Extending the Domain of Imprecise Jump Processes from Simple Variables to Measurable Ones

Alexander Erreygers Jasper De Bock

ISIPTA 2021

Ghent University, ELIS, Foundations Lab for imprecise probabilities

9th International Symposium on Imprecise Probability: Theories and Applications, Pescara, Italy, 2015

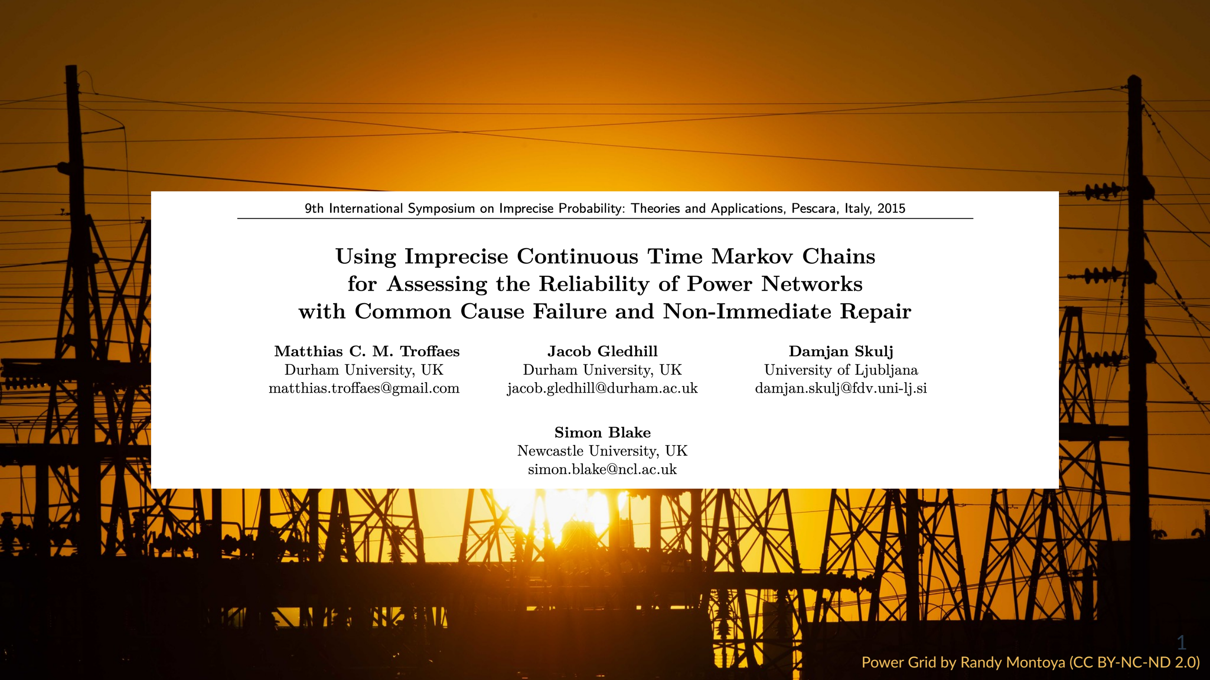
Using Imprecise Continuous Time Markov Chains for Assessing the Reliability of Power Networks with Common Cause Failure and Non-Immediate Repair

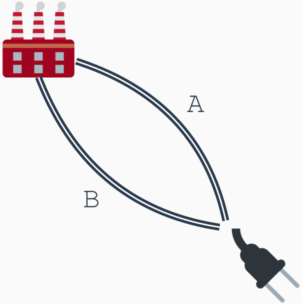
Matthias C. M. Troffaes
Durham University, UK
matthias.troffaes@gmail.com

Jacob Gledhill
Durham University, UK
jacob.gledhill@durham.ac.uk

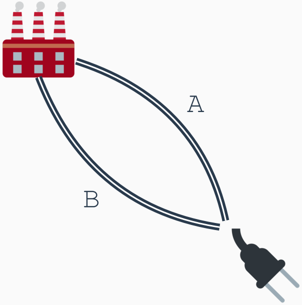
Damjan Skulj
University of Ljubljana
damjan.skulj@fdv.uni-lj.si

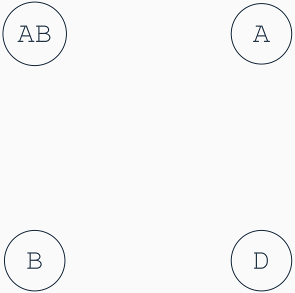
Simon Blake
Newcastle University, UK
simon.blake@ncl.ac.uk



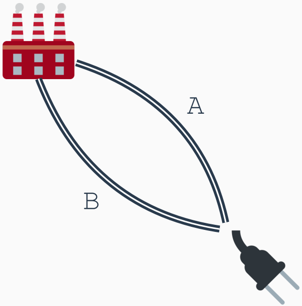


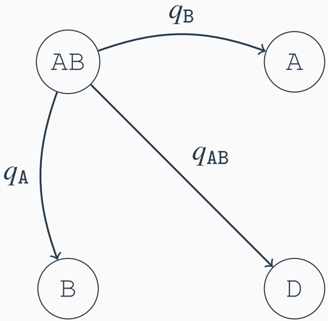
How long is the network down over 10 years?



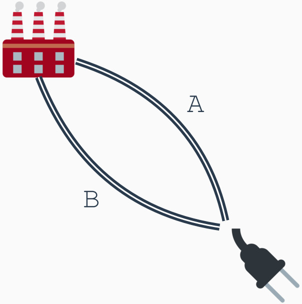


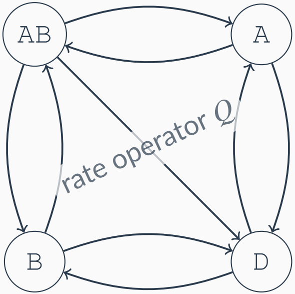
How long is the network down over 10 years?





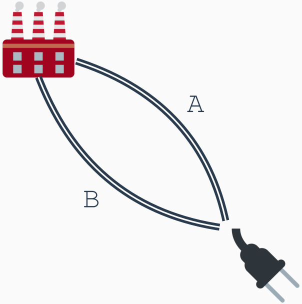
How long is the network down over 10 years?

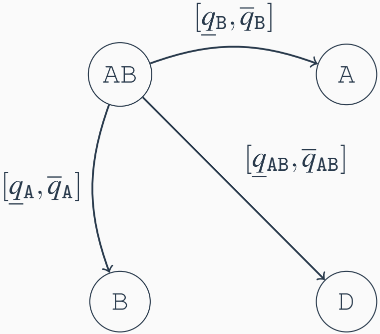




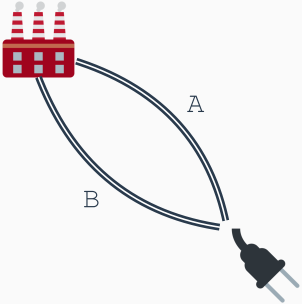
How long is the network down over 10 years?

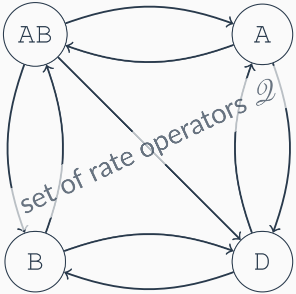
$$E_Q\left(\int_0^{10} \mathbb{I}_{\mathtt{D}}(X_t)\,\mathrm{d}t \,\middle|\, X_0 = \mathtt{AB}\right)$$





How long is the network down over 10 years?





How long is the network down over 10 years?

$$\overline{E}_{\mathscr{Q}}\left(\int_0^{10} \mathbb{I}_{\mathtt{D}}(X_t)\,\mathrm{d}t \,\middle|\, X_0 = \mathtt{AB}\right)$$

Applied Mathematics and Computation 250 (2015) 165–180

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Efficient computation of the bounds of continuous time imprecise Markov chains

Damjan Škulj

International Journal of Approximate Reasoning 88 (2017) 452–528

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Imprecise continuous-time Markov chains

Thomas Krak[a,*], Jasper De Bock[b], Arno Siebes[a]

Received: 16 October 2019 | Accepted: 12 September 2020

DOI: 10.1111/mafi.12289

MATHEMATICAL FINANCE An International Journal of Mathematics, Statistics and Financial Economics

WILEY

ORIGINAL ARTICLE

Markov chains under nonlinear expectation

Max Nendel

Applied Mathematics and Computation 250 (2015) 165–180

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Efficient computation of the bounds of continuous time imprecise Markov chains

Damjan Škulj

$$\overline{E}_{\mathscr{Q}}\left(f(X_t)\,\middle|\,X_0 = x\right)$$

International Journal of Approximate Reasoning 88 (2017) 452–528

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Imprecise continuous-time Markov chains

Thomas Krak [a,*], Jasper De Bock [b], Arno Siebes [a]

$$\overline{E}_{\mathscr{Q}}\left(f(X_{t_1},\dots,X_{t_n})\,\middle|\,X_0 = x\right)$$

Applied Mathematics and Computation 250 (2015) 165–180
Contents lists available at ScienceDirect
Applied Mathematics and Computation
journal homepage: www.elsevier.com/locate/amc

Efficient computation of the bounds of continuous time imprecise Markov chains

Damjan Škulj

International Journal of Approximate Reasoning 88 (2017) 452–528
Contents lists available at ScienceDirect
International Journal of Approximate Reasoning
www.elsevier.com/locate/ijar

Imprecise continuous-time Markov chains

Thomas Krak[a,*], Jasper De Bock[b], Arno Siebes[a]

$$\overline{E}_{\mathscr{Q}}\left(f(X_t)\,\middle|\,X_0 = x\right)$$

$$\overline{E}_{\mathscr{Q}}\left(f(X_{t_1},\ldots,X_{t_n})\,\middle|\,X_0 = x\right)$$

$$\overline{E}_{\mathscr{Q}}\left(f\left((X_t)_{t\in[s,r]}\right)\,\middle|\,X_0 = x\right)$$

Consequence of the *Ergodic Theorem*

$$\lim_{T\to+\infty}\frac{1}{T}E_Q\left(\int_0^T f(X_t)\,\mathrm{d}t \,\middle|\, X_0=x\right)=\lim_{t\to+\infty}E_Q\left(f(X_t)\,\middle|\,X_0=x\right)$$

$$\downarrow$$

$$E_Q\left(\int_0^{10}\mathbb{I}_{\mathtt{D}}(X_t)\,\mathrm{d}t\,\middle|\,X_0=x\right)\approx 10\left(\lim_{t\to+\infty}E_Q\left(\mathbb{I}_{\mathtt{D}}(X_t)\,\middle|\,X_0=x\right)\right)$$

Heuristic

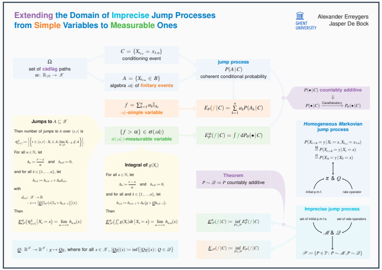
$$\underbrace{\overline{E}_{\mathscr{Q}}\left(\int_0^{10}\mathbb{I}_{\mathtt{D}}(X_t)\,\mathrm{d}t\,\middle|\,X_0=x\right)}_{\text{not defined}}\approx 10\left(\lim_{t\to+\infty}\overline{E}_{\mathscr{Q}}\left(\mathbb{I}_{\mathtt{D}}(X_t)\,\middle|\,X_0=x\right)\right)$$

Our contributions

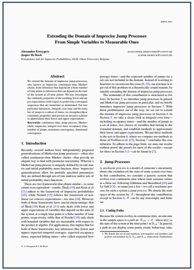
- We extend the domain of $\overline{E}_{\mathscr{Q}}$ to measurable (non-finitary) variables.
- We provide algorithms to compute $\overline{E}_{\mathscr{Q}}(f \,|\, X_s = x)$ for
  - Riemann integrals: $f = \int_s^r g(X_t)\,\mathrm{d}t$;
  - the number of jumps to $A \subseteq \mathscr{X}$: $f = \left|\left\{t \in (s, r] : X_t \in A, \lim_{\Delta \searrow 0} X_{t-\Delta} \notin A\right\}\right|$;
  - ...

Our contributions

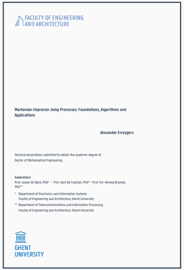
- We extend the domain of $\overline{E}_{\mathscr{Q}}$ to measurable (non-finitary) variables.
- We provide algorithms to compute $\overline{E}_{\mathscr{Q}}(f \,|\, X_s = x)$ for
  - Riemann integrals: $f = \int_s^r g(X_t)\,\mathrm{d}t$;
  - the number of jumps to $A \subseteq \mathscr{X}$: $f = \left|\left\{t \in (s,r] : X_t \in A, \lim_{\Delta \searrow 0} X_{t-\Delta} \notin A\right\}\right|$;
  - ...



$\subset$



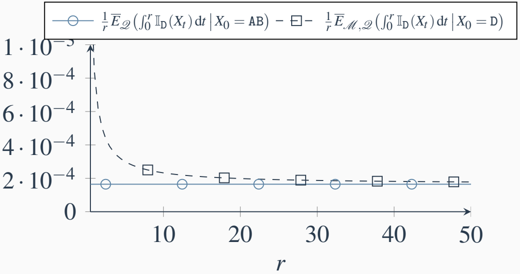
$\subset$



Heuristic

$$\underbrace{\overline{E}_{\mathscr{Q}}\left(\int_0^{10} \mathbb{I}_{\mathtt{D}}(X_t)\,\mathrm{d}t \,\middle|\, X_0 = x\right)}_{\text{defined!}} \approx 10\left(\lim_{t\to+\infty} \overline{E}_{\mathscr{Q}}\left(\mathbb{I}_{\mathtt{D}}(X_t) \,\middle|\, X_0 = x\right)\right)$$

	$x = \mathtt{AB}$	$x = \mathtt{D}$
heuristic	1.647×10^{-3}	1.647×10^{-3}
correct value	1.647×10^{-3}	2.332×10^{-3}



Extending the Domain of Imprecise Jump Processes from Simple Variables to Measurable Ones

GHENT UNIVERSITY

Alexander Erreygers
Jasper De Bock

Ω
set of **càdlàg** paths
$\omega\colon \mathbb{R}_{\geq 0} \to \mathscr{X}$

$C = \{X_{t_{1:n}} = x_{1:n}\}$
conditioning event

$A = \{X_{t_{1:m}} \in B\}$
algebra $\mathscr{A}_C$ of **finitary events**

jump process
$P(A \mid C)$
coherent conditional probability

$f = \sum_{k=1}^{n} a_k \mathbb{I}_{A_k}$
$\mathscr{A}_C$-simple variable

$E_P(f \mid C) = \sum_{k=1}^{n} a_k P(A_k \mid C)$

$\{f > \alpha\} \in \sigma(\mathscr{A}_C)$
$\sigma(\mathscr{A}_C)$-measurable variable

$E_P^\sigma(f \mid C) = \int f \, \mathrm{d}P_\sigma(\bullet \mid C)$

$P(\bullet \mid C)$ **countably additive**
$\Downarrow$
$P(\bullet \mid C) \xrightarrow{\text{Carathéodory}} P_\sigma(\bullet \mid C)$

Homogeneous Markovian jump process

$$P(X_{t+\Delta} = y \mid X_t = x, X_{t_{1:n}} = x_{1:n})$$
$$\overset{\mathrm{M}}{=} P(X_{t+\Delta} = y \mid X_t = x)$$
$$\overset{\mathrm{H}}{=} P(X_\Delta = y \mid X_0 = x)$$
$\updownarrow$
π & Q
initial p.m.f. — rate operator

Jumps to $A \subseteq \mathscr{X}$

Then number of *jumps to* A over $(s,r]$ is

$$\eta_{(s,r]}^A := \left| \left\{ t \in (s,r] \colon X_t \in A, \lim_{\Delta \searrow 0} X_{t-\Delta} \notin A \right\} \right|.$$

For all $n \in \mathbb{N}$, let

$$\Delta_n := \frac{r-s}{n} \quad \text{and} \quad h_{n,0} := 0,$$

and for all $k \in \{1, \ldots, n\}$, let

$$h_{n,k} := h_{n,k-1} + \Delta_n d_{n,k},$$

with

$$d_{n,k} \colon \mathscr{X} \to \mathbb{R} \colon x \mapsto \left[\underline{Q}\left(\mathbb{I}_{A^c}(x)\mathbb{I}_A + h_{n,k-1}\right)\right](x).$$

Then

$$\underline{E}_{\mathscr{P}}^\sigma\left(\eta_{(s,r]}^A \,\middle|\, X_s = x\right) = \lim_{n \to +\infty} h_{n,n}(x)$$

Integral of $g(X_t)$

For all $n \in \mathbb{N}$, let

$$\Delta_n := \frac{r-s}{n} \quad \text{and} \quad h_{n,0} := 0,$$

and for all and $k \in \{1, \ldots, n\}$, let

$$h_{n,k} := h_{n,k-1} + \Delta_n\left(g + \underline{Q} h_{n,k-1}\right).$$

Then

$$\underline{E}_{\mathscr{P}}^\sigma\left(\int_s^r g(X_t)\mathrm{d}t \,\middle|\, X_s = x\right) = \lim_{n \to +\infty} h_{n,n}(x)$$

Theorem
$P \sim \mathscr{Q} \Rightarrow P$ countably additive

Imprecise jump process

set of initial p.m.f.s — set of rate operators
$\mathscr{M}$ & $\mathscr{Q}$
$\downarrow$
$\mathscr{P} := \{P \in \mathbb{P} \colon P \sim \mathscr{M}, P \sim \mathscr{Q}\}$

$$\underline{E}_{\mathscr{P}}^\sigma(f \mid C) := \inf_{P \in \mathscr{P}} E_P^\sigma(f \mid C)$$

$$\underline{E}_{\mathscr{P}}(f \mid C) := \inf_{P \in \mathscr{P}} E_P(f \mid C)$$

$\underline{Q} \colon \mathbb{R}^{\mathscr{X}} \to \mathbb{R}^{\mathscr{X}} \colon g \mapsto \underline{Q}g$, where for all $x \in \mathscr{X}$, $[\underline{Q}g](x) := \inf\{[Qg](x) \colon Q \in \mathscr{Q}\}$