# How to Make your Markov Chains Robust

## An Imprecise Probabilities Perspective

Jasper De Bock & Alexander Erreygers

GHENT UNIVERSITY

21 September 2023

Eindhoven

 

# FLip

Foundations Lab for imprecise probabilities

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So what are imprecise
probabilities?

# MODELLING UNCERTAINTY

belief functions

probability measures

sets of desirable gambles

sets of probability measures

logic

set theory

imprecise probabilities

random sets

So what are imprecise probabilities?

choice functions

lower and upper expectations

p-boxes

probability intervals

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probability measures

sets of desirable gambles

**sets of probability measures**

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So what are imprecise probabilities?

p-boxes

probability intervals

$$P \in \mathscr{P}$$

**sets of probability measures**

# imprecise probabilities

choice functions

lower and upper expectations

probability intervals

$$P \in \mathscr{P}$$

**sets of probability measures**

# imprecise probabilities

choice functions

lower and upper expectations

**probability intervals**

$$\overline{P}(A) = \sup_{P \in \mathscr{P}} P(A)$$

$$\underline{P}(A) = \inf_{P \in \mathscr{P}} P(A)$$

$$P \in \mathscr{P}$$

**sets of probability measures**

# imprecise probabilities

$$\overline{E}(f) = \sup_{P \in \mathscr{P}} E_P(f)$$

$$\underline{E}(f) = \inf_{P \in \mathscr{P}} E_P(f) = -\overline{E}(-f)$$

choice functions

**lower and upper expectations**

probability intervals

PRECISE
DECISION MAKING

choose the one with the highest probability

$P \in \mathscr{P}$

**sets of probability measures**

IMPRECISE
DECISION MAKING

choose the one(s) with the

- highest lower probability
- highest upper probability
- highest probability for at least one $P \in \mathscr{P}$
- …

imprecise probabilities

**choice functions**

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Why would I use this?

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**sets of probability measures**

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# MODELLING UNCERTAINTY

beliefs of groups

complete uncertainty

computational pragmatism

solve curse of dimensionality

partial knowledge

**imprecise probabilities**

small data sets

global sensitivity analysis

Why would I use this?

structural sensitivity analysis

# SIPTA.org

society for

imprecise probabilities

theories and applications




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# Uncertain processes

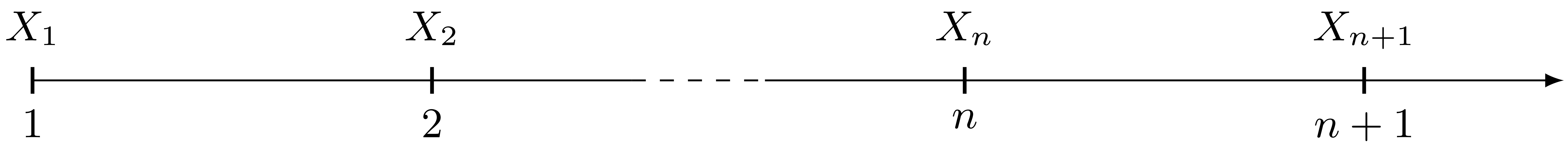
We consider a system whose uncertain state $X_t$

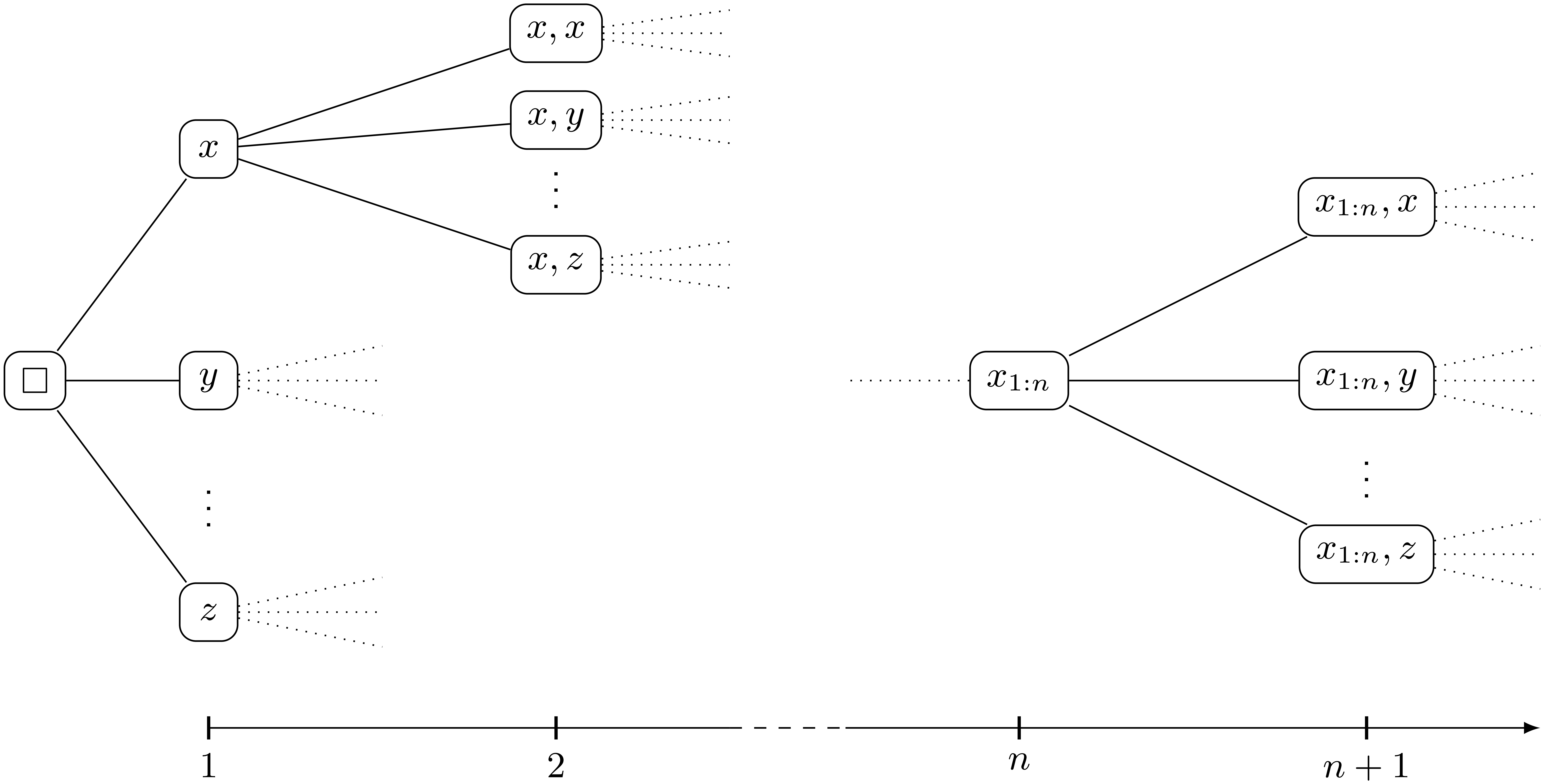
takes values in the state space $\mathscr{X}$ and

changes over time $t \in \mathbb{T}$.

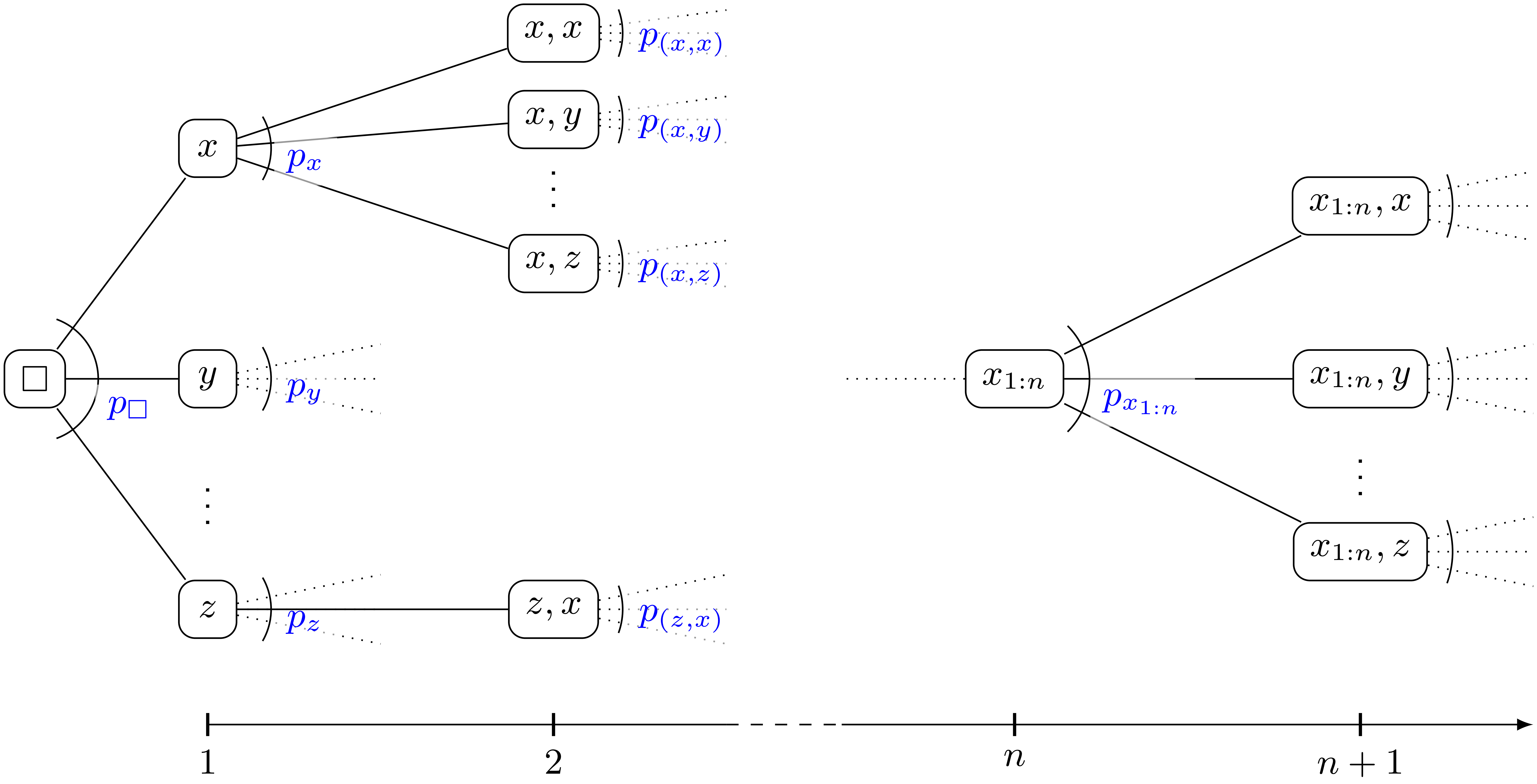
Let us focus on finite-state discrete-time uncertain processes, so with

$\mathscr{X}$ finite and $\mathbb{T} = \mathbb{N} = \{1,2,3\ldots\}$.


$X_1$
$X_2$
$X_n$
$X_{n+1}$
1
2
$n$
$n+1$

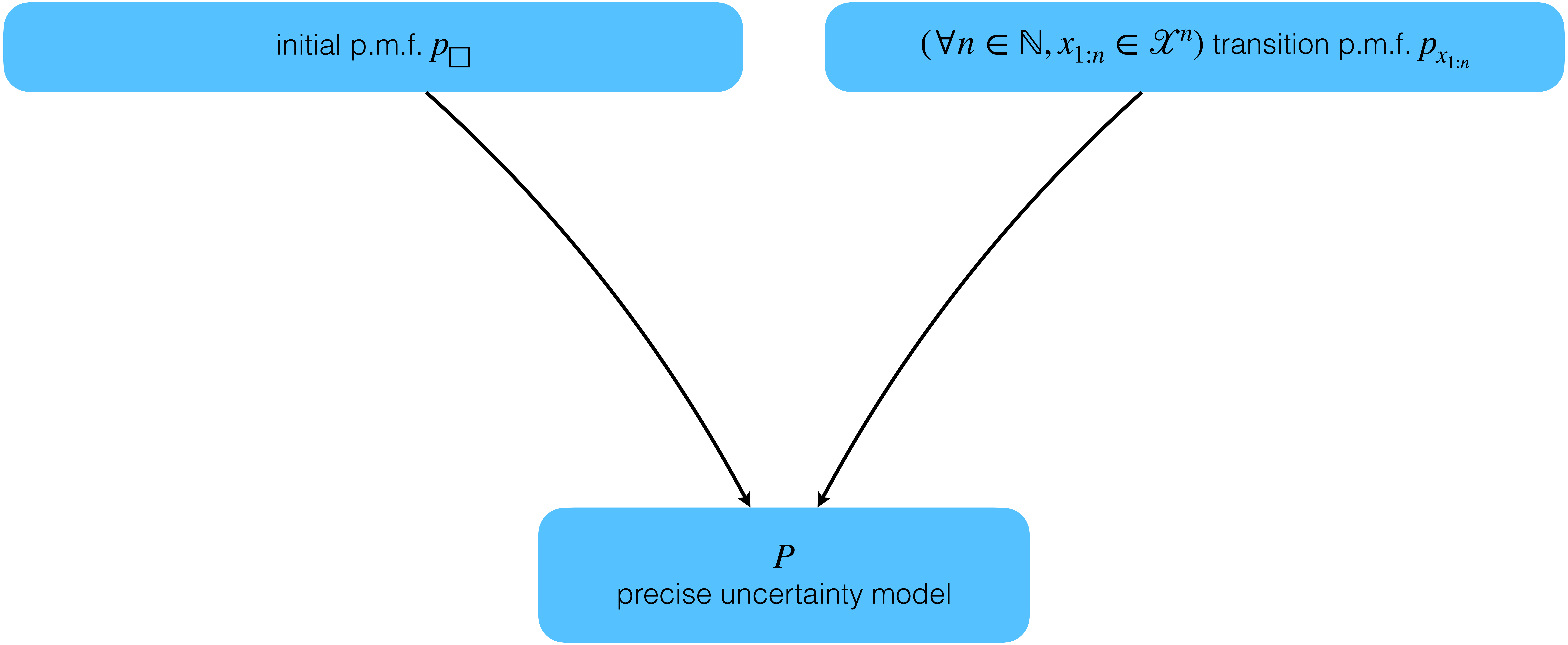


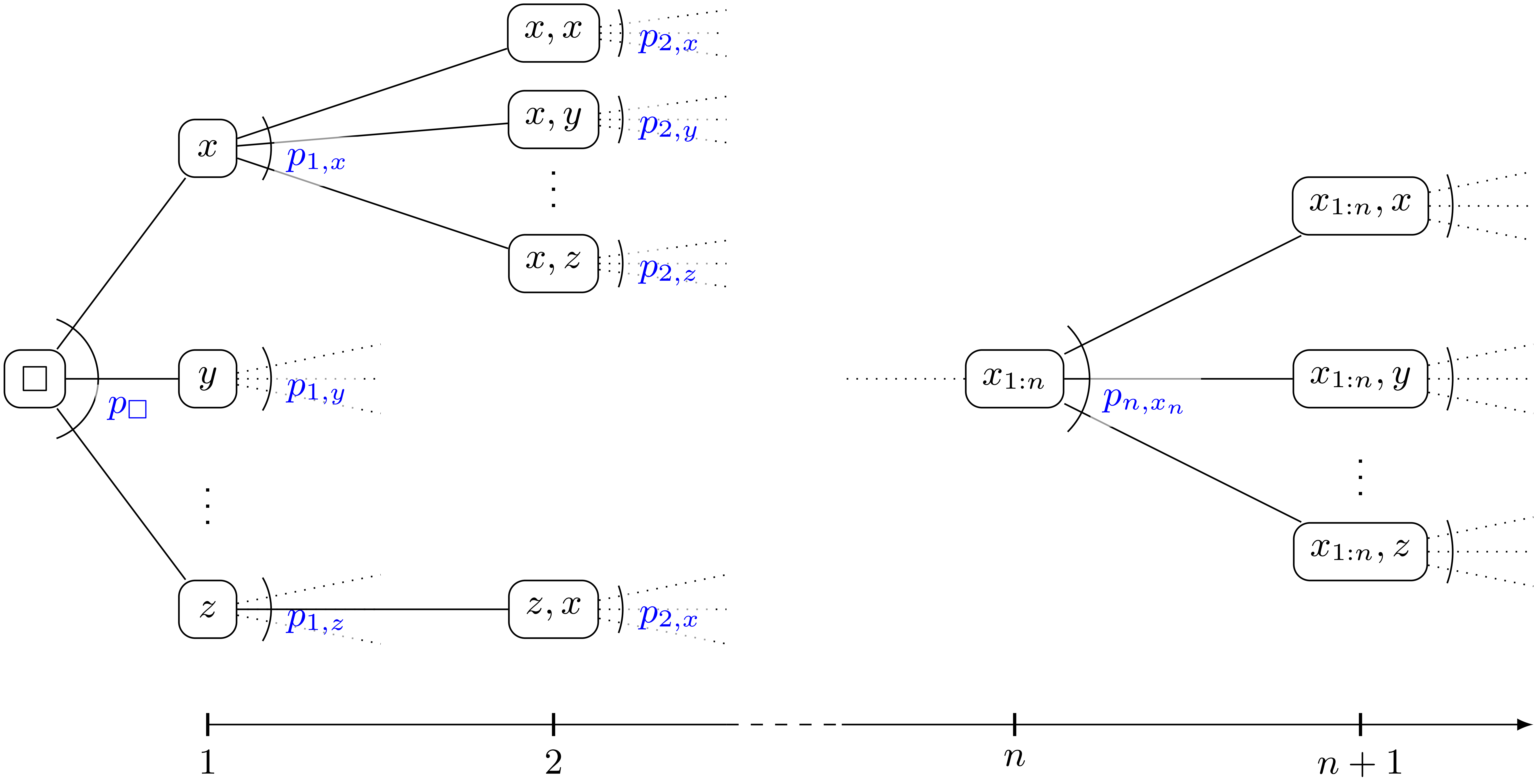
$\square$
$x$
$y$
$z$
$x, x$
$x, y$
$x, z$
$x_{1:n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
$n$
$n+1$



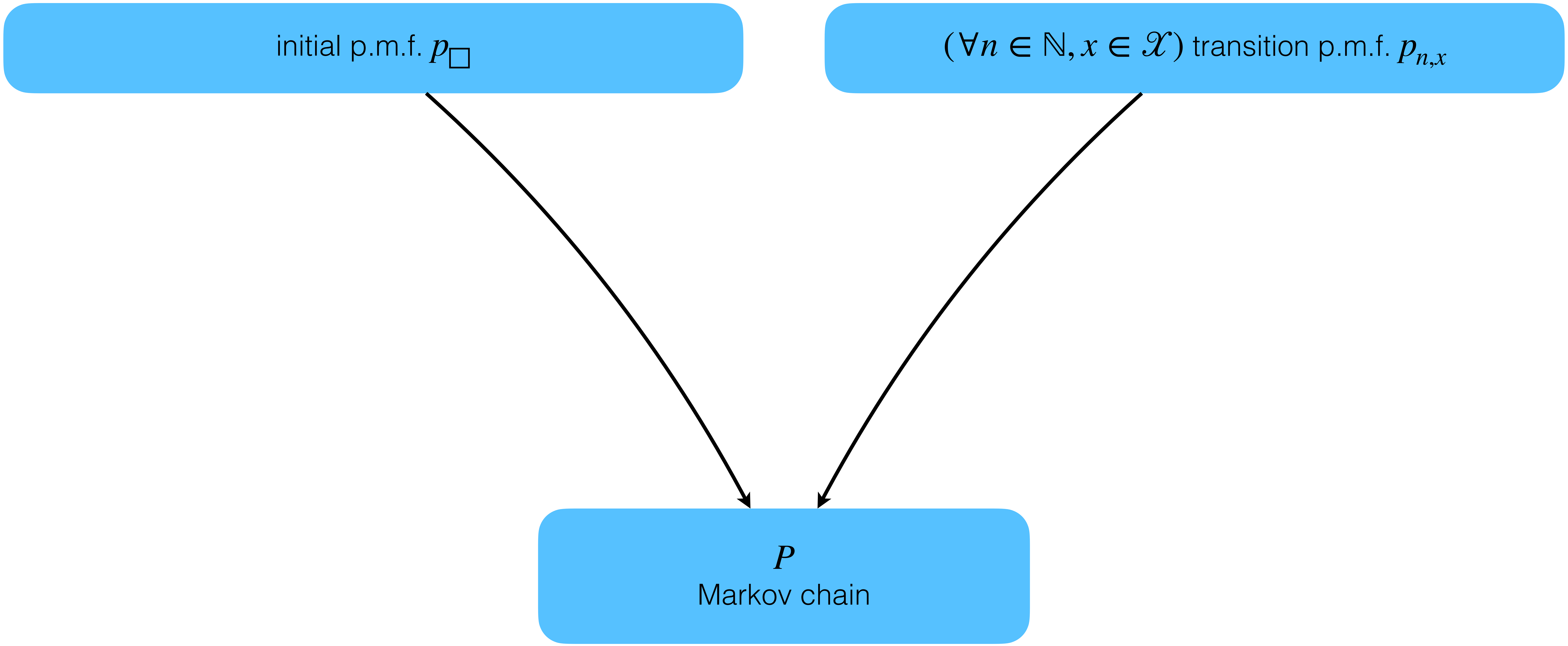
$x, x$
$p_{(x,x)}$
$x, y$
$p_{(x,y)}$
$x$
$p_x$
$x, z$
$p_{(x,z)}$
$x_{1:n}, x$
$\square$
$p_\square$
$y$
$p_y$
$x_{1:n}$
$p_{x_{1:n}}$
$x_{1:n}, y$
$x_{1:n}, z$
$z$
$p_z$
$z, x$
$p_{(z,x)}$
1
2
$n$
$n+1$

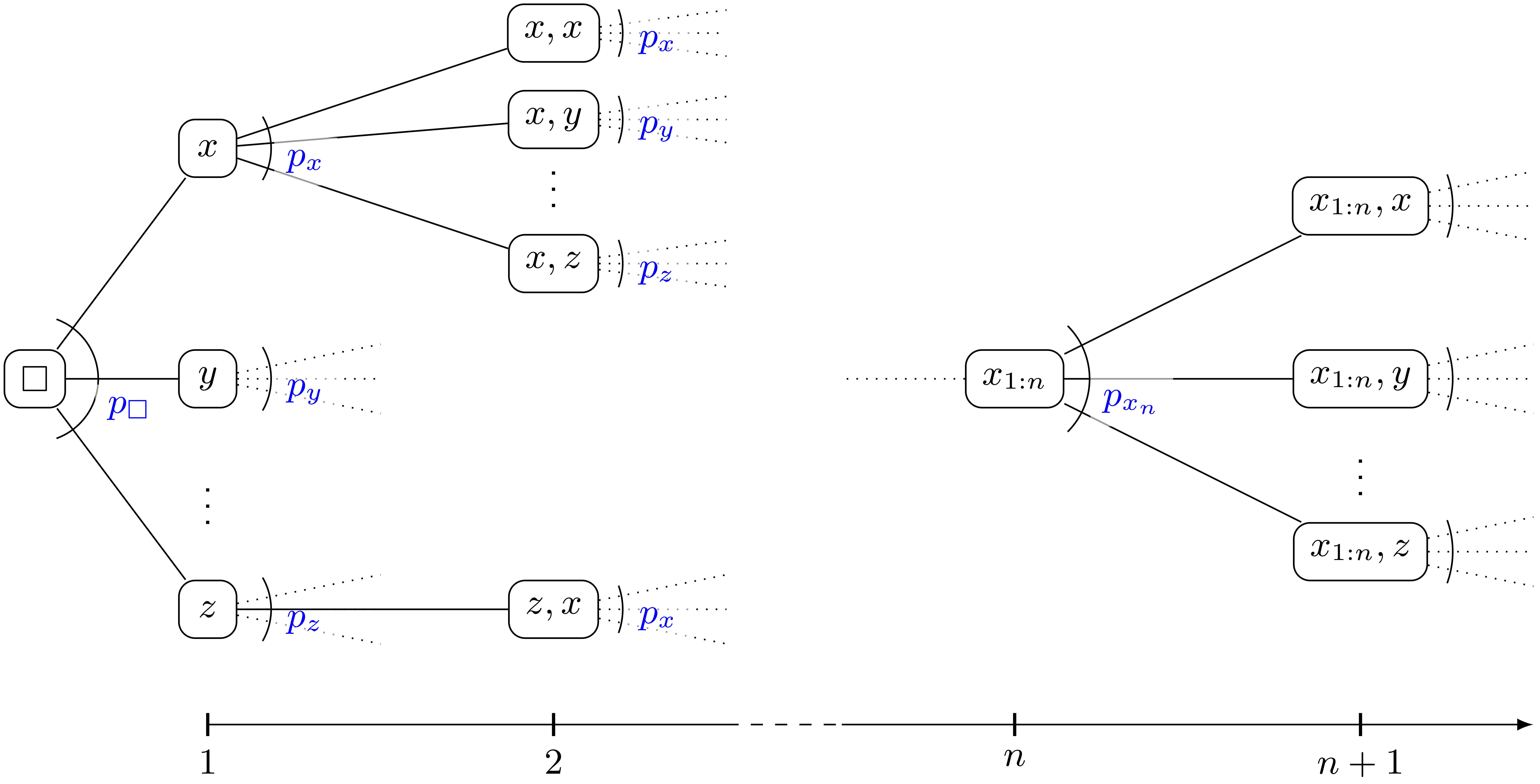
# Precise uncertainty model




□
$p_{\square}$
$x$
$y$
$z$
$p_{1,x}$
$p_{1,y}$
$p_{1,z}$
$x, x$
$x, y$
$x, z$
$z, x$
$p_{2,x}$
$p_{2,y}$
$p_{2,z}$
$p_{2,x}$
$x_{1:n}$
$p_{n,x_n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
$n$
$n+1$

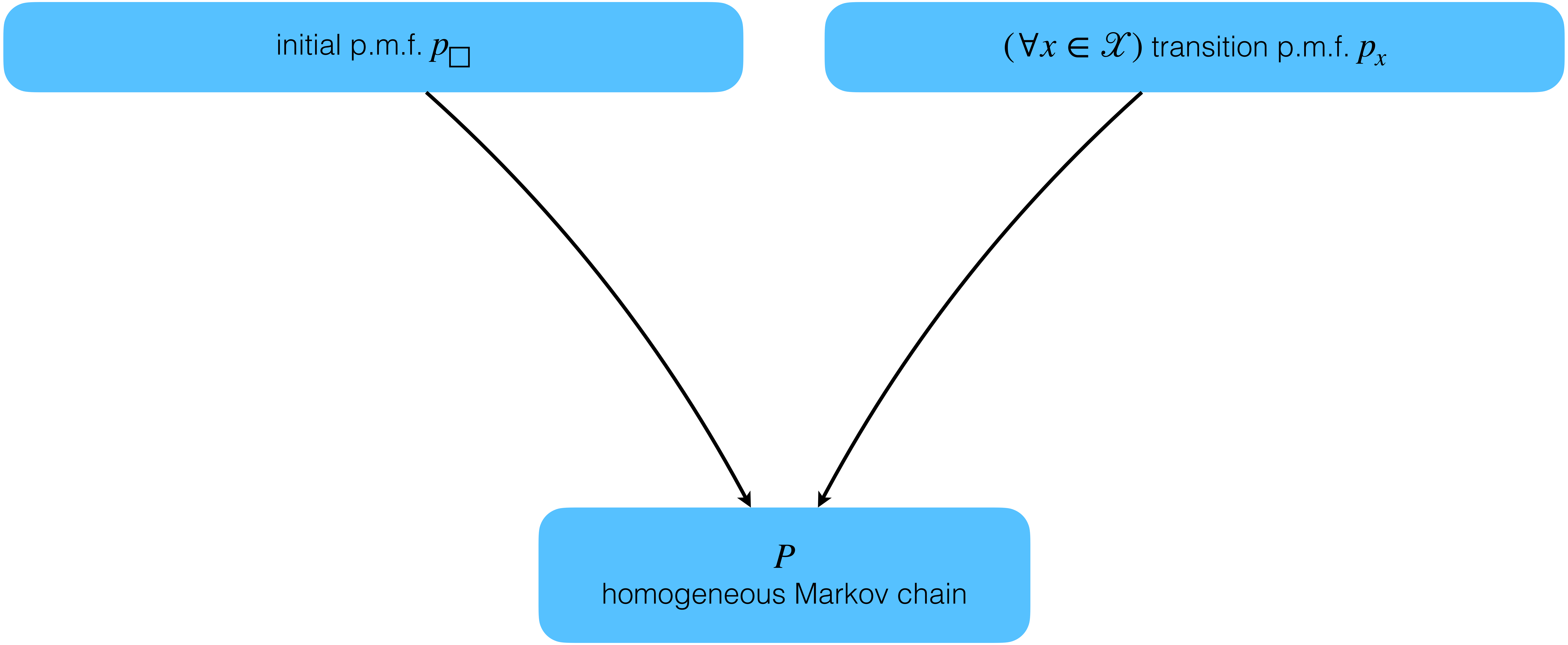
# Markov chain





$\square$
$p_\square$
$x$
$y$
$z$
$p_x$
$p_y$
$p_z$
$x, x$
$x, y$
$x, z$
$z, x$
$p_x$
$p_y$
$p_z$
$p_x$
$x_{1:n}$
$p_{x_n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
$n$
$n+1$

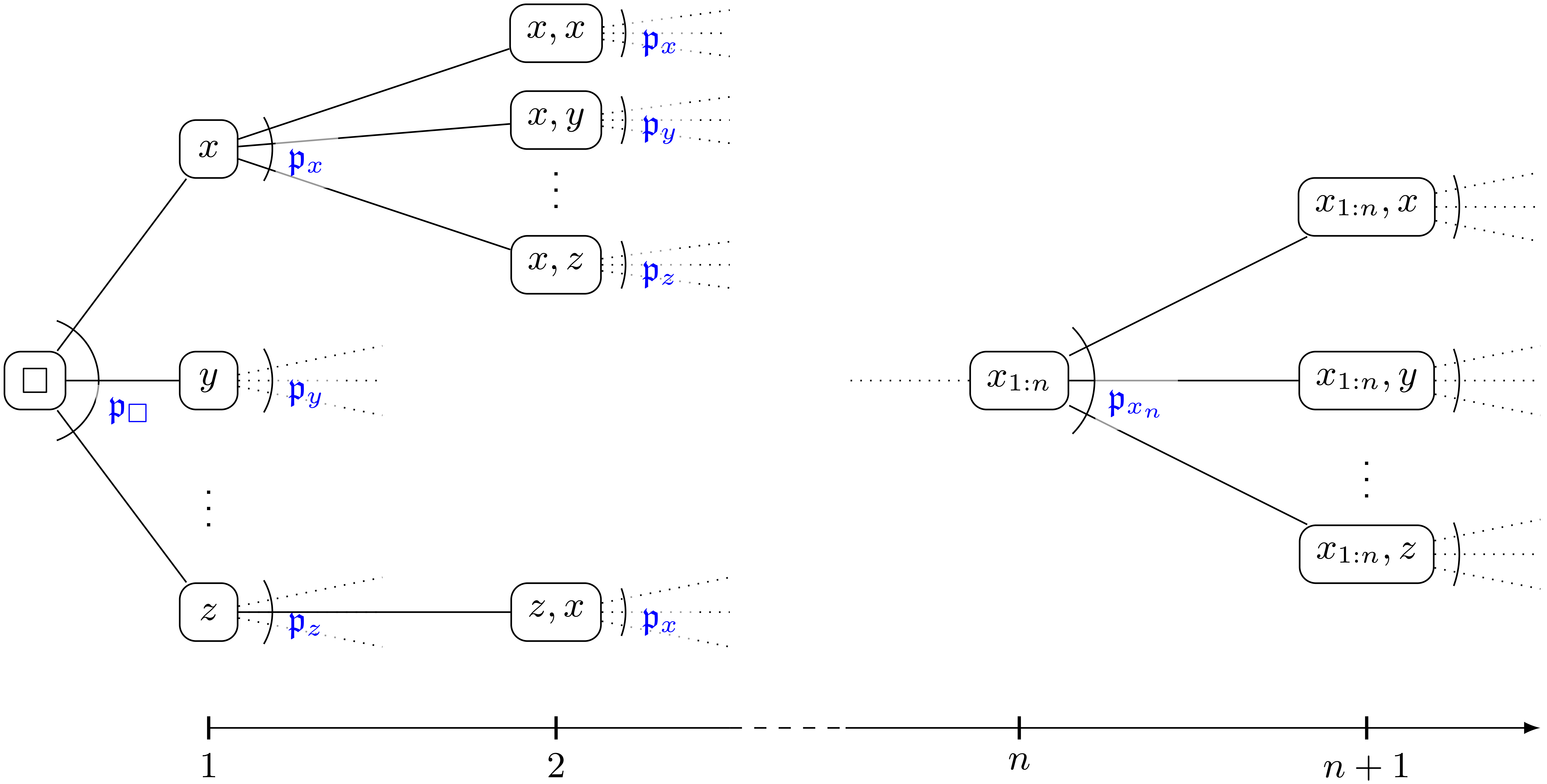
# Homogeneous Markov chain



# Imprecise Markov chain

set of initial p.m.f.s $\mathfrak{p}_\square$

$(\forall x \in \mathscr{X})$ a set of transition p.m.f.s $\mathfrak{p}_x$


$\square$
$\mathfrak{p}_\square$
$x$
$\mathfrak{p}_x$
$y$
$\mathfrak{p}_y$
$z$
$\mathfrak{p}_z$
$x, x$
$\mathfrak{p}_x$
$x, y$
$\mathfrak{p}_y$
$x, z$
$\mathfrak{p}_z$
$z, x$
$\mathfrak{p}_x$
$x_{1:n}$
$\mathfrak{p}_{x_n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
$n$
$n+1$

# Imprecise Markov chain

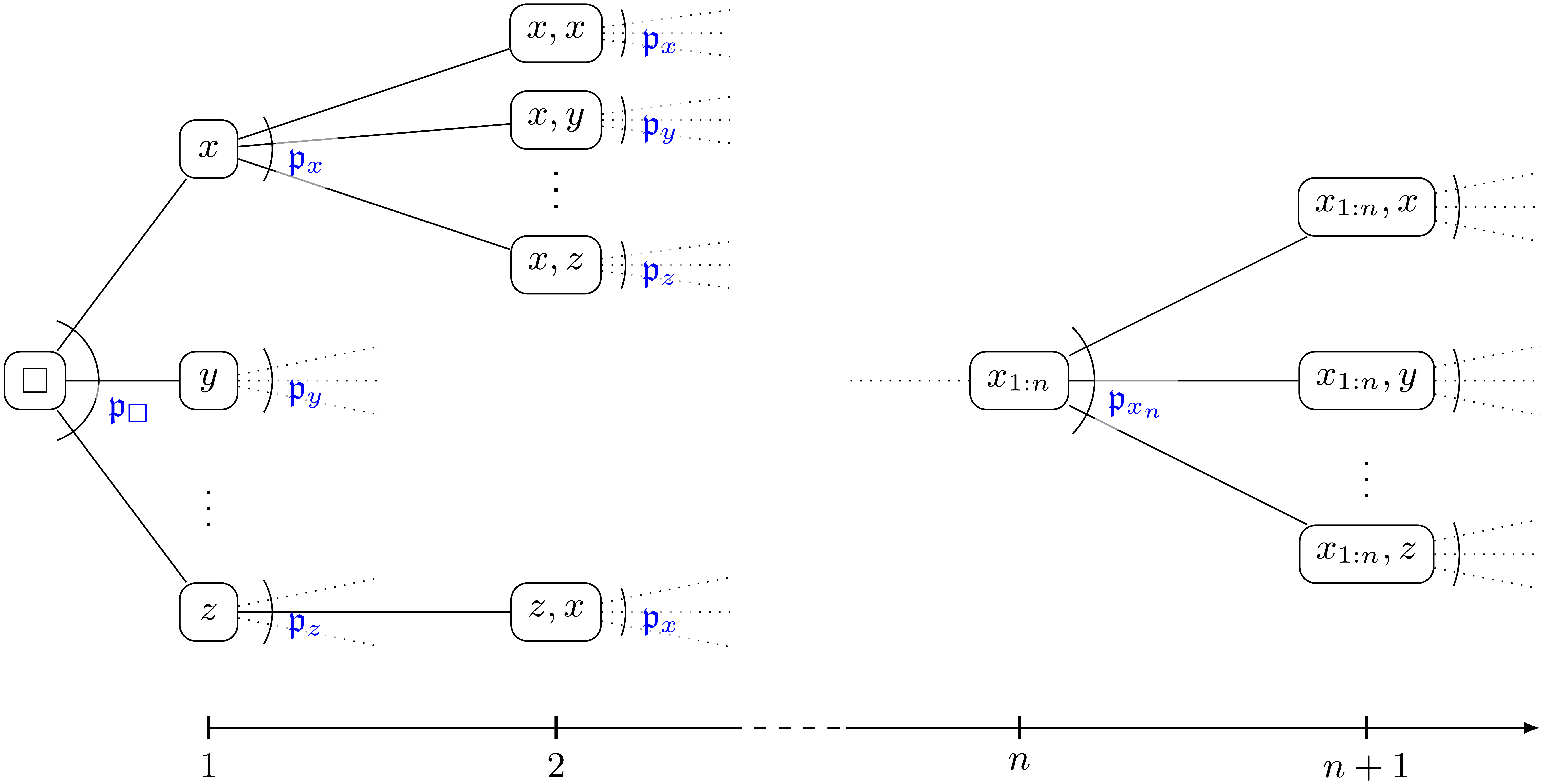
set of initial p.m.f.s $\mathfrak{p}_\square$

$(\forall x \in \mathscr{X})$ a set of transition p.m.f.s $\mathfrak{p}_x$

homogeneous Markov chain $P$ is **compatible** if $p_\square \in \mathfrak{p}_\square$ and $(\forall x \in \mathscr{X})\ p_x \in \mathfrak{p}_x$

$\mathscr{P}^{\mathrm{HM}}$

set of compatible homogeneous Markov chains


□
$\mathfrak{p}_\square$
$x$
$y$
$z$
$\mathfrak{p}_x$
$\mathfrak{p}_y$
$\mathfrak{p}_z$
$x, x$
$x, y$
$x, z$
$z, x$
$x_{1:n}$
$\mathfrak{p}_{x_n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
$n$
$n+1$

# Imprecise Markov chains

set of initial p.m.f.s $\mathfrak{p}_\square$

$(\forall x \in \mathscr{X})$ a set of transition p.m.f.s $\mathfrak{p}_x$

$P$ is **compatible** if $p_\square \in \mathfrak{p}_\square$ and $(\forall n \in \mathbb{N}, x_{1:n} \in \mathscr{X}^n)\ p_{x_{1:n}} \in \mathfrak{p}_{x_n}$

$\mathscr{P}$
set of compatible
precise uncertainty models

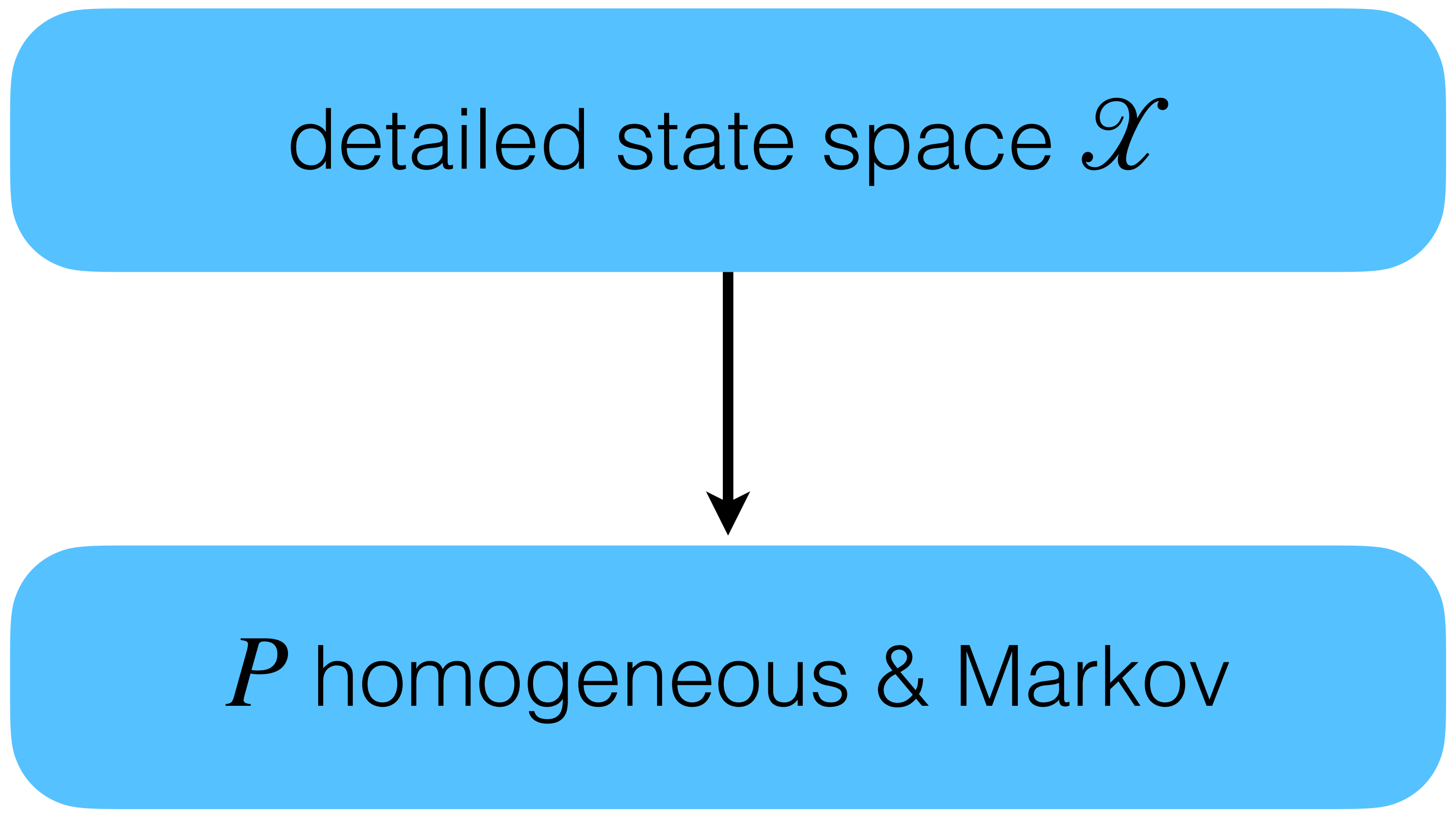
$\supseteq$

$\mathscr{P}^{\mathrm{M}}$
set of compatible
(inhomogeneous) Markov chains

$\supseteq$

$\mathscr{P}^{\mathrm{HM}}$
set of compatible
homogeneous Markov chains

# Lumping



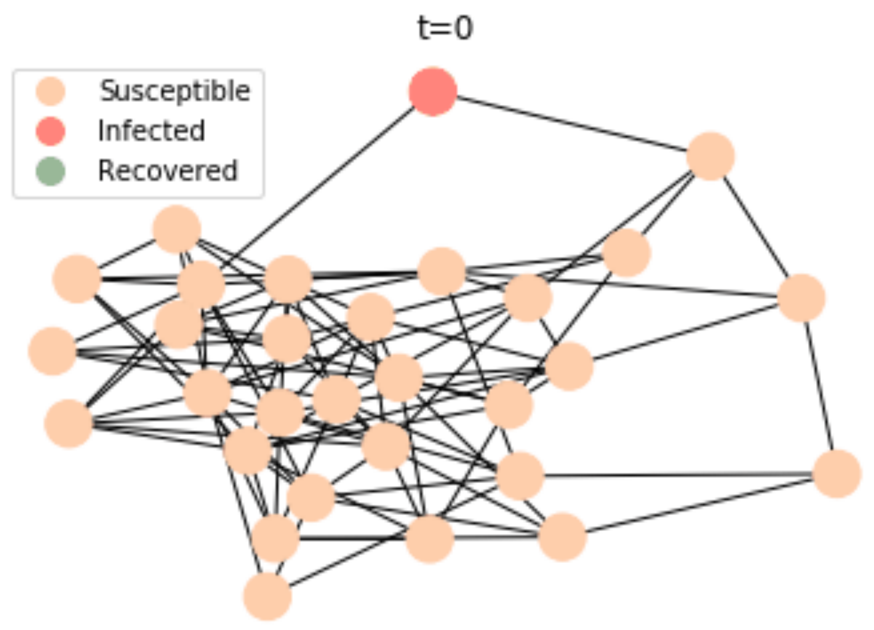
# Lumping

S/I/R per individual

detailed state space $\mathscr{X}$

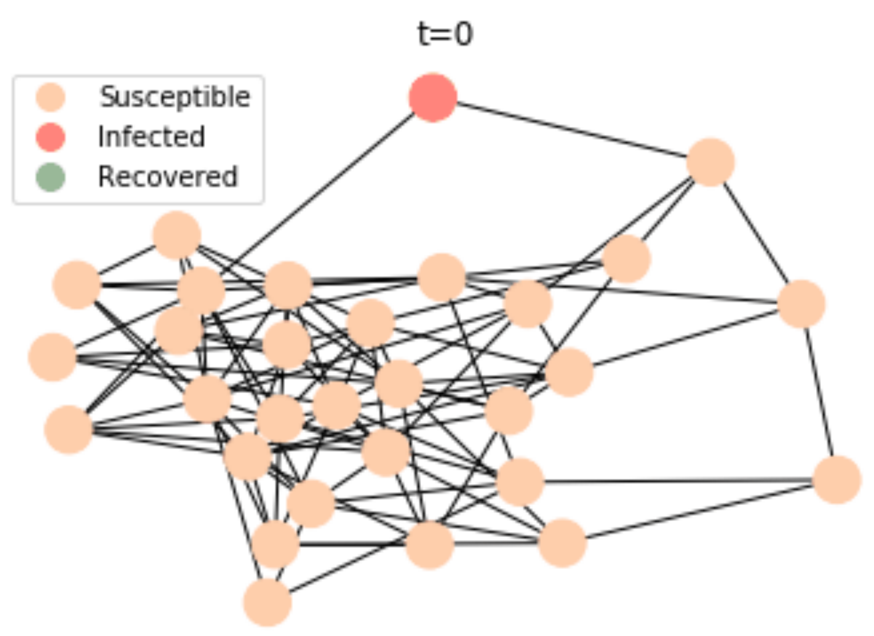
↓

$P$ homogeneous & Markov




https://je-suis-tm.github.io/graph-theory/epidemic-outbreak/

# Lumping




https://je-suis-tm.github.io/graph-theory/epidemic-outbreak/

S/I/R per individual

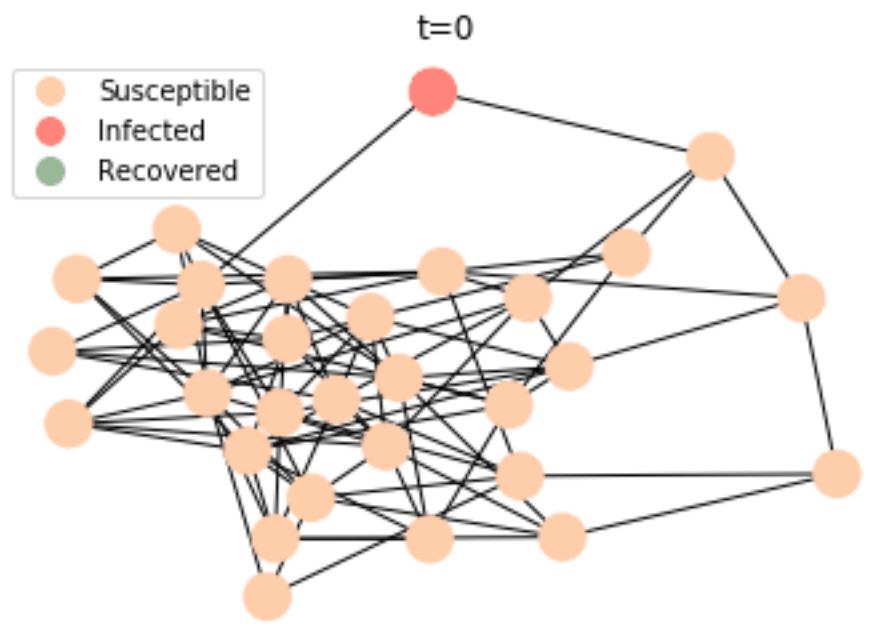
detailed state space $\mathscr{X}$

smaller lumped state space $\hat{\mathscr{X}}$

$P$ homogeneous & Markov

# Lumping

S/I/R per individual

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# individuals S/I/R

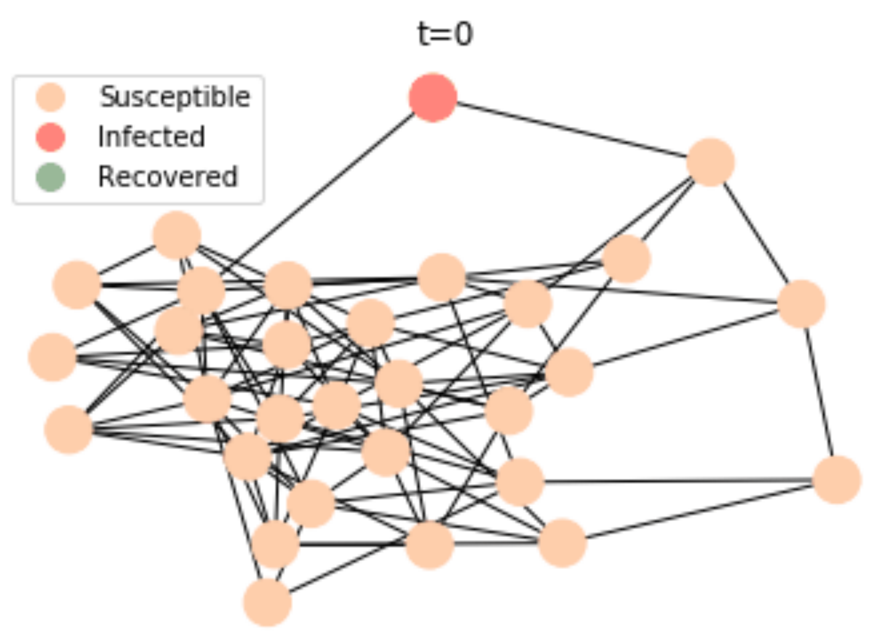
detailed state space $\mathscr{X}$

smaller lumped state space $\hat{\mathscr{X}}$

$P$ homogeneous & Markov

# Lumping

S/I/R per individual

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# individuals S/I/R

detailed state space $\mathscr{X}$

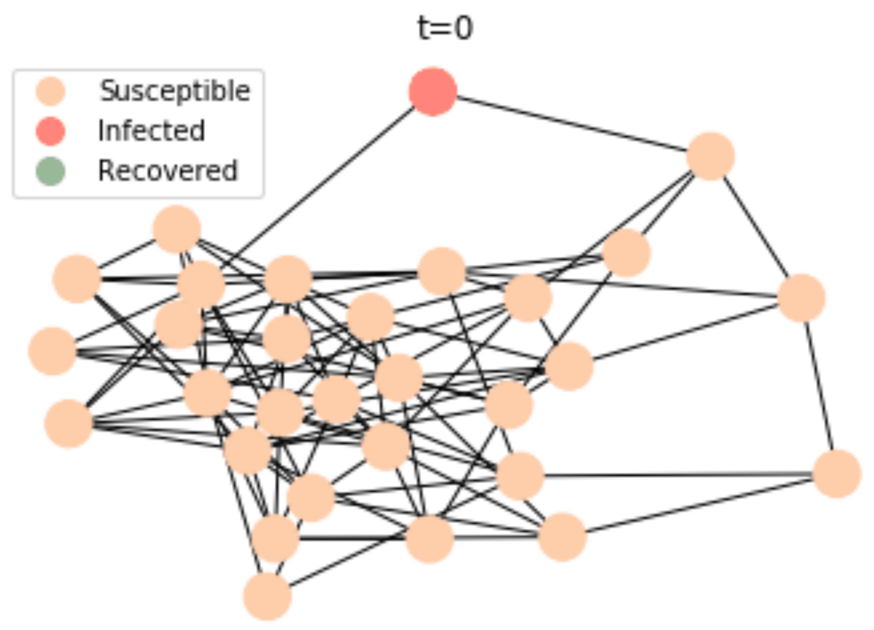
smaller lumped state space $\hat{\mathscr{X}}$

$P$ homogeneous & Markov

$\hat{P}$ not even Markov

# Lumping

S/I/R per individual

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# individuals S/I/R

detailed state space $\mathscr{X}$

smaller lumped state space $\hat{\mathscr{X}}$

$P$ homogeneous & Markov

$\hat{P}$ not even Markov

$p_\square \to \hat{\mathfrak{p}}_\square$

$p_x, p_y, \dots \to \hat{\mathfrak{p}}_{\hat{x}}$

$\hat{\mathscr{P}}$

# Inference for imprecise Markov chains

consider a (measurable) variable $f\colon \mathscr{X}^{\mathbb{N}} \to \overline{\mathbb{R}}$

$$\overline{E}(f) = \sup_{P \in \mathscr{P}} E_P(f) \geq \overline{E}^{\mathrm{M}}(f) = \sup_{P \in \mathscr{P}^{\mathrm{M}}} E_P(f) \geq \overline{E}^{\mathrm{HM}}(f) = \sup_{P \in \mathscr{P}^{\mathrm{HM}}} E_P(f)$$

# Inference for imprecise Markov chains

consider a (measurable) variable $f\colon \mathscr{X}^{\mathbb{N}} \to \overline{\mathbb{R}}$

$$\overline{E}(f) = \sup_{P \in \mathscr{P}} E_P(f) \geq \overline{E}^{\mathrm{M}}(f) = \sup_{P \in \mathscr{P}^{\mathrm{M}}} E_P(f) \geq \overline{E}^{\mathrm{HM}}(f) = \sup_{P \in \mathscr{P}^{\mathrm{HM}}} E_P(f)$$

**finitary variables**

$$g(X_{1:n})$$

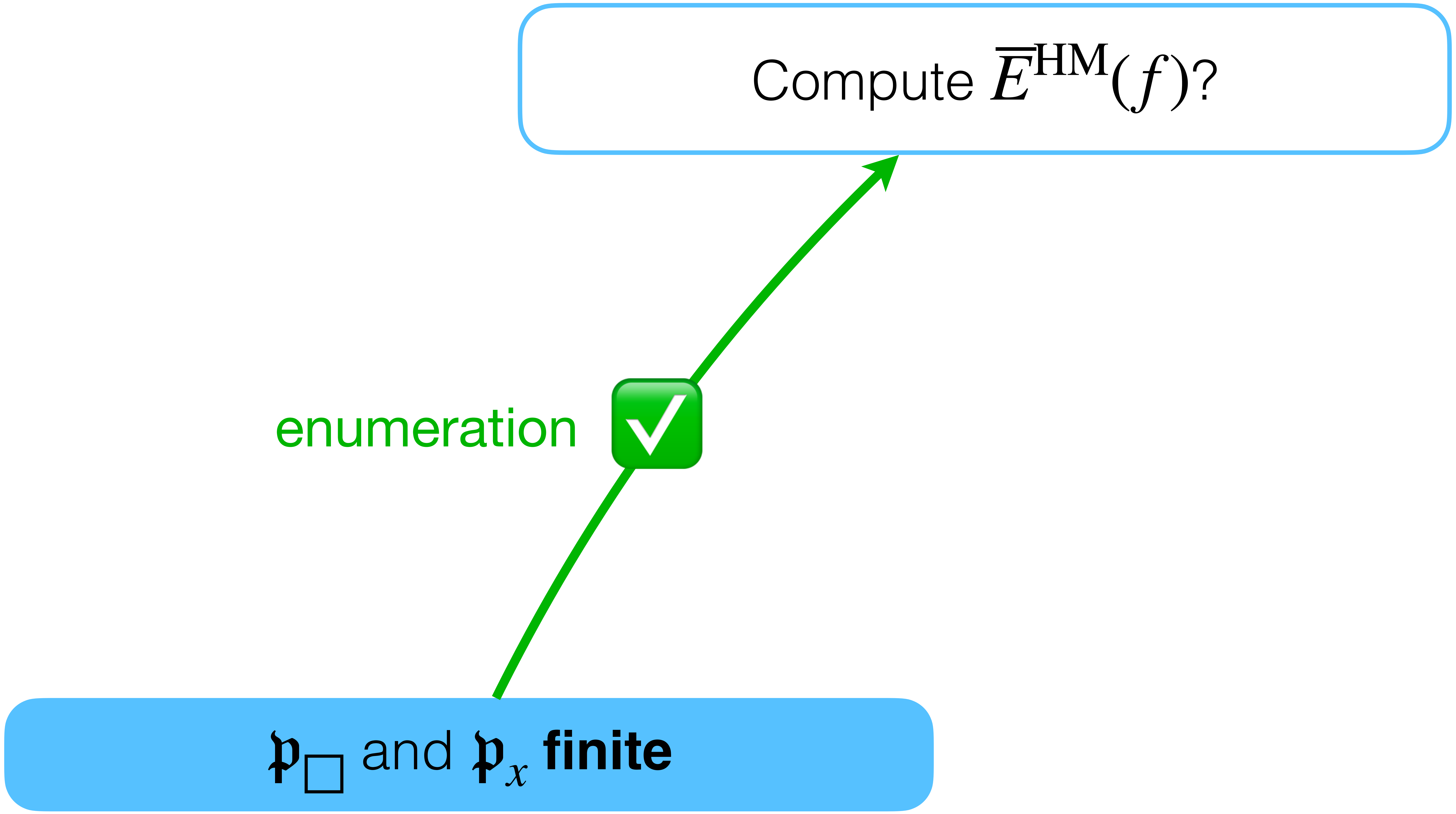
**non-finitary variables**

$$\tau_G\colon \mathscr{X}^{\mathbb{N}} \to \mathbb{R}_{\geq 0} \cup \{+\infty\} \colon (x_n)_{n\in\mathbb{N}} \mapsto \inf\{n \in \mathbb{N} \colon x_n \in G\}$$

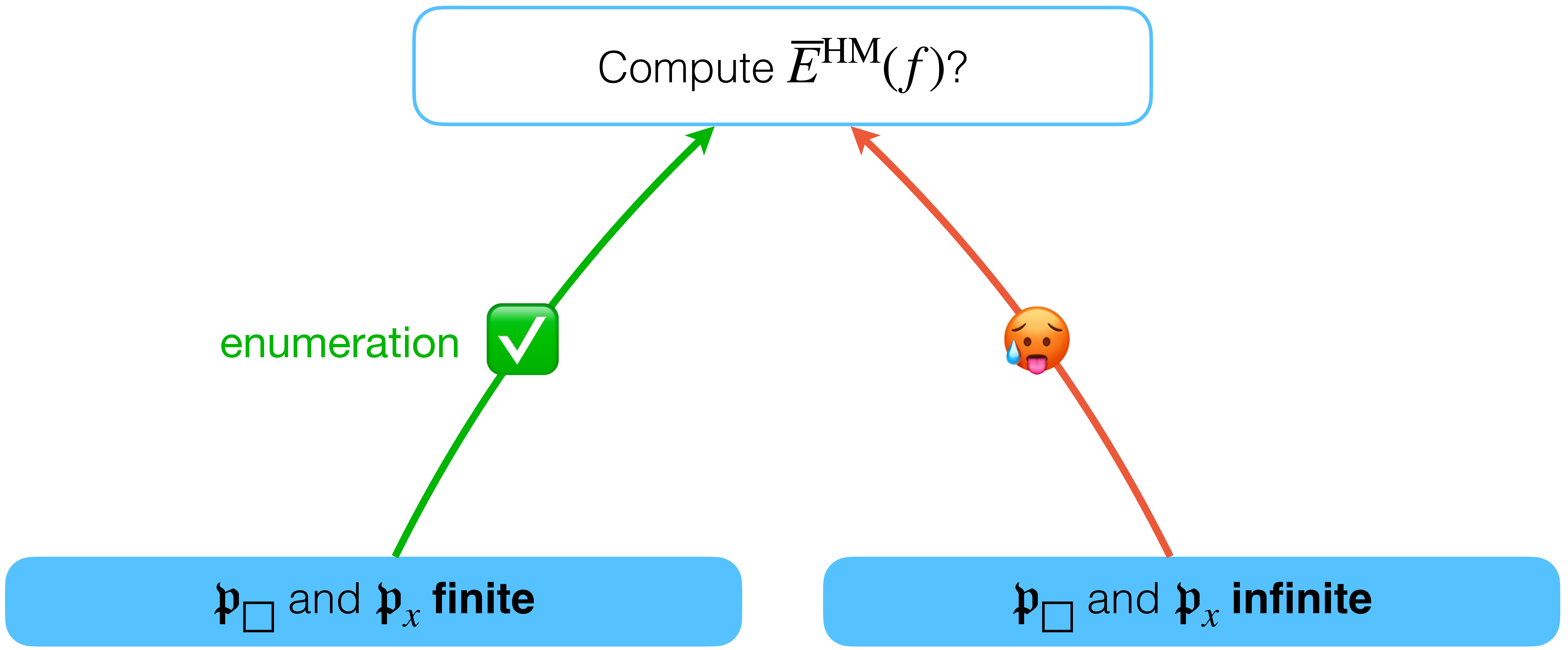
# Inference for $\mathscr{P}^{\mathrm{HM}}$

Compute $\overline{E}^{\mathrm{HM}}(f)$?

# Inference for $\mathscr{P}^{\mathrm{HM}}$



# Inference for $\mathscr{P}^{\mathrm{HM}}$



# Inference for $\mathscr{P}^{\mathrm{M}}$

Compute $\overline{E}^{\mathrm{M}}(f)$?

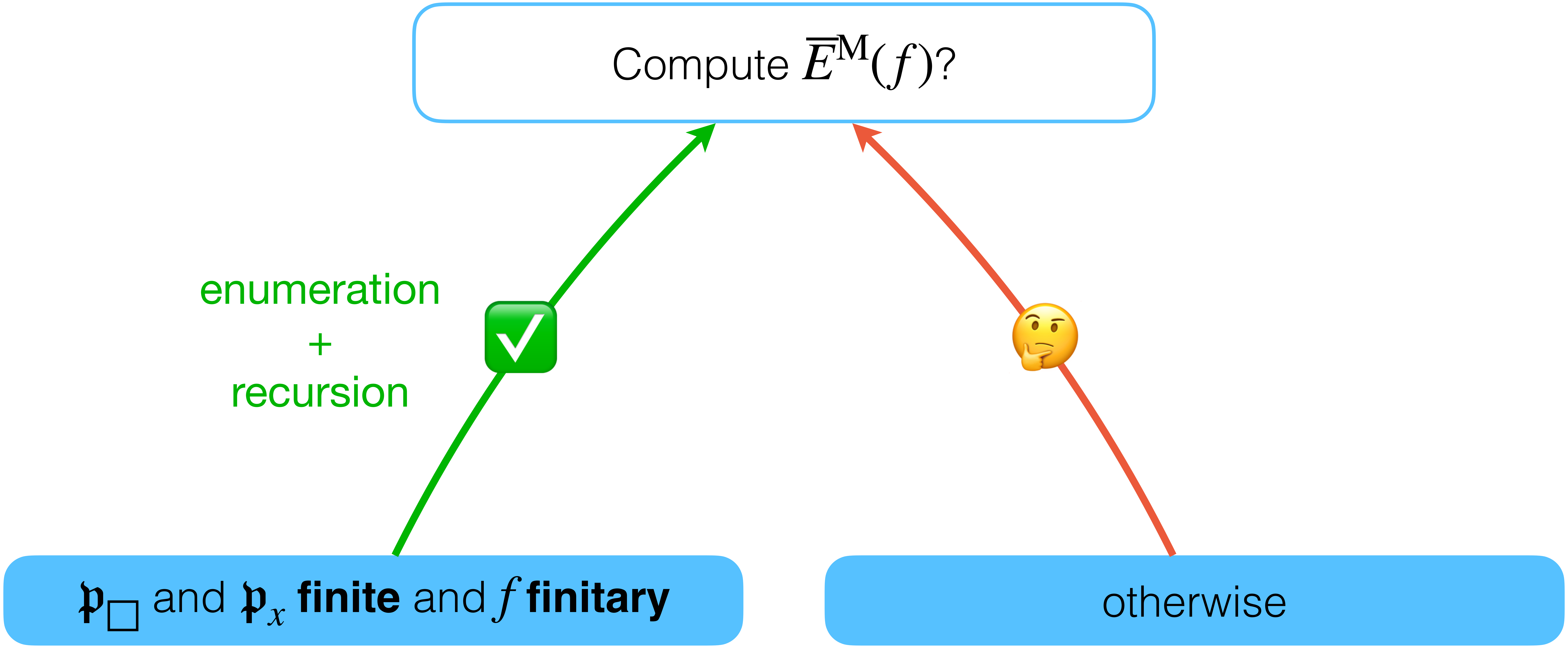
# Inference for $\mathscr{P}^{\mathrm{M}}$

Compute $\overline{E}^{\mathrm{M}}(f)$?

enumeration
+
recursion

$\mathfrak{p}_{\square}$ and $\mathfrak{p}_{x}$ **finite** and $f$ **finitary**

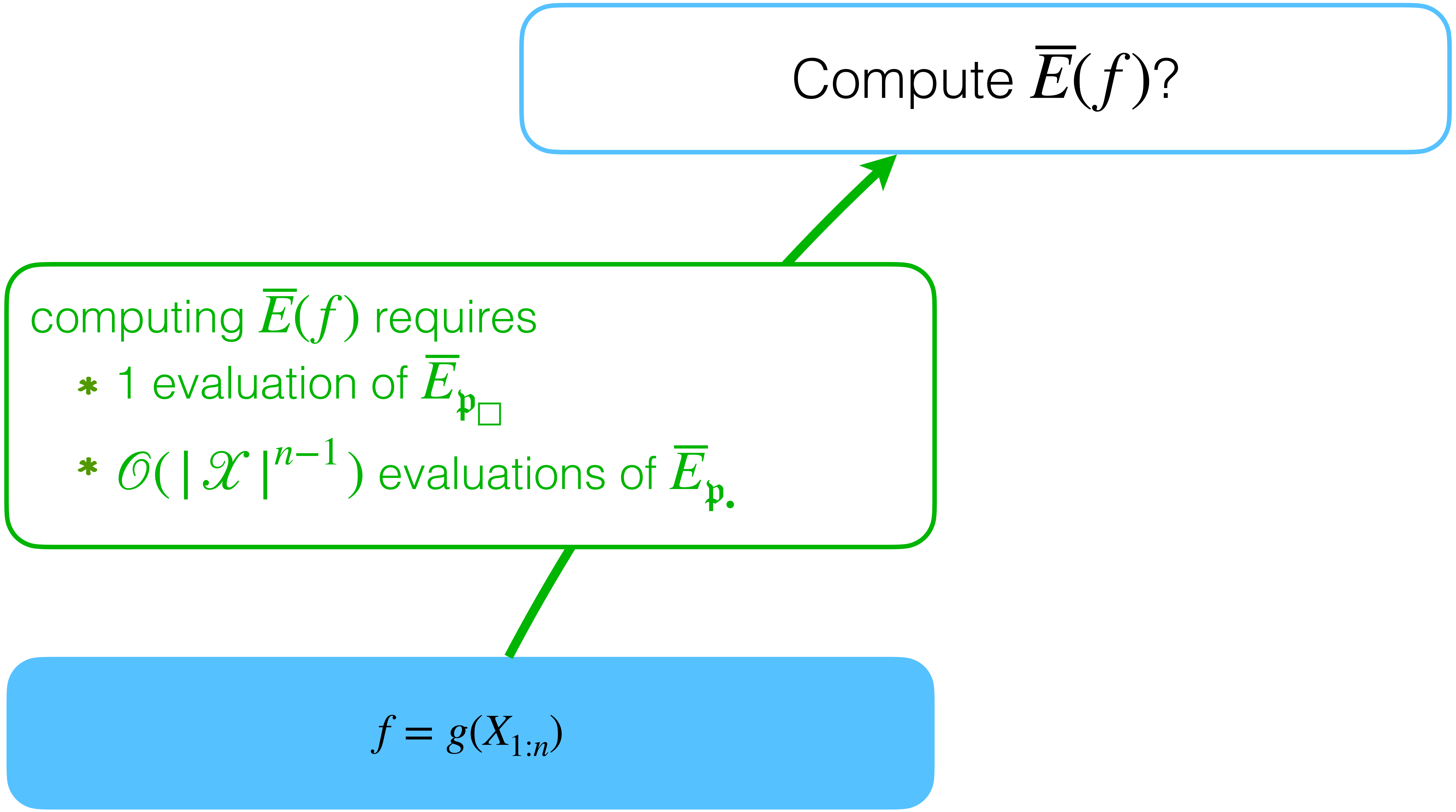
# Inference for $\mathscr{P}^{\mathrm{M}}$



# Inference for $\mathscr{P}$

Compute $\overline{E}(f)$?

# Inference for $\mathscr{P}$



$f$ has a **sum-product decomposition** if

$$f = g_1(X_1) + g_2(X_2)h_1(X_1) + \cdots + g_n(X_n)h_{n-1}(X_{n-1})\cdots h_1(X_1) = \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

$f$ has a **sum-product decomposition** if

$$f = g_1(X_1) + g_2(X_2)h_1(X_1) + \cdots + g_n(X_n)h_{n-1}(X_{n-1})\cdots h_1(X_1) = \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

$g(X_n)$

finite time average $\frac{1}{n}\sum_{k=1}^{n} g(X_k)$

finite horizon reward $\sum_{k=1}^{n} r_k(X_k)$

$\mathbb{1}_{\{S\mathcal{U}^{\leq n}G\}}$ for the time-bounded until event $\{S\mathcal{U}^{\leq n}G\} = \{k \leq n, X_k \in G$ and $X_1, \ldots, X_{k-1} \in S\}$

# Inference for $\mathscr{P}$

Compute $\overline{E}(f)$?

computing $\overline{E}(f)$ requires

- 1 evaluation of $\overline{E}_{\mathfrak{p}_\square}$
- $\mathcal{O}(|\mathscr{X}|^{n-1})$ evaluations of $\overline{E}_{\mathfrak{p}_\bullet}$

computing $\overline{E}(f)$ requires

- 1 evaluation of $\overline{E}_{\mathfrak{p}_\square}$
- $(n-1)|\mathscr{X}|$ evaluations of $\overline{E}_{\mathfrak{p}_\bullet}$

$$f = g(X_{1:n})$$

$$f = \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

# Inference for $\mathscr{P}$

Compute $\overline{E}(f)$?

computing $\overline{E}(f)$ requires

- 1 evaluation of $\overline{E}_{\mathfrak{p}_\square}$
- $\mathcal{O}(|\mathcal{X}|^{n-1})$ evaluations of $\overline{E}_{\mathfrak{p}_\bullet}$

$$h_1, \ldots, h_{n-1} \geq 0$$
$$\Downarrow$$
$$\overline{E}(f) = \overline{E}^{\mathrm{M}}(f)$$

$$f = g(X_{1:n})$$

$$f = \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

$$f = \lim_{n \to +\infty} \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

⚠️ $\Rightarrow$

$$\overline{E}(f) = \lim_{n \to +\infty} \overline{E}\left( \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell) \right)$$

$$f = \lim_{n \to +\infty} \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

⚠️ $\Rightarrow$

$$\overline{E}(f) = \lim_{n \to +\infty} \overline{E}\left( \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell) \right)$$

time average $\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} g(X_k)$

discounted reward $\lim_{n \to +\infty} \sum_{k=1}^{n} \lambda^{k-1} r(X_k)$

$\mathbb{I}_{\{S \mathscr{U} G\}}$ for the until event $\{S \mathscr{U} G\} = \{X_k \in G$ and $X_1, \ldots, X_{k-1} \in S\}$

hitting time $\tau_G = \inf\{n \in \mathbb{N} \colon X_n \in G\}$

$$f = \lim_{n\to+\infty} \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

⚠️ $\Rightarrow$

$$\overline{E}(f) = \lim_{n\to+\infty} \overline{E}\left( \sum_{k=1}^{n} g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell) \right)$$

time average $\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} g(X_k)$

...rd $\lim_{n\to+\infty} \sum_{k=1}^{n} \lambda^{k-1} r(X_k)$

$\mathbb{I}_{\{S\mathcal{U}G\}}$ for ... $\{X_k \in G$ and $X_1, \dots, X_{k-1} \in S\}$

$h_1, h_2, \ldots \geq 0$

$\Downarrow$

$\overline{E}(f)$ 👩‍🎓 $= \overline{E}^{\mathrm{M}}(f)$ ⚙️ $= \overline{E}^{\mathrm{HM}}(f)$

hitting time $\tau_G = \inf\{n \in \mathbb{N} \colon X_n \in G\}$

# How to Make your Markov Chains Robust

## An Imprecise Probabilities Perspective

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# Continuous-time imprecise Markov chains

$$p_{x_{1:n}} : \mathscr{X} \to [0,1] : y \mapsto P(X_{n+1} = y \mid X_1 = x_1, \ldots, X_n = x_n)$$

$$q_{x_{1:n}}^{t_{1:n}} : \mathscr{X} \to \mathbb{R} : y \mapsto \frac{\partial^+}{\partial \Delta} P(X_{t_n+\Delta} = y \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n)\Big|_{\Delta=0}$$

$P$ is **compatible** if $p_\square \in \mathfrak{p}_\square$ and $(\forall n \in \mathbb{N}, t_{1:n} \in \mathbb{R}^n_{\geq 0}, x_{1:n} \in \mathscr{X}^n)\ q_{x_{1:n}}^{t_{1:n}} \in \mathfrak{q}_{x_n}$

$\mathscr{P}$
set of compatible
precise uncertainty models

$\mathscr{P}^{\mathrm{M}}$
set of compatible
(inhomogeneous) Markov chains

$\mathscr{P}^{\mathrm{HM}}$
set of compatible
homogeneous Markov chains

# Limit distributions

Ergodicity

$$\overline{E}_\infty(f) = \lim_{n\to+\infty} \overline{E}(f(X_n) \,|\, X_1 = x)$$

$$\parallel$$

$$\overline{E}^{\mathrm{M}}_\infty(f) = \lim_{n\to+\infty} \overline{E}^{\mathrm{M}}(f(X_n) \,|\, X_1 = x)$$

$$\geq$$

$$\overline{E}^{\mathrm{HM}}_\infty(f) = \lim_{n\to+\infty} \overline{E}^{\mathrm{HM}}(f(X_n) \,|\, X_1 = x)$$

$$\pi_\infty(y) = \lim_{n\to+\infty} P(X_n = y \,|\, X_1 = x)$$

$$E_\infty(f) = \sum_{y\in\mathcal{X}} \pi_\infty(y) f(y)$$

# Limit distributions

| Ergodicity | | Weak ergodicity |
|---|---|---|

$$\overline{E}_{\infty}(f) = \lim_{n\to+\infty} \overline{E}(f(X_n)\,|\,X_1 = x) \quad \geq \quad \overline{E}_{\mathrm{av},\infty}(f) = \lim_{n\to+\infty} \overline{E}\left( \frac{1}{n}\sum_{i=1}^{n} f(X_i) \,\middle|\, X_1 = x \right)$$

$$\rotatebox{90}{=} \qquad\qquad\qquad\qquad \rotatebox{90}{=}$$

$$\overline{E}^{\mathrm{M}}_{\infty}(f) = \lim_{n\to+\infty} \overline{E}^{\mathrm{M}}(f(X_n)\,|\,X_1 = x) \quad \geq \quad \overline{E}^{\mathrm{M}}_{\mathrm{av},\infty}(f) = \lim_{n\to+\infty} \overline{E}^{\mathrm{M}}\left( \frac{1}{n}\sum_{i=1}^{n} f(X_i) \,\middle|\, X_1 = x \right)$$

$$\rotatebox{90}{\geq} \qquad\qquad\qquad\qquad \rotatebox{90}{=}$$

$$\overline{E}^{\mathrm{HM}}_{\infty}(f) = \lim_{n\to+\infty} \overline{E}^{\mathrm{HM}}(f(X_n)\,|\,X_1 = x) \quad = \quad \overline{E}^{\mathrm{HM}}_{\mathrm{av},\infty}(f) = \lim_{n\to+\infty} \overline{E}^{\mathrm{HM}}\left( \frac{1}{n}\sum_{i=1}^{n} f(X_i) \,\middle|\, X_1 = x \right)$$

# Pointwise ergodic theorem

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i) = E_\infty(f) = \sum_{x\in\mathscr{X}} \pi_\infty(x)f(x)$$ almost surely (with probability one)

For **all three types** of imprecise Markov chains:

$$\underline{E}_\infty(f) \leq \underline{E}_{\mathrm{av},\infty}(f) \leq \liminf_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i) \leq \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i) \leq \overline{E}_{\mathrm{av},\infty}(f) \leq \overline{E}_\infty(f)$$

almost surely (with lower probability one)

# How to Make your Markov Chains Robust

## An Imprecise Probabilities Perspective

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