One way to define an imprecise-probabilistic version of the Poisson process

Alexander Erreygers
Foundations Lab for imprecise probabilities – Ghent University

SIPTA Seminar – 16/06/2023


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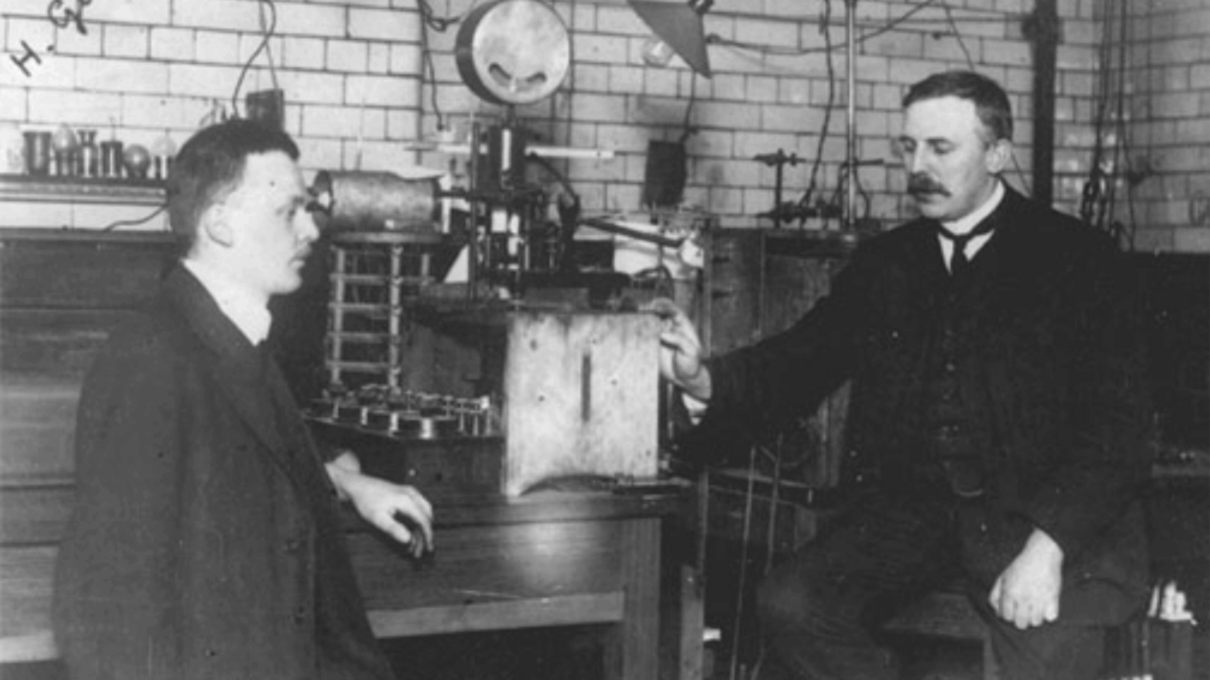


i Planen, som af to givne Keglesnitsbundter skæres i den samme Involution. Vinkelspidserne i den Trekant, de danner, er de tre Skæringspunkter mellem to Keglesnit, et af hvert Bundt, og det fjerde Skæringspunkt er Restpunktet til de 4 + 4 Punkter, der bestemmer Bundterne. Endepunkterne af hver Trekantside danner et Punktpar i Involutionen paa samme Side.

Sandsynlighedsregning og Telefonsamtaler.

Af A. K. Erlang.

Skønt der i Telefonien paa flere Punkter opstaar Spørgsmaal, hvis Løsning hører under Sandsynlighedsregningen, er denne, saavidt man kan se, hidtil ikke bleven brugt meget paa dette Omraade. I saa Henseende danner det københavnske Telefonselskab en Undtagelse, idet Hr. Telefondirektør F. Johannsen i flere Aar har benyttet Sandsynlighedsregningens Metoder til Løsning af forskellige Opgaver af praktisk Betydning og ligeledes sat andre i Arbejde med Undersøgelser af lignende Art. Da et og andet heraf maaske kan være af Interesse, og da der til Forstaaelsen aldeles ikke kræves særligt Kendskab til Telefonsager, vil jeg meddele det her.

1. Sandsynligheden for et givet Antal Opringninger i et Tidsrum af given Længde.

Det forudsættes, at der ikke er større Sandsynlighed for Opringning paa det ene Tidspunkt end paa ethvert andet. Lad a være den givne Tid, n Middelantallet af Opringninger i Tidsenheden. Vi vil søge Sandsynligheden S_0 for 0 Opringninger i Tiden a og derefter Sandsynligheden S_x for netop x Opringninger i Tiden a. Da

$\frac{na}{r}$

I.

APPROXIMERAD FRAMSTÄLLNING AF SANNOLIKHETSFUNKTIONEN

II.

ÅTERFÖRSÄKRING AF KOLLEKTIVRISKER

AKADEMISK AFHANDLING

SOM MED TILLSTÅND AF

FILOSOFISKA FAKULTETENS I UPPSALA

MATEMATISK-NATURVETENSKAPLIGA SEKTION

FÖR FILOSOFISK DOKTORSGRADS VINNANDE

TILL OFFENTLIG GRANSKNING FRAMSTÄLLES

Å LÄROSALEN N:O II

LÖRDAGEN DEN 7 NOVEMBER 1903 KL. 10 F. M.

AF

FILIP LUNDBERG

FILOSOFIE LICENTIAT AF STOCKHOLMS NATION

UPPSALA 1903

ALMQVIST & WIKSELLS BOKTRYCKERI-A.-B.

the scintillations were as bright if not brighter than those from a thin film of uranium. Boltwood has found that the range of the α particle from ionium is 2·8 cms., so that it appeared probable that the range of the α particles from uranium had been overestimated. This conclusion was confirmed by finding that the α rays from a thin film of uranium were more readily absorbed by aluminium than those from ionium. By a special method, the range of the α particle from uranium has been measured and found to be about 2·7 cms., while the range of the α particle from ionium is a millimetre or two longer. Further experiments are in progress to determine the range of the α particle from uranium accurately, and to examine carefully whether two sets of α particles of different range can be detected.

University of Manchester,
July 1910.

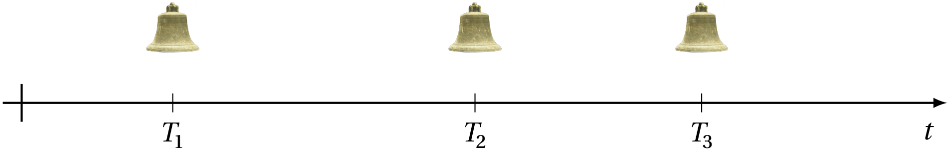
LXXVI. *The Probability Variations in the Distribution of α Particles. By* Professor E. RUTHERFORD, *F.R.S., and* H. GEIGER, *Ph.D. With a Note by* H. BATEMAN *.

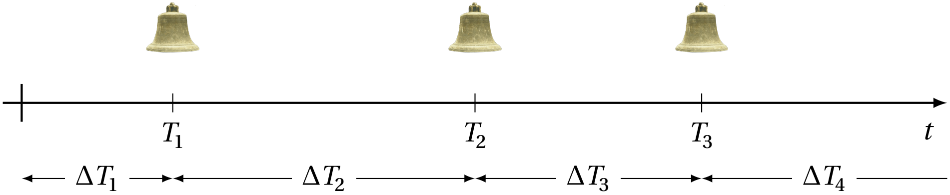
IN counting the α particles emitted from radioactive substances either by the scintillation or electric method, it is observed that, while the average number of particles from a steady source is nearly constant, when a large number is counted, the number appearing in a given short interval is subject to wide fluctuations. These variations are especially noticeable when only a few scintillations appear per minute. For example, during a considerable interval it may happen that no α particle appears; then follows a group of α particles in rapid succession; then an occasional α particle, and so on. It is of importance to settle whether these variations in distribution are in agreement with the laws of probability, *i. e.* whether the distribution of α particles on an average is that to be anticipated if the α particles are expelled at random both in regard to space and time. It might be conceived, for example, that the emission of an α particle might precipitate the disintegration of neighbouring atoms, and so lead to a distribution of α particles at variance with the simple probability law.

The magnitude of the probability variations in the number of α particles was first drawn attention to by E. v. Schweidler †. He showed that the average error from the mean number of α particles was $\sqrt{N.t}$, where N was the number of particles emitted per second and t the interval under consideration. This conclusion has been experimentally verified by several

* Communicated by the Authors.

† v. Schweidler, Congrès Internationale de Radiologie, Liège, 1905.

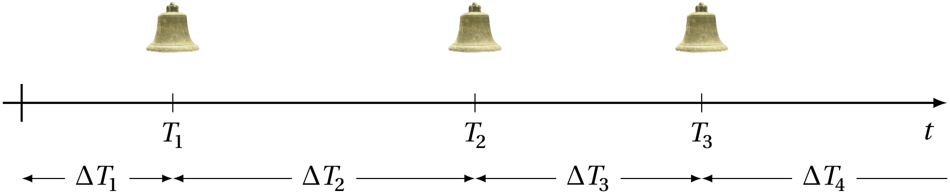

T_1
T_2
T_3
t

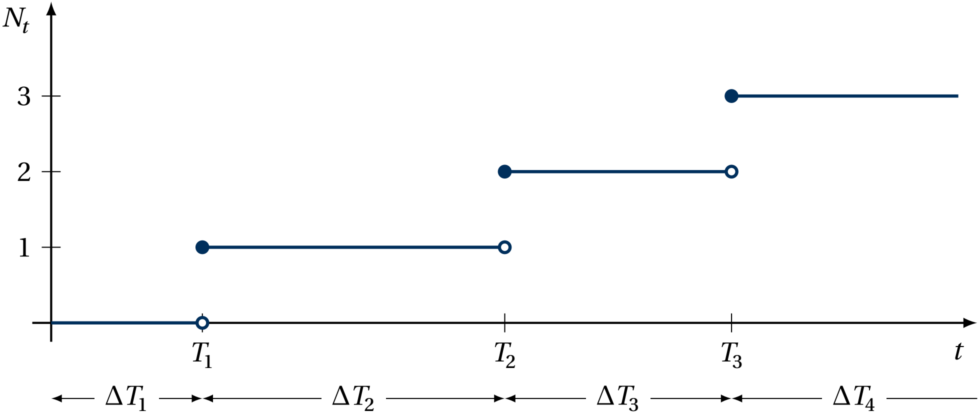

T_1
T_2
T_3
t
ΔT_1
ΔT_2
ΔT_3
ΔT_4

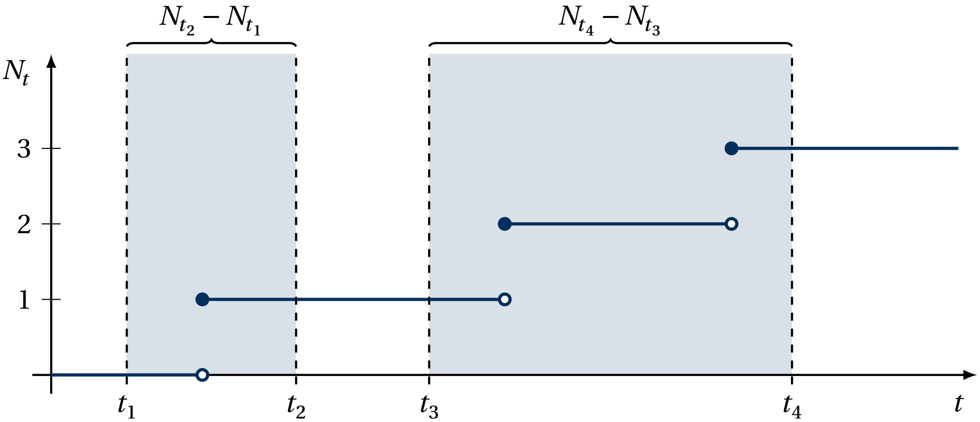
The Poisson process

à la Probability 101

The inter- times ΔT_1, ΔT_2, ... are independent and identically distributed; ΔT_k is **exponentially** distributed with mean $1/\lambda$.


T_1
T_2
T_3
t
ΔT_1
ΔT_2
ΔT_3
ΔT_4


N_t
3
2
1
T_1
T_2
T_3
t
ΔT_1
ΔT_2
ΔT_3
ΔT_4



$N_{t_2} - N_{t_1}$
$N_{t_4} - N_{t_3}$
N_t
3
2
1
t_1
t_2
t_3
t_4
t

The Poisson process

à la Probability 101

The inter- times ΔT_1, ΔT_2, ... are independent and identically distributed;
ΔT_k is **exponentially** distributed with mean $1/\lambda$.

↕

the process starts in 0, so $N_0 = 0$;
the increments $N_{t_2} - N_{t_1}$ and $N_{t_4} - N_{t_3}$ are independent whenever $t_1 \leq t_2 \leq t_3 \leq t_4$;
$N_{t_2} - N_{t_1}$ is **Poisson** distributed with mean $\lambda(t_2 - t_1)$.

A **stochastic process** P with state space $\mathbb{N}$ is completely defined—under some technical conditions—by the probabilities of the form

$$P(N_{t_1} = n_1, \ldots, N_{t_k} = n_k)$$

with $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k \in \mathbb{R}_{\geq 0}$ and $n_1, \ldots, n_k \in \mathbb{N}$.

The Poisson process
as a stochastic process

The **Poisson process** P **with rate** λ is defined by

$$P(N_{t_1} = n_1, \ldots, N_{t_k} = n_k) = \prod_{\ell=1}^{k} \psi_{\lambda(t_\ell - t_{\ell-1})}(n_\ell - n_{\ell-1})$$

with $t_0 := 0$, $n_0 := 0$ and $\psi_{\lambda(t_\ell - t_{\ell-1})} : \mathbb{Z} \to [0, 1]$ the Poisson distribution with mean $\lambda(t_\ell - t_{\ell-1})$.

What if we do not know the rate λ exactly?

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First Steps Towards an Imprecise Poisson Process

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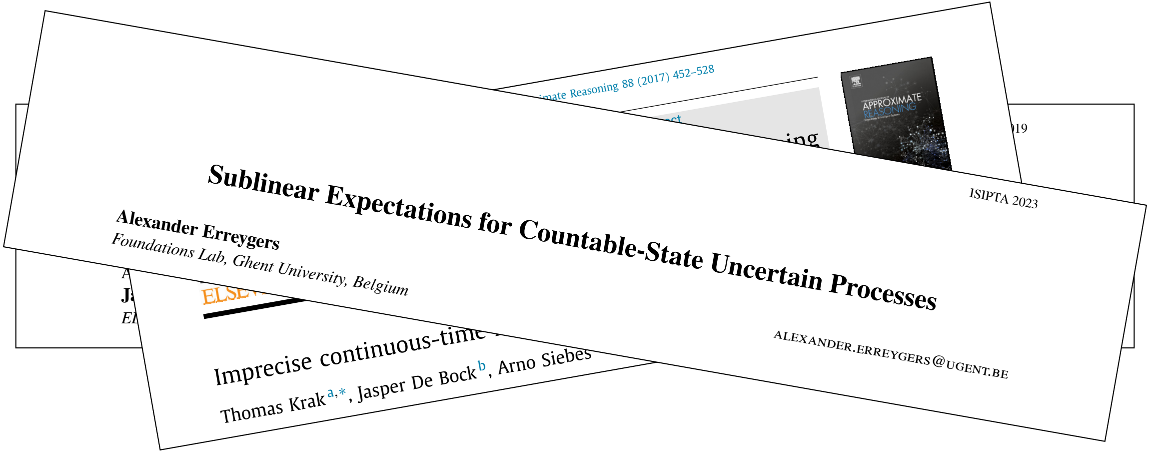
Imprecise continuous-time Markov chains

Thomas Krak [a,*], Jasper De Bock [b], Arno Siebes [a]


Sublinear Expectations for Countable-State Uncertain Processes
ISIPTA 2023
Alexander Erreygers
Foundations Lab, Ghent University, Belgium
ALEXANDER.ERREYGERS@UGENT.BE
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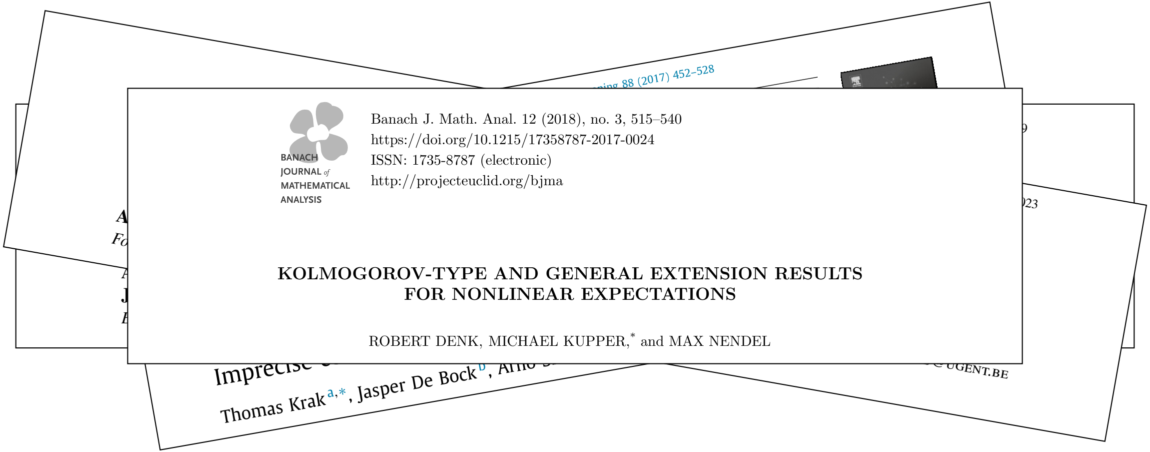
APPROXIMATE
REASONING
Imprecise continuous-time
Thomas Krak a,*, Jasper De Bock b, Arno Siebes


Banach J. Math. Anal. 12 (2018), no. 3, 515–540
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ISSN: 1735-8787 (electronic)
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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

ROBERT DENK, MICHAEL KUPPER,* and MAX NENDEL

A **stochastic process** P with state space $\mathbb{N}$ is completely defined—under some technical conditions—by the probabilities of the form

$$P(N_{t_1} = n_1, \dots, N_{t_k} = n_k)$$

with $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k \in \mathbb{R}_{\geq 0}$ and $n_1, \dots, n_k \in \mathbb{N}$.

$\updownarrow$

A **stochastic process** P with state space $\mathbb{N}$ is completely defined—under some technical conditions—by its corresponding expectations of the form

$$E_P\big(f(N_{t_1}, \dots, N_{t_k})\big)$$

with $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k \in \mathbb{R}_{\geq 0}$, $f \in \mathscr{L}(\mathbb{N}^k)$.

A **stochastic process** P with state space $\mathbb{N}$ is completely defined—under some techn [countable additivity] tions—by the probabilities of the form

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with $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k \in \mathbb{R}_{\geq 0}$ and $n_1, \ldots, n_k \in \mathbb{N}$.

Daniell–Stone Theorem

A **stochastic process** P with state space $\mathbb{N}$ is completely defined—under some techn [downward continuity] tions—by its corresponding expectations of the form

$$E_P\big(f(N_{t_1}, \ldots, N_{t_k})\big)$$

with $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k \in \mathbb{R}_{\geq 0}$, $f \in \mathscr{L}(\mathbb{N}^k)$.

The Poisson process

as a stochastic process … revisited

The **Poisson process** P **with rate** λ is the unique process corresponding to the expectation E_P on $\mathscr{D}$ defined recursively by

$$E_P\Big(f\big(N_t\big)\Big) = \sum_{k=0}^{+\infty} \psi_{\lambda t}(k) f(k) \quad \text{for all } t \in \mathbb{R}_{\geq 0}, f \in \mathscr{L}(\mathbb{N}).$$

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and for all $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k < t_{k+1} \in \mathbb{R}_{\geq 0}$ and $f \in \mathscr{L}(\mathbb{N}^{k+1})$

$$E_P\Big(f\big(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}\big)\Big) = E_P\Big(E_P\big(f(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}) \,\big|\, N_{t_1}, \ldots, N_{t_k}\big)\Big)$$

with $\Delta := t_{k+1} - t_k$ and

$$E_P\big(f(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}) \,\big|\, n_1, \ldots, n_k\big) = \sum_{\ell=0}^{+\infty} \psi_{\lambda\Delta}(\ell) f(n_1, \ldots, n_k, n_k + \ell).$$

For any $s \in \mathbb{R}_{\geq 0}$, let $\mathrm{T}_s : \mathscr{L}(\mathbb{N}) \to \mathscr{L}(\mathbb{N})$ be defined by

$$\mathrm{T}_s[f](n) := \sum_{k=0}^{+\infty} \psi_{\lambda s}(\ell) f(n+\ell) \quad \text{for all } f \in \mathscr{L}(\mathbb{N}), n \in \mathbb{N}.$$

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Then it is 'well-known' that for all $s \in \mathbb{R}_{\geq 0}$, $f \in \mathscr{L}(\mathbb{N})$ and $n \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{T}_s[f](n) = \lambda \mathrm{T}_s[f](n+1) - \lambda \mathrm{T}_s[f](n).$$

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$$\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{T}_s[f](n) = \mathrm{Q}_\lambda \mathrm{T}_s[f](n),$$

where $\mathrm{Q}_\lambda : \mathscr{L}(\mathbb{N}) \to \mathscr{L}(\mathbb{N})$ is defined by

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Then for all $s \in \mathbb{R}_{\geq 0}$

$$\mathrm{T}_s = \lim_{k \to +\infty} \left(\mathrm{I} + \frac{s}{k} \mathrm{Q}_\lambda \right)^k \eqqcolon e^{s \mathrm{Q}_\lambda}.$$

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Then for all $s \in \mathbb{R}_{\geq 0}$

$$\mathrm{T}_s = \lim_{k \to +\infty} \left(\mathrm{I} + \frac{s}{k} \mathrm{Q}_\lambda \right)^k =: e^{s\mathrm{Q}_\lambda},$$

and $(e^{s\mathrm{Q}_\lambda})_{s \in \mathbb{R}_{\geq 0}}$ is a ‘semigroup of transition operators’.

Operator semigroups

An **operator** S is a map from $\mathscr{L}(\mathbb{N})$ to $\mathscr{L}(\mathbb{N})$.

Operator semigroups

An operator S is a map from $\mathscr{L}(\mathbb{N})$ to $\mathscr{L}(\mathbb{N})$.

A semigroup is a family $(\mathrm{S}_s)_{s\in\mathbb{R}_{\geq 0}}$ of operators such that

- 👁 $\mathrm{S}_0 = \mathrm{I}$;
- 👨‍👩‍👦 $\mathrm{S}_{s_1+s_2} = \mathrm{S}_{s_1}\mathrm{S}_{s_2}$ for all $s_1, s_2 \in \mathbb{R}_{\geq 0}$.

Operator semigroups

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A **linear transition operator** S is an operator that is

- constant preserving, so $\mathrm{S}[\alpha] = \alpha$ for all constant $\alpha \in \mathscr{L}(\mathbb{N})$,
- isotone, so $\mathrm{S}[f] \leq \mathrm{S}[g]$ for all $f, g \in \mathscr{L}(\mathbb{N})$ such that $f \leq g$, and
- linear, so $\mathrm{S}[\mu f + g] = \mu\mathrm{S}[f] + \mathrm{S}[g]$ for all $f, g \in \mathscr{L}(\mathbb{N})$ and $\mu \in \mathbb{R}$.

The Poisson process

as a stochastic process … revisited

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and for all $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k < t_{k+1} \in \mathbb{R}_{\geq 0}$ and $f \in \mathscr{L}(\mathbb{N}^{k+1})$

$$E_P\Big(f\big(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}\big)\Big) = E_P\Big(E_P\big(f(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}) \mid N_{t_1}, \ldots, N_{t_k}\big)\Big)$$

with $\Delta := t_{k+1} - t_k$ and

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$$E_P\big(f(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}) \,\big|\, n_1, \ldots, n_k\big) = e^{\Delta \mathrm{Q}_\lambda}\big[f(n_1, \ldots, n_k, \bullet)\big](n_k).$$

What if we do not know the rate λ exactly?

Given is a rate interval $[\underline{\lambda}, \overline{\lambda}]$.

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Consider the set $\{E_\lambda : \lambda \in [\underline{\lambda}, \overline{\lambda}]\}$ of corresponding Poisson processes, and take lower/upper envelopes:

$$\underline{E}\Big(f\big(N_{t_1}, \dots, N_{t_k}\big)\Big) = \inf_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} E_\lambda\Big(f\big(N_{t_1}, \dots, N_{t_k}\big)\Big) \quad \text{and} \quad \overline{E}\Big(f\big(\dots\big)\Big) = \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} E_\lambda\Big(f\big(\dots\big)\Big).$$

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Consider the set $\{E_\lambda : \lambda \in [\underline{\lambda}, \overline{\lambda}]\}$ of corresponding Poisson processes, and take lower/upper envelopes:

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Then

- 😃 computing lower/upper probabilities and expectations is essentially a one-parameter optimisation problem, but
- 😵‍💫 in general there is no iterative way to compute $\overline{E}\Big(f(N_{t_1}, \dots, N_{t_k})\Big)$.

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First Steps Towards an Imprecise Poisson Process

Alexander Erreygers ALEXANDER.ERREYGERS@UGENT.BE
Jasper De Bock JASPER.DEBOCK@UGENT.BE
ELIS – FLip, Ghent University, Belgium

Let $\overline{\mathrm{Q}} \colon \mathscr{L}(\mathbb{N}) \to \mathscr{L}(\mathbb{N})$ be defined for all $f \in \mathscr{L}(\mathbb{N})$ and $n \in \mathbb{N}$ by

$$\overline{\mathrm{Q}}[f](n) := \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \mathrm{Q}_\lambda[f](n) = \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \lambda f(n+1) - \lambda f(n).$$

Let $\overline{\mathrm{Q}}\colon \mathscr{L}(\mathbb{N}) \to \mathscr{L}(\mathbb{N})$ be defined for all $f \in \mathscr{L}(\mathbb{N})$ and $n \in \mathbb{N}$ by

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Then for all $s \in \mathbb{R}_{\geq 0}$,

$$e^{s\overline{\mathrm{Q}}} := \lim_{k \to +\infty} \left(\mathrm{I} + \frac{s}{k}\overline{\mathrm{Q}}\right)^k$$

is a sublinear transition operator, so an operator S that is

- constant preserving, so $\mathrm{S}[\alpha] = \alpha$ for all constant $\alpha \in \mathscr{L}(\mathbb{N})$,
- isotone, so $\mathrm{S}[f] \leq \mathrm{S}[g]$ for all $f, g \in \mathscr{L}(\mathbb{N})$ such that $f \leq g$, and
- sublinear, so $\mathrm{S}[\mu f + g] \leq \mu \mathrm{S}[f] + \mathrm{S}[g]$ for all $f, g \in \mathscr{L}(\mathbb{N})$ and $\mu \in \mathbb{R}_{\geq 0}$.

Let $\overline{\mathrm{Q}}\colon \mathscr{L}(\mathbb{N}) \to \mathscr{L}(\mathbb{N})$ be defined for all $f \in \mathscr{L}(\mathbb{N})$ and $n \in \mathbb{N}$ by

$$\overline{\mathrm{Q}}[f](n) := \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \mathrm{Q}_\lambda[f](n) = \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \lambda f(n+1) - \lambda f(n).$$

Then for all $s \in \mathbb{R}_{\geq 0}$,

$$e^{s\overline{\mathrm{Q}}} := \lim_{k \to +\infty} \left(\mathrm{I} + \frac{s}{k} \overline{\mathrm{Q}} \right)^k$$

is a sublinear transition operator.

Furthermore, $(e^{s\overline{\mathrm{Q}}})_{s \in \mathbb{R}_{\geq 0}}$ is a semigroup, and

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{s\overline{\mathrm{Q}}} = \overline{\mathrm{Q}} e^{s\overline{\mathrm{Q}}}.$$

The sublinear Poisson process

The **sublinear Poisson process with rate interval** $[\underline{\lambda}, \overline{\lambda}]$ is the sublinear expectation $\overline{E}$ on $\mathscr{D}$ defined recursively by

$$\overline{E}\Big(f(N_t)\Big) := e^{t\overline{\mathrm{Q}}}[f](0) \quad \text{for all } t \in \mathbb{R}_{\geq 0}, f \in \mathscr{L}(\mathbb{N})$$

and for all $k \in \mathbb{N}_{>0}$, $t_1 < \cdots < t_k < t_{k+1} \in \mathbb{R}_{\geq 0}$ and $f \in \mathscr{L}(\mathbb{N}^{k+1})$

$$\overline{E}\Big(f\big(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}\big)\Big) := \overline{E}\Big(\overline{E}\big(f\big(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}\big) \,\big|\, N_{t_1}, \ldots, N_{t_k}\big)\Big)$$

with $\Delta := t_{k+1} - t_k$ and

$$\overline{E}\big(f\big(N_{t_1}, \ldots, N_{t_k}, N_{t_{k+1}}\big) \,\big|\, n_1, \ldots, n_k\big) := e^{\Delta\overline{\mathrm{Q}}}\big[f(n_1, \ldots, n_k, \bullet)\big](n_k).$$

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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

ROBERT DENK, MICHAEL KUPPER,* and MAX NENDEL

- ‘convex expectations’ instead of only ‘sublinear expectations’
- state space can be a Polish space

Fix some set of **paths** $\Omega \subseteq \mathbb{N}^{\mathbb{R}_{\geq 0}}$.

We investigate (sub)linear expectations on the set of **finitary bounded variables**

$$\mathscr{D} := \left\{ f \circ \pi_{\{t_1,\dots,t_k\}} \colon k \in \mathbb{N}_{>0}, t_1 < \cdots < t_k \in \mathbb{R}_{\geq 0}, f \in \mathscr{L}(\mathbb{N}^k) \right\},$$

where $\pi_{\{t_1,\dots,t_k\}} \colon \Omega \to \mathbb{N}^k$ maps any path $\omega \in \Omega$ to $\big(\omega(t_1), \dots, \omega(t_k)\big)$.

Fix some set of paths $\Omega \subseteq \mathbb{N}^{\mathbb{R}_{\geq 0}}$.

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🚨 **Many interesting variables are *not* included in $\mathscr{D}$!** 🚨

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For the (sublinear) Poisson process, for example, $\mathscr{D}$ does *not* include

#️⃣ the number of 🔔 between t_1 and t_2, so

$$N_{t_2} - N_{t_1} : \Omega \to \mathbb{Z} : \omega \mapsto \omega(t_2) - \omega(t_1).$$

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$$N_{t_2} - N_{t_1} : \Omega \to \mathbb{Z} : \omega \mapsto \omega(t_2) - \omega(t_1);$$

- ⏱ the time until the first 🔔, so

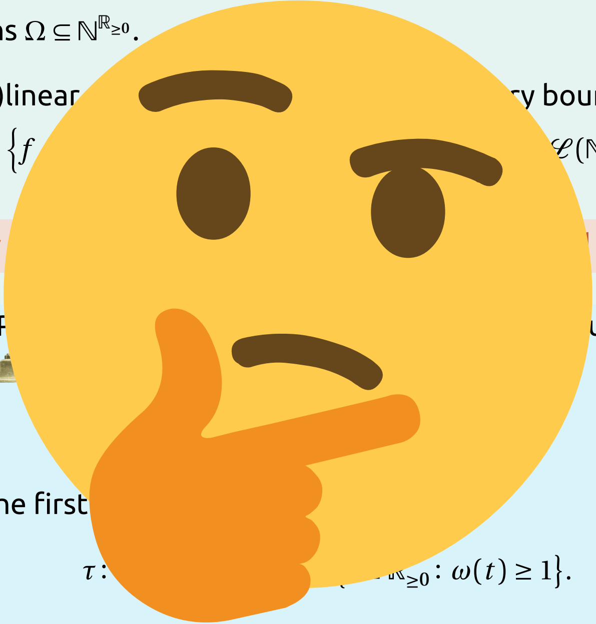
$$\tau : \Omega \to \overline{\mathbb{R}}_{\geq 0} : \omega \mapsto \inf\{t \in \mathbb{R}_{\geq 0} : \omega(t) \geq 1\}.$$

Fix some set of paths $\Omega \subseteq \mathbb{N}^{\mathbb{R}_{\geq 0}}$.

We investigate (sublinear ... ry bounded variables

$$\mathscr{D} := \{ f \ \ldots \ \mathscr{L}(\mathbb{N}^k) \}.$$

Many ... in $\mathscr{D}$!

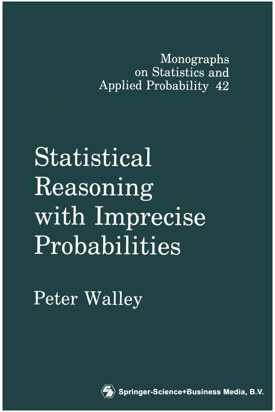


For the (sublinear) F ... ude

\# the number of ...

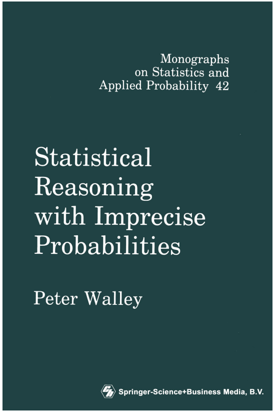
the time until the first ...

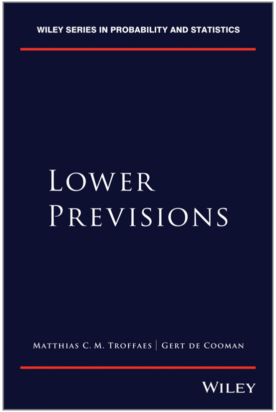
$$\tau : \ \ldots \ \mathbb{R}_{\geq 0} : \omega(t) \geq 1 \}.$$



Natural extension

- 🙅 only to bounded variables
- 😞 often overly conservative





Natural extension

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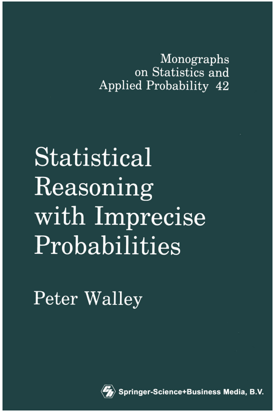
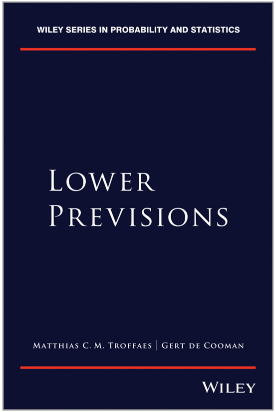
- 😞 often overly conservative

Previsibility

- 🙅 from all bounded variables
- 😞 to some real variables





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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

ROBERT DENK, MICHAEL KUPPER,* and MAX NENDEL

Communicated by N.-C. Wong

ABSTRACT. We provide extension procedures for nonlinear expectations to the space of all bounded measurable functions. We first discuss a maximal extension for convex expectations which have a representation in terms of finitely additive measures. One of the main results of this article is an extension procedure for convex expectations which are continuous from above and therefore admit a representation in terms of countably additive measures. This can be seen as a nonlinear version of the Daniell–Stone theorem. From this, we deduce a robust Kolmogorov extension theorem which is then used to extend nonlinear kernels to an infinite-dimensional path space. We then apply this theorem to construct nonlinear Markov processes with a given family of nonlinear transition kernels.

1. Introduction

Given a set M of bounded measurable functions $X\colon \Omega \to \mathbb{R}$ which contains the constants, a nonlinear expectation is a functional $\mathcal{E}\colon M \to \mathbb{R}$ which satisfies $\mathcal{E}(X) \le \mathcal{E}(Y)$ whenever $X(\omega) \le Y(\omega)$ for all $\omega \in \Omega$, and $\mathcal{E}(\alpha 1_\Omega) = \alpha$ for all $\alpha \in \mathbb{R}$. If a nonlinear expectation $\mathcal{E}$ is in addition sublinear, then $\rho(X) := \mathcal{E}(-X)$, $X \in M$, is a coherent monetary risk measure as introduced by Artzner et al. [1] and Delbaen [12], [13] (see also Föllmer and Schied [24] for an overview of convex monetary risk measures). Other prominent examples of nonlinear expectations

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Keywords. nonlinear expectations, extension results, Kolmogorov's extension theorem, nonlinear kernels.
515

Natural extension

- 🙅 only to bounded variables
- 😒 often overly conservative

Previsibility

- 🙅 from all bounded variables
- 😒 to some real variables

Kolmogorov-type extension

- 👍 sublinear processes
- 🤔 downward continuity

A sublinear expectation $\overline{E}$ on $\mathscr{D}$ is called downward continuous on $\mathscr{D}$ if

$$\lim_{k \to +\infty} \overline{E}(f_k) = \overline{E}(f) \quad \text{for all } \mathscr{D}^{\mathbb{N}} \ni (f_k)_{k \in \mathbb{N}} \searrow f \in \mathscr{D}.$$

the sublinear expectation $\overline{E}$ on $\mathscr{D}$ is downward continuous on $\mathscr{D}$

$\updownarrow$

every dominated linear expectation in

$$\mathscr{M}(\overline{E}) := \{E \text{ a linear expectation on } \mathscr{D} \colon (\forall f \in \mathscr{D})\ E(f) \leq \overline{E}(f)\}$$

is downward continuous on $\mathscr{D}$

the sublinear expectation $\overline{E}$ on $\mathscr{D}$ is downward continuous on $\mathscr{D}$

$\updownarrow$

for all $E \in \mathscr{M}(\overline{E})$, there is a unique probability measure P_E on $\sigma(\mathscr{D})$ such that

$$E(f) = \int f \,\mathrm{d}P_E \quad \text{for all } f \in \mathscr{D}$$

the sublinear expectation $\overline{E}$ on $\mathscr{D}$ is downward continuous on $\mathscr{D}$

$\updownarrow$

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let $\mathscr{D}_{\mathrm{b}}^{\star} := \{f \in \mathbb{R}^{\Omega} \colon f \text{ bounded and } \sigma(\mathscr{D})\text{-measurable}\}$ and

$$\overline{E}_{\mathrm{b}}^{\star}(f) := \sup_{E \in \mathscr{M}(\overline{E})} \int f \, \mathrm{d}P_E \quad \text{for all } f \in \mathscr{D}_{\mathrm{b}}^{\star}$$

$\downarrow$

$\overline{E}_{\mathrm{b}}^{\star}$ is the unique sublinear expectation on $\mathscr{D}_{\mathrm{b}}^{\star}$ that extends $\overline{E}$,
is downward continuous on $\mathscr{D}_{\delta} \cap \mathscr{D}_{\mathrm{b}}^{\star}$ and upward continuous on $\mathscr{D}_{\mathrm{b}}^{\star}$

the sublinear expectation $\overline{E}$ on $\mathscr{D}$ is downward continuous on $\mathscr{D}$

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for all $E \in \mathscr{M}(\overline{E})$, there is a unique probability measure P_E on $\sigma(\mathscr{D})$ such that

$$E(f) = \int f \, \mathrm{d}P_E \quad \text{for all } f \in \mathscr{D}$$

let $\mathscr{D}^\star := \{f \in \overline{\mathbb{R}}^\Omega : f$ bounded below or above and $\sigma(\mathscr{D})$-measurable$\}$ and

$$\overline{E}^\star(f) := \sup_{E \in \mathscr{M}(\overline{E})} \int f \, \mathrm{d}P_E \quad \text{for all } f \in \mathscr{D}^\star$$

$\downarrow$

$\overline{E}^\star$ is **a** sublinear expectation on $\mathscr{D}^\star$ that extends $\overline{E}$,
is downward continuous on $\mathscr{D}_\delta \cap \mathscr{D}^\star_{\mathrm{b}}$ and upward continuous on $\mathscr{D}^\star$

Suppose Ω is the set of **all** paths $\omega\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$. Then

- ✓ every linear expectation E on $\mathscr{D}$ is trivially downward continuous.

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- ✗ the extended domain $\mathscr{D}^{\star}$ is not sufficiently rich!

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- ✓ every linear expectation E on $\mathscr{D}$ is trivially downward continuous, but
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Let Ω be the set of **càdlàg** paths $\omega\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$, so those that are continuous from the right and have left-sided limits. Then

- ✓ the extended domain $\mathscr{D}^\star$ *is* sufficiently rich, but
- ⚙ a linear expectation E on $\mathscr{D}$ is not necessarily downward continuous.

The sublinear Poisson process

ISIPTA 2023

Sublinear Expectations for Countable-State Uncertain Processes

Alexander Erreygers ALEXANDER.ERREYGERS@UGENT.BE
Foundations Lab, Ghent University, Belgium

😎 $\overline{E}$ is downward continuous! 😎

↓

🥳 **$\overline{E}$ extends to a ↘️ & ↗️ sublinear expectation $\overline{E}^{\star}$ on $\mathscr{D}^{\star}$.** 🥳

A sublinear Markov process

$\overline{E}_0$ on $\mathscr{L}(\mathbb{N})$ & semigroup $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators

↓

sublinear expectation $\overline{E}$ on $\mathscr{D}$

A sublinear Markov process

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🤨 **Is $\overline{E}$ downward continuous?** 🤨

A sublinear Markov process

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↓

sublinear expectation $\overline{E}$ on $\mathscr{D}$

↘ & ⚙

↓

🤓 Is $\overline{E}$ downward continuous? **Yes it is!** 🤓

↓

🎉 **$\overline{E}$ extends to a ↘ & ↗ sublinear expectation $\overline{E}^\star$ on $\mathscr{D}^\star$.** 🎉

Uniform continuity

NEW

A semigroup $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators is called **uniformly continuous** if

$$\lim_{s\searrow 0}\overline{\mathrm{T}}_s = \mathrm{I} \Leftrightarrow \lim_{s\searrow 0}\|\overline{\mathrm{T}}_s - \mathrm{I}\| = 0.$$

Uniform continuity

NEW

A semigroup $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators is **uniformly continuous** if and only if

$$\overline{\mathrm{T}}_s = \lim_{k\to+\infty}\left(\mathrm{I}+\frac{s}{k}\overline{\mathrm{Q}}\right)^k \quad \text{for all } s\in\mathbb{R}_{\geq 0}$$

for some 'bounded sublinear generator' $\overline{\mathrm{Q}}\colon \mathscr{L}(\mathbb{N})\to\mathscr{L}(\mathbb{N})$.

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$$\frac{\mathrm{d}}{\mathrm{d}s}\overline{\mathrm{T}}_s = \overline{\mathrm{Q}}\,\overline{\mathrm{T}}_s \quad \text{for all } s\in\mathbb{R}_{\geq 0}.$$

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A 'sublinear generator' $\overline{\mathrm{Q}} \colon \mathscr{L}(\mathbb{N}) \to \mathscr{L}(\mathbb{N})$ is **bounded** if and only if there is a 'uniformly bounded' set $\mathscr{Q}$ of 'linear generators' such that

$$\overline{\mathrm{Q}}[f](n) = \sup_{\mathrm{Q}\in\mathscr{Q}} \mathrm{Q}[f](n) \quad \text{for all } f \in \mathscr{L}(\mathbb{N}), n \in \mathbb{N}.$$

- 🧐 Do the 2019 and 2023 approaches yield the same upper/sublinear expectation?
- 🐟 How could one generalise the inter-🔔 definition?
- 🎲 What about Watanabe's martingale characterization of Poisson processes?

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- 🐟 How could one generalise the inter-🔔 definition?
- 🎲 What about Watanabe's martingale characterization of Poisson processes?

- 👮‍♀️ What if $(\overline{\mathrm{T}}_s)_{s\in\mathbb{R}_{\geq 0}}$ is not uniformly continuous?
- ♾️ What about uncountably infinite state spaces?