Imprecise (Markov) processes

Beyond finitary variables

Alexander Erreygers
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ImPRooF – 20/09/2022

Model for a system whose *state* X_t

takes values in some *finite* state space $\mathscr{X}$

and changes over time t in some infinite time domain $\mathbb{T} \subseteq \mathbb{R}$ in an **uncertain** manner.

Some notation

1. Fix some set of paths $\Omega \subseteq \mathscr{X}^{\mathbb{T}}$.
2. For all $\mathscr{T} \subseteq \mathbb{T}$, let
$$X_{\mathscr{T}} : \Omega \to \mathscr{X}^{\mathscr{T}} : \omega \mapsto \omega|_{\mathscr{T}}.$$
3. Let $\mathfrak{U}$ the set of all non-empty and finite subsets of $\mathbb{T}$.

The starting point

We consider a set $\mathscr{P}$ of **probability charges** on the algebra of *cylinder events*

$$\mathscr{F} := \left\{ \{X_{\mathscr{U}} \in \tilde{A}\} \colon \mathscr{U} \in \mathfrak{U}, \tilde{A} \in \wp\left(\mathscr{X}^{\mathscr{U}}\right) \right\} \subseteq \wp(\Omega).$$

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$$\updownarrow$$

We consider a set $\mathscr{M}$ of **expectations** on the vector lattice of *finitary variables*

$$\mathbb{F} := \operatorname{span}\left(\{\mathbb{I}_A : A \in \mathscr{F}\}\right) = \left\{f \circ X_{\mathscr{U}} : \mathscr{U} \in \mathfrak{U}, f \in \mathbb{R}^{\mathscr{X}^{\mathscr{U}}}\right\} \subseteq \mathbb{R}^{\Omega}.$$

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$$\downarrow$$

We are interested in the corresponding **upper expectation**

$$\overline{E}_{\mathscr{M}} \colon \mathbb{F} \to \mathbb{R} \colon f \mapsto \overline{E}_{\mathscr{M}}(f) := \sup\{E(f) \colon E \in \mathscr{M}\}.$$

The issue

Many interesting variables are not included in $\mathbb{F}$!

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For example, $\mathbb{F}$ does not include

- the *hitting time* of $H \subseteq \mathscr{X}$, so

$$\tau_H \colon \Omega \to \overline{\mathbb{R}} \colon \omega \mapsto \inf\{t \in \mathbb{T} \colon \omega(t) \in H\}.$$

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For example, $\mathbb{F}$ does not include

- 📈 if $\mathbb{T} = \mathbb{N}$, (the indicator of) the event that 'the limit of the average of $h(X_t)$ exists', so

$$\left\{\omega \in \Omega \colon \limsup_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} h\big(\omega(k)\big) = \liminf_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} h\big(\omega(k)\big)\right\}.$$

The issue

Many interesting variables are not included in $\mathbb{F}$!

For example, $\mathbb{F}$ does not include

- if $[s, r] \subseteq \mathbb{T}$, the 'average of $h(X_t)$ over $[s, r]$', so

$$\frac{1}{r-s}\int_s^r h(X_t)\,\mathrm{d}t\colon \Omega \to \mathbb{R}\colon \omega \mapsto \frac{1}{r-s}\int_s^r h\big(\omega(t)\big)\,\mathrm{d}t$$

  – in fact, this Riemann integral may not even exist!

Two possible solutions

1. Extend every $E \in \mathscr{M}$ to the same larger domain and then take the upper envelope of these extensions.
2. Directly extend $\overline{E}$ to some larger domain.

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Robust modelling and optimisation in stochastic processes using imprecise probabilities, with an application to queueing theory

Stavros Lopatatzidis

Promotoren: prof. dr. ir. G. de Cooman, dr. ir. J. De Bock, prof. dr. ir. S. De Vuyst
Proefschrift ingediend tot het behalen van de graad van
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Upper Expectations for Discrete-Time Imprecise Stochastic Processes: In Practice, They Are All the Same!

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Daniell's extension

An expectation E on $\mathbb{F}$ is called **continuous from above at 0** if

$$\lim_{n \to +\infty} E(f_n) = 0 \quad \text{for all } \mathbb{F}^{\mathbb{N}} \ni (f_n)_{n \in \mathbb{N}} \searrow 0.$$

Daniell's extension

If an expectation E on $\mathbb{F}$ is continuous from above at 0, then it has an extension

$$E^{\mathrm{D}}(f) := \begin{cases} \sup\left\{\lim_{n\to+\infty} E(h_n) \colon \mathbb{F}^{\mathbb{N}} \ni (h_n)_{n\in\mathbb{N}} \searrow \leq f\right\} \\ \inf\left\{\lim_{n\to+\infty} E(h_n) \colon \mathbb{F}^{\mathbb{N}} \ni (h_n)_{n\in\mathbb{N}} \nearrow \geq f\right\} \end{cases} \quad \text{for all } f \in \mathbb{D}$$

on some domain $\mathbb{F} \subseteq \mathbb{D} \subseteq \overline{\mathbb{R}}^{\Omega}$ such that

- 🎲 E^{D} is linear – on the part of $\mathbb{D}$ where this makes sense;
- 🎰 $\inf f \leq E^{\mathrm{D}}(f) \leq \sup f$ for all $f \in \mathbb{D}$.

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- 🎰 $\inf f \leq E^{\mathrm{D}}(f) \leq \sup f$ for all $f \in \mathbb{D}$;
- ↗️ $\mathbb{D}$ and E^{D} are continuous from below, meaning that

$$\lim_{n\to+\infty} E^{\mathrm{D}}(f_n) = E^{\mathrm{D}}\Big(\lim_{n\to+\infty} f_n\Big) \quad \text{for all } \mathbb{D}^{\mathbb{N}} \ni (f_n) \nearrow \text{ such that } E^{\mathrm{D}}(f_1) > -\infty;$$

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on some domain $\mathbb{F} \subseteq \mathbb{D} \subseteq \overline{\mathbb{R}}^{\Omega}$ such that

- 📏 $\mathbb{D}$ includes all $\sigma(\mathscr{F})$-measurable variables that are bounded above or below.

If every $E \in \mathscr{M}$ is continuous from above at 0, we can consider the upper envelope

$$\overline{E}^{\mathrm{D}}_{\mathscr{M}} \colon \mathbb{M}_{\mathrm{b}} \cup \mathbb{M}^{\mathrm{b}} \to \overline{\mathbb{R}} \colon f \mapsto \sup\{E^{\mathrm{D}}(f) \colon E \in \mathscr{M}\},$$

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which

- 🎲 is sublinear on the part of $\mathbb{D}$ where this makes sense;
- 🎰 dominates $\inf$ and is dominated by $\sup$.

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which

- ↗ is continuous from below – provided $\overline{E}^{\mathrm{D}}_{\mathscr{M}}(f_1) > -\infty$;
- ↘ converges conservatively from above:

$$\lim_{n \to +\infty} \overline{E}^{\mathrm{D}}_{\mathscr{M}}(f_n) \geq \overline{E}^{\mathrm{D}}_{\mathscr{M}}(f) \quad \text{for all } (f_n)_{n \in \mathbb{N}} \searrow f \text{ such that } \overline{E}^{\mathrm{D}}_{\mathscr{M}}(f_1) < +\infty.$$

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GSP

Full conglomerability

Enrique Miranda [a] and Marco Zaffalon [b]

[a]Department of Statistics and Operations Research, University of Oviedo, Oviedo, Spain; [b]Istituto Dalle Molle di Studi sull'Intelligenza Artificiale (IDSIA), Lugano, Switzerland

If $\Omega = \mathscr{X}^{\mathbb{N}}$, then every expectation E on $\mathbb{F}$ is trivially continuous from above at 0!

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- ⏱ the *hitting time* of $H \subseteq \mathscr{X}$;
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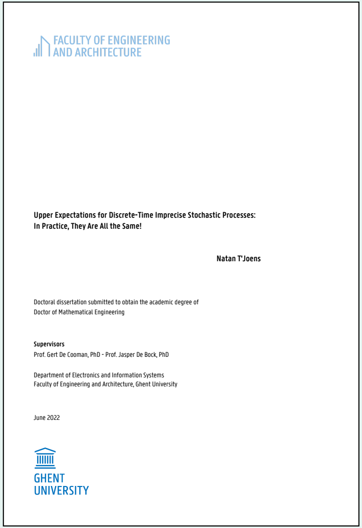

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Under some conditions on $\mathscr{M}$, $\overline{E}^{\mathrm{D}}_{\mathscr{M}}$ is

continuous from above on $\mathbb{F}$.

Proceedings of Machine Learning Research 103:265–275, 2019 ISIPTA 2019

Hitting Times and Probabilities for Imprecise Markov Chains

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updates

A Recursive Algorithm for Computing Inferences in Imprecise Markov Chains

Natan T'Joens(✉), Thomas Krak, Jasper De Bock, and Gert de Cooman

ELIS – FLip, Ghent University, Ghent, Belgium

Sum-Product Laws and Efficient Algorithms for Imprecise Markov Chains

Jasper De Bock[1] **Alexander Erreygers**[1] **Thomas Krak**[2]

$\mathbb{T} = \mathbb{R}_{\geq 0}$

If $\Omega = \mathscr{X}^{\mathbb{R}_{\geq 0}}$, then every expectation E on $\mathbb{F}$ is trivially continuous from above at 0!

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The reason for this is that

$$A \in \sigma(\mathscr{F}) \Leftrightarrow A = \{X_{\mathscr{C}} \in \tilde{C}\} \text{ for countable } \mathscr{C} \subseteq \mathbb{T} \text{ and } \tilde{C} \in \bigtimes_{\mathscr{C}} \wp(\mathscr{X}) \subseteq \wp\left(\mathscr{X}^{\mathscr{C}}\right).$$

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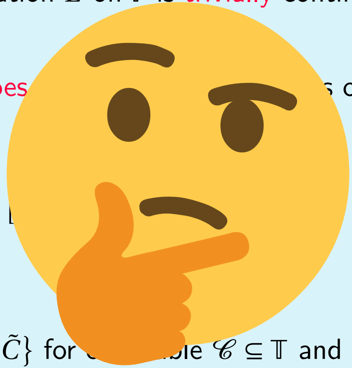

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A *càdlàg* path $\omega \in \mathscr{X}^{\mathbb{R}_{\geq 0}}$ is completely defined by its values on a countable dense subset of $\mathbb{T}$.

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Hence, if $\Omega = \text{càdlàg}(\mathscr{X}^{\mathbb{R}_{\geq 0}})$, then $\mathbb{M}_\mathrm{b} \cup \mathbb{M}^\mathrm{b}$ does include ‘most’ of the variables of interest; for example, it then includes

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🚨 **An expectation E on $\mathbb{F}$ may not be continuous from above at 0!** 🚨

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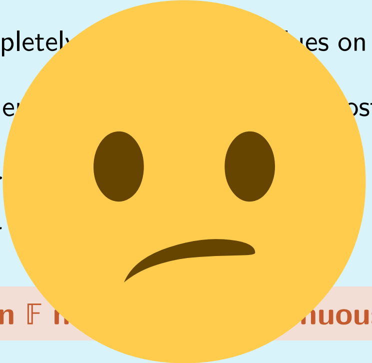

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An expectation E on $\mathbb{F}$ n … nuous from above at 0!

$\mathbb{T} = \mathbb{R}_{\geq 0}, \Omega = \text{càdlàg}(\mathscr{X}^{\mathbb{R}_{\geq 0}})$

Consider an expectation E on $\mathbb{F}$. If there is some $\lambda \in \mathbb{R}_{\geq 0}$ such that

$$\limsup_{s \to t} \frac{E\left(\mathbb{I}_{\{X_t \neq X_s\}}\right)}{|s - t|} \leq \lambda \quad \text{for all } t \in \mathbb{R}_{\geq 0},$$

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NEW Recently, we found a necessary and sufficient condition for this.

$\mathbb{T} = \mathbb{R}_{\geq 0}, \Omega = \text{càdlàg}(\mathscr{X}^{\mathbb{R}_{\geq 0}})$

If there is some $\lambda \in \mathbb{R}_{\geq 0}$ such that

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then every $E \in \mathscr{M}$ is continuous from above at 0.

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ELSEVIER

Markovian imprecise jump processes: Extension to measurable variables, convergence theorems and algorithms

Check for updates

Alexander Erreygers *, Jasper De Bock

Hitting Times for Continuous-Time Imprecise-Markov Chains

Thomas Krak[1]

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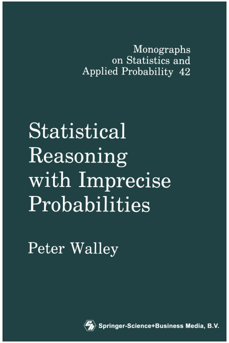
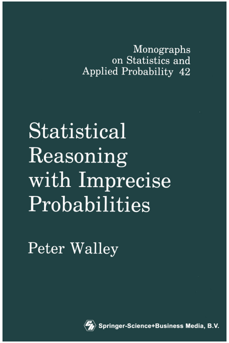
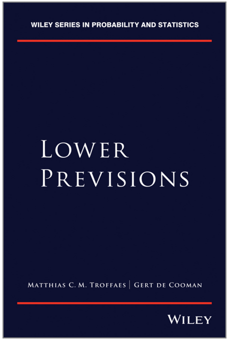


Natural extension

- limited to bounded variables and
- often overly conservative.





Natural extension

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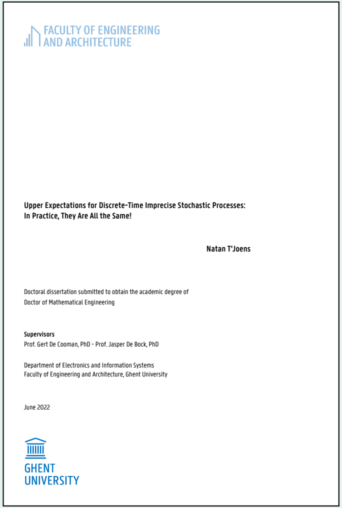
Extension to previsible real variables

- 🛑 limited to real variables and
- 🚧 starts from $\{f \in \mathbb{R}^\Omega : \sup|f| < +\infty\}$.

$\mathbb{T} = \mathbb{N}, \Omega = \mathscr{X}^{\mathbb{T}}$



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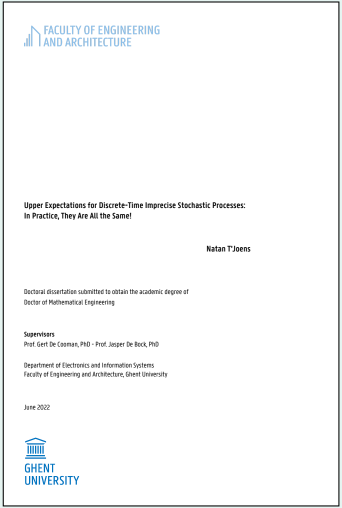
Essentially starting from an upper expectation $\overline{E}$ on $\mathbb{F}$, he argues that

- an extension of $\overline{E}$ to $\overline{\mathbb{R}}^{\Omega}$ should have some desirable continuity properties.

$\mathbb{T} = \mathbb{N}, \Omega = \mathscr{X}^{\mathbb{T}}$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Department of Electronics and Information Systems
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June 2022

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Essentially starting from an upper expectation $\overline{E}$ on $\mathbb{F}$, he argues that

- 😍 an extension of $\overline{E}$ to $\overline{\mathbb{R}}^{\Omega}$ should have some desirable continuity properties;
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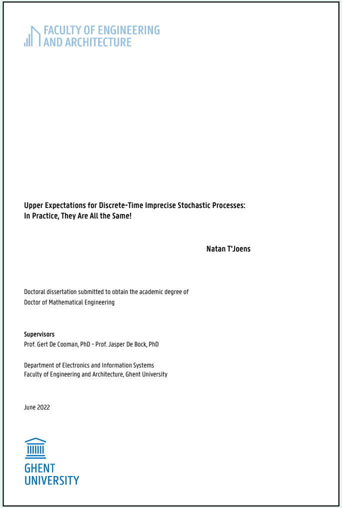
FACULTY OF ENGINEERING AND ARCHITECTURE

Upper Expectations for Discrete-Time Imprecise Stochastic Processes: In Practice, They Are All the Same!

Natan T'Joens

Doctoral dissertation submitted to obtain the academic degree of Doctor of Mathematical Engineering

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$$\overline{E}^{\mathrm{A}}(f) = \inf\left\{ \liminf_{n\to+\infty} \overline{E}(f_n) : \mathbb{F}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \to_{\geq} f, \inf_{n\in\mathbb{N}} \inf f_n > -\infty \right\}.$$

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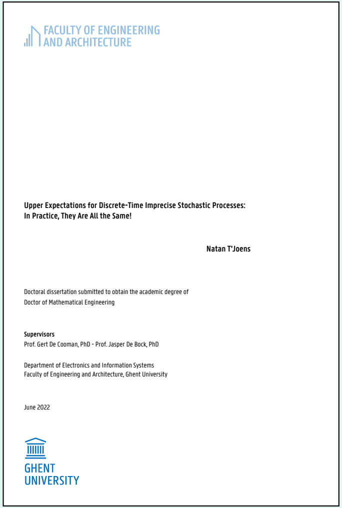

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- ⚗️ in particular, for all $f \in \mathbb{M}_{\mathrm{b}} \cup \mathbb{F}_{\delta}$,

$$\overline{E}^{\mathrm{A}}(f) = \sup\Big\{\lim_{n\to+\infty} \overline{E}(f_n) : \mathbb{F}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow_{\leq} f\Big\}.$$

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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

ROBERT DENK, MICHAEL KUPPER,* and MAX NENDEL

- ‘convex expectations’ instead of only ‘sublinear expectations’
- state space $\mathscr{X}$ can be a Polish space

A robust version of Daniell's extension

An upper expectation $\overline{E}$ on $\mathbb{F}$ is called **continuous from above at 0** if

$$\lim_{n\to+\infty} \overline{E}(f_n) = 0 \quad \text{for all } \mathbb{F}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow 0.$$

A robust version of Daniell's extension

If an upper expectation $\overline{E}$ on $\mathbb{F}$ is continuous from above at 0, then there is a (unique) extension to $\mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}}$ that is

- sublinear
- bounded below by $\inf$ and above by $\sup$
- continuous from below on $\mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}}$ and
- continuous from above on $\mathbb{F}_{\delta,\mathrm{b}} := \{f \in \mathbb{F}_{\delta} \colon \inf f > -\infty\}$,

and this extension is given by

$$\overline{E}^{\star}(f) = \sup\left\{\lim_{n\to+\infty} \overline{E}(f_n) \colon \mathbb{F}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow \leq f\right\} \quad \text{for all } f \in \mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}}.$$

Outline of the proof

the upper expectation $\overline{E}$ on $\mathbb{F}$ is continuous from above at 0

$\updownarrow$

every dominated expectation in

$$\mathscr{M}_{\overline{E}} := \{E \text{ an expectation on } \mathbb{F} \colon (\forall f \in \mathbb{F})\ E(f) \leq \overline{E}(f)\}$$

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Outline of the proof

the upper expectation $\overline{E}$ on $\mathbb{F}$ is continuous from above at 0

$$\Updownarrow$$

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$$\Downarrow$$

$\overline{E}^{\star} \colon \mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}} \colon f \mapsto \sup\{E^{\mathrm{D}}(f) \colon E \in \mathscr{M}_{\overline{E}}\}$ is 🎲, 🎰, ↗️ on $\mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}}$ and ↘️ on $\mathbb{F}_{\delta,\mathrm{b}}$

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unicity of $\overline{E}^\star$ follows from Choquet's Capacitibility Theorem

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is contin… …ove at 0

$\overline{E}^{\star} : \mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}} : f \mapsto \sup\{E^{\mathrm{D}}(f) : E \in$ … 🎰, ↗ on $\mathbb{M}_{\mathrm{b}} \cap \mathbb{M}^{\mathrm{b}}$ and ↘ on $\mathbb{F}_{\delta,\mathrm{b}}$

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NEW

C1. $\mathscr{M} = \{E$ an expectation on $\mathbb{F}$: $(\forall f \in \mathbb{F})\ E(f) \leq \overline{E}_{\mathscr{M}}(f)\}$

C2. every $E \in \mathscr{M}$ is continuous from above at 0

↓

$\overline{E}^{\mathrm{D}}_{\mathscr{M}}$ is ↘ on $\mathbb{F}_\delta$

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If Ω is the set of all paths, (C2) is always satisfied!

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If Ω is the set of all paths, (C2) is always satisfied!

If Ω is the set of all **càdlàg** paths, (C2) is **not** always satisfied!

C1. $\mathscr{M} = \{E \text{ an expectation on } \mathbb{F} \colon (\forall f \in \mathbb{F})\ E(f) \leq \overline{E}_{\mathscr{M}}(f)\}$

C2. every $E \in \mathscr{M}$ is continuous from above at 0

↓

$\overline{E}^{\mathrm{D}}_{\mathscr{M}}$ is $\searrow$ on $\mathbb{F}_{\delta}$

↓

$$\overline{E}^{\mathrm{D}}_{\mathscr{M}}(f) = \sup\Big\{\lim_{n\to+\infty} \overline{E}(f_n) \colon \mathbb{F}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow\leq f\Big\} \text{ for all } f \in \mathbb{M}_{\mathrm{b}}$$

- Can we also extend an upper expectation $\overline{E}$ on $\mathbb{F}$ directly in case $\mathbb{T} = \mathbb{R}_{\geq 0}$?
- What about the case of non-finite state spaces and non-bounded variables?
- Are modifications or path regularity a thing in robust finance?

$\mathbb{T} = \mathbb{R}_{\geq 0}, \Omega = \text{càdlàg}(\mathscr{X}^{\mathbb{R}_{\geq 0}})$

NEW

Consider a countable state space $\mathscr{X}$.

For all $\mathscr{U} = \{t_1, \dots, t_n\} \in \mathfrak{U}$ – with $t_1 < \cdots < t_n$ – let

$$\eta_{\mathscr{U}} := \sum_{k=2}^{n} \mathbb{I}_{\{X_{t_{k-1}} \neq X_{t_k}\}} \in \mathbb{F}.$$

Then an expectation E on $\mathbb{F}$ is continuous from above at 0 if and only if

R1. for all $t \in \mathbb{R}_{\geq 0}$,

$$\lim_{r \searrow t} P_E(\{X_t \neq X_r\}) = 0;$$

R2. for all $n \in \mathbb{N}$,

$$\lim_{k \to +\infty} \sup\{P_E(\{\eta_{\mathscr{U}} \geq k\}) : \mathscr{U} \in \mathfrak{U}, \max \mathscr{U} \leq n\} = 0,$$

where P_E is the corresponding probability charge on $\mathscr{F}$.