Convex expectations for countable-state uncertain processes with càdlàg sample paths

Alexander Erreygers[a]

[a] *Ghent University, Technologiepark-Zwijnaarde 125, 9000, Ghent, Belgium*

Abstract

This work investigates convex expectations, mainly in the setting of uncertain processes with countable state space. In the general setting it shows how, under the assumption of downward continuity, a convex expectation on a linear lattice of bounded functions can be extended to a convex expectation on the measurable extended real functions. This result is especially relevant in the setting of uncertain processes: there, an easy way to obtain a convex expectation on the linear lattice of finitary bounded functions is to combine an initial convex expectation with a convex transition semigroup. Crucially, this work presents a sufficient condition on this semigroup which guarantees that the induced convex expectation is downward continuous, so that it can be extended to the set of measurable extended real functions. To conclude, this work looks at existing results on convex transition semigroups from the point of view of the aforementioned sufficient condition, in particular to construct a sublinear Poisson process.

Keywords: downward continuity, Daniell–Kolmogorov Extension Theorem, sublinear Poisson process

1. Introduction

This article is part of the recent push towards generalising the theory of (continuous-time) Markov processes from the measure-theoretic framework [1–5] to the two closely-related frameworks of nonlinear expectations and imprecise probabilities, which both intend to model uncertainty in a (more) robust manner.

The framework of nonlinear—sublinear or even convex—expectations was initially put forward by Peng [6], and has gained a lot of traction in the setting of robust (mathematical) finance. It dealt with Markov processes from its conception, and since then the following (and quite possibly more) Markov processes have been generalised to this framework: those with finite state space [7], countable state space [8] and $\mathbb{R}^d$ as state space [6], Lévy processes with $\mathbb{R}^d$ as state space [9–11] and Feller processes with a Polish state space [12]. Most authors only consider bounded functions on the set of *all* paths that are measurable with respect to the product σ-algebra, which is a domain that is not very rich. Neufeld and Nutz [10] are a notable exception to this; while they do assume càdlàg paths, they essentially never go beyond functions of the state in a single time point.

The theory of imprecise probabilities, which is actually a collection of frameworks including those of coherent lower/upper previsions and sets of (coherent conditional) probabilities, was popularised by Walley [13]. This theory often comes with a subjectivist interpretation in terms of betting behaviour, but can also be thought of as allowing partial specification of linear (or precise) uncertainty models. In this framework much has been done for Markov processes with finite state space [14–18], but to the best of my knowledge, the only work regarding a non-finite state space is mine on the Poisson process [19].

This work builds on and continues the aforementioned work by investigating convex expectations for countable-state uncertain processes in general and Markov processes in particular, with a particular focus

Email address: alexander.erreygers@ugent.be (Alexander Erreygers)

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on the usefulness of their domain. On the one hand, it builds on and extends some of the results in the framework of nonlinear expectations, while on the other hand it essentially extends the approach in [17] from the finite-state and sublinear setting to the countable-state and convex one. The remainder of this work is structured as follows.

Section 2 rids the extension results in [20, Section 3] for downward continuous convex expectations from the restriction that all functions in the extended domain are bounded. The key result there is Theorem 1, which improves on Theorem 3.10 in [20]. Theorem 2, the final result in Section 3, establishes a more useful version of the robust Daniell–Kolmogorov Extension Theorem [20, Section 4] in the specific case of a countable state space: the domain is changed from the bounded measurable functions on the set of *all* paths to the much richer set of extended real measurable functions on the set of *càdlàg* paths.

Section 4 investigates how to construct a 'consistent collection of finite-dimensional (downward continuous) convex expectations', which forms the starting point in Theorem 2. This then leads to the final result in Theorem 4 that given an initial downward continuous convex expectation and a convex Markov semigroup that has uniformly bounded rate, there is a corresponding convex expectation on (a large part of) the set of extended real measurable functions on the set of càdlàg paths.

The present article is a significantly extended version of an earlier conference contribution by the author [21]. Section 2 is largely the same as its counterpart in the conference contribution, although the proofs there were hidden away in the Supplementary Material. In contrast, Sections 3 and 4 differ significantly from their counterparts in [21]: instead of being limited to sublinear expectations, all results now deal with the more general convex expectations. Section 5 has been expanded to incorporate some of my recent results on uniformly continuous semigroups of transition operators [22], which in its turn allowed me to simplify the exposition in Section 6.

2. Nonlinear expectations and their extensions

A (linear) expectation is a linear, normed and order preserving real—or even extended real—functional on a suitable domain. Examples include the Lebesgue integral with respect to a probability measure [4, Chapter 8] and (linear) previsions in the style of de Finetti [23]—see also [24]. A nonlinear expectation, then, generalises this notion by relaxing the requirement of linearity. Two examples of such classes of functionals are the convex expectations that appear in (robust) mathematical finance [6] and the coherent upper previsions that are at the core of the theory of imprecise probabilities [13, 25]; see [26] and references therein for connections between the two. In Section 2.1 we will formally introduce these well-known notions for domains that may include unbounded and even extended real functions, while Section 2.2 deals with extending these nonlinear expectations in such a way that their properties are (partially) preserved. First, however, we introduce some necessary terminology and notation.

Consider some (non-empty) sets $\mathcal{Y}, \mathcal{Z}$. We denote the set of all maps from $\mathcal{Y}$ to $\mathcal{Z}$ by $\mathcal{Z}^{\mathcal{Y}}$. For any real-valued function f on $\mathcal{Y}$, we let

$$\|f\|_\infty := \sup|f| = \sup\{|f(y)|\colon y \in \mathcal{Y}\},$$

and we say that f is *bounded* whenever $\|f\|_\infty < +\infty$. We collect the bounded real-valued functions on $\mathcal{Y}$ in $\mathcal{L}(\mathcal{Y}) \subseteq \mathbb{R}^{\mathcal{Y}}$. The bounded real-valued functions on $\mathcal{Y}$ include the indicator functions: for any subset Y of $\mathcal{Y}$, the corresponding *indicator* $1_Y \in \mathcal{L}(\mathcal{Y})$ maps $y \in \mathcal{Y}$ to 1 if $y \in Y$ and to 0 otherwise; for any $y \in \mathcal{Y}$, we shorten $1_{\{y\}}$ to 1_y.

Fix some non-empty subset $\mathcal{D}$ of $\overline{\mathbb{R}}^{\mathcal{Y}}$. An (extended real) *functional* Φ (on $\mathcal{D}$) is an extended real map on $\mathcal{D}$. We call a functional Φ *positively homogeneous* if $\Phi(\mu f) = \mu\Phi(f)$ for all $f \in \mathcal{D}$ and $\mu \in \mathbb{R}_{\geq 0}$ such that $\mu f \in \mathcal{D}$, and *homogeneous* if the same equality holds for all $f \in \mathcal{D}$ and $\mu \in \mathbb{R}$ such that $\mu f \in \mathcal{D}$. Furthermore, we call a functional Φ *convex* if $\Phi(\lambda f + (1-\lambda)g) \leq \lambda\Phi(f) + (1-\lambda)\Phi(g)$ for all $f, g \in \mathcal{D}$ and $\lambda \in [0,1]$ such that $\lambda f + (1-\lambda)g$ is meaningful[2] and in $\mathcal{D}$ and $\lambda\Phi(f) + (1-\lambda)\Phi(g)$ is meaningful, *subadditive*

[2]We follow the standard conventions in that the sum or difference of two extended reals is meaningful if it does not lead to $(+\infty) - (+\infty)$ or $(+\infty) + (-\infty)$—see for example [25, Appendix D], [4, Definition 4.8] or [27, Table 8.1].

if $\Phi(f+g) \leq \Phi(f)+\Phi(g)$ for all $f,g \in \mathcal{D}$ such that $f+g$ is meaningful and in $\mathcal{D}$ and $\Phi(f)+\Phi(g)$ is meaningful, and *additive* if this inequality always holds with equality whenever it is meaningful. Finally, Φ is said to be *order preserving* (sometimes also 'monotone' or 'isotone') if $\Phi(f) \leq \Phi(g)$ for all $f,g \in \mathcal{D}$ such that $f \leq g$.

We are especially interested in the behaviour of (order-preserving) functionals with respect to monotone sequences. A sequence $(f_n)_{n\in\mathbb{N}}$ of extended real functions on $\mathcal{Y}$ is said to be *monotone* if it is either increasing or decreasing, where the sequence is *increasing* if $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$. Any monotone sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a limit $\lim_{n\to+\infty} f_n \in \overline{\mathbb{R}}^{\mathcal{Y}}$; $(f_n)_{n\in\mathbb{N}} \nearrow f$ denotes an increasing sequence with pointwise limit f, while $(f_n)_{n\in\mathbb{N}} \searrow f$ denotes a sequence that decreases to f.

Now consider a functional Φ on $\mathcal{D}$. If Φ is order preserving, then for any monotone sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$, the derived sequence $(\Phi(f_n))_{n\in\mathbb{N}}$ is also monotone, and therefore converges to a limit; if furthermore the pointwise limit f of $(f_n)_{n\in\mathbb{N}}$ belongs to the domain $\mathcal{D}$, then

$$\lim_{n\to+\infty} \Phi(f_n) = \inf\{\Phi(f_n)\colon n \in \mathbb{N}\} \geq \Phi(f).$$

We call the (not necessarily order-preserving) functional Φ *downward continuous*[3] *on* $\mathcal{C} \subseteq \mathcal{D}$ if for all $\mathcal{C}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow f \in \mathcal{C}$, $\lim_{n\to+\infty} \Phi(f_n) = \Phi(f)$, and simply *downward continuous* if it is downward continuous on $\mathcal{D}$. Similarly, we call Φ *upward continuous*[4] *on* $\mathcal{C} \subseteq \mathcal{D}$ if for all $\mathcal{C}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \nearrow f \in \mathcal{C}$, $\lim_{n\to+\infty} \Phi(f_n) = \Phi(f)$, and simply *upward continuous* if it is upward continuous on $\mathcal{D}$.

2.1. Nonlinear, convex, sublinear and linear expectations

A *nonlinear expectation* is a functional Φ whose domain $\mathcal{D}$, a subset of $\overline{\mathbb{R}}^{\mathcal{Y}}$, includes all constant real functions on $\mathcal{Y}$ and which is order preserving and *constant preserving* (or 'normalised'), meaning that $\Phi(\mu) = \mu$[5] for all $\mu \in \mathbb{R}$. Due to its properties, any nonlinear expectation Φ dominates the infimum and is dominated by the supremum; hence, a nonlinear expectation Φ with domain $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ is a real functional.

This contribution exclusively deals with nonlinear expectations that are convex, sublinear or even linear. A *convex expectation* is a convex nonlinear expectation, a *sublinear expectation* is a sublinear—so subadditive and positively homogeneous—nonlinear expectation and a *(linear) expectation* is a nonlinear expectation that is, well, linear; we will always denote convex, sublinear and linear expectations by (variants on) $\check{E}$, $\overline{E}$ and E, respectively. Clearly, any linear expectation is a sublinear expectation as well, and any sublinear expectation is also a convex one. While these notions are well-established in the theory of nonlinear expectations—see for example [6, Section 2.2], [20, Section 1] or [8, Section 2]—they are, to the best of my knowledge, always stated for domains $\mathcal{D}$ of bounded real functions.

Example 1. For all examples in this section, we'll assume $\mathcal{Y} = \mathbb{N} = \{1, 2, \dots\}$ and consider the set of bounded functions for the domain $\mathcal{D} \coloneqq \mathcal{L}(\mathbb{N})$. For any natural number n, we define the corresponding real functionals

$$\check{E}_n\colon \mathcal{L}(\mathbb{N}) \to \mathbb{R}\colon f \mapsto \sup\{f(m)\colon m \in \mathbb{N}, m \leq n\} \cup \{f(m) - \arctan(m)\colon m \in \mathbb{N}, m > n\},$$
$$\overline{E}_n\colon \mathcal{L}(\mathbb{N}) \to \mathbb{R}\colon f \mapsto \sup\{f(m)\colon m \in \mathbb{N}, m \leq n\}$$

and

$$E_n\colon \mathcal{L}(\mathbb{N}) \to \mathbb{R}\colon f \mapsto f(n).$$

These are clearly constant and order preserving, and therefore nonlinear expectations. Furthermore, it is not difficult to verify that $\check{E}_n$ is convex, $\overline{E}_n$ is sublinear and E_n is linear, so these are convex, sublinear and linear expectations, respectively. Note that for all $n \in \mathbb{N}$ and $f \in \mathcal{L}(\mathbb{N})$, $E_n(f) \leq \overline{E}_n(f) \leq \check{E}_n(f)$.

[3] The term 'continuous from above' is also used frequently.
[4] Some authors prefer the terminology 'continuous from below'.
[5] We identify any $\mu \in \mathbb{R}$ with the corresponding constant function 'μ' on $\mathcal{Y}$ whose range is $\{\mu\}$.

Instead of a convex or sublinear expectation $\check{E}$ on $\mathcal{D}$, it may be more convenient or appropriate to investigate the corresponding conjugate functional Φ on $-\mathcal{D} := \{-f : f \in \mathcal{D}\}$, defined by the conjugacy relation

$$\Phi(f) := -\check{E}(-f) \quad \text{for all } f \in -\mathcal{D}.$$

Clearly, the conjugate functional of a convex expectation is a nonlinear expectation that is concave instead of convex, while that of a sublinear expectation is a nonlinear expectation that is superlinear instead of sublinear.

If the domain $\mathcal{D}$ of the sublinear expectation $\check{E}$ is a linear space of real functions, its properties make it a 'coherent upper prevision'.[6] Coherent upper previsions have an interesting interpretation (and definition) in terms of rationality requirements on supremum selling prices of bets, but pursuing this would lead us too far astray; I invite the interested reader to consult the excellent manuscripts by Walley [13] or Troffaes and de Cooman [25], see also [28] for a more gentle introduction. In the context of this article, the following definition in terms of three coherence properties will suffice [25, Theorems 4.15 and 13.34]; note that for this definition, it is essential that the domain $\mathcal{D}$ is a linear space of real functions.

Definition 1. Let $\mathcal{D} \subseteq \mathbb{R}^{\mathcal{Y}}$ be a linear space. A *coherent upper prevision* $\overline{E}$ *(on* $\mathcal{D}$*)* is a sublinear real functional on $\mathcal{D}$ such that $\overline{E}(f) \leq \sup f$ for all $f \in \mathcal{D}$.

Lemma 1. *Let* $\mathcal{D} \subseteq \mathbb{R}^{\mathcal{Y}}$ *be a linear space that contains all constant functions. A real functional* $\overline{E}$ *on* $\mathcal{D}$ *is a sublinear expectation if and only if it is a coherent upper prevision.*

Proof. That any coherent upper prevision is a sublinear expectation follows immediately from [25, Theorem 4.13 and Theorem 13.31].

For the converse implication, assume that $\overline{E}$ is a sublinear expectation. Then $\overline{E}$ is a coherent upper prevision because it is sublinear and bounded by the supremum because $\overline{E}$ is order and constant preserving:

$$\overline{E}(f) \leq \overline{E}(\sup f) = \sup f \quad \text{for all } f \in \mathcal{D}.$$

Example 2. Since $\mathcal{L}(\mathbb{N})$ is a linear space that contains all constant functions, for all $n \in \mathbb{N}$ the sublinear expectation $\overline{E}_n$ as defined in Example 1 is a coherent upper prevision by Lemma 1. Another example is the so-called *vacuous coherent upper prevision* [13, Section 2.3.7], which in general is given by

$$\mathcal{D} \to \mathbb{R} : f \mapsto \sup f.$$

In the remainder of this section, we fix some linear space $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ that contains all constant real functions. Recall from before that this means that any convex expectation $\check{E}$ on $\mathcal{D}$ is real valued. Moreover, it turns out that a convex expectation is then always convergent with respect to uniform convergence—for proofs, see [25, Theorem 4.13 (xiii)] and [8, Lemma 2.5].

Lemma 2. *Suppose* $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ *is a linear space that includes all constant functions, and consider a convex expectation* $\check{E}$ *on* $\mathcal{D}$*. Then for any sequence* $(f_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ *that converges uniformly to some* $f \in \mathcal{D}$*, meaning that* $\lim_{n\to+\infty} \|f_n - f\|_\infty = 0$*,*

$$\lim_{n\to+\infty} \check{E}(f_n) = \check{E}(f).$$

Pointwise continuity is equivalent to uniform continuity whenever $\mathcal{Y}$ is finite, so in that case Lemma 2 implies that any convex expectation $\check{E}$ on $\mathcal{D}$ is downward and upward continuous. Things are more complicated when $\mathcal{Y}$ is infinite, but then we can resort to a necessary and sufficient condition for downward continuity; to state it, we need to introduce some additional notation. Let $\mathbb{E}_{\mathcal{D}}$ be the set of all linear expectations on $\mathcal{D}$. With any convex expectation $\check{E}$ on $\mathcal{D}$, we associate the conjugate function

$$\check{E}^* : \mathbb{E}_{\mathcal{D}} \to \mathbb{R}_{\geq 0} \cup \{+\infty\} : E \mapsto \sup\{E(f) - \check{E}(f) : f \in \mathcal{D}\},$$

[6] Its conjugate functional is what is known as a 'coherent lower prevision'. Note that for many reference works in the field of imprecise probabilities, including [13, 25, 28], 'coherent lower previsions' are the primal object under study.

and we collect all linear expectations on $\mathcal{D}$ whose image under this conjugate function is real in

$$\mathbb{E}_{\check{E}} := \{E \in \mathbb{E}_{\mathcal{D}} : \check{E}^*(E) < +\infty\}.$$

This conjugate function $\check{E}^*$ and the derived set $\mathbb{E}_{\check{E}}$ play an important role in the following two results.

The first result, taken from [20, Lemma 3.2], links the downward continuity of a convex expectation $\check{E}$ to the downward continuity of the linear expectations E in $\mathbb{E}_{\check{E}}$—so those with $\check{E}^*(E) < +\infty$; Miranda and Zaffalon [29, Proposition 5.1.2] give a similar result for the particular case of sublinear expectations.

Lemma 3. *Suppose $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ is a linear space that includes all constant functions. A convex expectation $\check{E}$ on $\mathcal{D}$ is downward continuous if and only if every linear expectation $E \in \mathbb{E}_{\check{E}}$ is downward continuous.*

Lemma 3 is crucial in the proof of Theorem 3.10 in [20], and consequently also in our extension of it. We will see why in due course.

The second result in which we encounter $\mathbb{E}_{\check{E}}$ is the following representation result, taken from [20, Lemma 2.4]; for sublinear expectations, it is known as the *Upper Envelope Theorem* [25, Theorem 4.38].

Lemma 4. *Suppose $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ is a linear space that includes all constant functions. For any convex expectation $\check{E}$ on $\mathcal{D}$,*

$$\check{E}(f) = \max\big\{E(f) - \check{E}^*(E) : E \in \mathbb{E}_{\check{E}}\big\} \quad \text{for all } f \in \mathcal{D}.$$

Moreover, for any sublinear expectation $\overline{E}$ on $\mathcal{D}$,

$$\overline{E}(f) = \max\big\{E(f) : E \in \mathbb{E}_{\overline{E}}\big\} \quad \text{for all } f \in \mathcal{D},$$

where now

$$\mathbb{E}_{\overline{E}} = \{E \in \mathbb{E}_{\mathcal{D}} : \overline{E}^*(E) = 0\} = \{E \in \mathbb{E}_{\mathcal{D}} : (\forall f \in \mathcal{D})\ E(f) \leq \overline{E}(f)\}.$$

Lemma 4 leads to the sensitivity analysis interpretation of convex and sublinear expectations. On this interpretation, we assume that our uncertainty can be accurately modelled by a linear expectation, but we do not have enough information to determine or specify this expectation exactly. For a convex expectation $\check{E}$, we can then think of $\check{E}^*(E)$ as a penalty which indicates our degree of belief that the linear expectation E is the 'correct' one. For a sublinear expectation $\overline{E}$, then, we can think of $\mathbb{E}_{\overline{E}}$ as the set of all 'feasible' linear expectations. Note that we can also go the other way around: we can define a sublinear (or convex) expectation by specifying some set $\mathcal{E} \subseteq \mathbb{E}_{\mathcal{D}}$ of linear expectations (and some penalty function $\alpha : \mathcal{E} \to \mathbb{R}_{\geq 0}$)—as we effectively did in Example 1. For more details, I refer the interested reader to [13, Section 3.3], [25, Section 4.6] and [30, Section 4.2].

Example 3. Fix some $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ with $m \leq n$,

$$E_m(f) - \check{E}_n(f) \leq E_m(f) - E_m(f) = 0 \quad \text{for all } f \in \mathcal{L}(\mathbb{N}),$$

and therefore $\check{E}_n^*(E_m) = 0 < +\infty$. Similarly, for all $m \in \mathbb{N}$ with $m > n$,

$$E_m(f) - \check{E}_n(f) \leq E_m(f) - E_m(f) + \arctan(m) = \arctan(m) \quad \text{for all } f \in \mathcal{L}(\mathbb{N}),$$

whence in this case $\check{E}_n^*(E_m) = \arctan(m) < +\infty$. So for all $m \in \mathbb{N}$, $E_m \in \mathbb{E}_{\check{E}_n}$. In fact, in this case it even holds that $\mathbb{E}_{\check{E}_n} = \mathbb{E}_{\mathcal{L}(\mathbb{N})}$. To see why, fix any $E \in \mathbb{E}_{\mathcal{L}(\mathbb{N})}$. Then for all $f \in \mathcal{L}(\mathbb{N})$ and $\epsilon \in \mathbb{R}_{>0}$, there is some $k \in \mathbb{N}$ such that $0 \leq \sup f - f(k) < \epsilon$, and therefore

$$E(f) - \check{E}_n(f) \leq \sup f - \check{E}_n(f);$$

if $k \leq n$, then $E(f) - \check{E}_n(f) \leq \sup f - f(k) < \epsilon$, while if $k > n$ then $E(f) - \check{E}_n(f) \leq \sup f - f(k) + \arctan(k) < \epsilon + \frac{\pi}{2}$. Since $f \in \mathcal{L}(\mathbb{N})$ and $\epsilon \in \mathbb{R}_{>0}$ were arbitrary, we conclude that $\check{E}_n^*(E) \leq \frac{\pi}{2}$, so indeed $E \in \mathbb{E}_{\check{E}_n}$.

Similarly, it's easy to see that $\overline{E}_n^*(E_m) = 0$ for all $m \in \mathbb{N}$ with $m \leq n$. In contrast, $\overline{E}_n^*(E_m) = +\infty$ for $m \in \mathbb{N}$ with $m > n$, as $\alpha 1_m \in \mathcal{L}(\mathbb{N})$ for any $\alpha \in \mathbb{R}_{>0}$ and

$$E_m(\alpha 1_m) - \overline{E}_n(\alpha 1_m) = \alpha 1_m(m) - \sup\{\alpha 1_m(k) \in \mathbb{N}, k \leq n\} = \alpha - 0 = \alpha.$$

Here too, it is possible to determine the set $\mathbb{E}_{\overline{E}_n}$: we leave it to the reader to verify that

$$\mathbb{E}_{\overline{E}_n} = \left\{ \sum_{k=1}^{n} \alpha_k E_k \colon \alpha_k \in \mathbb{R}_{\geq 0}, \sum_{k=1}^{n} \alpha_k = 1 \right\} \subsetneq \mathbb{E}_{\mathcal{L}(\mathbb{N})} = \mathbb{E}_{\check{E}_n}. \tag{1}$$

Now any $E = \sum_{k=1}^{n} \alpha_k E_k \in \mathbb{E}_{\overline{E}_n}$ is downward continuous: for any $\mathcal{L}(\mathbb{N})^{\mathbb{N}} \ni (f_m)_{m\in\mathbb{N}} \searrow f \in \mathcal{L}(\mathbb{N})$,

$$\lim_{m\to+\infty} E(f_m) = \lim_{m\to+\infty} \sum_{k=1}^{n} \alpha_k E_k(f_m) = \lim_{m\to+\infty} \sum_{k=1}^{n} \alpha_k f_m(k) = \sum_{k=1}^{n} \alpha_k f(k) = E(f).$$

Hence, it follows from Lemmas 3 and 4 that $\overline{E}_n$ is downward continuous. In contrast, $\check{E}_n$ is *not* downward continuous, as $\mathbb{E}_{\check{E}_n} = \mathbb{E}_{\mathcal{L}(\mathbb{N})}$ definitely contains expectations that are not downward continuous—for but one example, see [31, Examples 2.1.3 (8)]. Alternatively, take the sequence $(f_k)_{k\in\mathbb{N}} \in \mathcal{L}(\mathbb{N})$ where f_k maps $m \in \mathbb{N}$ to $f_k(m) = 2^{m-k}$ if $m \leq k$ and $f_k(m) = 2$ otherwise. Then for all $k \in \mathbb{N}$,

$$\begin{aligned}
\check{E}_n(f_k) &= \sup\{f_k(m)\colon m \in \mathbb{N}, m \leq n\} \cup \left\{ f_k(m) - \frac{1}{m} \colon m \in \mathbb{N}, m > n \right\} \\
&= \sup\left\{2^{m-k}\colon m \leq n, k\right\} \cup \{2\colon k < m \leq n\} \\
&\qquad \cup \left\{2^{m-k} - \arctan(m)\colon n < m \leq k\right\} \cup \{2 - \arctan(m)\colon n, k < m\} \\
&\geq 2 - \frac{\pi}{2}.
\end{aligned}$$

Since $(f_k)_{k\in\mathbb{N}} \searrow 0 \in \mathcal{L}(\mathbb{N})$ but

$$\lim_{k\to+\infty} \check{E}_n(f_k) \geq 2 - \frac{\pi}{2} > 0 = \check{E}_n(0),$$

this also shows that $\check{E}_n$ is not downward continuous.

2.2. Extending convex expectations

For the remainder of this section, we assume that the linear subspace $\mathcal{D}$ of $\mathcal{L}(\mathcal{Y})$ not only includes all constant functions, but is also closed under pointwise minima (and then maxima), meaning that for all $f, g \in \mathcal{D}$, $f \wedge g \in \mathcal{D}$ (and then $f \vee g = -((-f) \wedge (-g)) \in \mathcal{D}$); this makes $\mathcal{D}$ a linear lattice [25, Definition 1.1]. We let $\sigma(\mathcal{D})$ denote the smallest σ-algebra Σ in $\mathcal{Y}$ such that every $f \in \mathcal{D}$ is $\Sigma/\mathcal{B}(\mathbb{R})$-measurable—see, for example, [32, Definition I.5 b)].

Due to our assumptions, the Daniell–Stone Theorem [32, Theorem III.35] establishes that if a linear expectation E on $\mathcal{D}$ is downward continuous, then there is a unique probability measure P_E on $\sigma(\mathcal{D})$ such that

$$E(f) = \int f \mathrm{d}P_E \quad \text{for all } f \in \mathcal{D}.$$

This means that we can extend E to the domain of the Lebesgue integral with respect to P_E. In this work we'll use the Lebesgue integral as defined by Fristedt and Gray [4, Definition 4.8 and Definition 8.2], so the domain of the Lebesgue integral with respect to a probability measure P on $\sigma(\mathcal{D})$ is the set of *P-integrable functions*

$$\mathcal{I}(P) := \left\{ f \in \mathcal{M}(\mathcal{D})\colon \int f^+ \,\mathrm{d}P - \int f^- \,\mathrm{d}P \text{ meaningful} \right\},$$

where $\mathcal{M}(\mathcal{D})$ collects all extended real functions on $\mathcal{Y}$ that are $\sigma(\mathcal{D})/\mathcal{B}(\overline{\mathbb{R}})$-measurable;[7] the integral is allowed to take the values $+\infty$ and $-\infty$, and is real-valued on the set of *absolute P-integrable functions*

$$\mathcal{A}(P) := \left\{ f \in \mathcal{M}(\mathcal{D})\colon \int f^+ \,\mathrm{d}P, \int f^- \,\mathrm{d}P < +\infty \right\} = \left\{ f \in \mathcal{M}(\mathcal{D})\colon \int |f| \,\mathrm{d}P < +\infty \right\}.$$

[7] $\mathcal{B}(\overline{\mathbb{R}})$ denotes the Borel σ-algebra generated by the 'usual' topology on $\overline{\mathbb{R}}$; for more details, see [4, Chapter 2 and Appendix C.2] or [27, Chapter 8].

It is easy to verify [see 17, Lemma 24] that the set $\mathcal{I}(P)$ of P-integrable functions definitely includes those extended real functions f on $\mathcal{Y}$ that are $\sigma(\mathcal{D})/\mathcal{B}(\overline{\mathbb{R}})$-measurable and either *bounded below*, meaning that $\inf f > -\infty$, or *bounded above*, meaning that $\sup f < +\infty$. For this reason, we collect all $\sigma(\mathcal{D})/\mathcal{B}(\overline{\mathbb{R}})$-measurable extended real functions on $\mathcal{Y}$ that are bounded below in $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$ and those that are bounded above in $\mathcal{M}^{\mathrm{b}}(\mathcal{D})$. Note that $\mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ includes

$$\mathcal{D}_{\delta,\mathrm{b}} \coloneqq \left\{ f \in \mathcal{L}(\mathcal{Y}) \colon \left(\exists (f_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}\right) \ (f_n)_{n\in\mathbb{N}} \searrow f \right\}.$$

Example 4. Note that $\mathcal{L}(\mathbb{N})$ is a linear lattice that includes all constant functions, and that $\sigma(\mathcal{L}(\mathbb{N})) = \wp(\mathbb{N})$ is the power set of $\mathbb{N}$. Then for any $n \in \mathbb{N}$, $P_n \coloneqq P_{E_n}$ is the *Dirac measure on* $\{n\}$, given by

$$P_n(A) = P_{E_n}(A) = 1_A(n) \quad \text{for all } A \in \wp(\mathbb{N}).$$

Consequently, any function $f \in \overline{\mathbb{R}}^{\mathbb{N}}$ is P_n-integrable, while the absolute P_n-integrable functions are exactly those functions $f \in \overline{\mathbb{R}}^{\mathbb{N}}$ for which $f(n)$ is real-valued.

The following extension result generalises the extension result in Theorem 3.10 of Denk et al. [20] by extending the domain of the extension from bounded measurable functions to—most but not all—extended-real measurable functions. The proof remains largely the same, which is why I've chosen to relegate it to Appendix A.

Theorem 1. *Let $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ be a linear lattice that includes all constant functions, and consider a convex expectation $\check{E}$ on $\mathcal{D}$ that is downward continuous. Let*

$$\mathcal{I}(\check{E}) \coloneqq \bigcap_{E \in \mathbb{E}_{\check{E}}} \mathcal{I}(P_E) \supseteq \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cup \mathcal{M}^{\mathrm{b}}(\mathcal{D})$$

and

$$\check{E}^{\odot} \colon \mathcal{I}(\check{E}) \to \overline{\mathbb{R}} \colon f \mapsto \check{E}^{\odot}(f) \coloneqq \sup\left\{ \int f \mathrm{d}P_E - \check{E}^*(E) \colon E \in \mathbb{E}_{\check{E}} \right\}.$$

Then $\check{E}^{\odot}$ is a convex expectation that extends $\check{E}$, is downward continuous on $\mathcal{D}_{\delta,\mathrm{b}}$ and upward continuous on

$$\mathcal{I}_{\mathrm{uc}}(\check{E}) \coloneqq \left\{ f \in \mathcal{I}(\check{E}) \colon (\forall E \in \mathbb{E}_{\check{E}}) \ \int f \mathrm{d}P_E > -\infty \right\} \supseteq \mathcal{M}_{\mathrm{b}}(\mathcal{D}).$$

If $\check{E}$ is a sublinear expectation, then so is $\check{E}^{\odot}$.

Example 5. Fix some $n \in \mathbb{N}$. Since the sublinear expectation $\overline{E}_n$ is downward continuous [Example 3], we can use Theorem 1 to obtain the sublinear expectation $\overline{E}_n^{\odot}$ on $\mathcal{I}(\overline{E}_n)$. With the help of Eqn. (1) in Example 3, the reader will have no difficulty in verifying that

$$\mathcal{I}(\overline{E}_n) = \left\{ f \in \overline{\mathbb{R}}^{\mathbb{N}} \colon f(1), \dots, f(n) < +\infty \right\} \cup \left\{ f \in \overline{\mathbb{R}}^{\mathbb{N}} \colon f(1), \dots, f(n) > -\infty \right\}$$

and

$$\overline{E}_n^{\odot}(f) = \sup\{ f(m) \colon m \in \mathbb{N}, m \leq n \} \quad \text{for all } f \in \mathcal{I}(\overline{E}_n).$$

The continuity properties in Theorem 1 are relevant because they often allows us to determine $\check{E}^{\odot}(f)$ for functions $f \in \mathcal{I}(\check{E}) \setminus \mathcal{D}$ without resorting to its definition. Indeed, we can derive $\check{E}^{\odot}(f)$ 'directly' from $\check{E}$ in at least two cases: (i) f is bounded below and the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$; and (ii) f is the pointwise limit of an increasing sequence $(f_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$. In both of these cases,

$$\check{E}^{\odot}(f) = \lim_{n\to+\infty} \check{E}^{\odot}(f_n) = \lim_{n\to+\infty} \check{E}(f_n).$$

We can, of course, continue this scheme, but now with functions f that are the pointwise limit of increasing sequences $(f_n)_{n\in\mathbb{N}}$ of functions for which we have determined $\check{E}^{\odot}(f_n)$, and so on.

Example 6. We again fix some $n \in \mathbb{N}$ and consider the downward-continuous sublinear expectation $\overline{E}_n$ on $\mathcal{L}(\mathbb{N})$ from Example 1 and its extension $\overline{E}_n^{\odot}$ from Example 5. For all $k \in \mathbb{N}$, let $f_k \in \mathcal{L}(\mathbb{N})$ be the function given by $f_k(m) := k \wedge m$ for all $m \in \mathbb{N}$. Then $(f_k)_{k\in\mathbb{N}}$ is clearly increasing, and its pointwise limit is the identity function $f \colon \mathbb{N} \to \mathbb{N} \colon m \mapsto m$, which is not bounded but is bounded below, that is, which is not in $\mathcal{L}(\mathbb{N})$ but is in $\mathcal{M}_{\mathrm{b}}(\mathcal{D}) \subseteq \mathcal{I}(\overline{E}_n)$. As per the explanation above,

$$\overline{E}_n^{\odot}(f) = \lim_{k\to+\infty} \overline{E}_n^{\odot}(f_k) = \lim_{k\to+\infty} \overline{E}_n(f_k) = \lim_{k\to+\infty} \sup\{k \wedge m \colon m \in \mathbb{N}, m \leq n\} = \lim_{k\to+\infty} k \wedge n = n.$$

Alternatively, from the expression for $\overline{E}_n^{\odot}$ in Example 5 it also follows that

$$\overline{E}_n^{\odot}(f) = \sup\{f(m) \colon m \in \mathbb{N}, m \leq n\} = \sup\{m \colon m \in \mathbb{N}, m \leq n\} = n.$$

The extension $\check{E}^{\odot}$ in Theorem 1 is the unique one with the given properties, at least when we restrict its domain to the bounded below functions. This follows almost immediately from Theorem 3.10 in [20]. In our 'generalisation' of this result, we write $\mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow\leq f$ to mean any decreasing $(f_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ such that $\lim_{n\to+\infty} f_n \leq f$.

Lemma 5. *Consider a convex expectation $\check{E}$ on $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$ that is downward continuous on $\mathcal{D}_{\delta,\mathrm{b}}$ and upward continuous. Then for all $f \in \mathcal{M}_{\mathrm{b}}(\mathcal{D})$,*

$$\check{E}(f) = \sup\left\{\lim_{n\to+\infty} \check{E}(f_n) \colon \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow\leq f\right\}.$$

Proof. From Theorem 3.10 in [20]—or the functional version of Choquet's Capacitibility Theorem, see [33, Proposition 2.1]—it follows that for all $f \in \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D}) = \mathcal{M}(\mathcal{D}) \cap \mathcal{L}(\mathcal{Y})$,

$$\check{E}(f) = \sup\left\{\lim_{n\to+\infty} \check{E}(f_n) \colon \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow\leq f\right\}. \tag{2}$$

It remains for us to prove the equality in the statement for all $f \in \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \setminus \mathcal{M}^{\mathrm{b}}(\mathcal{D})$, so let us fix any such f. Then $(f \wedge k)_{k\in\mathbb{N}}$ is an increasing sequence in $\mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ that converges pointwise to f, and therefore

$$\check{E}(f) = \lim_{k\to+\infty} \check{E}(f \wedge k) = \sup\{\check{E}(f \wedge k) \colon k \in \mathbb{N}\}.$$

Because $f \wedge k \in \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ for all $k \in \mathbb{N}$, it follows from this equality and Eq. (2) that

$$\begin{aligned}\check{E}(f) &= \sup\left\{\lim_{n\to+\infty} \check{E}(f_n) \colon k \in \mathbb{N}, \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow\leq f \wedge k\right\} \\ &= \sup\left\{\lim_{n\to+\infty} \check{E}(f_n) \colon \mathcal{D}^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \searrow\leq f\right\},\end{aligned}$$

as required.

3. Countable-state uncertain processes

With Theorem 1 under our belts, we can move from the general setting of convex expectations to the more specific setting of convex expectations for uncertain processes, that is, 'systems'—or phenomena—that assume a state in each point in time. A 'subject' is uncertain about the state as it evolves over time, and we want to model their uncertainty by means of a convex expectation.

As announced by the title, we will focus on uncertain processes whose state takes values in a countable *state space* $\mathscr{X}$. An outcome or realisation of such an uncertain process is a map from the time set $\mathbb{T}$ to $\mathscr{X}$, which we call a *path*.

In the discrete-time setting it is customary to choose $\mathbb{T} = \mathbb{N}$, which leads to $\mathscr{X}^{\mathbb{N}}$ as set of paths (and hence outcome space). The set of bounded real functions on $\mathscr{X}^{\mathbb{N}}$ that depend on the state of the path in a finite number of time points then form a linear lattice of bounded functions that includes the constants. To construct a convex expectation on this linear lattice, one can follow the approach in [20, Example 5.3], which leads to a 'convex Markov chain'—see also [34–36] for the finite-state sublinear case. It is not all too difficult to obtain a sufficient condition under which the constructed convex expectation is downward continuous, but we will not pursue this matter. That said, in the particular setting of sublinear expectations, it would be interesting to compare the resulting extended sublinear expectation to the sublinear expectation as defined in the game-theoretic framework of Shafer and Vovk [37]; T'Joens [36] has essentially done so in the particular case of a finite state space.

Things get more involved in the setting of continuous time, though, and this will be our main focus in the remainder. Henceforth, we assume that time is indexed by the positive real numbers: $\mathbb{T} = \mathbb{R}_{\geq 0}$. We make—or our subject makes—the common assumptions that (i) after it transitions to a new state, the system stays in its new state for some time; and (ii) the number of state transitions in every bounded interval is finite. This is equivalent to assuming that the outcomes are *càdlàg* paths, meaning that they are continuous from the right and have left-sided limits—see, for example, [2, Chapter 3, Section 5] or [4, Section 31.1]. Thus, our outcome space is the set of all càdlàg paths, which we denote by $\Omega \subseteq \mathscr{X}^{\mathbb{R}_{\geq 0}}$. As will become clear at the end of this section, the assumption of càdlàg paths ensures that the domain of the constructed convex expectation is sufficiently rich.

Our starting point will be a convex expectation on the set of bounded functions on Ω that are *finitary*, meaning that they only depend on the states that a path $\omega \in \Omega$ assumes in a finite set of time points. To formalise this, it will be convenient to introduce some notation.

For all $U \subseteq V \subseteq \mathbb{R}_{\geq 0}$, we let $\pi_U^V \colon \mathscr{X}^V \to \mathscr{X}^U$ be the projection that maps any $\mathscr{X}$-valued function on V to its restriction to U [see 3, Section II.25]; if $V = \mathbb{R}_{\geq 0}$, we simply write π_U and moreover restrict the domain of π_U to $\Omega \subsetneq \mathscr{X}^{\mathbb{R}_{\geq 0}}$. Let $\mathscr{U}$ be the set of finite subsets of $\mathbb{R}_{\geq 0}$. Then given some $U = \{t_1, \dots, t_n\} \in \mathscr{U}$, we will sometimes identify $x = (x_t)_{t \in U} \in \mathscr{X}^U$ with the n-tuple $(x_1, \dots, x_n) \coloneqq (x(t_1), \dots, x(t_n)) \in \mathscr{X}^n$ of its values.

With this notation, we can formally define the set of finitary bounded functions on Ω:

$$\mathscr{D} \coloneqq \big\{ f \circ \pi_U \colon U \in \mathscr{U}, f \in \mathcal{L}(\mathscr{X}^U) \big\}. \tag{3}$$

One easily verifies that $\mathscr{D} \subseteq \mathcal{L}(\Omega)$ is a linear lattice that includes all constants, so the results in Section 2.2 apply.

Before we get down to using Theorem 2, we should establish that a convex expectation $\check{E}$ on $\mathscr{D}$ is in one-to-one correspondence with a 'consistent collection of finite-dimensional convex expectations'; especially in Section 4.2 further on, it will be more convenient to work with the latter. This notion, essentially taken from [20, Definition 4.2], is the counterpart of the well-known notion of a 'consistent collection of finite-dimensional distributions' for probability measures [3, Section II.29].

Definition 2. A *collection of finite-dimensional convex/sublinear/linear expectations* is a collection $(\check{E}_U)_{U \in \mathscr{U}}$ such that for all $U \in \mathscr{U}$, $\check{E}_U$ is a convex/sublinear/linear expectation on $\mathcal{L}(\mathscr{X}^U)$. Such a collection is *consistent* if for all $U, V \in \mathscr{U}$ with $U \subseteq V$,

$$\check{E}_U(f) = \check{E}_V\big(f \circ \pi_U^V\big) \quad \text{for all } f \in \mathcal{L}(\mathscr{X}^U).$$

Proposition 1. *Consider a consistent collection $(\check{E}_U)_{U \in \mathscr{U}}$ of finite-dimensional convex/sublinear/linear expectations. Then there is a unique convex/sublinear/linear expectation $\check{E}$ on $\mathscr{D}$ such that*

$$\check{E}(f \circ \pi_U) = \check{E}_U(f) \quad \textit{for all } U \in \mathscr{U}, f \in \mathcal{L}(\mathscr{X}^U).$$

Conversely, if $\check{E}$ is a convex/sublinear/linear expectation on $\mathscr{D}$ and for all $U \in \mathscr{U}$ we let

$$\check{E}_U \colon \mathcal{L}(\mathscr{X}^U) \to \mathbb{R} \colon f \mapsto \check{E}_U(f) \coloneqq \check{E}(f \circ \pi_U),$$

then $(\check{E}_U)_{U \in \mathscr{U}}$ is a consistent collection of finite-dimensional convex/sublinear/linear expectations.

Proof. The first part of the statement follows almost immediately from the consistency of the collection $(\check{E}_U)_{U\in\mathscr{U}}$—see for example the proof of Proposition 4.4 in [20]. The second part of the statement follows immediately from the properties of convex/sublinear/linear expectations.

Example 7. In all the examples in this section, we'll assume $\mathscr{X} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ and we'll fix some sequence $(\omega_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ of càdlàg paths—take, for example, $\omega_n \colon \mathbb{R}_{\geq 0} \to \mathscr{X} : t \mapsto n$. For any $U \in \mathscr{U}$, we define the corresponding finite-dimensional convex expectation

$$\check{E}_U \colon \mathcal{L}(\mathscr{X}^U) \to \mathbb{R} \colon f \mapsto \sup\{[f \circ \pi_U](\omega_n) - n + 1 \colon n \in \mathbb{N}\}.$$

The resulting collection $(\check{E}_U)_{U\in\mathscr{U}}$ of finite-dimensional convex expectations is clearly consistent: for all $U, V \in \mathscr{U}$ with $U \subseteq V$ and $f \in \mathcal{L}(\mathscr{X}^U)$,

$$\check{E}_U(f) = \sup\{[f \circ \pi_U](\omega_n) - n + 1 \colon n \in \mathbb{N}\} = \sup\big\{[(f \circ \pi_U^V) \circ \pi_V](\omega_n) - n + 1 \colon n \in \mathbb{N}\big\} = \check{E}_V(f \circ \pi_U^V).$$

Hence, it follows from Proposition 1 that there is a unique convex expectation $\check{E}$ on $\mathscr{D}$ corresponding to $(\check{E}_U)_{U\in\mathscr{U}}$, and one can readily see that this is given by

$$\check{E}(f) = \sup\{f(\omega_n) - n + 1 \colon n \in \mathbb{N}\} \quad \text{for all } f \in \mathscr{D}.$$

Our aim is to use Theorem 1 to extend the convex expectation $\check{E}$ on the set $\mathscr{D}$ of finitary bounded functions to (as much of) the set $\mathcal{M}(\mathscr{D})$ of $\sigma(\mathscr{D})/\mathcal{B}(\overline{\mathbb{R}})$-measurable functions (as possible). To this end, we need to check whether the convex expectation $\check{E}$ on $\mathscr{D}$ is downward continuous. Let us do so for the convex expectation obtained in Example 7.

Example 8. To verify that $\check{E}$ is downward continuous, we fix any $\mathscr{D}^{\mathbb{N}} \ni (f_k)_{k\in\mathbb{N}} \searrow f \in \mathscr{D}$, and set out to show that

$$\lim_{k\to+\infty} \check{E}(f_k) = \check{E}(f). \tag{4}$$

Since $f_1 \geq f_2 \geq \cdots \geq f$ are all bounded, there is some $m \in \mathbb{N}$ such that $\sup f_1 - \inf f + 1 \leq m$. Then for all $k \in \mathbb{N}$ and $n > m$, $f_k(\omega_n) - n + 1 \leq \sup f_k - n + 1 \leq \sup f_1 - n + 1 < \inf f \leq f_k(\omega_1)$, whence

$$\check{E}(f_k) = \sup\{f_k(\omega_n) - n + 1 \colon n \in \mathbb{N}\} = \max\{f_k(\omega_n) - n + 1 \colon n \in \mathbb{N}, n \leq m\}.$$

A similar argument shows that

$$\check{E}(f) = \sup\{f(\omega_n) - n + 1 \colon n \in \mathbb{N}\} = \max\{f(\omega_n) - n + 1 \colon n \in \mathbb{N}, n \leq m\}.$$

Since $(f_k)_{k\in\mathbb{N}}$ converges pointwise to f, the limit statement in Eqn. (4) follows from these two equalities.

While we can directly check downward continuity for simple convex expectations as the one in Example 7, this approach becomes quite unwieldy for more complex convex expectations. For this reason, we set out to find a sufficient condition for downward continuity that is easier to verify, and we'll find this in the realm of measure theory.

3.1. The classical measure-theoretic setting

Rather than a (convex) expectation on a sufficiently large set of functions on Ω, the measure-theoretical approach aims to obtain a probability measure on the σ-algebra generated by the cylinder events, which we'll introduce now. A subset F of Ω is a *cylinder event* [3, Definition II.25.4][8] if there are some $U \in \mathscr{U}$ and $A \subseteq \mathscr{X}^U$ such that

$$F = \pi_U^{-1}(A) := \big\{\omega \in \Omega \colon \pi_U(\omega) \in A\big\}.$$

[8] In general, this definition assumes $A \in \mathcal{X}^U$, where $\mathcal{X}$ is a σ-algebra on $\mathscr{X}$. Since $\mathscr{X}$ is countable, $\mathcal{X} = \wp(\mathscr{X})$ and $\mathcal{X}^U = \wp(\mathscr{X}^U)$.

We collect all cylinder events in $\mathscr{F}$, and note that this can also be found as the trace σ-algebra on Ω of the product σ-algebra on $\mathscr{X}^{\mathbb{R}_{\geq 0}}$.

The reader who is familiar with (measure-theoretic) stochastic processes will probably wonder whether the σ-algebra $\sigma(\mathscr{D})$ generated by $\mathscr{D}$ is equal to $\sigma(\mathscr{F})$, the one generated by the cylinders. It is easy to verify that

$$\mathscr{F} = \big\{F \in \wp(\Omega) \colon 1_F \in \mathscr{D}\big\}, \tag{5}$$

and that $\mathscr{F}$ is an algebra of subsets of Ω; since $\mathscr{D}$ is a linear space, this implies that

$$\operatorname{span}(\{1_F \colon F \in \mathscr{F}\}) \subseteq \mathscr{D}.$$

Let us now verify that indeed

$$\sigma(\mathscr{F}) = \sigma(\mathscr{D}). \tag{6}$$

Due to Lemma 8.1 (and Lemma 8.3) in [27], $\sigma(\mathscr{D})$ is generated by the collection of level sets

$$\mathcal{C} \coloneqq \big\{\{\omega \in \Omega \colon f(\omega) \geq \alpha\} \colon f \in \mathscr{D}, \alpha \in \mathbb{R}\big\}.$$

Hence, it follows from Eq. (5) that every cylinder $F \in \mathscr{F}$ belongs to $\mathcal{C}$, and therefore also to $\sigma(\mathscr{D})$. Consequently, $\sigma(\mathscr{F}) \subseteq \sigma(\mathscr{D})$. To prove that $\sigma(\mathscr{D}) \subseteq \sigma(\mathscr{F})$, it suffices to verify that any level set in $\mathcal{C}$ is a cylinder. To this end, we fix any $f \in \mathscr{D}$ and $\alpha \in \mathbb{R}$. By definition of $\mathscr{D}$, there are some $U \in \mathscr{U}$ and $g \in \mathcal{L}(\mathscr{X}^U)$ such that $f = g \circ \pi_U$. Let $A \coloneqq \{x \in \mathscr{X}^U \colon g(x) \geq \alpha\}$. Then clearly

$$\{\omega \in \Omega \colon f(\omega) \geq \alpha\} = \{\omega \in \Omega \colon \pi_U(\omega) \in A\}, \tag{7}$$

so this level set is indeed a cylinder.

There even is a third way of getting to $\sigma(\mathscr{D})$: since the countable set $\mathscr{X}$ equipped with the discrete metric is a Polish space, Ω equipped with the so-called 'Skorokhod metric' is a Polish space, and the Borel σ-algebra induced by this metric space is precisely $\sigma(\mathscr{F}) = \sigma(\mathscr{D})$; the interested reader may consult Section 31.1 in [4] or Section 5 in [2, Chapter 3] for the details.

Rather than extending a (convex) expectation on the set of finitary bounded functions, the measure-theoretical approach seeks to extend a probability charge P—in the sense of [31, Definition 2.1.1], often also called a finitely additive probability measure—on the cylinders, which is equivalent to a collection $(P^U)_{U \in \mathscr{U}}$ of 'finite-dimensional charges' [38, Sections 2.2 and 2.3], where P^U is a probability charge on $\wp(\mathscr{X}^U)$. If we were to use the set of all paths $\mathscr{X}^{\mathbb{R}_{\geq 0}}$ as possibility space instead of the set of càdlàg paths Ω, then this probability charge P can be extended to a probability measure on the σ-algebra generated by the cylinder events if and only if for all $U \in \mathscr{U}$, the corresponding probability charge $P^U \colon \wp(\mathscr{X}^U) \to [0,1] \colon A \mapsto P(\pi_U^{-1}(A))$ is countably additive. This result is known as the Daniell–Kolmogorov Extension Theorem [3, Theorem II.34], see also [38, Theorem 2].

With the set of càdlàg paths Ω as possibility space, things are more difficult though, and this problem has been studied intensively. For example, Billingsley [39, Section 13] treats this issue thoroughly, although they consider $\mathbb{R}$ as state space and $[0,1]$ as time axis, and Stroock and Varadhan [5, Chapter 2] treat a related issue with $\mathbb{R}^d$ as state space and $\mathbb{R}_{\geq 0}$ as time axis but continuous rather than càdlàg paths. Here too, one needs that the probability charge P on the cylinders (for the set of all paths) is countably additive, and uses Caratheodory's Extension Theorem to extend it to a probability measure P' on the σ-algebra generated by the cylinders on the set of all paths. The final and crucial step, then, is to construct a modification of the canonical coordinate process that does have càdlàg sample paths (P' almost surely); alternatively, it suffices to show that the set of càdlàg paths Ω has P' outer measure 1,[9] as this then allows one to obtain a probability measure P on $\sigma(\mathscr{D}) = \sigma(\mathscr{F})$ [5, Lemma 2.1.1].

A more direct way of showing that the probability charge P on $\mathscr{F}$ is countably additive (and therefore extendable to a probability measure on $\sigma(\mathscr{F})$), without the need to first construct a probability measure

[9] That is, for any sequence $(F_n)_{n \in \mathbb{N}}$ of measurable subsets of the set of all paths such that $\Omega \subseteq \bigcup_{n \in \mathbb{N}} F_n$, $\sum_{n \in \mathbb{N}} P'(F_n) = 1$.

on the set of all paths, can be found in [38]. Again, it's necessary but not sufficient that for all $U \in \mathscr{U}$, the corresponding finite-dimensional charge P^U is downward continuous. For sufficiency, there is an extra condition which essentially requires that the finite-dimensional charges agree with the càdlàg nature of the paths [38, Theorem 10]. However, for the sake of simplicity, here we will rely on the sufficient (but not necessary) condition in [38, Corollary 13]—comparable to the condition in what is known as Kolmogorov's Continuity Theorem, see [39, Theorem 13.6] or [5, Corollary 2.1.5]. The next result contains a translation of this result to the context of the present article, with the help of Proposition 1, Eq. (5) and Theorem 10 in [38]. In it, as well as in Theorem 2 further on, we avail ourselves of the following notation: for all $U = \{t_1, t_2\} \in \mathscr{U}$, let $d_U^{\neq} \in \mathcal{L}(\mathscr{X}^U)$ be defined for all $x \in \mathscr{X}^U$ by $d_U^{\neq}(x) := 1$ if $x(t_1) \neq x(t_2)$ and $d_U^{\neq}(x) := 0$ otherwise; similarly, let $D_U^{\neq} := \{x \in \mathscr{X}^U : x(t_1) \neq x(t_2)\}$ and note that $d_U^{\neq} = 1_{D_U^{\neq}}$.

Lemma 6. *Consider a probability charge $R\colon \mathscr{F} \to [0,1]$. If*

(i) *for all $U \in \mathscr{U}$, the derived set function*

$$R^U : \wp(\mathscr{X}^U) \to [0,1] \colon A \mapsto R\big(\pi_U^{-1}(A)\big)$$

is countably additive; and

(ii) *for all $n \in \mathbb{N}$, there is some $\lambda_n \in \mathbb{R}_{\geq 0}$ such that*

$$\limsup_{s \to t} \frac{R^{\{t,s\}}(D^{\neq}_{\{t,s\}})}{|s-t|} \leq \lambda_n \quad \text{for all } t \in [0,n];$$

then R is countably additive.

Rather than from a probability charge on $\mathscr{F}$, we start from a linear expectation E on $\mathscr{D}$. In order to avail ourselves of results in the measure-theoretical literature, we focus on the induced set-function

$$R_E \colon \mathscr{F} \to [0,1] \colon A \mapsto R_E(A) := E(1_A). \tag{8}$$

It is not difficult to verify that this derived set-function is a probability charge. With a bit more work, one can show that E is downward continuous if and only if R_E is countably additive. This result follows relatively easily from standard measure-theoretical results, but I could not immediately find a reference for it; I give a formal proof here for the sake of completeness.

Lemma 7. *Consider a linear expectation E on $\mathscr{D}$. Then the derived set function*

$$R_E \colon \mathscr{F} \to [0,1] \colon F \mapsto E(1_F)$$

is a probability charge. Furthermore, E is downward continuous if and only if R_E is countably additive; whenever this is the case, $R_E = P_E|_{\mathscr{F}}$, where P_E is the probability measure corresponding to E according to the Daniell–Stone Theorem.

Proof. That R_E is positive and finitely additive with $R_E(\Omega) = 1$ follows immediately because E is a linear expectation. Hence, we focus on the second part of the statement.

First, we assume that E is downward continuous. Then it follows immediately from the Daniell–Stone Theorem [32, Theorem III.35] that $R_E = P_E|_{\mathscr{F}}$, and therefore R_E is countably additive.

Second, we assume that R_E is countably additive. Then it is well known—see for example Proposition 9 in [4, Chapter 7] or Lemma 4.3 in [3, Chapter II]—that for any decreasing $(F_n)_{n\in\mathbb{N}} \in \mathscr{F}^{\mathbb{N}}$—meaning that $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{N}$—with $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$,

$$\lim_{n \to +\infty} R_E(F_n) = 0. \tag{9}$$

To show that E is downward continuous, we fix any $f \in \mathcal{D}$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{D}^{\mathbb{N}}$ that converges pointwise to f. Then

$$E(f_n) - E(f) = E(f_n - f) \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

Obviously, $(f_n - f)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathscr{D}$ that converges pointwise to 0.

Fix any $\epsilon \in \mathbb{R}_{>0}$, and let $\beta := \|f_1 - f\|_\infty = \sup f_1 - f$. Then for all $n \in \mathbb{N}$, we let $F_n := \{y \in \mathcal{Y}\colon f_n(y) - f(y) > \epsilon\}$; since this is a level set for a function in $\mathscr{D}$, we know from around Eqn. (7) that $F_n \in \mathscr{F}$. This way, $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathscr{F}$ with $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$, and for all $n \in \mathbb{N}$, $f_n - f \leq \epsilon + \beta 1_{F_n}$ and therefore

$$E(f_n - f) \leq \epsilon + E(\beta 1_{F_n}) = \epsilon + \beta R_E(F_n).$$

It follows from this and Eq. (9) that

$$\limsup_{n\to+\infty} E(f_n - f) \leq \lim_{n\to+\infty} \epsilon + \beta R_E(F_n) = \epsilon.$$

Since this inequality holds for any strictly positive real number ϵ, we infer from it and the one in Eq. (10) that

$$\lim_{n\to+\infty} E(f_n) = E(f),$$

as required.

3.2. Back to convex expectations

It is time to return to convex expectations after our little excursion to the land of measure theory. Recall that our starting point is a consistent collection $(\check{E}_U)_{U\in\mathscr{U}}$ of finite-dimensional convex expectations that are downward continuous. The question at hand is whether we can use Theorem 1 to extend the unique corresponding convex expectation $\check{E}$ on $\mathscr{D}$ of Proposition 1, or equivalently, whether $\check{E}$ is downward continuous.

To check this, we essentially follow the same strategy as the one Denk et al. [20] use to prove their Theorem 4.6: by Lemmas 4 and 3 it suffices to show that every linear expectation $E \in \mathbb{E}_{\check{E}}$—so any $E \in \mathbb{E}_{\mathcal{D}}$ with $\check{E}^*(E) < +\infty$—is downward continuous. The path forward is clear: we will show that every $E \in \mathbb{E}_{\check{E}}$ is downward continuous with the help of Lemmas 7 and 6.

Theorem 2. *Consider a consistent collection $(\check{E}_U)_{U\in\mathscr{U}}$ of finite-dimensional convex expectations that are downward continuous. If for all $n \in \mathbb{N}$ there is some $\lambda_n \in \mathbb{R}_{\geq 0}$ such that*

$$\limsup_{s\to t} \check{E}_{\{s,t\}}\left(\frac{d^{\neq}_{\{s,t\}}}{|s-t|}\right) \leq \lambda_n \quad \textit{for all } t \in [0, n],$$

then the corresponding convex expectation $\check{E}$ on $\mathscr{D}$ of Proposition 1 is downward continuous.

Proof. To prove that $\check{E}$ is downward continuous, we recall from Proposition 1 that $\check{E}$ is a convex expectation, and from right after Eq. (3) that its domain $\mathscr{D}$ is a linear lattice of bounded functions that includes the constants. By Lemmas 4 and 3, it suffices to verify that every $E \in \mathbb{E}_{\check{E}}$—so every linear expectation E on $\mathscr{D}$ with $\check{E}^*(E) < +\infty$—is downward continuous. So fix any $E \in \mathbb{E}_{\check{E}}$, and consider the induced set function

$$R_E\colon \mathscr{F} \to [0,1]\colon F \mapsto E(1_F).$$

We know from Lemma 7 that R_E is a probability charge, and that E is downward continuous if and only if R_E is countably additive. In its turn, Proposition 6 provides a sufficient condition for the countable additivity of R_E, which we set out to verify.

For the first part of the condition, fix any $U \in \mathscr{U}$, and let

$$E_U : \mathcal{L}(\mathscr{X}^U) \to \mathbb{R} : g \mapsto E(g \circ \pi_U).$$

Then E_U is a linear expectation [Proposition 1], and it is easy to see that its corresponding set function is

$$R_E^U : \wp(\mathscr{X}^U) \to [0,1] : A \mapsto E_U(1_A) = E\big(1_A \circ \pi_U\big) = R_E\big(\pi_U^{-1}(A)\big).$$

Furthermore, it follows from the definition of E_U and Eq. (3) that

$$\begin{aligned}\check{E}_U^*(E_U) &= \sup\big\{E_U(g) - \check{E}_U(g) : g \in \mathscr{X}^U\big\} \\ &= \sup\big\{E(g \circ \pi_U) - \check{E}(g \circ \pi_U) : g \in \mathscr{X}^U\big\} \\ &\leq \sup\big\{E(f) - \check{E}(f) : f \in \mathscr{D}\big\} \\ &= \check{E}^*(E) \\ &< +\infty.\end{aligned}$$

Since $\check{E}_U$ is downward continuous by the assumptions in the statement, it follows from this inequality and Lemma 3 that E_U is downward continuous. It now follows from Lemma 7—with $\mathcal{Y} = \mathscr{X}^U$ and $\mathcal{D} = \mathcal{L}(\mathscr{X}^U)$—that R_E^U is countably additive.

For the second part of the condition, we fix some $n \in \mathbb{N}$ and $t \in [0, n]$ Then for all $s \in \mathbb{R}_{\geq 0} \setminus \{t\}$, it follows from the preceding that

$$\frac{R_E^{\{t,s\}}\big(D_{\{t,s\}}^{\neq}\big)}{|s-t|} = \frac{E_{\{t,s\}}\big(d_{\{t,s\}}^{\neq}\big)}{|s-t|} = E_{\{t,s\}}\left(\frac{d_{\{t,s\}}^{\neq}}{|s-t|}\right) \leq \check{E}_{\{t,s\}}\left(\frac{d_{\{t,s\}}^{\neq}}{|s-t|}\right) + \check{E}_{\{t,s\}}^*(E_{\{t,s\}}).$$

We have shown before that $\check{E}_{\{t,s\}}^*(E_{\{t,s\}}) \leq \check{E}^*(E) < +\infty$ for all $s \in \mathbb{R}_{\geq 0} \setminus \{t\}$. Hence, it follows from the preceding inequality, and with $\lambda_n' := \lambda_n + \check{E}^*(E) \in \mathbb{R}_{\geq 0}$, that

$$\limsup_{s \to t} \frac{R_E^{\{t,s\}}(D_{\{t,s\}}^{\neq})}{|s-t|} \leq \limsup_{s \to t} \check{E}_{\{t,s\}}\left(\frac{d_{\{t,s\}}^{\neq}}{|s-t|}\right) + \check{E}_{\{t,s\}}^*(E_{\{t,s\}}) \leq \lambda_n',$$

which verifies the second condition in Lemma 6 and finishes our proof.

I would like to emphasise that while the assumption that the outcomes are càdlàg paths makes it more difficult to prove downward continuity, it is also crucial to ensure that the 'domain' $\mathcal{M}_{\mathrm{b}}(\mathscr{D}) \cup \mathcal{M}^{\mathrm{b}}(\mathscr{D})$ of the extensions is sufficiently rich. Indeed, it is well-known—see, for example, [38, Section 3.2] or [3, Lemma II.25.9]—that with the set $\mathscr{X}^{\mathbb{R}_{\geq 0}}$ of all paths as outcome space, the σ-algebra $\sigma(\mathscr{D}) = \sigma(\mathscr{F})$ generated by the cylinder events only contains events that depend on the state of the system in a countable subset of $\mathbb{R}_{\geq 0}$. This then implies that $\mathcal{M}_{\mathrm{b}}(\mathscr{D}) \cup \mathcal{M}^{\mathrm{b}}(\mathscr{D})$ essentially only contains functions that depend on the state of the system in a countable subset of $\mathbb{R}_{\geq 0}$, excluding important functions like stopping times and time averages.

Let us use Theorem 2 to check whether the convex expectation obtained in Example 7 is downward continuous.

Example 9. If all of the paths in the sequence $(\omega_n)_{n \in \mathbb{N}}$ are constant, then for all $s, t \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}$, $\omega_n(t) = \omega_n(s)$ and therefore $[d_{\{s,t\}}^{\neq} \circ \pi_{\{s,t\}}](\omega_n) = 0$. Hence, for all $s, t \in \mathbb{R}_{\geq 0}$ with $s \neq t$,

$$\check{E}_{\{s,t\}}\left(\frac{d_{\{s,t\}}^{\neq}}{|s-t|}\right) = \sup\left\{\frac{[d_{\{s,t\}}^{\neq} \circ \pi_{\{s,t\}}](\omega_n)}{|s-t|} - n + 1 : n \in \mathbb{N}\right\} = \sup\{-n + 1 : n \in \mathbb{N}\} = 0,$$

so it follows from Theorem 2 that $\check{E}$ is downward continuous.

If on the other hand there is some path ω_n that is not constant, then there is some time point $t \in \mathbb{R}_{>0}$ and $\delta \in]0, t[$ such that for all $s \in]t-\delta, t[$, $\omega_n(t) \neq \omega_n(s)$ and therefore $[d^{\neq}_{\{s,t\}} \circ \pi_{\{s,t\}}](\omega_n) = 1$. Consequently, for all $s \in]t-\delta, t[$,

$$\check{E}_{\{s,t\}}\left(\frac{d^{\neq}_{\{s,t\}}}{|s-t|}\right) = \sup\left\{\frac{[d^{\neq}_{\{s,t\}} \circ \pi_{\{s,t\}}](\omega_n)}{|s-t|} - n + 1 \colon n \in \mathbb{N}\right\} \geq \frac{1}{|s-t|} - n + 1.$$

Taking the limit superior for $s \nearrow t$, we learn from this inequality that the condition in Theorem 2 cannot be satisfied. However, the condition in Theorem 2 is only a sufficient one, so this does not mean that $\check{E}$ cannot be downward continuous. In fact, we've already done so in Example 8!

4. Constructing a collection of finite-dimensional convex expectations

Up to now, we've gone about constructing a consistent collection of finite-dimensional convex expectations in a very weird, technical and rather unnatural manner. A more natural and elegant way to construct such a collection is to glue together an 'initial convex expectation' and a 'convex Markov semigroup'. This approach, which was essentially put forward by Denk et al. [20] and is also used by Nendel [7, 8], is inspired by and a generalisation of the well-known construction method for Markov processes, where an 'initial distribution' is glued to a 'Markov semigroup' to end up with a consistent collection of finite-dimensional distributions (and then a probability measure)—see, for example, [3, Section III.7] or [4, Chapter 31] for more details.

We will get to this construction method in Section 4.2 further on, but only after we have discussed operators in general and convex Markov semigroups in particular in Section 4.1.

4.1. Convex Markov semigroups

To simplify our notation, let us write $\mathscr{L} \coloneqq \mathcal{L}(\mathscr{X})$. An *operator* A is a map from $\mathscr{L}$ to $\mathscr{L}$, or differently put, a transformation of $\mathscr{L}$. One important operator is the *identity operator* I, which maps any $f \in \mathscr{L}$ to itself.

We extend (almost) all of the terminology regarding properties of functionals to operators in the obvious 'componentwise' manner; for example, an operator $\mathrm{A} \in \mathfrak{O}$ is called subadditive if for all $x \in \mathscr{X}$, the corresponding component functional

$$[\mathrm{A}\bullet](x) \colon \mathscr{L} \to \mathbb{R} \colon f \mapsto [\mathrm{A}f](x)$$

is subadditve.

Definition 3. A *convex transition operator*[10] $\check{\mathrm{T}}$ is an operator that is order preserving, constant preserving and convex, or equivalently, such that for all $x \in \mathscr{X}$, the corresponding component functional $[\check{\mathrm{T}}\bullet](x)$ on $\mathscr{L}$ is a convex expectation. *Sublinear transition operators* and (linear) *transition operators* are defined in a similar manner.

The identity operator I is a trivial example of a (convex/sublinear/linear) transition operator. We will be particularly concerned with convex transition operators that are downward continuous, of which the aforementioned identity operator I is the easiest example. Recall from Lemma 2 that this requirement of downward continuity is always satisfied whenever the state space $\mathscr{X}$ is finite.

We'll never investigate a convex transition operator by itself, but will always consider a family of convex transition operators indexed by $\mathbb{R}_{\geq 0}$. Families of operators indexed by $\mathbb{R}_{\geq 0}$ have been investigated thoroughly, usually in the following setting [41–46]: a *semigroup* is a family $(\mathrm{S}_t)_{t \in \mathbb{R}_{\geq 0}}$ of operators such that

SG1. $\mathrm{S}_{s+t} = \mathrm{S}_s\mathrm{S}_t$ for all $s, t \in \mathbb{R}_{\geq 0}$, and

[10]The term 'convex transition operator' is a combination of the terminology for two related notions: the '(upper) transition operators' as introduced by Whittle [24, Chapter 9] and generalised by de Cooman and Hermans [40, Section 8] and the 'convex kernels' as preferred by Denk et al. [20, Definition 5.1].

SG2. $\mathrm{S}_0 = \mathrm{I}$.

We're only concerned with semigroups $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ of convex/sublinear/linear transition operators, which we'll briefly call *convex/sublinear/linear transition semigroups*; in this context, the semigroup property (SG1) is often called the 'Chapman–Kolmogorov Equation.' A *convex/sublinear/linear Markov semigroup*,[11] then, is a convex/sublinear/linear transition semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ such that for all $t \in \mathbb{R}_{\geq 0}$, $\check{\mathrm{T}}_t$ is downward continuous.

It is customary to impose a notion of continuity on semigroups, and we will to the same. The most prevalent notion—see for example [44, Chapter 3], [45, Chapter III] or [47]—is *strong continuity*, in the sense that

$$\lim_{s\to t}\|\mathrm{S}_s f - \mathrm{S}_t f\|_\infty = 0 \quad \text{for all } t \in \mathbb{R}_{\geq 0}, f \in \mathscr{L};$$

another common notion is that of *uniform continuity*, meaning that

$$\lim_{s\to t}\|\mathrm{S}_s - \mathrm{S}_t\| = 0 \quad \text{for all } t \in \mathbb{R}_{\geq 0},$$

where $\|\bullet\|$ is an operator norm—see for example [22, Section 2.1]. That said, we mainly need a—at least at first sight—different requirement.

Definition 4. A convex transition semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ *has uniformly bounded rate*[12] if

$$\limsup_{t\searrow 0}\sup\left\{\left[\check{\mathrm{T}}_t\left(\frac{1-1_x}{t}\right)\right](x)\colon x\in\mathscr{X}\right\} < +\infty.$$

The condition of uniformly bounded rate might seem rather strong, but we are not the first to encounter its usefulness and/or necessity. The strength of this condition will become clear in Section 5 further on, where we go over some examples of linear, sublinear and even convex Markov semigroups that satisfy this condition

4.2. From convex transition semigroup to finite-dimensional convex expectations

Now that we have properly introduced the necessary concepts, we can get down to glueing an initial convex expectation to a convex transition semigroup in such a way that we end up with a consistent collection of finite-dimensional convex expectations.

Proposition 2. *Consider a convex expectation $\check{E}_0$ on $\mathscr{L}$ and a convex transition semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$. Then there is a unique consistent collection $(\check{E}_U)_{U\in\mathscr{U}}$ of finite-dimensional convex expectations such that*

(i) *$\check{E}_{\{0\}}(f) = \check{E}_0(f)$ for all $f \in \mathcal{L}(\mathscr{X})$; and*

(ii) *for all $U = \{s_1, \dots, s_n\} \in \mathscr{U}$, $t \in \mathbb{R}_{\geq 0}$ such that $s_1 < \cdots < s_n < t$ and $f \in \mathcal{L}(\mathscr{X}^{U\cup\{t\}})$,*

$$\check{E}_{U\cup\{t\}}(f) = \check{E}_U\big(\check{\mathrm{T}}_{t-s_n} f\big),$$

where $\check{\mathrm{T}}_{t-s_n} f \in \mathcal{L}(\mathscr{X}^U)$ maps $x = (x_s)_{s\in U} \in \mathscr{X}^U$ to

$$[\check{\mathrm{T}}_{t-s_n} f](x) \coloneqq \big[\check{\mathrm{T}}_{t-s_n} f(x_{s_1}, \dots, x_{s_n}, \bullet)\big](x_{s_n}).$$

Proof. The argument is the same as in the first part of the proof of Theorem 5.6 in [20]. The collection $(\check{E}_U)_{U\in\mathscr{U}}$ is constructed recursively, and the semigroup property (SG1) ensures that the constructed family satisfies the condition for consistency in Definition 2.

[11] This definition generalises the notion of a 'Markov semigroup' in [3, Section III.3] to the convex case.
[12] After Anderson [48, Chapter 2, Eqn. (2.35)].

The consistent collection of finite-dimensional convex expectations in Proposition 2 corresponds to a unique convex expectation $\check{E}$ on $\mathscr{D}$ due to Proposition 1. Now the question naturally arises whether $\check{E}$ is downward continuous, because then we can invoke Theorem 1 to extend $\check{E}$ to a domain that includes $\mathcal{M}_{\mathrm{b}}(\mathscr{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathscr{D})$. With the help of Theorem 2 we get the following sufficient condition, which I believe to be one of the main results in this article

Theorem 3. *Consider a downward continuous convex expectation $\check{E}_0$ on $\mathscr{L}$ and a convex Markov semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ with uniformly bounded rate. Then the unique corresponding sublinear expectation $\check{E}$ on $\mathscr{D}$ induced by Propositions 2 and 1 is downward continuous.*

Proof. With $\check{E}$ the convex expectation of Proposition 1, it suffices to verify that the corresponding consistent collection of finite dimensional convex expectations $(\check{E}_U)_{U\in\mathscr{U}}$ [Proposition 1] satisfies the conditions in Theorem 2.

First, we prove that for all $U \in \mathscr{U}$, $\check{E}_U$ is downward continuous. For $U = \{0\}$, $\check{E}_U = \check{E}_0$ by Proposition 1 (i), so $\check{E}_U$ is downward continuous because $\check{E}_0$ is downward continuous as per the assumptions in the statement. Next, we fix any $U = \{t_0, t_1, \dots, t_n\} \in \mathscr{U}$ with $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$.

For all $h \in \mathcal{L}(\mathscr{X}^U)$, it follows from repeated application of Proposition 1 (ii) and a single application of Proposition 1 (i) that

$$\overline{E}_U(h) = \overline{E}_0(g_0), \tag{11}$$

where the sequence $g_0, \dots, g_n$ is derived recursively from the initial condition $g_n \coloneqq h$ and, for all $k \in \{0, \dots, n-1\}$, $g_k \in \mathcal{L}(\mathscr{X}^{\{t_0,\dots,t_k\}})$ is defined recursively for all $x = (x_0, \dots, x_k) \in \mathscr{X}^{\{t_0,\dots,t_k\}}$ by

$$g_k(x) \coloneqq \big[\check{\mathrm{T}}_{(t_{k+1}-t_k)} g_{k+1}(x_0, \dots, x_k, \bullet)\big](x_k). \tag{12}$$

Fix any $\mathcal{L}(\mathscr{X}^U)^{\mathbb{N}} \ni (f_\ell)_{\ell\in\mathbb{N}} \searrow f \in \mathcal{L}(\mathscr{X}^U)$. Then by Eq. (11) for $h = f$, $\overline{E}_U(f) = \overline{E}_0(g_0)$ with $(g_0, \dots, g_n)$ the sequence as defined in Eq. (12); similarly, for all $\ell \in \mathbb{N}$, $\overline{E}_U(f_\ell) = \overline{E}_0(g_{\ell,0})$ with $(g_{\ell,0}, \dots, g_{\ell,n})$ defined as in Eq. (12) with initial condition $g_{\ell,n} = f_\ell$. Then for $k = n-1$, and subsequently for $k = n-2$ to $k = 0$, it follows from the downward continuity of $\check{\mathrm{T}}_{(t_{k+1}-t_k)}$ that $(g_{\ell,k})_{\ell\in\mathbb{N}} \searrow g_k$. From this and the downward continuity of $\check{E}_0$, we infer that

$$\lim_{\ell\to+\infty} \check{E}_U(f_\ell) = \lim_{\ell\to+\infty} \check{E}_0(g_{\ell,0}) = \check{E}_0(g_0) = \check{E}_U(f),$$

as required. Finally, for all $U = \{t_1, \dots, t_n\} \in \mathscr{U}$ with $0 < t_1 < \cdots < t_n$, the downward continuity of $\check{E}_{\{0\}\cup U}$—which we have just proved—implies that of $\check{E}_U$ because $(\check{E}_V)_{V\in\mathscr{U}}$ is consistent.

Second, we set out to prove that for all $n \in \mathbb{N}$, there is some $\lambda_n \in \mathbb{R}_{\geq 0}$ such that

$$\limsup_{s\to t} \check{E}_{\{s,t\}}\left(\frac{d^{\neq}_{\{s,t\}}}{|s-t|}\right) \leq \lambda_n \quad \text{for all } t \in [0, n]. \tag{13}$$

Since $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate, it follows from Definition 4 that for all $t \in \mathbb{R}_{\geq 0}$,

$$\lambda \coloneqq \limsup_{s\to t} \sup\left\{\left[\check{\mathrm{T}}_{|s-t|}\left(\frac{1-1_x}{|s-t|}\right)\right](x) \colon x \in \mathscr{X}\right\} < +\infty. \tag{14}$$

For all $t_1, t_2 \in \mathbb{R}_{\geq 0}$ such that $t_1 < t_2$ and $(x_{t_1}, x_{t_2}) \in \mathscr{X}^{\{t_1,t_2\}}$,

$$\frac{d^{\neq}_{\{t_1,t_2\}}(x_{t_1}, x_{t_2})}{t_2 - t_1} = \frac{1 - 1_{x_{t_1}}(x_{t_2})}{t_2 - t_1},$$

so it follows from Proposition 1 that, with $\Delta \coloneqq t_2 - t_1$,

$$\check{E}_{\{t_1,t_2\}}\left(\frac{d^{\neq}_{\{t_1,t_2\}}}{\Delta}\right) = \check{E}_0\left(\check{\mathrm{T}}_{t_1}\left(\mathscr{X} \to \mathbb{R} \colon x \mapsto \left[\check{\mathrm{T}}_\Delta\left(\frac{1-1_x}{\Delta}\right)\right](x)\right)\right).$$

Since $\check{\mathrm{T}}_{t_1}$ and $\check{E}_0$ are bounded above by the supremum, it follows more or less immediately from the preceding equality and Eq. (14) that

$$\limsup_{s \to t} \check{E}_{\{s,t\}} \left(\frac{d^{\neq}_{\{s,t\}}}{|s-t|} \right) \leq \lambda \quad \text{for all } t \in \mathbb{R}_{\geq 0}.$$

So for all $n \in \mathbb{N}$, Eq. (13) holds with $\lambda_n = \lambda$.

By combining Theorem 3 with Theorem 1 and Lemma 5, we get what I believe to be the second main result in this article. In order to highlight the similarity to [20, Theorem 5.6], [7, Theorem 2.5] and [8, Definition 5.5], for all $t \in \mathbb{R}_{\geq 0}$ we let $X_t \colon \Omega \to \mathscr{X}$ be the projector that maps $\omega \in \Omega$ to $\omega(t)$. To simplify the statement, we denote the set of $\sigma(\mathscr{D})/\mathcal{B}(\overline{\mathbb{R}})$-measurable functions that are bounded below by $\mathscr{M}_{\mathrm{b}} \coloneqq \mathcal{M}_{\mathrm{b}}(\mathscr{D})$ and the set of those that are bounded above by $\mathscr{M}^{\mathrm{b}} \coloneqq \mathcal{M}^{\mathrm{b}}(\mathscr{D})$; furthermore, we also let $\mathscr{M} \coloneqq \mathscr{M}_{\mathrm{b}} \cup \mathscr{M}^{\mathrm{b}}$.

Theorem 4. *Consider a convex expectation $\check{E}_0$ that is downward continuous and a convex Markov semigroup $(\check{\mathrm{T}}_t)_{t \in \mathbb{R}_{\geq 0}}$ with uniformly bounded rate. Then there is a convex expectation $\check{E}$ on $\mathscr{M}$ such that*

(i) *for all $f \in \mathcal{L}(\mathscr{X})$, $\check{E}(f(X_0)) = \check{E}_0(f)$;*

(ii) *for all $U = \{s_1, \dots, s_n\} \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}_{\geq 0}$ such that $s_1 < \cdots < s_n < t$ and all $f \in \mathcal{L}(\mathscr{X}^{n+1})$,*

$$\check{E}(f(X_{s_1}, \dots, X_{s_n}, X_t)) = \check{E}\big([\check{\mathrm{T}}_{t-s_n} f](X_{s_1}, \dots, X_{s_n})\big);$$

(iii) *$\check{E}$ is downward continuous on $\mathscr{D}_{\delta,\mathrm{b}}$; and*

(iv) *$\check{E}$ is upward continuous on $\mathscr{M}_{\mathrm{b}}$.*

Moreover, the restriction of $\check{E}$ to $\mathscr{M}_{\mathrm{b}}$ is the unique convex expectation that has these four properties.

Proof. Follows from Theorem 3, Propositions 2 and 1, Theorem 1 and Lemma 5.

In comparison to Theorem 5.6 in [20], Theorem 4 has a more limited scope, since the former involves a Polish space as state space and a 'two-parameter semigroup'. Theorem 4 is more useful though, since the domain $\mathscr{M}$ of the convex expectation in this result is much richer than the one in [20, Theorem 5.6]—which only includes bounded functions on the set of all paths that are measurable with respect to the (for many purposes inadequate) product σ-algebra. A similar comparison can be made between Theorem 4 on the one hand and Theorem 2.5 in [7] and Section 5 in [8]—which rely on Theorem 5.6 in [20]—on the other hand, the difference with before being that all of these results assume a countable state space and start from a one-parameter semigroup.

5. Constructing convex Markov semigroups

Theorem 4 is only useful if we can actually construct and/or determine a convex, sublinear or linear Markov semigroup. Fortunately, there are plenty of existing results which allows us to do exactly that.

5.1. Linear Markov semigroups

Consider a (linear) *Markov semigroup* $(\mathrm{T}_t)_{t \in \mathbb{R}_{\geq 0}}$, and fix some ordering $(x_1, x_2, \dots)$ of the state space $\mathscr{X}$. Then it follows from our assumptions—and the Daniell–Stone Theorem—that the matrix representation (with respect to the obvious basis $\{1_{x_1}, 1_{x_2}, \dots\}$) of $(\mathrm{T}_t)_{t \in \mathbb{R}_{\geq 0}}$, given by

$$\mathrm{T}_t(x, y) = [\mathrm{T}_t 1_y](x) \quad \text{for all } t \in \mathbb{R}_{\geq 0}, x, y \in \mathscr{X},$$

is in one-to-one correspondence with what is known as a transition (matrix) function—sometimes also 'transition matrix', see [48, § 1.1], [3, Example III.3.6], [41, Section 23.10] and [1, Part II, §1]. The semigroup $(\mathrm{T}_t)_{t \in \mathbb{R}_{\geq 0}}$ has uniformly bounded rate if and only if it is uniformly continuous—see for example [41, Section 23.11] or [1, Section II.19, Theorem 2]—and under this condition, the Markov semigroup $(\mathrm{T}_t)_{t\in\mathbb{R}_{\geq 0}}$ is generated by some operator Q, in the sense that

$$\mathrm{T}_t = e^{t\mathrm{Q}} = \lim_{n\to+\infty}\left(\mathrm{I} + \frac{t}{n}\mathrm{Q}\right)^n = \sum_{n=0}^{+\infty}\frac{t^n\mathrm{Q}^n}{n!} \quad \text{for all } t \in \mathbb{R}_{\geq 0}. \tag{15}$$

This operator Q is unique and given by

$$\mathrm{Q} = \lim_{t\searrow 0}\frac{\mathrm{T}_t - \mathrm{I}}{t};$$

moreover, it is (i) linear; (ii) downward continuous; (iii) bounded, in the sense that

$$\|\mathrm{Q}\| = \sup\left\{\frac{\|\mathrm{Q}f\|_\infty}{\|f\|_\infty} : f \in \mathscr{L}, f \neq 0\right\} < +\infty; \tag{16}$$

and (iv) a rate operator, meaning that

R1. $\mathrm{Q}\mu = 0$ for all $\mu \in \mathbb{R}$;

R2. $[\mathrm{Q}f](x) \leq 0$ for all $f \in \mathscr{L}$ and $x \in \mathscr{X}$ such that $f(x) = \sup f \geq 0$.

Conversely, any bounded linear rate operator Q—so a linear operator satisfying (R1), (R2) and Eq. (16)—generates a semigroup $(e^{t\mathrm{Q}})_{t\in\mathbb{R}_{\geq 0}}$ of transition operators that has uniformly bounded rate. Finally, it can be verified that $(e^{t\mathrm{Q}})_{t\in\mathbb{R}_{\geq 0}}$ is a Markov semigroup—or equivalently, that for all $t \in \mathbb{R}_{\geq 0}$, $e^{t\mathrm{Q}}$ is downward continuous—if (and then only if) Q is downward continuous. For more details regarding linear (uniformly continuous) semigroups and exponentials of bounded linear operators, we refer the reader to [41, 46] and references therein.

5.2. Convex Markov semigroups with a finite state space

Suppose that the state space $\mathscr{X}$ is finite. Under this condition, Nendel [7, Theorem 2.5] shows that a convex operator $\check{\mathrm{Q}}$ is a rate operator—so satisfies (R1) and (R2)—if and only if there is a convex transition semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ such that

$$\check{\mathrm{Q}}f = \lim_{t\searrow 0}\frac{\check{\mathrm{T}}_t f - f}{t} \quad \text{for all } f \in \mathscr{L}. \tag{17}$$

Whenever this is the case, the semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is strongly continuous because for all $f \in \mathscr{L}$, $\check{\mathrm{T}}_\bullet f : \mathbb{R}_{\geq 0} \to \mathscr{L}$ is the unique solution in $\mathrm{C}^1(\mathbb{R}_{\geq 0}, \mathscr{L})$ to the initial value problem

$$\begin{cases} g'(t) = \check{\mathrm{Q}}g(t) & \text{for all } t \in \mathbb{R}_{\geq 0}, \\ g(0) = f. \end{cases}$$

Since the solution to this initial value problem remains bounded, it can be approximated numerically by means of Runge–Kutta methods [7, Remark 2.6.(g)]. Nendel also shows the following analogon to Lemma 4.

Lemma 8. *An operator $\check{\mathrm{Q}}$ is a convex rate operator if and only if there is some set $\mathcal{Q}$ of linear rate operators and a family $(\alpha_\mathrm{Q})_{\mathrm{Q}\in\mathcal{Q}} \in \mathscr{L}^\mathcal{Q}$ with $\alpha_\mathrm{Q} \geq 0$ for all $\mathrm{Q} \in \mathcal{Q}$ and $\alpha_\mathrm{Q} = 0$ for some $\mathrm{Q} \in \mathcal{Q}$ such that*

$$[\check{\mathrm{Q}}f](x) = \sup\{[\mathrm{Q}f](x) - \alpha_\mathrm{Q}(x) : \mathrm{Q} \in \mathcal{Q}\} \quad \textit{for all } f \in \mathscr{L}. \tag{18}$$

Of course, the question now is whether such a convex transition semigroup $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ (i) is Markov, meaning that $\check{\mathrm{T}}_t$ is downward continuous for all $t \in \mathbb{R}_{\geq 0}$; and (ii) has uniformly bounded rate. Since $\mathscr{X}$ is finite, pointwise convergence in $\mathscr{L}$ is equivalent to uniform convergence, so $\check{\mathrm{T}}_t$ is downward continuous due to Lemma 2. Answering the second question isn't as easy, unfortunately. From Eq. (17), we infer that

$$\lim_{t\searrow 0}\frac{[\check{\mathrm{T}}_t(1 - 1_x)](x)}{t} = [\check{\mathrm{Q}}(1 - 1_x)](x) < +\infty \quad \text{for all } x \in \mathscr{X}. \tag{19}$$

Unfortunately, the lack of positive homogeneity does not allow us to use this to draw conclusions about

$$\limsup_{t\searrow 0} \sup\biggl\{ \biggl[\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr)\biggr](x) \colon x\in\mathscr{X}\biggr\}.$$

That said, we can use Nendel's Lemma 4.5 [7] to obtain the following nice result.

Proposition 3. *Consider a convex rate operator $\check{\mathrm{Q}}$, denote the corresponding convex Markov semigroup by $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ and let $\mathcal{Q}$ and $(\alpha_{\mathrm{Q}})_{\mathrm{Q}\in\mathcal{Q}}$ be as in Lemma 8. If*

$$\sup\bigl\{\|\alpha_{\mathrm{Q}}\|_\infty \colon \mathrm{Q}\in\mathcal{Q}\bigr\} < +\infty,$$

then $\mathcal{Q}$ is bounded in the sense that

$$\sup\bigl\{\|\mathrm{Q}\| \colon \mathrm{Q}\in\mathcal{Q}\bigr\} < +\infty,$$

and $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate.

Proof. Nendel [7, Remark 4.3] proves that $\mathcal{Q}$ is bounded, so it remains for us to prove that $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate. To this end, note that for all $t\in\mathbb{R}_{>0}$ and $x\in\mathscr{X}$,

$$\biggl[\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr)\biggr](x) = \biggl[\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr)\biggr](x) - \biggl[\frac{1-1_x}{t}\biggr](x) \leq \biggl\|\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr) - \biggl(\frac{1-1_x}{t}\biggr)\biggr\|_\infty.$$

Because $(\alpha_{\mathrm{Q}})_{\mathrm{Q}\in\mathcal{Q}}$ is bounded by the assumptions in the statement, it follows from Lemma 4.5 in [7] that for all $t\in\mathbb{R}_{>0}$ and $x\in\mathscr{X}$

$$\biggl\|\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr) - \biggl(\frac{1-1_x}{t}\biggr)\biggr\|_\infty \leq t\sup\biggl\{\biggl\|\mathrm{Q}\biggl(\frac{1-1_x}{t}\biggr) - \alpha_{\mathrm{Q}}\biggr\|_\infty \colon \mathrm{Q}\in\mathcal{Q}\biggr\}.$$

Using some properties of norms, we see that for all $t\in\mathbb{R}_{>0}$ and $x\in\mathscr{X}$, and briefly using different notation for the supremum,

$$\begin{aligned} t\sup_{\mathrm{Q}\in\mathcal{Q}}\biggl\|\mathrm{Q}\biggl(\frac{1-1_x}{t}\biggr) - \alpha_{\mathrm{Q}}\biggr\|_\infty &\leq t\sup_{\mathrm{Q}\in\mathcal{Q}}\biggl(\biggl\|\mathrm{Q}\biggl(\frac{1-1_x}{t}\biggr)\biggr\|_\infty + \|\alpha_{\mathrm{Q}}\|_\infty\biggr) \\ &\leq t\sup_{\mathrm{Q}\in\mathcal{Q}}\biggl\|\mathrm{Q}\biggl(\frac{1-1_x}{t}\biggr)\biggr\|_\infty + t\sup_{\mathrm{Q}\in\mathcal{Q}}\|\alpha_{\mathrm{Q}}\|_\infty \\ &\leq t\sup_{\mathrm{Q}\in\mathcal{Q}}\|\mathrm{Q}\|\biggl\|\frac{1-1_x}{t}\biggr\|_\infty + t\sup_{\mathrm{Q}\in\mathcal{Q}}\|\alpha_{\mathrm{Q}}\|_\infty \\ &\leq \sup_{\mathrm{Q}\in\mathcal{Q}}\|\mathrm{Q}\| + t\sup_{\mathrm{Q}\in\mathcal{Q}}\|\alpha_{\mathrm{Q}}\|_\infty. \end{aligned}$$

From all this, we infer that for all $t\in\mathbb{R}_{>0}$,

$$\sup\biggl\{\biggl[\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr)\biggr](x) \colon x\in\mathscr{X}\biggr\} \leq \sup\{\|\mathrm{Q}\| \colon \mathrm{Q}\in\mathcal{Q}\} + t\sup\{\|\alpha_{\mathrm{Q}}\|_\infty \colon \mathrm{Q}\in\mathcal{Q}\}.$$

Since $\sup\{\|\alpha_{\mathrm{Q}}\|_\infty \colon \mathrm{Q}\in\mathcal{Q}\} < +\infty$ by the assumptions in the statement and $\sup\{\|\mathrm{Q}\| \colon \mathrm{Q}\in\mathcal{Q}\} < +\infty$ by the first part in the statement, it follows that

$$\limsup_{t\searrow 0}\sup\biggl\{\biggl[\check{\mathrm{T}}_t\biggl(\frac{1-1_x}{t}\biggr)\biggr](x) \colon x\in\mathscr{X}\biggr\} \leq \sup\{\|\mathrm{Q}\| \colon \mathrm{Q}\in\mathcal{Q}\} < +\infty,$$

so $(\check{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ does indeed have uniformly bounded rate.

We can get stronger results if we consider the sublinear instead of the convex case. In fact, all of the aforementioned results in the linear case carry over to the sublinear case. In my doctoral dissertation [49, Theorem 3.75 and Lemma 3.76], I show that a sublinear transition semigroup $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate if and only if it is uniformly continuous, and in that case

$$\overline{\mathrm{Q}} := \lim_{t\searrow 0} \frac{\overline{\mathrm{T}}_t - \mathrm{I}}{t}$$

is the unique sublinear rate operator such that

$$\overline{\mathrm{T}}_t = e^{t\overline{\mathrm{Q}}} := \lim_{n\to+\infty} \left(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\right)^n \quad \text{for all } t \in \mathbb{R}_{\geq 0},$$

where the limits are with respect to a norm on the set $\mathfrak{O}_{\mathrm{ph}}$ of 'bounded' positively homogeneous operators given by

$$\|\bullet\| : \mathfrak{O}_{\mathrm{ph}} \to \mathbb{R}_{\geq 0} : \mathrm{A} \mapsto \sup\left\{ \frac{\|\mathrm{A}f\|_\infty}{\|f\|_\infty} : f \in \mathscr{L}, f \neq 0 \right\}.$$

Conversely, for any sublinear rate operator $\overline{\mathrm{Q}}$, the corresponding family of exponentials $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$ is a sublinear Markov semigroup that has uniformly bounded rate (and therefore is uniformly continuous) [14–16]. The latter also follows from Proposition 3 and the following analogon to (the sublinear part of) Lemma 4; while this result is taken from [16, Propositions 7.5 and 7.6], it can be seen as a consequence of Lemma 8 as well.

Lemma 9. *An operator $\overline{\mathrm{Q}}$ is a sublinear rate operator if and only if there is some bounded set $\mathcal{Q}$ of linear rate operators such that*

$$[\overline{\mathrm{Q}}f](x) = \sup\{[\mathrm{Q}f](x) : \mathrm{Q} \in \mathcal{Q}\} \quad \textit{for all } f \in \mathscr{L}, x \in \mathscr{X}.$$

Consider a sublinear rate operator $\overline{\mathrm{Q}}$, and let $\mathcal{Q}$ be a bounded set of linear rate operators as in Lemma 9. Then Nendel [7, Theorem 2.5] shows that $(e^{t\overline{\mathrm{Q}}})_{t\in\mathbb{R}_{\geq 0}}$ is the pointwise smallest semigroup such that $e^{t\mathrm{Q}}f \leq e^{t\overline{\mathrm{Q}}}f$ for all $\mathrm{Q} \in \mathscr{Q}$ and $t \in \mathbb{R}_{\geq 0}$.

5.3. Convex Markov semigroups with a countably infinite state space

The case that $\mathscr{X}$ is countably infinite has received less attention. To the best of my knowledge, no work has been done on convex transition groups; sublinear transition semigroups have been studied, though [8, 19, 22].

Nendel [8, Section 5] constructs a strongly continuous sublinear transition semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ as follows. His starting point is a set $\{(\mathrm{T}^i_t)_{t\in\mathbb{R}_{\geq 0}} : i \in \mathcal{I}\}$ of (linear) Markov semigroups, from which he constructs the induced *Nisio semigroup* $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$. He shows that this Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is a sublinear transition semigroup, and the pointwise smallest semigroup that dominates each of the Markov semigroups $(\mathrm{T}^i_t)_{t\in\mathbb{R}_{\geq 0}}$. Further on, he mentions that since $\mathscr{X}$ is infinite, establishing that $\overline{\mathrm{S}}_t$ is downward continuous 'is not trivial and leads to restrictions' on $\{(\mathrm{T}^i_t)_{t\in\mathbb{R}_{\geq 0}} : i \in \mathcal{I}\}$.

In the special case that $(\mathrm{T}^i_t)_{t\in\mathbb{R}_{\geq 0}}$ has uniformly bounded rate for all $i \in \mathcal{I}$, we collect their generators in

$$\mathscr{R} := \left\{ \mathrm{Q}^i := \lim_{t\searrow 0} \frac{\mathrm{T}^i_t - \mathrm{I}}{t} : i \in \mathcal{I} \right\}.$$

Under the assumption that this set is bounded, in the sense that $\sup\{\|\mathrm{Q}^i\| : i \in \mathcal{I}\} < +\infty$, Nendel [8, Remark 5.6] shows that

$$\lim_{t\searrow 0} \frac{\overline{\mathrm{S}}_t f - f}{t} = \overline{\mathrm{R}}f \quad \text{for all } f \in \mathscr{L}, \tag{20}$$

where $\overline{\mathrm{R}}$ is the pointwise upper envelope of the (bounded) set $\mathscr{R}$ of generators of the Markov semigroups, and that for all $f \in \mathscr{L}$, $\overline{\mathrm{S}}_{\bullet} f \colon \mathbb{R}_{\geq 0} \to \mathscr{L}$ is the unique solution in $\mathrm{C}^1(\mathbb{R}_{\geq 0}, \mathscr{L})$ to the initial value problem

$$\begin{cases} g'(t) = \overline{\mathrm{R}} g(t) & \text{for all } t \in \mathbb{R}_{\geq 0}, \\ g(0) = f. \end{cases}$$

He does not investigate whether the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t \in \mathbb{R}_{\geq 0}}$ has uniformly bounded rate, but this is easy to show using Eq. (20).

Lemma 10. *Consider a uniformly bounded set $\mathcal{Q}$ of linear rate operators that are downward continuous. Then the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t \in \mathbb{R}_{\geq 0}}$ corresponding to $\{(e^{t\mathrm{Q}})_{t \in \mathbb{R}_{\geq 0}} \colon \mathrm{Q} \in \mathcal{Q}\}$ has uniformly bounded rate.*

Proof. First, observe that for all $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathscr{X}$,

$$\begin{aligned} \left[\overline{\mathrm{S}}_t \left(\frac{1 - 1_x}{t}\right)\right](x) &= \frac{[\overline{\mathrm{S}}_t(1 - 1_x)](x)}{t} \\ &= \frac{[\overline{\mathrm{S}}_t(1 - 1_x)](x) - [1 - 1_x](x)}{t} \\ &\leq \left\| \frac{[\overline{\mathrm{S}}_t(1 - 1_x)](x) - [1 - 1_x](x)}{t} \right\|_\infty \\ &\leq \left\| \frac{\overline{\mathrm{S}}_t(1 - 1_x) - (1 - 1_x)}{t} - \overline{\mathrm{R}}(1 - 1_x) \right\|_\infty + \|\overline{\mathrm{R}}(1 - 1_x)\|_\infty. \end{aligned}$$

Second, fix some $x \in \mathscr{X}$ and $\epsilon \in \mathbb{R}_{>0}$. Then there are some $y \in \mathscr{X}$ and $\mathrm{Q} \in \mathcal{Q}$ such that

$$\|\overline{\mathrm{R}}(1 - 1_x)\|_\infty - \frac{\epsilon}{2} < |[\overline{\mathrm{R}}(1 - 1_x)](y)| \quad \text{and} \quad |[\overline{\mathrm{R}}(1 - 1_x)](y) - [\mathrm{Q}(1 - 1_x)](y)| < \frac{\epsilon}{2},$$

from which we infer that

$$\|\overline{\mathrm{R}}(1 - 1_x)\|_\infty - \epsilon < |[\mathrm{Q}(1 - 1_x)](y)| \leq \|\mathrm{Q}(1 - 1_x)\|_\infty \leq \|\mathrm{Q}\|.$$

Since this inequality holds for arbitrary $\epsilon \in \mathbb{R}_{>0}$, we can conclude from it that

$$\|\overline{\mathrm{R}}(1 - 1_x)\|_\infty \leq \sup\{\|\mathrm{Q}\| \colon \mathrm{Q} \in \mathcal{Q}\}.$$

Finally, it follows from our preceding two observations and Eq. (20) that

$$\limsup_{t \searrow 0} \sup\left\{ \left[\overline{\mathrm{S}}_t \left(\frac{1 - 1_x}{t}\right)\right](x) \colon x \in \mathscr{X} \right\} \leq \sup\{\|\mathrm{Q}\| \colon \mathrm{Q} \in \mathcal{Q}\}.$$

Since $\mathcal{Q}$ is bounded, this proves that the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t \in \mathbb{R}_{\geq 0}}$ has uniformly bounded rate.

More recently, I've shown that as in the linear case and the sublinear finite-state case, (i) uniform continuity is necessary and sufficient for a sublinear transition semigroup to be generated by a bounded sublinear rate operator; (ii) this semigroup is Markov if and only if the corresponding bounded sublinear rate operator is downward continuous; and (iii) uniform continuity is equivalent to uniformly bounded rate. The interested reader can find the details in [22], but here I'll summarise the more salient points. More formally, Theorem 1 there establishes that a sublinear transition semigroup $(\overline{\mathrm{T}}_t)_{t \in \mathbb{R}_{\geq 0}}$ is uniformly continuous if and only if there is a bounded sublinear rate operator $\overline{\mathrm{Q}}$ such that

$$\overline{\mathrm{T}}_t = e^{t\overline{\mathrm{Q}}} = \lim_{n \to +\infty} \left(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\right)^n \quad \text{for all } n \in \mathbb{N}.$$

Furthermore, from Proposition 24 in [22] we learn that the family $(e^{t\overline{\mathrm{Q}}})_{t \in \mathbb{R}_{\geq 0}}$ is a sublinear Markov semigroup—meaning that the sublinear transition operator $e^{t\overline{\mathrm{Q}}}$ is downward continuous for all $t \in \mathbb{R}_{\geq 0}$—if and only if $\overline{\mathrm{Q}}$ is downward continuous. To establish the equivalence between uniform continuity and uniformly bounded rate, we simply need to string together a couple of results in [22].

Lemma 11. *A sublinear transition semigroup $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous if and only if it has uniformly bounded rate.*

Proof. Proposition 19 in [22] tells us that $(\overline{\mathrm{T}}_t)_{t\in\mathbb{R}_{\geq 0}}$ is uniformly continuous if and only if

$$\limsup_{t\searrow 0} \sup\left\{\frac{1}{\|f\|_\infty}\left\|\left(\frac{\overline{\mathrm{T}}_t-\mathrm{I}}{t}\right)f\right\|_\infty : f\in\mathscr{L}, f\neq 0\right\} < +\infty.$$

It therefore suffices to observe that for all $t\in\mathbb{R}_{>0}$,

$$\begin{aligned}\left\{\frac{1}{\|f\|_\infty}\left\|\left(\frac{\overline{\mathrm{T}}_t-\mathrm{I}}{t}\right)f\right\|_\infty : f\in\mathscr{L}, f\neq 0\right\} &= 2\sup\left\{\left[\left(\frac{\overline{\mathrm{T}}_t-\mathrm{I}}{t}\right)(1-\mathbb{1}_x)\right](x) : x\in\mathscr{X}\right\}\\ &= 2\sup\left\{(\overline{\mathrm{T}}_t-\mathrm{I})\left(\frac{1-\mathbb{1}_x}{t}\right)(x) : x\in\mathscr{X}\right\}\\ &= 2\sup\left\{\overline{\mathrm{T}}_t\left(\frac{1-\mathbb{1}_x}{t}\right)(x) : x\in\mathscr{X}\right\},\end{aligned}$$

where for the first equality we used Proposition 9 in [22], which is relevant since $(\overline{\mathrm{T}}_t-\mathrm{I})/t$ is a (bounded) sublinear rate operator for all $t\in\mathbb{R}_{>0}$ [22, Lemma 10], for the second equality we used the positive homogeneity of $\overline{\mathrm{T}}_t-\mathrm{I}$ and for the third equality we used that $[1-\mathbb{1}_x](x)=0$ for all $x\in\mathscr{X}$.

To compare these results with the aforementioned results by Nendel, I would like to mention that in [22, Proposition 28, Lemma 31 and Proposition 18] I show that a sublinear rate operator is 'bounded' if and only if it is the pointwise upper envelope of a uniformly bounded set or rate operators, and that $e^{\bullet\overline{\mathrm{Q}}}\colon \mathbb{R}_{\geq 0}\to\mathscr{L}\colon t\mapsto e^{t\overline{\mathrm{Q}}}f$ is the unique solution in $\mathrm{C}^1(\mathbb{R}_{\geq 0},\mathscr{L})$ to the initial value problem

$$\begin{cases} g'(t)=\overline{\mathrm{Q}}g(t) & \text{for all } t\in\mathbb{R}_{\geq 0},\\ g(0)=f.\end{cases}$$

Hence, the Nisio semigroup $(\overline{\mathrm{S}}_t)_{t\in\mathbb{R}_{\geq 0}}$—corresponding to $\{(e^{t\mathrm{Q}})_{t\in\mathbb{R}_{\geq 0}}\colon \mathrm{Q}\in\mathscr{R}\}$ with $\mathscr{R}$ a uniformly bounded set of rate operators that are downward continuous—coincides with the sublinear transition semigroup $(e^{t\overline{\mathrm{R}}})_{t\in\mathbb{R}_{\geq 0}}$ generated by the pointwise upper envelope $\overline{\mathrm{R}}$ of $\mathscr{R}$; consequently this Nisio semigroup is a sublinear Markovian semigroup if and only if this upper envelope $\overline{\mathrm{R}}$ is downward continuous.

6. The sublinear Poisson process

To wrap things up, let us revisit my initial approach towards constructing/defining sublinear Poisson processes in [19] with the novel results that we have at our disposal now. Henceforth, we let $\mathscr{X}:=\mathbb{Z}_{\geq 0}$, and we fix some rate interval $\Lambda:=[\underline{\lambda},\overline{\lambda}]\subset\mathbb{R}_{\geq 0}$. We define the *sublinear Poisson generator* $\overline{\mathrm{G}}\colon\mathscr{L}\to\mathscr{L}$ for all $f\in\mathscr{L}$ by

$$\overline{\mathrm{G}}f\colon \mathbb{Z}_{\geq 0}\to\mathbb{R}\colon z\mapsto \max\{\lambda(f(z+1)-f(z))\colon \lambda\in\Lambda\}.$$

It is easy to verify that $\overline{\mathrm{G}}$ is a sublinear rate operator that is downward continuous. Additionally, it is obvious that the sublinear Poisson generator $\overline{\mathrm{G}}$ is the pointwise upper envelope of a bounded set of bounded rate operators, for example the sets $\{\mathrm{G}_\lambda\colon \lambda\in\Lambda\}$ and $\{\mathrm{G}_{\underline{\lambda}},\mathrm{G}_{\overline{\lambda}}\}$ of Poisson generators, where for all $\lambda\in\mathbb{R}_{\geq 0}$, the corresponding Poisson generator G_λ is the linear rate operator that is defined for all $f\in\mathscr{L}$ by

$$\mathrm{G}_\lambda f\colon \mathbb{Z}_{\geq 0}\to\mathbb{R}\colon z\mapsto\lambda(f(z+1)-f(z)).$$

I leave it to the reader to check that $\|\mathrm{G}_\lambda\|=2\lambda$; with this equality, we see that the sets $\{\mathrm{G}_\lambda\colon\lambda\in\Lambda\}$ and $\{\mathrm{G}_{\underline{\lambda}},\mathrm{G}_{\overline{\lambda}}\}$ are uniformly bounded by $2\overline{\lambda}$, so $\overline{\mathrm{G}}$ is 'bounded'.

Thanks to these properties of $\overline{\mathrm{G}}$ and the aforementioned general results, we know that $(e^{t\overline{\mathrm{G}}})_{t\in\mathbb{R}_{\geq 0}}$ is a sublinear Markov transition semigroup that has uniformly bounded rate. Note that in this particular case, this was essentially already shown by Erreygers and De Bock [19, 50]. There, we defined a sublinear version of the Poisson process in the setting of a sensitivity analysis: we (i) consider the 'set of all counting/Poisson processes that are consistent with Λ', and (ii) define the sublinear expectation as the upper envelope of the expectations corresponding to the processes in this set. In Section 6 there, we then explain how we can use $(e^{t\overline{\mathrm{G}}})_{t\in\mathbb{R}_{\geq 0}}$ to determine the upper/sublinear expectation of any function of the form $f(X_t)$, but we do not go beyond that. While the results in Section 4 allow us to deal with more involved variables, note that there's a crucial difference between the approach there and the one here: there the (conditional) sublinear expectations are a derived notion and can be conditional, while here we construct a sublinear expectation directly.

To construct the sublinear Poisson process by means of Theorem 4, we define the initial sublinear expectation

$$E_0\colon \mathscr{L} \to \mathbb{R}\colon f \mapsto E_0(f) \coloneqq f(0),$$

and recall that $(e^{t\overline{\mathrm{G}}})_{t\in\mathbb{R}_{\geq 0}}$ is a sublinear Markov semigroup with uniformly bounded rate. Theorem 4 now tells us that there is a unique corresponding sublinear expectation $\overline{E}$ on $\mathscr{M}$.

If $\underline{\lambda} = \overline{\lambda} = \lambda$, then $(e^{t\overline{\mathrm{G}}})_{t\in\mathbb{R}_{\geq 0}} = (e^{t\mathrm{G}_\lambda})_{t\in\mathbb{R}_{\geq 0}}$ is a Markov semigroup and $\overline{E}$ is linear. It is essentially well-known—see for example [50, Proposition 69] or [51, Section 2.4]—that

$$[e^{t\mathrm{G}_\lambda} f](z) = \sum_{y\geq z} f(y)\psi_{\lambda t}(y-z) \quad \text{for all } f \in \mathscr{L}, z \in \mathbb{Z}_{\geq 0}.$$

It follows from this equality and the construction in Proposition 2 that $\overline{E}$ is a linear expectation that coincides with the one that is obtained through the classical definition—for example in [52, Section 2.1]—in terms of the distributions of the increments $X_{t_k} - X_{s_k}$ for $s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n$.

If $\underline{\lambda} < \overline{\lambda}$, then $\overline{E}$ can be thought of as a sublinear Poisson process. Besides the similar construction as the Poisson process, the rate interval $[\underline{\lambda}, \overline{\lambda}]$ has a similar interpretation as the rate λ of the Poisson process. For example, for all $s, t \in \mathbb{R}_{\geq 0}$ with $s < t$,

$$\overline{E}(X_t - X_s) = \overline{\lambda}(t-s);$$

this is because

$$\overline{E}(f(X_s, X_t)) = \overline{E}([e^{(t-s)\overline{\mathrm{Q}}} f](X_s)) \quad \text{with } f(X_s, X_t) = X_t - X_s$$

by Theorem 4 and $[e^{(t-s)\overline{\mathrm{Q}}} f](X_s) = \overline{\lambda}(t-s)$ by [19, Theorem 15 and Corollary 18]. For the Poisson process, we also have that $1/\lambda$ is equal to the expected time between two arrivals/events. To investigate this for $\overline{E}$, one could use the construction and monotonicity properties of $\overline{E}$. As this would lead us too far, we'll not go into this here; note, though, that it should be feasible by combining the results in [19, Sections 5.3 and 5.4] regarding the computation of $e^{t\overline{\mathrm{G}}}$ with the approach used to approximate hitting times in [49, Section 6.3].

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Appendix A. Proof of Theorem 1

Proof of Theorem 1. As $\check{E}$ is downward continuous, Lemma 3 guarantees that every $E \in \mathbb{E}_{\check{E}}$ is downward continuous. Consequently, it follows from the Daniell–Stone Theorem that for all $E \in \mathbb{E}_{\check{E}}$, $E = E^{\odot}|_{\mathcal{D}}$ with

$$E^{\odot} : \mathcal{I}(P_E) \to \overline{\mathbb{R}} : g \mapsto \int g \mathrm{d}P_E.$$

It follows immediately from this and Lemma 4 that $\check{E}^{\odot}$ is well defined and extends $\check{E}$.

On several occasions, we will need that for all $f \in \mathcal{I}(\check{E})$ and $E \in \mathbb{E}_{\check{E}}$, $\check{E}^*(E) \in \mathbb{R}$ (due to Lemma 4) and

$$E^{\odot}(f) \leq \check{E}^{\odot}(f) + \check{E}^*(E). \tag{A.1}$$

Next, we show that $\check{E}^{\odot}$ is a convex expectation. The extension $\check{E}^{\odot}$ is a nonlinear expectation: (i) $\mathcal{I}(\check{E})$ includes all constant real functions because $\mathcal{D} \subseteq \mathcal{I}(\check{E})$ and $\mathcal{D}$ includes all constant real functions; (ii) $\check{E}^{\odot}$ is order preserving because the Lebesgue integral is order preserving [4, Theorem 4.2 (iv) or Theorem 8.5 (iv)]; and (iii) $\check{E}^{\odot}$ is constant preserving because it extends $\check{E}$ and $\check{E}$ is constant preserving. To verify that $\check{E}^{\odot}$ is convex, we fix some $f, g \in \mathcal{I}(\check{E})$ and $\lambda \in [0,1]$ such that $f+g$ is meaningful and in $\mathcal{I}(\check{E})$ and $\lambda\check{E}^{\odot}(f) + (1-\lambda)\check{E}^{\odot}(g)$ is meaningful. If $\lambda = 0$ or $\lambda = 1$, clearly $\check{E}^{\odot}(\lambda f + (1-\lambda)f) = \lambda\check{E}^{\odot}(f) + (1-\lambda)\check{E}^{\odot}(g)$; hence, without loss of generality we may assume that $0 < \lambda < 1$. Due to symmetry, and because $\lambda\check{E}^{\odot}(f)+(1-\lambda)\check{E}^{\odot}(g)$ is meaningful, we need to distinguish three cases: (i) $\check{E}^{\odot}(f) = +\infty$ and $\check{E}^{\odot}(g) > -\infty$; (ii) $\check{E}^{\odot}(f)$ and $\check{E}^{\odot}(g)$ both real; and (iii) $\check{E}^{\odot}(f) = -\infty$ and $\check{E}^{\odot}(g) < +\infty$. In the first case, the required inequality holds trivially. In the second case, it follows from Eq. (A.1) that for all $E \in \mathbb{E}_{\check{E}}$, $E^{\odot}(f) < +\infty$ and $E^{\odot}(g) < +\infty$, so $\lambda E^{\odot}(f) + (1-\lambda)E^{\odot}(g)$ is meaningful and, due to the linearity of $E^{\odot}$ [4, Theorem 4.9 (i) or Theorem 8.5 (i)], equal to $E^{\odot}(\lambda f + (1-\lambda)g)$. Similarly, in the third case, it follows from Eq. (A.1) that for all $E \in \mathbb{E}_{\check{E}}$, $E^{\odot}(f) = -\infty$ and $E^{\odot}(g) < +\infty$, so $\lambda E^{\odot}(f) + (1-\lambda)E^{\odot}(g)$ is meaningful and, due to the linearity of $E^{\odot}$, equal to $E^{\odot}(\lambda f + (1-\lambda)g)$. Consequently, in the last two cases,

$$\begin{aligned}
\check{E}^{\odot}(\lambda f + (1-\lambda)g) &= \sup\bigl\{E^{\odot}(\lambda f + (1-\lambda)g) - \check{E}^*(E) : E \in \mathbb{E}_{\check{E}}\bigr\} \\
&= \sup\bigl\{\lambda E^{\odot}(f) + (1-\lambda)E^{\odot}(g) - \check{E}^*(E) : E \in \mathbb{E}_{\check{E}}\bigr\} \\
&\leq \lambda \sup\bigl\{E^{\odot}(f) - \check{E}^*(E) : E \in \mathbb{E}_{\check{E}}\bigr\} \\
&\qquad + (1-\lambda)\sup\bigl\{E^{\odot}(g) - \check{E}^*(E) : E \in \mathbb{E}_{\check{E}}\bigr\} \\
&= \lambda\check{E}^{\odot}(f) + (1-\lambda)\check{E}^{\odot}(g),
\end{aligned}$$

as required.

Denk et al. [20, Theorem 3.10] show that the restriction of $\check{E}^{\odot}$ to $\mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ is downward continuous on $\mathcal{D}_{\delta,\mathrm{b}}$. Since $\mathcal{D}_{\delta,\mathrm{b}} \subseteq \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cap \mathcal{M}^{\mathrm{b}}(\mathcal{D})$, this implies that $\check{E}^{\odot}$ is downward continuous on $\mathcal{D}_{\delta,\mathrm{b}}$ as well.

Proving the upward continuity on $\mathcal{I}_{\mathrm{uc}}(\check{E})$ is straightforward. To this end, we fix any $(\mathcal{I}_{\mathrm{uc}}(E))^{\mathbb{N}} \ni (f_n)_{n\in\mathbb{N}} \nearrow f \in \mathcal{I}_{\mathrm{uc}}(E)$. For all $E \in \mathbb{E}_{\check{E}}$, $E^{\odot}$ is upward continuous on

$$\bigl\{f \in \mathcal{I}(P_E) : E^{\odot}(f) > -\infty\bigr\} \supseteq \mathcal{M}_{\mathrm{b}}(\mathcal{D})$$

due to the Monotone Convergence Theorem—see for example [4, Corollary 4.13]—and therefore

$$\lim_{n\to+\infty} E^{\odot}(f_n) = \sup_{n\in\mathbb{N}} E^{\odot}(f_n) = E^{\odot}(f).$$

From this it follows that, since $\check{E}^{\odot}$ is order preserving,

$$\begin{aligned}
\lim_{n\to+\infty} \check{E}^{\odot}(f_n) &= \sup\big\{\check{E}^{\odot}(f_n)\colon n\in\mathbb{N}\big\} \\
&= \sup\big\{\sup\big\{E^{\odot}(f_n) - \check{E}^{*}(E)\colon E\in\mathbb{E}_{\check{E}}\big\}\colon n\in\mathbb{N}\big\} \\
&= \sup\big\{\sup\big\{E^{\odot}(f_n) - \check{E}^{*}(E)\colon n\in\mathbb{N}\big\}\colon E\in\mathbb{E}_{\check{E}}\big\} \\
&= \sup\big\{E^{\odot}(f) - \check{E}^{*}(E)\colon E\in\mathbb{E}_{\check{E}}\big\} \\
&= \check{E}^{\odot}(f),
\end{aligned}$$

as required.

To prove the second part of the statement, we assume that $\check{E}$ is sublinear. Recall from Lemma 4 that $\check{E}^{*}(E) = 0$ for all $E \in \mathbb{E}_{\check{E}}$ and that $\mathbb{E}_{\check{E}}$ is the set of dominated linear expectations (on $\mathcal{D}$). Hence, to see that $\check{E}^{\odot}$ is positively homogeneous, it suffices to additionally realise that $E^{\odot}$ is homogeneous [4, Theorem 4.9 (i) or Theorem 8.5 (i)] for all $E \in \mathbb{E}_{\check{E}}$. That $\check{E}^{\odot}$ is subadditve follows immediately from its convexity and positive homogeneity.