Sublinear Expectations for Countable-State Uncertain Processes

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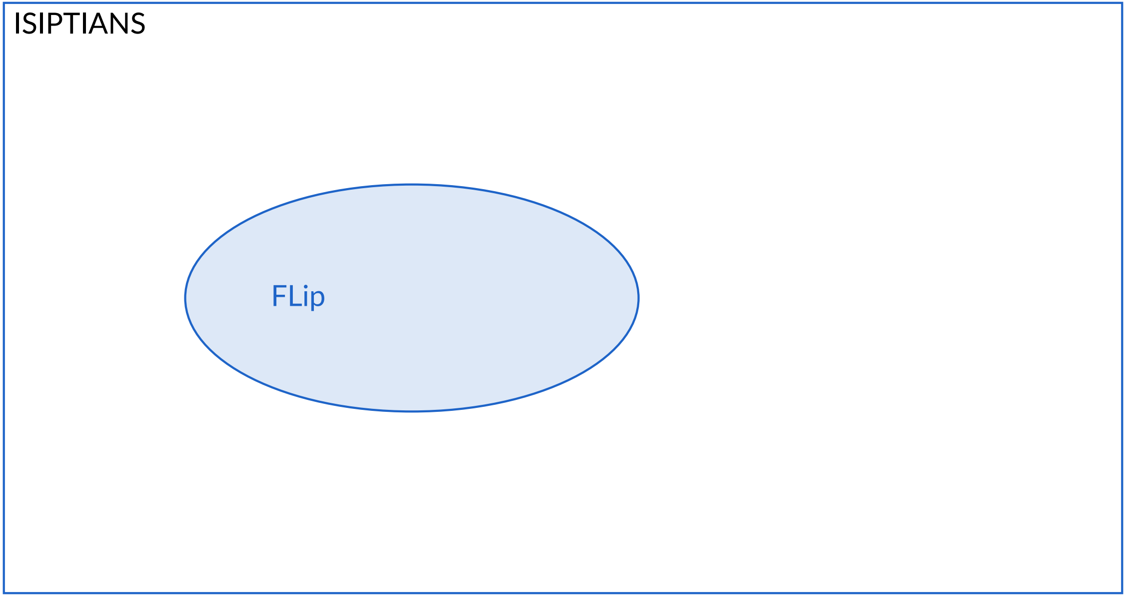
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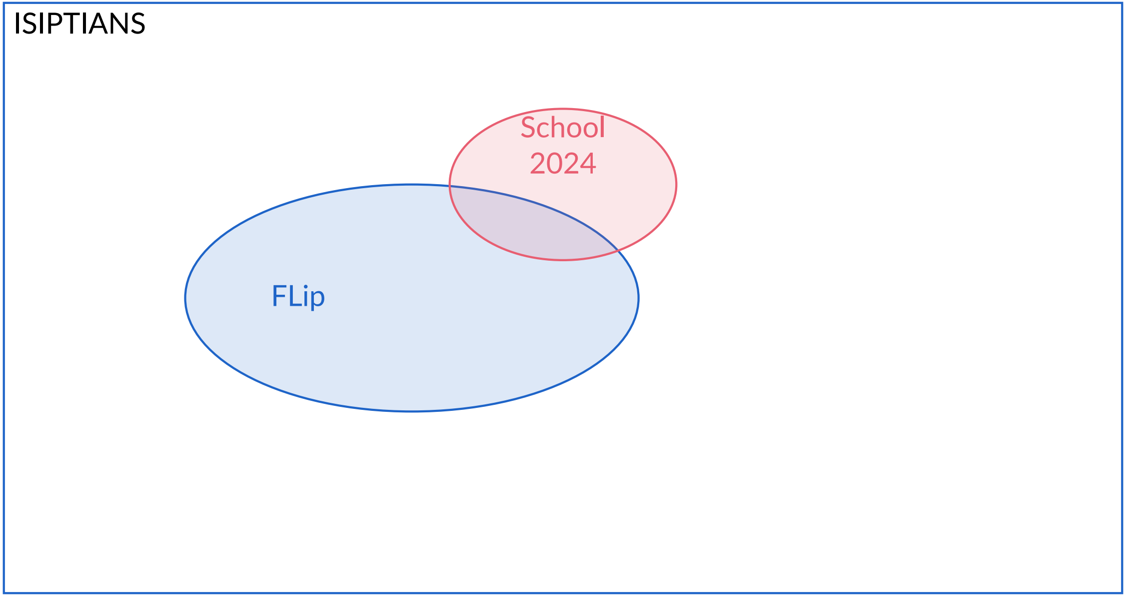
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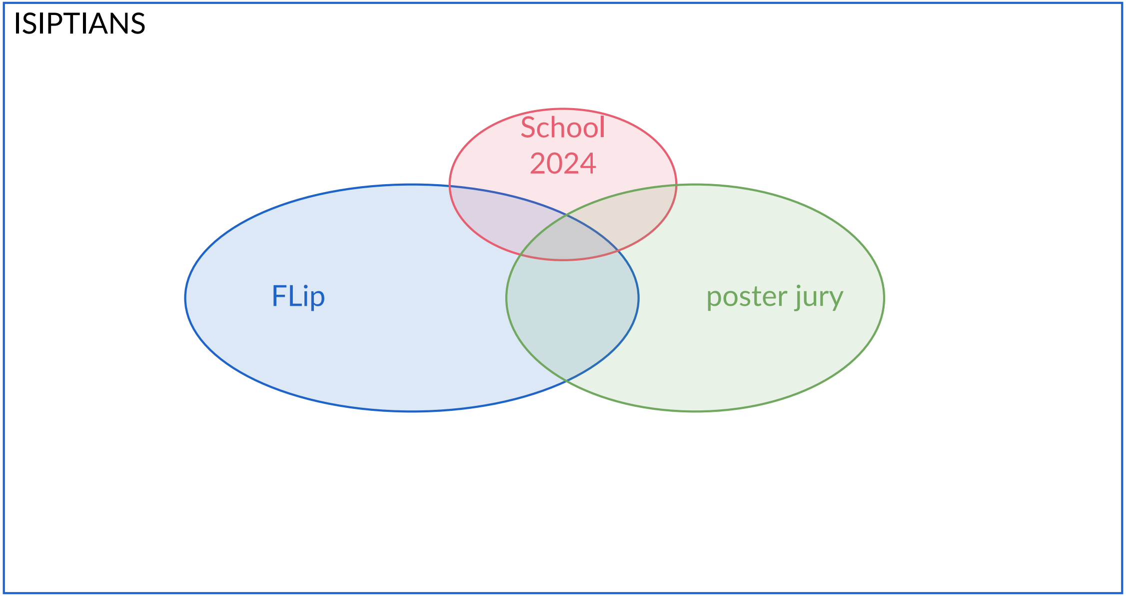
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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

ROBERT DENK, MICHAEL KUPPER,* and MAX NENDEL

Sublinear Expectations for Countable-State Uncertain Processes

sublinear expectation $\overline{E}\colon \mathcal{D} \subseteq \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

$\{\alpha \in \mathbb{R}^{\Omega} : \alpha \text{ constant}\} \subseteq \mathcal{D}$

constant preserving:
$\overline{E}(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$

isotone:
$\overline{E}(f) \leq \overline{E}(g)$ for all $f \leq g \in \mathcal{D}$

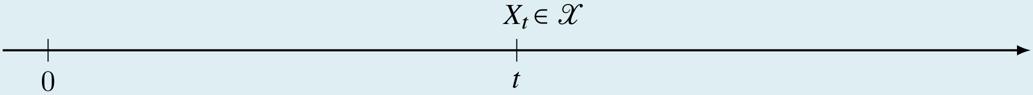
sublinear:
$\overline{E}(\mu f + g) \leq \mu \overline{E}(f) + \overline{E}(g)$
for all $\mu \in \mathbb{R}_{\geq 0}$ and $f, g, \mu f + g \in \mathcal{D}$

sublinear expectation $\overline{E}\colon \mathcal{D} \subseteq \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

$\Updownarrow$ $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ is a linear space

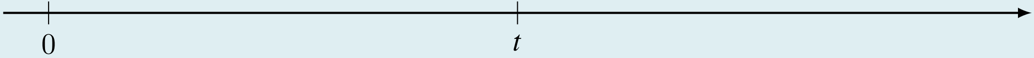
$\overline{E}$ is a coherent upper prevision

Sublinear Expectations for Countable-State Uncertain Processes


$X_t \in \mathscr{X}$
0
t

$\Omega \subseteq \mathscr{X}^{\mathbb{R}_{\geq 0}}$ ‘some’ set of paths $\omega \colon \mathbb{R}_{\geq 0} \to \mathscr{X}$

$$X_t : \Omega \to \mathscr{X} : \omega \mapsto \omega(t)$$



$\Omega \subseteq \mathcal{X}^{\mathbb{R}_{\geq 0}}$ 'some' set of paths $\omega\colon \mathbb{R}_{\geq 0} \to \mathcal{X}$

$$\mathcal{D} \coloneqq \left\{g(X_{t_1}, \ldots, X_{t_n})\colon n \in \mathbb{N}, t_1 < \cdots < t_n \in \mathbb{R}_{\geq 0}, g \in \mathcal{L}(\mathcal{X}^n)\right\}$$

$\Omega \subseteq \mathscr{X}^{\mathbb{R}_{\geq 0}}$ 'some' set of paths $\omega \colon \mathbb{R}_{\geq 0} \to \mathscr{X}$

🤔 ¿sublinear expectation $\overline{E}$ on $\mathcal{D}$? 🤔

$\mathcal{D} := \{g(X_{t_1}, \dots, X_{t_n}) \colon n \in \mathbb{N}, t_1 < \dots < t_n \in \mathbb{R}_{\geq 0}, g \in \mathcal{L}(\mathscr{X}^n)\}$

$\Omega \subseteq \mathscr{X}^{\mathbb{R}_{\geq 0}}$ 'some' set of paths $\omega \colon \mathbb{R}_{\geq 0} \to \mathscr{X}$

¡sublinear Markov process!

¿sublinear ... $\overline{E}$ on $\mathcal{D}$? 🤔

$g($ ... X ...

0 ... t_n

$\mathcal{D} \coloneqq \left\{ g(X_{t_1}, \dots, X_{t_n}) \colon n \in \mathbb{N}, t_1 < \dots < t_n \in \mathbb{R}_{\geq 0}, g \in \mathcal{L}(\mathscr{X}^n) \right\}$

… for Countable-State Uncertain Processes

Let $\mathcal{X}$ denote the countable state space. The possibility space Ω is **some** set of *paths* $\omega\colon \mathbb{R}_{\geq 0} \to \mathcal{X}$, and the domain $\mathcal{D}$ are the **finitary bounded variables**:

$$\mathcal{D} := \{g(X_{t_1},\dots,X_{t_n}) : n\in\mathbb{N}, t_1<\dots<t_n \in \mathbb{R}_{\geq 0}, g\in\mathcal{L}(\mathcal{X}^n)\} \quad \text{with } X_t\colon \Omega\to\mathcal{X}\colon \omega\mapsto\omega(t).$$

sublinear expectation $\overline{E}_0$ on $\mathcal{L}(\mathcal{X})$

semigroup $(\overline{T}_t\colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X}))_{t\in\mathbb{R}_{\geq 0}}$ of 'sublinear transition operators':

(i) $\overline{T}_t[\bullet](x)$ is a sublinear expectation

(ii) $\overline{T}_0 = I$

(iii) $\overline{T}_{s+t} = \overline{T}_s \circ \overline{T}_t$

¿sublinear process $\overline{E}$ on $\mathcal{D}$?

¡sublinear Markov process!

$(\forall n\in\mathbb{N}; t_1<\dots t_n \in \mathbb{R}_{\geq 0}; x_1,\dots,x_n\in\mathcal{X})(\exists\omega\in\Omega)$
$\omega(t_1)=x_1,\dots,\omega(t_n)=x_n$

Theorem

There is a **unique sublinear expectation $\overline{E}$** on $\mathcal{D}$ such that

(i) $\overline{E}(g(X_0)) = \overline{E}_0(g)$ for all $g\in\mathcal{L}(\mathcal{X})$ and

(ii) for all $s_1<\dots<s_n<t\in\mathbb{R}_{\geq 0}$ and $g\in\mathcal{L}(\mathcal{X}^{n+1})$,

$$\overline{E}(g(X_{s_1},\dots,X_{s_n},X_t)) = \overline{E}(h(X_{s_1},\dots,X_{s_n}))$$

with $h\in\mathcal{L}(\mathcal{X}^{\{s_1,\dots,s_n\}})$ defined by

$$h(x_{s_1},\dots,x_{s_n}) := \overline{T}_{t-s_n}[g(x_{s_1},\dots,x_{s_n},\bullet)](x_{s_n}).$$

Is this corresponding $\overline{E}$ downward continuous on $\mathcal{D}$?

A semigroup $(\overline{T}_t)_{t\in\mathbb{R}_{\geq 0}}$ of sublinear transition operators …

… has **uniformly bounded rate** if

$$\limsup_{t\searrow 0}\frac{1}{t}\sup\{\overline{T}_t[1-\mathbb{I}_x](x) : x\in\mathcal{X}\} < +\infty.$$

$\Omega := \mathrm{cdlg}(\mathcal{X}^{\mathbb{R}_{\geq 0}}) \subseteq \mathcal{X}^{\mathbb{R}_{\geq 0}}$

$\overline{E}_0$ is downward continuous
&
$\overline{T}_t[\bullet](x)$ is downward continuous
&
$(\overline{T}_t)$ has uniformly bounded rate

$\Omega := \mathcal{X}^{\mathbb{R}_{\geq 0}}$

$\overline{E}_0$ is downward continuous
&
$\overline{T}_t[\bullet](x)$ is downward continuous

Many interesting variables are *not* included in $\mathcal{D}$!

Many interesting variables are *not* included in $\mathcal{D}$!

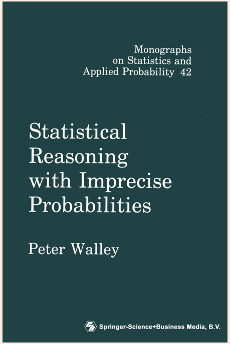
$\mathcal{D}$ does not include

- 📶 the average of $g(X_t)$ over $[0, T]$ for some $g \in \mathcal{L}(\mathscr{X})$, so

$$\frac{1}{T}\int_0^T g(X_t)\mathrm{d}t : \Omega \to \mathbb{R} : \omega \mapsto \frac{1}{T}\int_0^T g\big(\omega(t)\big)\,\mathrm{d}t.$$

- ⏱ the *hitting time of* $A \subseteq \mathscr{X}$, so

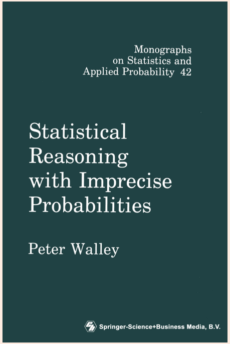
$$\tau_A : \Omega \to \overline{\mathbb{R}}_{\geq 0} : \omega \mapsto \inf\big\{t \in \mathbb{R}_{\geq 0} : \omega(t) \in A\big\}.$$

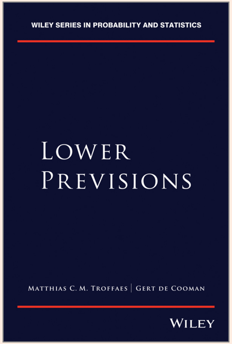


$$\mathcal{D} \subseteq \mathcal{L}(\Omega)$$

$$\downarrow$$

$$\mathcal{L}(\Omega)$$





$\mathcal{D} \subseteq \mathcal{L}(\Omega)$

↓

$\mathcal{L}(\Omega)$

$\mathcal{L}(\Omega)$

↓

'$\overline{E}$-previsible' $f \in \mathbb{R}^{\Omega}$

‘$\overline{E}$-previsible’ $f \in \mathbb{R}^{\Omega}$

sublinear expectation $\overline{E} \colon \mathcal{D} \subseteq \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

downward continuous on $\mathcal{S} \subseteq \mathcal{D}$ if

$$\lim_{n \to +\infty} \overline{E}(f_n) = \overline{E}(f) \text{ for all } \mathcal{S}^{\mathbb{N}} \ni (f_n)_{n \in \mathbb{N}} \searrow f \in \mathcal{S}$$

upward continuous on $\mathcal{S} \subseteq \mathcal{D}$ if

$$\lim_{n \to +\infty} \overline{E}(f_n) = \overline{E}(f) \text{ for all } \mathcal{S}^{\mathbb{N}} \ni (f_n)_{n \in \mathbb{N}} \nearrow f \in \mathcal{S}$$

sublinear expectation $\overline{E}\colon \mathcal{D} \subseteq \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

$\overline{E}$ is downward continuous on $\mathcal{D}$

$\mathcal{D} \subseteq \mathcal{L}(\Omega)$ is a linear lattice

$f \in \mathbb{R}^{\Omega}$ bounded & $\sigma(\mathcal{D})$-measurable

There is a unique sublinear expectation $\overline{E}^{\star}$ on $\mathcal{D}^{\star}$ that

- 🔰 extends $\overline{E}$,
- ↘ is downward continuous on $\mathcal{D}_{\delta} \cap \mathcal{L}(\Omega)$ and
- ↗ upward continuous on $\mathcal{D}^{\star}$.

sublinear expectation $\overline{E}\colon \mathcal{D} \subseteq \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

$\overline{E}$ is downward continuous on $\mathcal{D}$

$\mathcal{D} \subseteq \mathcal{L}(\Omega)$ is a linear lattice

$f \in \overline{\mathbb{R}}^{\Omega}$ bounded below/above
& $\sigma(\mathcal{D})$-measurable

There is a sublinear expectation $\overline{E}^{\sigma}$ on $\mathcal{D}^{\sigma}$ that

- extends $\overline{E}$,
- is downward continuous on $\mathcal{D}_{\delta} \cap \mathcal{L}(\Omega)$ and
- upward continuous on $\{f \in \mathcal{D}^{\sigma} : \inf f > -\infty\}$.

¿Markovian $\overline{E}$ downward continuous on $\mathcal{D}$?

¿$\mathcal{D}^{\sigma}$ sufficiently large?

Sublinear Expectations ...

For a domain $\mathcal{D} \subseteq \overline{\mathbb{R}}^{\Omega}$ which includes all constant functions, a sublinear/linear expectation on $\mathcal{D}$ is a functional $\overline{E}/E\colon \mathcal{D} \to \overline{\mathbb{R}}$ that is constant preserving, isotone and ...

... sublinear, meaning that

$$\overline{E}(\alpha f + g) \leq \alpha \overline{E}(f) + \overline{E}(g)$$

for all $f, g \in \mathcal{D}$ and $\alpha \in \mathbb{R}_{\geq 0}$ with $\alpha f + g \in \mathcal{D}$.

... linear, meaning that

$$E(\alpha f + g) = \alpha E(f) + E(g)$$

for all $f, g \in \mathcal{D}$ and $\alpha \in \mathbb{R}$ with $\alpha f + g \in \mathcal{D}$.

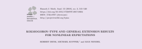
Such a sublinear expectation $\overline{E}$ is said to be **downward continuous** on $\mathcal{S} \subseteq \mathcal{D}$ if

$$\lim_{n \to +\infty} \overline{E}(f_n) = \overline{E}(f) \text{ for all } \mathcal{S}^{\mathbb{N}} \ni (f_n)_{n \in \mathbb{N}} \searrow f \in \mathcal{S}$$

and **upward continuous** on $\mathcal{S} \subseteq \mathcal{D}$ if

$$\lim_{n \to +\infty} \overline{E}(f_n) = \overline{E}(f) \text{ for all } \mathcal{S}^{\mathbb{N}} \ni (f_n)_{n \in \mathbb{N}} \nearrow f \in \mathcal{S}.$$

straightforward modification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KOLMOGOROV-TYPE AND GENERAL EXTENSION RESULTS FOR NONLINEAR EXPECTATIONS

ROBERT DENK, MICHAEL KUPPER,* and MAX NENDEL

Suppose $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ is a linear lattice.

$\overline{E}$ is downward (& then upward) continuous on $\mathcal{D}$ iff every dominated linear expectation in

$$\mathbb{E}_{\overline{E}} := \{E \in \mathbb{E}(\mathcal{D}) \colon (\forall f \in \mathcal{D})\ E(f) \leq \overline{E}(f)\}$$

is downward continuous.

E is downward (& then upward) continuous on $\mathcal{D}$ iff there is a unique probability measure P_E on $\sigma(\mathcal{D})$ such that

$$E(f) = \int f \,\mathrm{d}P_E \quad \text{for all } f \in \mathcal{D}.$$

Theorem

The sublinear expectation $\overline{E}^{\sigma}$ extends $\overline{E}$, is downward continuous on $\mathcal{D}_{\delta} \cap \mathcal{L}(\Omega)$ and upward continuous on $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$.

On $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$, this extension is unique.

Let $\mathcal{M}(\mathcal{D}) := \mathcal{M}_{\mathrm{b}}(\mathcal{D}) \cup \mathcal{M}^{\mathrm{b}}(\mathcal{D})$ be the set of $\sigma(\mathcal{D})$-measurable variables $f \in \overline{\mathbb{R}}^{\Omega}$ that are bounded below/above and let

$$\overline{E}^{\sigma}\colon \mathcal{M}(\mathcal{D}) \to \overline{\mathbb{R}}\colon f \mapsto \sup\left\{\int f \,\mathrm{d}P_E \colon E \in \mathbb{E}_{\overline{E}}\right\}.$$

... for Countable-State Uncertain Processes

Let $\mathcal{X}$ denote the countable state space. The possibility space Ω is **some** set of *paths* $\omega\colon \mathbb{R}_{\geq 0} \to \mathcal{X}$, and the domain $\mathcal{D}$ are the **finitary bounded variables**:

$$\mathcal{D} := \{g(X_{t_1}, \ldots, X_{t_n}) \colon n \in \mathbb{N}, t_1 < \cdots < t_n \in \mathbb{R}_{\geq 0}, g \in \mathcal{L}(\mathcal{X}^n)\} \quad \text{with } X_t\colon \Omega \to \mathcal{X}\colon \omega \mapsto \omega(t).$$

sublinear expectation $\overline{E}_0$ on $\mathcal{L}(\mathcal{X})$

semigroup $(\overline{\mathrm{T}}_t\colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X}))_{t \in \mathbb{R}_{\geq 0}}$ of 'sublinear transition operators':

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(iii) $\overline{\mathrm{T}}_{s+t} = \overline{\mathrm{T}}_s \circ \overline{\mathrm{T}}_t$

¿sublinear process $\overline{E}$ on $\mathcal{D}$?

¡sublinear Markov process!

$(\forall n \in \mathbb{N}; t_1 < \cdots t_n \in \mathbb{R}_{\geq 0}; x_1, \ldots, x_n \in \mathcal{X})(\exists \omega \in \Omega)\ \omega(t_1) = x_1, \ldots, \omega(t_n) = x_n$

Theorem

There is a unique sublinear expectation $\overline{E}$ on $\mathcal{D}$ such that

(i) $\overline{E}(g(X_0)) = \overline{E}_0(g)$ for all $g \in \mathcal{L}(\mathcal{X})$ and

(ii) for all $s_1 < \cdots < s_n < t \in \mathbb{R}_{\geq 0}$ and $g \in \mathcal{L}(\mathcal{X}^{n+1})$,

$$\overline{E}(g(X_{s_1}, \ldots, X_{s_n}, X_t)) = \overline{E}(h(X_{s_1}, \ldots, X_{s_n}))$$

with $h \in \mathcal{L}(\mathcal{X}^{\{s_1, \ldots, s_n\}})$ defined by

$$h(x_{s_1}, \ldots, x_{s_n}) := \overline{\mathrm{T}}_{t-s_n}[g(x_{s_1}, \ldots, x_{s_n}, \bullet)](x_{s_n}).$$

Is this corresponding $\overline{E}$ downward continuous on $\mathcal{D}$?

A semigroup $(\overline{\mathrm{T}}_t)_{t \in \mathbb{R}_{\geq 0}}$ of sublinear transition operators ...

... has **uniformly bounded rate** if

$$\limsup_{t \searrow 0} \frac{1}{t} \sup\{\overline{\mathrm{T}}_t[1 - \mathbb{I}_x](x) \colon x \in \mathcal{X}\} < +\infty,$$

or NEW equivalently, $\limsup_{t \searrow 0} \frac{1}{t} \|\overline{\mathrm{T}}_t - \mathrm{I}\| < +\infty$ NEW.

⇕ NEW

... is **uniformly continuous** if

$$\lim_{t \searrow 0} \|\overline{\mathrm{T}}_t - \mathrm{I}\| = 0.$$

$\Omega := \mathrm{cdlg}(\mathcal{X}^{\mathbb{R}_{\geq 0}}) \subsetneq \mathcal{X}^{\mathbb{R}_{\geq 0}}$

$\overline{E}_0$ is downward continuous
&
$\overline{\mathrm{T}}_t[\bullet](x)$ is downward continuous
&
$(\overline{\mathrm{T}}_t)_{t \in \mathbb{R}_{\geq 0}}$ has uniformly bounded rate

⇓

$\overline{E}$ is downward continuous on $\mathcal{D}$

$\mathcal{M}(\mathcal{D})$ is sufficiently rich

$\Omega := \mathcal{X}^{\mathbb{R}_{\geq 0}}$

$\overline{E}_0$ is downward continuous
&
$\overline{\mathrm{T}}_t[\bullet](x)$ is downward continuous

⇓

$\overline{E}$ is downward continuous on $\mathcal{D}$

$\mathcal{M}(\mathcal{D})$ is not sufficiently rich

⇕ NEW

For some 'bounded sublinear rate operator' $\overline{\mathrm{Q}}\colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X})$,

$$\overline{\mathrm{T}}_t = e^{t\overline{\mathrm{Q}}} := \lim_{n \to +\infty} \left(\mathrm{I} + \frac{t}{n}\overline{\mathrm{Q}}\right)^n \text{ for all } t \in \mathbb{R}_{\geq 0};$$

whenever this is the case,

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathrm{T}}_t := \lim_{s \to t} \frac{\overline{\mathrm{T}}_s - \overline{\mathrm{T}}_t}{|s - t|} = \overline{\mathrm{Q}}\,\overline{\mathrm{T}}_t \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

sublinear Poisson process

Fix some rate interval $[\underline{\lambda}, \overline{\lambda}] \subset \mathbb{R}_{\geq 0}$, and take

$$\mathcal{X} := \mathbb{Z}_{\geq 0}, \quad \overline{E}_0(g) := g(0) \quad \& \quad (\overline{\mathrm{T}}_t)_{t \in \mathbb{R}_{\geq 0}} := (e^{t\overline{\mathrm{L}}})_{t \in \mathbb{R}_{\geq 0}},$$

where $\overline{\mathrm{L}}\colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X})$ maps $g \in \mathcal{L}(\mathcal{X})$ to

$$\mathcal{X} \to \mathbb{R}\colon x \mapsto \max\{\lambda(g(x+1) - g(x)) \colon \lambda \in [\underline{\lambda}, \overline{\lambda}]\}.$$

Cool, on $\mathcal{M}_{\mathrm{b}}(\mathcal{D})$ there is a ...... 'sufficiently continuous' extension of the downward continuous sublinear expectation $\overline{E}$ on $\mathcal{D}$!

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