with parameter $\lambda\Delta$, which explains the name.

Hence, the expected number of Poisson-events in any time-period is proportional to λ:

$$E_P(X_{t+\Delta} \mid X_t = x, X_u = x_u) = x + \lambda\Delta.$$

Furthermore, λ is the rate at which the probability that a single Poisson-event occurs in a time interval increases with the length of this time interval:

$$P(X_{t+\Delta} = x+1 \mid X_t = x, X_u = x_u) = \lambda\Delta + o(\Delta).$$

What if we only know that the rate λ belongs to the rate interval $[\underline{\lambda}, \overline{\lambda}]$?

Set of consistent counting processes

Another option is to consider the set $\mathscr{P}_{\mathrm{CP}}$ of all *counting* processes P that are **consistent** with the rate interval $[\underline{\lambda}, \overline{\lambda}]$, in the sense that

$$\underline{\lambda}\Delta + o(\Delta) \leq P(X_{t+\Delta} = x+1 \mid X_t = x, X_u = x_u) \leq \overline{\lambda}\Delta + o(\Delta).$$

As every Poisson process is a counting process, this set is more general than the set of Poisson processes:

$$\mathscr{P}_{\mathrm{PP}} \subseteq \mathscr{P}_{\mathrm{CP}};$$

this inclusion is in fact strict!

We let $\underline{E}_{\mathrm{CP}}(\cdot \mid \cdot)$ denote the lower envelope of the expectations $E_P(\cdot \mid \cdot)$ with respect to all P in $\mathscr{P}_{\mathrm{CP}}$. Then clearly,

$$\underline{E}_{\mathrm{CP}}(\cdot \mid \cdot) \leq \underline{E}_{\mathrm{PP}}(\cdot \mid \cdot) \leq \overline{E}_{\mathrm{PP}}(\cdot \mid \cdot) \leq \overline{E}_{\mathrm{CP}}(\cdot \mid \cdot).$$

At first sight, computing the lower expectation $\underline{E}_{\mathrm{CP}}$ requires the explicit construction of and subsequent optimisation over the set $\mathscr{P}_{\mathrm{CP}}$; a *non-trivial* optimisation problem!

However, we show that

$$\underline{E}_{\mathrm{CP}}(f(X_{t+\Delta}) \mid X_t = x, X_u = x_u) = [\underline{T}_\Delta f](x),$$

a tractable optimisation problem!

From this, it follows that—quite remarkably—the lower expectation $\underline{E}_{\mathrm{CP}}(\cdot \mid \cdot)$ satisfies the imprecise versions of (PP1)–(PP3) as well as Equations (1) and (2), just like $\underline{E}_{\mathrm{PP}}(\cdot \mid \cdot)$.

Let $\mathscr{L}$ be the real vector space of all bounded real-valued functions on $\mathbb{Z}_{\geq 0}$. Essential to our approach is the **generator** $\underline{Q}\colon \mathscr{L} \to \mathscr{L}$, defined as

$$[\underline{Q}f](x) := \min_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \lambda f(x+1) - \lambda f(x).$$

We show that

$$\Phi_{\Delta,n} := \left(I + \frac{\Delta}{n}\underline{Q}\right)^n$$

converges to a transformation on $\mathscr{L}$ in the limit for $n \to +\infty$. Hence, we can define

$$\underline{T}_\Delta := \lim_{n \to +\infty} \Phi_{\Delta,n}.$$

For functions f such that

$$f(y) = f(y)\mathbb{I}_{\leq \overline{x}}(y) + f(\overline{x})\mathbb{I}_{> \overline{x}}(y),$$

we can determine $[\underline{T}_\Delta f](x)$ by means of transformations on the vector space of real-valued functions on the *finite* set

$$\{y \in \mathbb{Z}_{\geq 0} \colon y \leq \overline{x}\}.$$

This is extremely useful in practice because, for general bounded functions f,

$$[\underline{T}_\Delta f](x) = \lim_{\overline{x} \to +\infty} [\underline{T}_\Delta(\mathbb{I}_{\leq \overline{x}} f + f(\overline{x})\mathbb{I}_{> \overline{x}})](x).$$

Similar limit techniques also work for functions that are only bounded below.

See `arXiv:1905.05734` **for all details!**

Set of Poisson processes

One option is to consider the set $\mathscr{P}_{\mathrm{PP}}$ of all Poisson processes with a rate that belongs to the rate interval $[\underline{\lambda}, \overline{\lambda}]$.

We let $\underline{E}_{\mathrm{PP}}(\cdot \mid \cdot)$ denote the lower envelope of the expectations $E_P(\cdot \mid \cdot)$ with respect to all P in $\mathscr{P}_{\mathrm{PP}}$. Clearly, we can compute this lower expectation by means of a one-parameter optimisation problem.

This lower expectation $\underline{E}_{\mathrm{PP}}(\cdot \mid \cdot)$ satisfies imprecise versions of (PP1)–(PP3):

1. Markovianity:

$$\underline{E}_{\mathrm{PP}}(f(X_{t+\Delta}) \mid X_t = x, X_u = x_u) = \underline{E}_{\mathrm{PP}}(f(X_{t+\Delta}) \mid X_t = x);$$

2. time-homogeneity:

$$\underline{E}_{\mathrm{PP}}(f(X_{t+\Delta}) \mid X_t = x) = \underline{E}_{\mathrm{PP}}(f(X_\Delta) \mid X_0 = x);$$

3. state-homogeneity:

$$\underline{E}_{\mathrm{PP}}(f(X_\Delta - X_0) \mid X_0 = x) = \underline{E}_{\mathrm{PP}}(f(X_\Delta) \mid X_0 = 0).$$

Furthermore,

$$\underline{E}_{\mathrm{PP}}(X_{t+\Delta} \mid X_t = x, X_u = x_u) = x + \underline{\lambda}\Delta \quad (1)$$

and

$$\overline{E}_{\mathrm{PP}}(X_{t+\Delta} \mid X_t = x, X_u = x_u) = x + \overline{\lambda}\Delta. \quad (2)$$

However, assuming (PP1)–(PP3) is not always justified!

Numerical example

Below, we have depicted tight lower and upper bounds—with respect to both sets—on the probability of having no Poisson-event or a single Poisson-event in a time period of length Δ for the rate interval $[\underline{\lambda}, \overline{\lambda}] = [1, 2]$.