Extending the Domain of Imprecise Jump Processes from Simple Variables to Measurable Ones

GHENT UNIVERSITY

Alexander Erreygers
Jasper De Bock

Ω
set of **càdlàg** paths
$\omega\colon \mathbb{R}_{\geq 0} \to \mathscr{X}$

$C = \{X_{t_{1:n}} = x_{1:n}\}$
conditioning event

$A = \{X_{t_{1:m}} \in B\}$
algebra $\mathscr{A}_C$ of **finitary events**

jump process
$P(A \,|\, C)$
coherent conditional probability

$P(\bullet \,|\, C)$ **countably additive**

$\Downarrow$

$P(\bullet \,|\, C) \xrightarrow{\text{Carathéodory}} P_\sigma(\bullet \,|\, C)$

$f = \sum_{k=1}^{n} a_k \mathbb{I}_{A_k}$
$\mathscr{A}_C$-simple variable

$$E_P(f \,|\, C) = \sum_{k=1}^{n} a_k P(A_k \,|\, C)$$

$\{f > \alpha\} \in \sigma(\mathscr{A}_C)$
$\sigma(\mathscr{A}_C)$-measurable variable

$$E_P^\sigma(f \,|\, C) = \int f \,\mathrm{d}P_\sigma(\bullet \,|\, C)$$

Jumps to $A \subseteq \mathscr{X}$

Then number of *jumps to* A over $(s, r]$ is

$$\eta^A_{(s,r]} := \left|\left\{t \in (s,r] \colon X_t \in A, \lim_{\Delta \searrow 0} X_{t-\Delta} \notin A\right\}\right|.$$

For all $n \in \mathbb{N}$, let

$$\Delta_n := \frac{r-s}{n} \quad \text{and} \quad h_{n,0} := 0,$$

and for all $k \in \{1, \dots, n\}$, let

$$h_{n,k} := h_{n,k-1} + \Delta_n d_{n,k},$$

with

$$d_{n,k} \colon \mathscr{X} \to \mathbb{R} \colon x \mapsto \left[\underline{Q}\left(\mathbb{I}_{A^c}(x)\mathbb{I}_A + h_{n,k-1}\right)\right](x).$$

Then

$$\underline{E}^\sigma_{\mathscr{P}}\left(\eta^A_{(s,r]} \,\middle|\, X_s = x\right) = \lim_{n \to +\infty} h_{n,n}(x)$$

Integral of $g(X_t)$

For all $n \in \mathbb{N}$, let

$$\Delta_n := \frac{r-s}{n} \quad \text{and} \quad h_{n,0} := 0,$$

and for all and $k \in \{1, \dots, n\}$, let

$$h_{n,k} := h_{n,k-1} + \Delta_n\left(g + \underline{Q} h_{n,k-1}\right).$$

Then

$$\underline{E}^\sigma_{\mathscr{P}}\left(\int_s^r g(X_t)\mathrm{d}t \,\middle|\, X_s = x\right) = \lim_{n \to +\infty} h_{n,n}(x)$$

$\underline{Q}\colon \mathbb{R}^{\mathscr{X}} \to \mathbb{R}^{\mathscr{X}} \colon g \mapsto \underline{Q}g$, where for all $x \in \mathscr{X}$, $[\underline{Q}g](x) := \inf\{[Qg](x) \colon Q \in \mathscr{Q}\}$

Homogeneous *M*arkovian jump process

$$P(X_{t+\Delta} = y \,|\, X_t = x, X_{t_{1:n}} = x_{1:n}) \overset{\text{M}}{=} P(X_{t+\Delta} = y \,|\, X_t = x) \overset{\text{H}}{=} P(X_\Delta = y \,|\, X_0 = x)$$

$\updownarrow$

π & Q

initial p.m.f.   rate operator

Theorem
$P \sim \mathscr{Q} \Rightarrow P$ countably additive

Imprecise jump process

set of initial p.m.f.s   set of rate operators

$\mathscr{M}$ & $\mathscr{Q}$

$\downarrow$

$\mathscr{P} := \{P \in \mathbb{P} \colon P \sim \mathscr{M}, P \sim \mathscr{Q}\}$

$$\underline{E}^\sigma_{\mathscr{P}}(f \,|\, C) := \inf_{P \in \mathscr{P}} E^\sigma_P(f \,|\, C)$$

$$\underline{E}_{\mathscr{P}}(f \,|\, C) := \inf_{P \in \mathscr{P}} E_P(f \,|\, C)$$