# Allowing for Imprecision in the Game-theoretic Characterisation of the Poisson Process

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**Abstract.** In their 1993 paper 'Forecasting point and continuous processes: Prequential analysis' in *Test*, Vovk put forward a game-theoretic definition of the Poisson process. A key assumption therein is that the rate of the Poisson process is known or specified exactly. In contrast, I replace this assumption with the less stringent—and arguably more realistic—one that the available information about the process takes the form of bounds on the rate rather than a single, exact value. The resulting process has properties similar to the standard, 'precise' Poisson process, albeit with an imprecise flavour to them, thus justifying the moniker 'imprecise Poisson process'.

**Keywords:** Counting process · Capital process · Bid–ask spread

## 1 Introduction

More than twenty years ago, Vovk [14] put forward a 'prequential' definition of the Poisson and Wiener processes,[1] with an (upper) expectation operator that is derived from the capital in a gambling game. Nowadays, this approach towards modelling uncertainty is known as the *game-theoretic* one, as laid out and popularised by Shafer & Vovk in their seminal monographs [11,12]. Discrete-time processes have been studied extensively in this game-theoretic framework, often allowing for imprecision in the local uncertainty models. Continuous-time processes have also received quite some attention in this framework; in contrast to discrete-time processes, I'm not aware of work in the setting of continuous time that allows for imprecision. This is why in this contribution, I set out to allow for imprecision, or a bid–ask spread, in Vovk's [14] game-theoretic definition of the Poisson process.

I lay down the foundation for doing so in Section 2, in the form of basic notation and terminology regarding (counting) paths, variables and processes. Section 3 introduces, in a relatively general manner, the basics of Shafer & Vovk's game-theoretic foundations for modelling uncertainty regarding a continuous-time process. Next, I specialise this to the imprecise Poisson process in Section 4, and investigate the properties of the resulting conditional upper expectation operator in Section 5. Section 6 concludes this contribution.

[1] More generally, he considers the reduction of continuous martingales to these processes through a change of time.

This manuscript is accepted for publication in the proceedings of IPMU 2026. This extended arXiv version differs from the conference version in some parts, the biggest difference being that includes Appendix A, to which I've relegated the proofs for most of the results; the sole exceptions are Propositions 5 and 7 and Theorem 2. Unless mentioned otherwise, these relegated proofs are modifications of the proofs of related results in [14].

## 2 Counting Paths, Variables & Processes

The set-up for this contribution is essentially standard, and follows Vovk's [14] rather closely; for the reader's sake, I feel it's nonetheless necessary to take some care in introducing it clearly. Let $\Omega$ be the set of all *counting paths*: those paths $\omega\colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ that start in 0, are increasing with *unit* jumps and right-continuous; these paths also have limits from the left, whence they form a subset of the well-known càdlàg paths [5, Ch. 3, § 5].

An *extended real partial variable* is a map from a subset of $\Omega$ to the set $\overline{\mathbb{R}}$ of extended real numbers.[2] We drop the adjective 'partial' if the domain is $\Omega$ and/or the adjective 'extended' if the codomain is $\mathbb{R}$. So $\overline{\mathbb{V}} \coloneqq \overline{\mathbb{R}}^{\Omega}$ is the set of extended real variables, and for any time point $t \in \mathbb{R}_{\geq 0}$, the corresponding *coordinate variable*

$$N_t\colon \Omega \to \mathbb{Z}_{\geq 0}\colon \omega \mapsto \omega(t)$$

is a real variable; with $g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$ and $t \in \mathbb{R}_{\geq 0}$, the composition

$$g(N_t) \coloneqq g \circ N_t\colon \Omega \to \mathbb{R}\colon \omega \mapsto g(\omega(t))$$

is a real variable that is bounded if and only if $g$ is bounded. Indicator variables are also bounded real variables: for any *event* $A \subseteq \Omega$, its corresponding *indicator* $\mathbb{I}_A$ is the $\{0,1\}$-valued variable that is 1 on $A$ and 0 elsewhere.

An essential class of extended real variables are the *stopping times*: this class $\mathfrak{T}$ consists of the positive extended real variables $\tau$—that is, maps from $\Omega$ to $\overline{\mathbb{R}}_{\geq 0}$—such that for all $\omega_1, \omega_2 \in \Omega$, if $\omega_1|_{[0,\tau(\omega_1)]\cap\mathbb{R}_{\geq 0}} = \omega_2|_{[0,\tau(\omega_1)]\cap\mathbb{R}_{\geq 0}}$ then $\tau(\omega_1) = \tau(\omega_2)$. Note that the constant real variable $t\colon \omega \mapsto t$ is a stopping time, and that for all stopping times $\tau_1, \tau_2 \in \mathfrak{T}$, their pointwise minimum $\tau_1 \wedge \tau_2$ and maximum $\tau_1 \vee \tau_2$ are also stopping times. For any stopping time $\tau \in \mathfrak{T}$ and path $\omega$, we let $\mathcal{I}_\tau(\omega)$ be the set of counting paths that agree with $\omega$ on $[0, \tau(\omega)] \cap \mathbb{R}_{\geq 0}$:

$$\mathcal{I}_\tau(\omega) \coloneqq \{\varpi \in \Omega\colon \varpi|_{[0,\tau(\omega)]\cap\mathbb{R}_{\geq 0}} = \omega|_{[0,\tau(\omega)]\cap\mathbb{R}_{\geq 0}}\}.$$

An extended real partial variable $f$ is *$\tau$-measurable* if for all $\omega_1, \omega_2 \in \operatorname{dom} f$,

$$\omega_1|_{[0,\tau(\omega_1)]\cap\mathbb{R}_{\geq 0}} = \omega_2|_{[0,\tau(\omega_1)]\cap\mathbb{R}_{\geq 0}} \implies f(\omega_1) = f(\omega_2);$$

differently put, $f$ is constant on $\mathcal{I}_\tau(\omega)$ for all $\omega \in \operatorname{dom} f$. Although a stopping time $\tau$ can take the value $+\infty$, we are typically interested in the set of paths for

[2] I adhere to the same extension of addition and multiplication from $\mathbb{R}$ to $\overline{\mathbb{R}}$ as Shafer & Vovk [12, p. 420], so in particular $+\infty - \infty = +\infty$.

which it is finite/real; we denote this set by $\{\tau < +\infty\} \coloneqq \{\omega \in \Omega \colon \tau(\omega) < +\infty\}$. For example, for any stopping time $\tau$, we denote the set of extended real partial variables whose domain includes $\{\tau < +\infty\}$ by $\overline{\mathbb{V}}_\tau$, and for any $\omega \in \{\tau < +\infty\}$, we shorten $\omega(\tau(\omega))$ to $\omega(\tau)$.

Finally, a *process* is a family of real variables $\mathcal{S}_\bullet = (\mathcal{S}_t)_{t \in \mathbb{R}_{\geq 0}}$ such that $\mathcal{S}_t$ is $t$-measurable for all $t \in \mathbb{R}_{\geq 0}$. Such a process $\mathcal{S}_\bullet$ is *bounded below* if

$$\inf \mathcal{S}_\bullet \coloneqq \inf\{\mathcal{S}_t(\omega) \colon t \in \mathbb{R}_{\geq 0}, \omega \in \Omega\} > -\infty.$$

The canonical example of a process that is bounded below is the *coordinate process* $N_\bullet$. For any process $\mathcal{S}_\bullet$ and stopping time $\tau$, we'll often consider the 'stopped process' $\mathcal{S}_\tau \colon \{\tau < +\infty\} \to \mathbb{R} \colon \omega \mapsto \mathcal{S}_{\tau(\omega)}(\omega(\tau))$ and the extended real variable

$$\liminf \mathcal{S}_\bullet \colon \Omega \to \overline{\mathbb{R}} \colon \omega \mapsto \liminf_{t \to +\infty} \mathcal{S}_t(\omega);$$

note that if $\mathcal{S}_\bullet$ is bounded below, then so is $\liminf \mathcal{S}_\bullet$.

## 3 Game-theoretic Upper Expectations

In this contribution, we'll use Shafer & Vovk's game-theoretic approach to model the uncertain (future) evolution of the coordinate process $N_\bullet$. A lot can be said about their powerful game-theoretic framework, but we'll stick to what is necessary for the remainder of this contribution; we refer the interested reader to their monographs [11,12] and references therein for more background. In the setting of continuous time, Shafer & Vovk consider a game between two players, called *trader* and *market*: first trader announces their strategy to trade in a number of securities, then market determines the price path of the securities. As in discrete time, one can also imagine a third player, called *forecaster*: they choose the securities that are available to trader. In the context of this contribution, the securities are derived from the coordinate process $N_\bullet$ and market chooses one realisation $\omega$ from $\Omega$.

Trader's strategy has to obey some restrictions for this game to make sense. For example, it makes sense to assume that at time $t \in \mathbb{R}_{\geq 0}$, they are uncertain about the future prices of the securities, or differently put, only have exact information about the prices of the securities in the past. We'll get back to these restrictions in Section 4 further on. For now, we'll assume that each allowed trading strategy gives rise to a process $\mathcal{K}_\bullet$, which we'll call trader's *capital process* under this strategy; recall from Section 2 that for $\mathcal{K}_\bullet$ to be a process, it must be that $\mathcal{K}_t$ is $t$-measurable for all $t \in \mathbb{R}_{\geq 0}$, meaning that it's completely defined by the value of the paths on $[0, t]$—or, in other words, doesn't depend on the future values of the prices of the securities.

The set $\mathfrak{K}$ collects the capital processes corresponding to all allowed trading strategies. Then the conditional upper expectation $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau] \colon \overline{\mathbb{V}}_\tau \times \{\tau < +\infty\} \to \overline{\mathbb{R}}$ is defined for all $f \in \overline{\mathbb{V}}_\tau$ and $\omega \in \{\tau < +\infty\}$ by

$$\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \coloneqq \inf\big\{\mathcal{K}_\tau(\omega) \colon \mathcal{K}_\bullet \in \mathfrak{K}, \inf \mathcal{K}_\bullet > -\infty, \liminf \mathcal{K}_\bullet \geq_{\mathcal{I}_\tau(\omega)} f\big\},$$

where here and in the remainder, we write '$\liminf \mathcal{K}_\bullet \geq_{\mathcal{I}_\tau(\omega)} f$' as a shorthand for '$\liminf \mathcal{K}_\bullet(\varpi) \geq f(\varpi)$ for all $\varpi \in \mathcal{I}_\tau(\omega)$'. This conditional expectation $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ is the infimum capital trader needs at time $\tau(\omega)$ to ('in the long run') superhedge $f$ in 'all possible futures' $\mathcal{I}_\tau(\omega)$ using an allowed trading strategy for which the losses are bounded below a priori. Differently put, if trader has capital $c > \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ at $\tau(\omega)$ (and knowing the partial counting path $\omega|_{[0,\tau(\omega)]}$), they can trade according to an allowed trading strategy such that they'll have $f(\varpi)$ in the long run, whatever the actual realisation $\varpi \in \mathcal{I}_\tau(\omega)$ turns out to be. Consequently, we can interpret $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ as trader's infimum selling price for the uncertain pay-off $f$, given the information they have at time $\tau(\omega)$. Since $\omega(0) = 0$ for all $\omega \in \Omega$, $\mathcal{K}_0$ is constant on $\Omega$, and the same is true for $\overline{\mathrm{E}}_{\mathfrak{K}}[f|0]$, which for this reason we'll shorten to $\overline{\mathrm{E}}_{\mathfrak{K}}[f]$ from now on; this can be interpreted as the infimum initial capital trader needs at time 0 to superhedge $f$ 'in the long run', or alternatively, their infimum selling price for the uncertain pay-off $f$.

For every stopping time $\tau \in \mathfrak{T}$ and extended variable $f \in \overline{\mathbb{V}}_\tau$, our definition ensures that the extended real partial variable $\omega \mapsto \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ is constant on $\mathcal{I}_\tau(\omega)$, and therefore $\tau$-measurable. Furthermore, for every extended variable $f \in \overline{\mathbb{V}}$ and path $\omega \in \Omega$, the function $t \mapsto \overline{\mathrm{E}}_{\mathfrak{K}}[f|t](\omega)$ is increasing because the map $t \mapsto \mathcal{I}_t(\omega)$ is decreasing.

Our calling $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\bullet]$ a (conditional) upper expectation—after [17]—is justified under some mild assumptions on $\mathfrak{K}$.

**Proposition 1.** *Suppose $\mathfrak{K}$ contains all constant processes and is a cone—that is, closed under pointwise addition and multiplication with positive scalars. Then for all $\tau \in \mathfrak{T}$, $f, g \in \overline{\mathbb{V}}_\tau$ and $\mu \in \mathbb{R}$,*

E1. $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] \leq \sup f$;
E2. $\overline{\mathrm{E}}_{\mathfrak{K}}[f+g|\tau] \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] + \overline{\mathrm{E}}_{\mathfrak{K}}[g|\tau]$;
E3. $\overline{\mathrm{E}}_{\mathfrak{K}}[\mu f|\tau] = \mu \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau]$ *whenever* $\mu > 0$;
E4. $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] \leq \overline{\mathrm{E}}_{\mathfrak{K}}[g|\tau]$ *whenever* $f \leq g$;
E5. $\overline{\mathrm{E}}_{\mathfrak{K}}[f+\mu|\tau] = \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau] + \mu$.

Following Vovk [14, p. 196], we call the set $\mathfrak{K}$ of allowed capital processes *coherent* if

$$\inf\{\liminf \mathcal{K}_\bullet(\varpi) \colon \varpi \in \mathcal{I}_t(\omega)\} \leq \mathcal{K}_t(\omega) \quad \text{for all } \mathcal{K}_\bullet \in \mathfrak{K}, t \in \mathbb{R}_{\geq 0}, \omega \in \Omega; \quad (1)$$

in other words, this coherence condition ensures that no trading strategy can result in a guaranteed profit for trader. That we demand coherence should come to no surprise to the reader who is familiar with coherent (conditional) upper expectations a la de Finetti [6], Williams [17] and/or Walley [15]—see also [13]. As is customary in that setting, we define the conditional lower expectation through conjugacy:

$$\underline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \coloneqq -\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) \quad \text{for all } \tau \in \mathfrak{T}, f \in \overline{\mathbb{V}}_\tau, \omega \in \{\tau < +\infty\};$$

the value $\underline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ can be thought of as trader's supremum buying price for $f$ given the information they have at $\tau(\omega)$. If $\mathfrak{K}$ is coherent, trader's buying price is never higher than their selling price.

**Proposition 2.** *If $\mathfrak{K}$ satisfies the conditions in Proposition 1 and is coherent, then for all $\tau \in \mathfrak{T}$, $f \in \overline{\mathbb{V}}_\tau$ and $\omega \in \{\tau < +\infty\}$,*

E6. $\inf f|_{\mathcal{I}_\tau(\omega)} \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \leq \sup f|_{\mathcal{I}_\tau(\omega)}$;
E7. $\underline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$.

## 4 Betting Strategies & Capital Processes for the Poisson Process

As announced in the Introduction, this contribution aims to extend Vovk's [14] game-theoretic description of the Poisson process to allow for imprecision. Their game-theoretic characterisation draws inspiration from Watanabe's martingale characterisation of the Poisson process [16, Theorem 2.3 and following Remark]: the coordinate process $N_\bullet$ is a Poisson process with rate $\lambda \in \mathbb{R}_{\geq 0}$ if and only if the compensated process $(N_t - \lambda t)_{t \in \mathbb{R}_{\geq 0}}$ is a martingale—that is, if conditional on the information up to the current time point $s \in \mathbb{R}_{\geq 0}$, for every future time point $t \in ]s, +\infty[$ the expectation of $N_t - \lambda t$ is $N_s - \lambda s$, or differently put, the expectation of the increment $N_t - N_s$ is $\lambda(t-s)$. Vovk's allowed trading strategies are exactly those that involve this compensated increment.

If forecaster puts forth some *rate* $\lambda \in \mathbb{R}_{\geq 0}$, trader is allowed to bet on a series of compensated increments of the form $N_{\tau_{k+1}} - N_{\tau_k} - \lambda(\tau_{k+1} - \tau_k)$, for stopping times $\tau_k \leq \tau_{k+1}$. More formally, a *two-sided elementary trading strategy* $G$ is an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_{n+1}$ and a sequence $h_1, \ldots, h_n$ of variables such that for all $k \in \{1, \ldots, n\}$, the *stake* $h_k$ is a (possibly partial) bounded real variable with $\operatorname{dom} h_k \supseteq \{\tau_k < +\infty\}$ that is $\tau_k$-measurable. We interpret such an elementary trading strategy $G$ as follows: at time $\tau_k$, trader puts stake $h_k$ on the increment $N_{\tau_{k+1}} - N_{\tau_k} - \lambda(\tau_{k+1} - \tau_k)$—or more accurately, they 'pay' $h_k(\omega)(\omega(\tau_k) - \lambda\tau_k(\omega))$ to receive $h_k(\omega)(\omega(\tau_{k+1}) - \lambda\tau_{k+1}(\omega))$ at time $\tau_{k+1}$ of their choice. Consequently, if trader starts with initial capital $c \in \mathbb{R}$ and follows the two-sided elementary trading strategy $G$, their capital at time $t \in \mathbb{R}_{\geq 0}$ is the real variable

$$\mathcal{K}_t^{G,c} := c + \sum_{k=1}^{n} h_k \cdot \big(N_{\tau_{k+1} \wedge t} - N_{\tau_k \wedge t} - \lambda(\tau_{k+1} \wedge t - \tau_k \wedge t)\big), \qquad (2)$$

where $\cdot$ indicates pointwise multiplication of two real variables and where we ignore the zero terms in the sum that arise when $\tau_k(\omega) \wedge t = \tau_{k+1}(\omega) \wedge t$. Our assumptions on the stopping times $\tau_k$ and stakes $h_k$ ensure that $(\mathcal{K}_t^{G,c})_{t \in \mathbb{R}_{\geq 0}}$ is a real process, which we call a *two-sided elementary capital process*. We collect these capital processes in the set $\mathfrak{K}_\lambda$, and denote the corresponding conditional upper and lower expectation by $\overline{\mathrm{E}}_\lambda[\bullet|\bullet]$ and $\underline{\mathrm{E}}_\lambda[\bullet|\bullet]$, respectively. It's easy to see that $\mathfrak{K}_\lambda$ includes the constant processes and is closed under scalar multiplication, but it takes a bit more work to verify that $\mathfrak{K}_\lambda$ is coherent and closed under pointwise addition; we don't prove this here separately, since it's a special case of Proposition 3 further on.

In the two-sided betting strategies, trader bets on the increment $N_{\tau_{k+1}} - N_{\tau_k}$ being 'large'—greater than $\lambda(\tau_{k+1} - \tau_k)$—when $h_k \geq 0$, and on $N_{\tau_{k+1}} - N_{\tau_k}$ being 'small'—smaller than $\lambda(\tau_{k+1} - \tau_k)$—when $h_k \leq 0$. In the one-sided betting strategies that we're going introduce next, there's a spread between the prices for these two bets. This time around, forecaster puts forth *rate bounds* $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_{\geq 0}$ such that $\underline{\lambda} \leq \overline{\lambda}$, and trader can take a *positive* stake on a series of bets $N_{\tau_{k+1}} - N_{\tau_k} - \overline{\lambda}(\tau_{k+1} - \tau_k)$ and $\underline{\lambda}(\tau_{k+1} - \tau_k) - (N_{\tau_{k+1}} - N_{\tau_k})$. More formally, a *one-sided elementary trading strategy* $G$ is an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_{n+1}$ and a sequence $\overline{h}_1, \underline{h}_1, \ldots, \overline{h}_n, \underline{h}_n$ of variables such that for all $k \in \{1, \ldots, n\}$, the *stakes* $\overline{h}_k$ and $\underline{h}_k$ are $\tau_k$-measurable (possibly partial) bounded *positive* real variables whose domain includes $\{\tau_k < +\infty\}$. If trader starts with initial capital $c \in \mathbb{R}$ and follows the one-sided elementary trading strategy $G$, their corresponding capital at time $t \in \mathbb{R}_{\geq 0}$ is the real variable

$$\begin{aligned}\mathcal{K}_t^{G,c} \coloneqq c + \sum_{k=1}^{n} \overline{h}_k \cdot \big(N_{\tau_{k+1}\wedge t} - N_{\tau_k \wedge t} - \overline{\lambda}(\tau_{k+1} \wedge t - \tau_k \wedge t)\big) \\ + \underline{h}_k \cdot \big(\underline{\lambda}(\tau_{k+1} \wedge t - \tau_k \wedge t) - N_{\tau_{k+1}\wedge t} + N_{\tau_k \wedge t}\big). \quad (3)\end{aligned}$$

Our assumptions on the stopping times $\tau_k$ and stakes $\overline{h}_k$ and $\underline{h}_k$ ensure that $(\mathcal{K}_t^{G,c})_{t\in\mathbb{R}_{\geq 0}}$ is a real process, which we call a *one-sided elementary capital process*. We collect these capital processes in the set $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, and denote the corresponding conditional upper and lower expectations by $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\bullet]$ and $\underline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\bullet]$, respectively.

For every capital process $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, index $k \in \{1, \ldots, n\}$, counting path $\omega \in \Omega$ and time points $t, r \in [\tau_k(\omega), \tau_{k+1}(\omega)] \cap \mathbb{R}_{\geq 0}$ such that $t \leq r$, it follows from Eq. (3) that, with $\underline{\overline{h}}_k \coloneqq \overline{h}_k - \underline{h}_k$,

$$\mathcal{K}_r^{G,c}(\omega) - \mathcal{K}_t^{G,c}(\omega) = \underline{\overline{h}}_k(\omega)\big(\omega(r) - \omega(t) - \underline{\lambda}(r - t)\big) - \overline{h}_k(\omega)(\overline{\lambda} - \underline{\lambda})(r - t). \quad (4)$$

This equality makes it obvious that for all $\lambda \in \mathbb{R}_{\geq 0}$, the set $\mathfrak{K}_{\{\lambda\}}$ of one-sided elementary capital processes for $[\lambda, \lambda] = \{\lambda\}$ is equal to the set $\mathfrak{K}_\lambda$ of two-sided elementary capital processes for $\lambda$.

Recall from Section 3 that it's crucial that the set of capital processes $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ satisfies the conditions in Proposition 2.

**Proposition 3.** *For all $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_{\geq 0}$ with $\underline{\lambda} \leq \overline{\lambda}$, the set $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ of one-sided elementary capital processes includes the constants, is a cone and is coherent.*

## 5 Properties of the Conditional Upper Expectation for the Poisson Process

In the remainder of this contribution, I list some properties of the conditional upper expectation operator $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\bullet]$ which I hope should convince the reader of the sensibility of the game-theoretic characterisation of the imprecise Poisson process.

### 5.1 Properties Related to the Rate Bounds

The first property looks at nested rate bounds: the conditional upper expectation for $[\underline{\lambda}_{\mathrm{i}}, \overline{\lambda}_{\mathrm{i}}]$ is dominated by the one for $[\underline{\lambda}_{\mathrm{o}}, \overline{\lambda}_{\mathrm{o}}]$ whenever $[\underline{\lambda}_{\mathrm{i}}, \overline{\lambda}_{\mathrm{i}}] \subseteq [\underline{\lambda}_{\mathrm{o}}, \overline{\lambda}_{\mathrm{o}}]$.

**Proposition 4.** *For all $\underline{\lambda}_{\mathrm{o}}, \underline{\lambda}_{\mathrm{i}}, \overline{\lambda}_{\mathrm{i}}, \overline{\lambda}_{\mathrm{o}} \in \mathbb{R}_{\geq 0}$ with $\underline{\lambda}_{\mathrm{o}} \leq \underline{\lambda}_{\mathrm{i}} \leq \overline{\lambda}_{\mathrm{i}} \leq \overline{\lambda}_{\mathrm{o}}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda}_{\mathrm{i}}, \overline{\lambda}_{\mathrm{i}}]}[f|\tau] \leq \overline{\mathrm{E}}_{[\underline{\lambda}_{\mathrm{o}}, \overline{\lambda}_{\mathrm{o}}]}[f|\tau] \quad \text{for all } \tau \in \mathfrak{T}, f \in \overline{\mathbb{V}}_{\tau}.$$

The second property involves the interpretation of the rate bounds $\overline{\lambda}$ and $\underline{\lambda}$ as proportionality constants for forecaster's selling and buying prices. Since, as explained in Section 3, $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_\tau - N_\sigma|\sigma](\omega)$ can be interpreted as trader's infimum selling price for the uncertain increment $N_\tau - N_\sigma$ given the information they have at time $\sigma(\omega)$, one would expect that like forecaster's price, this is proportional to $\overline{\lambda}$ ; the next result confirms that this is indeed the case, at least when these stopping times are *constant*.

**Proposition 5.** *For any two time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_t - N_s|s] = \overline{\lambda}(t-s) \quad \textit{and} \quad \underline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_t - N_s|s] = \underline{\lambda}(t-s).$$

*Proof.* We prove first that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[N_t - N_s|s] \leq \overline{\lambda}(t-s) \quad \text{and} \quad \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(N_t - N_s)|s] \leq -\underline{\lambda}(t-s). \tag{5}$$

Fix any $m \in \mathbb{N}$, let $c_m := \overline{\lambda}(t-s)/m$ and observe that for the trading strategy $G_m$ with stopping times $\tau_1 := s$ and $\tau_2 := t$ and stakes $\overline{h}_1 := 1/m$ and $\underline{h}_1 := 0$,

$$\mathcal{K}_r^{G_m,c_m} = \begin{cases} \frac{1}{m}\overline{\lambda}(t-s) & \text{if } r \leq s \\ \frac{1}{m}\big(\overline{\lambda}(t-r) + N_r - N_s\big) & \text{if } s < r < t \\ \frac{1}{m}(N_t - N_s) & \text{if } r \geq t \end{cases} \quad \text{for all } r \in \mathbb{R}_{\geq 0}.$$

Since $\mathcal{K}_\bullet^{G_m,c_m}$ is non-negative and superhedges $(N_t - N_s)/m$, we've shown that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[(N_t - N_s)/m|s] \leq \frac{1}{m}\overline{\lambda}(t-s).$$

Setting $m = 1$ gives the first inequality in (5).

Proving the second inequality is a bit more involved. Let $c := -\underline{\lambda}(t-s)$. For all $m \in \mathbb{N}$, consider the trading strategy $G'_m$ with $\tau_1^m = s$, $\overline{h}_1^m := 0$, $\underline{h}_1^m := 1$ and

$$\tau_2^m \colon \Omega \to \overline{\mathbb{R}}_{\geq 0} \colon \omega \mapsto \inf\{r \in [s, +\infty[ \colon \omega(r) \geq \omega(s) + m\} \cup \{t\}.$$

Then for all $\omega \in \Omega$ and $r \in \mathbb{R}_{\geq 0}$,

$$\mathcal{K}_r^{G'_m,c}(\omega) = \begin{cases} -\underline{\lambda}(t-s) & \text{if } r \leq s \\ -\underline{\lambda}(t-r) - \big(\omega(r) - \omega(s)\big) & \text{if } s < r < \tau_2^m(\omega) \\ -\big(\omega(t) - \omega(s)\big) & \text{if } r \geq \tau_2^m(\omega) = t \\ -\underline{\lambda}(t - \tau_2^m(\omega)) - m & \text{if } r \geq \tau_2^m(\omega) < t. \end{cases}$$

This equality tells us that $\mathcal{K}_\bullet^{G'_m,c}$ is bounded below and superhedges $-(N_t - N_s)$ on $\{\omega \in \Omega\colon \omega(t) < \omega(s) + m\}$. Since $\mathcal{K}_\bullet^{G_m,c_m}$ is bounded below and superhedges $(N_t - N_s)/m$, it follows that $\mathcal{K}_\bullet^{G'_m,c} + \underline{\lambda}(t-s)\mathcal{K}_\bullet^{G_m,c_m}$ is bounded below and superhedges $-(N_t - N_s)$. Consequently,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(N_t - N_s)|s] \leq \mathcal{K}_s^{G'_m,c} + \underline{\lambda}(t-s)\mathcal{K}_s^{G_m,c_m} = -\underline{\lambda}(t-s) + \frac{1}{m}\underline{\lambda}\overline{\lambda}(t-s)^2.$$

Since this inequality holds for arbitrary $m \in \mathbb{N}$, it implies the second inequality in (5).

If $\underline{\lambda} = \lambda = \overline{\lambda}$, we infer from the two inequalities in (5) that

$$\lambda(t-s) \leq \underline{\mathrm{E}}_\lambda[N_t - N_s|s] \leq \overline{\mathrm{E}}_\lambda[N_t - N_s|s] \leq \lambda(t-s).$$

If $\underline{\lambda} < \overline{\lambda}$, the equality follows from (5) and the equality above for $\lambda = \overline{\lambda}$ and $\lambda = \underline{\lambda}$ due to Proposition 4 [with $\underline{\lambda}_\mathrm{i} = \lambda = \overline{\lambda}_\mathrm{i}$, $\underline{\lambda}_\mathrm{o} = \underline{\lambda}$ and $\overline{\lambda}_\mathrm{o} = \overline{\lambda}$]. □

The next property we turn to is the alternative characterisation of the rate parameter $\lambda$ of the 'classical' Poisson process as the inverse of the expected time until the next increment. The time until the next increment after the stopping time $\tau \in \mathfrak{T}$ is the partial variable

$$\rho_\tau\colon \{\tau < +\infty\} \to \overline{\mathbb{R}}_{\geq 0}\colon \omega \mapsto \inf\{r \in ]\tau(\omega), +\infty[\colon \omega(r) > \omega(\tau)\};$$

note that we can extend the domain of $\rho_\tau$ to $\Omega$ by setting it equal to $+\infty$ on $\{\tau < +\infty\}^\mathrm{c}$, which makes this variable a stopping time. It's fairly easy to verify that the upper and lower expectation of $\rho_\tau - \tau$ conditional on $\tau$ are the inverse of the forecaster's rate bounds.

**Proposition 6.** *For every stopping time $\tau \in \mathfrak{T}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau] = \frac{1}{\underline{\lambda}} \quad \textit{and} \quad \underline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau] = \frac{1}{\overline{\lambda}},$$

*where here we follow the convention $1/0 = +\infty$.*

Since the proof for this result is similar to but a bit more involved as the one for Proposition 5, it's been relegated to Appendix A.4.

### 5.2 The Strong Markov Property

The next property of the 'classical' Poisson process on the list is that it satisfies the (strong) Markov property. We'll actually show a stronger 'memorylessness'-like result first: that the conditional upper expectation $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\tau]$ can be 'shifted' from the stopping time $\tau$ to 0—intuitively, this is a straightforward consequence of the fact that trading strategies can be 'shifted'. In the formal statement of this result, we need to stitch together two counting paths; following [10], we stitch together paths $\omega, \varpi \in \Omega$ at the stopping time $\tau \in \mathfrak{T}$ as follows:

$$\omega \oplus_\tau \varpi\colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}\colon t \mapsto \begin{cases} \omega(t) & \text{if } t < \tau(\omega), \\ \omega(\tau(\omega)) + \varpi(t - \tau(\omega)) & \text{if } t \geq \tau(\omega); \end{cases}$$

the reader will have no trouble verifying that $\omega \oplus_\tau \varpi$ is indeed a counting path. Similarly, for any time point $s \in \mathbb{R}_{\geq 0}$ and counting path $\omega \in \Omega$, we'll need to look at the derived counting path

$$\omega_{[s,+\infty[}\colon \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}\colon t \mapsto \omega(t+s) - \omega(s)$$

which looks only at the increments of $\omega$ starting from $s$.

**Proposition 7.** *For all $\tau \in \mathfrak{T}$, $f \in \overline{\mathbb{V}}_\tau$ and $\omega \in \{\tau < +\infty\}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f(\omega \oplus_\tau \bullet)] \quad \textit{where } f(\omega \oplus_\tau \bullet)\colon \Omega \to \overline{\mathbb{R}}\colon \varpi \mapsto f(\omega \oplus_\tau \varpi).$$

*Proof.* We'll verify the equality in the statement by proving that the left-hand side of the equality is lower than or equal to the right-hand side and vice versa.

First, we show that the left-hand side of the equality is lower than or equal to the right-hand side. Consider any bounded below capital process $\mathcal{K}^{G,c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges $f(\omega \oplus \bullet)$. Let us enumerate the stopping times and stakes of $G$ as $\tau_1, \ldots, \tau_{n+1}$ and $\overline{h}_1, \ldots, \underline{h}_{n+1}$, respectively. Consider the trading strategy $G'$ with stopping times $\tau'_1, \ldots, \tau'_{n+1}$ defined for all $k \in \{1, \ldots, n+1\}$ as

$$\tau'_k\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}\colon \varpi \mapsto \begin{cases} \tau(\omega) + \tau_k(\varpi_{[\tau(\omega),+\infty[}) & \text{if } \varpi \in \mathcal{I}_\tau(\omega) \\ +\infty & \text{if } \varpi \notin \mathcal{I}_\tau(\omega) \end{cases}$$

and, for all $k \in \{1, \ldots, n\}$, stakes defined on $\mathcal{I}_\tau(\omega)$ by

$$\overline{h}'_k(\varpi) \coloneqq \overline{h}_k(\varpi_{[\tau(\omega),+\infty[}) \quad \text{and} \quad \underline{h}'_k(\varpi) \coloneqq \underline{h}_k(\varpi_{[\tau(\omega),+\infty[}) \quad \text{for all } \varpi \in \mathcal{I}_\tau(\omega).$$

Our construction ensures that $\mathcal{K}^{G',c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is bounded below with $\mathcal{K}^{G',c}_\tau(\omega) = c$ and

$$\mathcal{K}^{G',c}_r(\varpi) = \mathcal{K}^{G,c}_{r-\tau(\omega)}(\varpi_{[\tau(\omega),+\infty[}) \quad \text{for all } \varpi \in \mathcal{I}_\tau(\omega), r \in [\tau(\omega), +\infty[.$$

For all $\varpi \in \mathcal{I}_\tau(\omega)$, we infer from all this that

$$\liminf \mathcal{K}^{G',c}_\bullet(\varpi) = \liminf \mathcal{K}^{G,c}_\bullet(\varpi_{[\tau(\omega),+\infty[}) \geq f(\omega \oplus_\tau \varpi_{[\tau(\omega),+\infty[}) = f(\varpi),$$

where for the last equality we used that $\varpi = \omega \oplus_\tau \varpi_{[\tau(\omega),+\infty[}$. So for any bounded below capital process $\mathcal{K}^{G,c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges $f(\omega \oplus_\tau \bullet)$, there is some bounded below capital process $\mathcal{K}^{G',c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ with $\mathcal{K}^{G'c}_\tau(\omega) = c$ that superhedges $f$ on $\mathcal{I}_\tau(\omega)$. From this, we infer that indeed

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau](\omega) \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f(\omega \oplus_\tau \bullet)].$$

The proof of the reverse inequality is similar. Consider any bounded below capital process $\mathcal{K}^{G,c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges $f$ on $\mathcal{I}_\tau(\omega)$ with stopping times

$\tau_1, \ldots, \tau_{n+1}$ and stakes $\overline{h}_1, \ldots, \underline{h}_{n+1}$. We now consider the trading strategy $G'$ with stopping times given for all $k \in \{1, \ldots, n+1\}$ by

$$\tau'_k \colon \Omega \to \overline{\mathbb{R}}_{\geq 0} \colon \varpi \mapsto \begin{cases} \tau_k(\omega \oplus_\tau \varpi) - \tau(\omega) & \text{if } \tau_k(\omega \oplus_\tau \varpi) \geq \tau(\omega) \\ 0 & \text{otherwise} \end{cases}$$

and stakes defined, for all $k \in \{1, \ldots, n\}$, by

$$\overline{h}'_k(\varpi) \coloneqq \overline{h}_k(\omega \oplus_\tau \varpi) \quad \text{and} \quad \underline{h}'_k(\varpi) \coloneqq \underline{h}_k(\omega \oplus_\tau \varpi) \quad \text{for all } \varpi \in \Omega.$$

With $c' \coloneqq \mathcal{K}^{G,c}_\tau(\omega)$, our construction ensures that

$$\mathcal{K}^{G',c'}_r(\varpi) = \mathcal{K}^{G,c}_{\tau(\omega)+r}(\omega \oplus_\tau \varpi) \quad \text{for all } \varpi \in \Omega, r \in \mathbb{R}_{\geq 0},$$

and therefore $\mathcal{K}^{G',c'}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is bounded below with

$$\liminf \mathcal{K}^{G',c'}_\bullet(\varpi) = \liminf \mathcal{K}^{G,c}_\bullet(\omega \oplus_\tau \varpi) \geq f(\omega \oplus_\tau \varpi) \quad \text{for all } \varpi \in \Omega.$$

Since $\mathcal{I}_\tau(\omega) = \{\omega \oplus_\tau \varpi \colon \varpi \in \Omega\}$, we've shown that for any bounded below capital process $\mathcal{K}^{G,c}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges $f$ on $\mathcal{I}_\tau(\omega)$, there is some bounded below capital process $\mathcal{K}^{G',c'}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ with $c' = \mathcal{K}^{G,c}_\tau(\omega)$ that superhedges $f(\omega \oplus_\tau \bullet)$. From this, we infer that indeed

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|\tau](\omega) \geq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f(\omega \oplus_\tau \bullet)].$$

□

The Markov property—see, for example, [5, Ch. 4, Eq. (1.2)]—follows almost immediately from this memoryless character of the conditional upper expectation: for all time points $t \in \mathbb{R}_{\geq 0}$ and time periods $\Delta \in \mathbb{R}_{\geq 0}$,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t+\Delta})|t](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(t) + N_\Delta)] \quad \text{for all } g \in \overline{\mathbb{R}}^{\mathbb{Z}_{\geq 0}}, \omega \in \Omega;$$

so does the strong Markov property [5, Ch. 4, Eq. (1.17)]: for all stopping times $\tau \in \mathfrak{T}$ and time periods $\Delta \in \mathbb{R}_{\geq 0}$,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{\tau+\Delta})|\tau](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(\tau) + N_\Delta)] \quad \text{for all } g \in \overline{\mathbb{R}}^{\mathbb{Z}_{\geq 0}}, \omega \in \{\tau < +\infty\}.$$

### 5.3 The Law of Iterated Upper Expectations

Next up is the crucial *law of iterated (upper) expectations*, also known as the tower property. For the sake of simplicity, we'll only establish a version of this law for constant stopping times and bounded and so-called finitary variables: those that depend only on the value of the counting path at finitely many time points. Obviously, a variable $f \in \overline{\mathbb{V}}$ is bounded and finitary if and only if $f = g(N_{t_1}, \ldots, N_{t_k})$ for some $k \in \mathbb{N}$, $t_1 \leq \cdots \leq t_k \in \mathbb{R}_{\geq 0}$ and some $g \in \mathbb{G}_k$, where here and in the remainder we let $\mathbb{G}_k$ be the set of bounded real functions on $(\mathbb{Z}_{\geq 0})^k$—for $k = 1$, we'll simply write $\mathbb{G}$.

**Theorem 1.** *For all natural numbers $k \in \mathbb{N}$, time points $t_1, \dots, t_{k+1} \in \mathbb{R}_{\geq 0}$ such that $t_1 < t_2 < \dots < t_{k+1}$ and gambles $g \in \mathbb{G}_{k+1}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\big[g(N_{t_1}, \dots, N_{t_{k+1}})\big|t_1\big] = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \dots, N_{t_{k+1}})|t_k]\Big|t_1\Big].$$

My proof is inspired by Shafer & Vovk's [11, Proposition 8.7] proof for a similar result in discrete time, but adds a move to ensure the 'cut' is finite. Because it's rather lengthy, I've relegated it to Appendix A.

One can turn the law of iterated upper expectations in Theorem 1 into a (theoretical) recursive computational scheme to compute $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \dots, N_{t_k})|t_1]$, at least once one realises that due to Proposition 7, the variable

$$\varpi \mapsto \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \dots, N_{t_{k+1}})|t_k](\varpi)$$

in the conditional upper expectation on the right-hand side of the equality in Theorem 1 is only functionally dependent on the values $\varpi$ takes in $t_1$, …, $t_k$ rather than on the values it takes on the entire interval $[0, t_k]$—so only on a countable set rather than an uncountable one.

**Corollary 1.** *For all natural numbers $k \in \mathbb{N}$, time points $t_1, \dots, t_{k+1} \in \mathbb{R}_{\geq 0}$ such that $t_1 < \dots < t_{k+1}$, bounded functions $g \in \mathbb{G}_{k+1}$ and paths $\omega \in \Omega$, and with $\Delta := t_{k+1} - t_k$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_{t_1}, \dots, N_{t_{k+1}})|t_k](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(t_1), \dots, \omega(t_k), \omega(t_k) + N_\Delta)].$$

### 5.4 Connection to the Sublinear Poisson Semigroup

Corollary 1 tells us that it's important to be able to compute conditional upper expectations of the form $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s]$ for bounded functions $g \in \mathbb{G}$ and time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s < t$. For the classical Poisson process with rate $\lambda$, and for Vovk's measure-theoretic Poisson process [14, Theorem 5], these conditional expectations are related to the Poisson distribution $\psi_{\lambda(t-s)}$ with parameter $\lambda(t - s)$:

$$\overline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = \underline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = \sum_{z \in \mathbb{Z}_{\geq 0}} g(\omega(s) + z)\psi_{\lambda(t-s)}(z).$$

To generalise this property to our imprecise setting, it will be elucidative to express the right-hand side of this equality using the so-called *Poisson semigroup* $(\mathrm{S}_\Delta)_{\Delta \in \mathbb{R}_{\geq 0}}$ for $\Delta = t - s$:

$$\overline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = \underline{\mathrm{E}}_\lambda[g(N_t)|s](\omega) = [\mathrm{S}_{(t-s)}g](\omega(s)).$$

This family $\mathrm{S}_\bullet$ of linear operators—linear maps from $\mathbb{G}$ to $\mathbb{G}$—is generated by the Poisson generator $\mathrm{G}\colon \mathbb{G} \to \mathbb{G}$, which maps any $g \in \mathbb{G}$ to

$$\mathrm{G}g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}\colon n \mapsto \lambda(g(n+1) - g(n)),$$

as follows:

$$\mathrm{S}_\Delta = e^{\Delta \mathrm{G}} = \lim_{k\to+\infty} \left(\mathrm{I} + \frac{\Delta}{k}\mathrm{G}\right)^k \quad \text{for all } \Delta \in \mathbb{R}_{\geq 0}. \tag{6}$$

Crucially, a similar result holds for the imprecise game-theoretic Poisson process, at least if we replace the Poisson semigroup $\mathrm{S}_\bullet$ by the sublinear Poisson semigroup $\overline{\mathrm{S}}_\bullet$. As explained in [3], this family of sublinear operators is generated by the sublinear Poisson generator $\overline{\mathrm{G}}\colon \mathbb{G} \to \mathbb{G}$, which maps any $g \in \mathbb{G}$ to

$$\overline{\mathrm{G}}g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}\colon n \mapsto \max\{\lambda\big(g(n+1) - g(n)\big)\colon \lambda \in \{\underline{\lambda}, \overline{\lambda}\}\},$$

through the operator exponential:

$$\overline{\mathrm{S}}_\Delta \coloneqq e^{\Delta \overline{\mathrm{G}}} = \lim_{k\to+\infty} \left(\mathrm{I} + \frac{\Delta}{k}\overline{\mathrm{G}}\right)^k \quad \text{for all } \Delta \in \mathbb{R}_{\geq 0}. \tag{7}$$

**Theorem 2.** *For all time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$ and $g \in \mathbb{G}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s] = [\overline{\mathrm{S}}_{t-s}g](N_s).$$

As an intermediary step towards proving this result, we establish the following 'generalisation'[3] of Vovk's [14] Theorem 5; the proof is a rather straightforward modification of the original one, essentially replacing the 'Bernouilli approximation' of Eq. (6) with that of Eq. (7) and replacing the derived two-sided trading strategy with the natural one-sided counterpart.

**Proposition 8.** *For all $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$ and $g \in \mathbb{G}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s] \leq \big[\overline{\mathrm{S}}_{t-s}g\big](N_s).$$

Proving Theorem 2 is now a matter of combining Proposition 8 with a couple of other previously obtained results.

*Proof (Proof of Theorem 2).* Recall from Proposition 7 that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(\omega(s) + N_{t-s})] \quad \text{for all } \omega \in \Omega.$$

Consequently, it suffices to show that

$$\overline{\mathrm{E}}[g(z + N_\Delta)] = [\overline{\mathrm{S}}_\Delta g](z) \quad \text{for all } g \in \mathbb{G}, z \in \mathbb{Z}_{\geq 0}, \Delta \in \mathbb{R}_{\geq 0}.$$

So for all $\Delta \in \mathbb{R}_{\geq 0}$, let $\overline{\mathrm{T}}_\Delta\colon \mathbb{G} \to \mathbb{G}$ map $g \in \mathbb{G}$ to

$$\overline{\mathrm{T}}_\Delta g\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}\colon z \mapsto \overline{\mathrm{E}}[g(z + N_\Delta)].$$

It follows immediately from the properties of the upper expectation that $\overline{\mathrm{T}}_\Delta$ is a sublinear transition operator, from Proposition 8 and [4, Lemma 53] that $\overline{\mathrm{T}}_\Delta$ is dominated by $\overline{\mathrm{S}}_\Delta$, and from (E6) that $\overline{\mathrm{T}}_0 = \mathrm{I}$.

[3] We only generalise their result in the particular case $A_t = \lambda t$.

Furthermore, we can use the law of iterated upper expectations [Theorem 1] to show that $\overline{\mathrm{T}}_\bullet$ is a semigroup. To this end, fix some $\Delta_1, \Delta_2 \in \mathbb{R}_{>0}$, $g \in \mathbb{G}$ and $z \in \mathbb{Z}_{\geq 0}$, and observe that

$$[\overline{\mathrm{T}}_{\Delta_1+\Delta_2} g](z) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z+N_{\Delta_1+\Delta_2})] = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z+N_{\Delta_1+\Delta_2})|\Delta_1]].$$

Now for all $\omega \in \Omega$, it follows from Proposition 7 that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z+N_{\Delta_1+\Delta_2})|\Delta_1](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z+\omega(\Delta_1)+N_{\Delta_2})] = [\overline{\mathrm{T}}_{\Delta_2} g](z+\omega(\Delta_1)).$$

Substituting into the preceding equality, we find that

$$[\overline{\mathrm{T}}_{\Delta_1+\Delta_2} g](z) = \overline{\mathrm{E}}\big[[\overline{\mathrm{T}}_{\Delta_2} g](z+N_{\Delta_1})\big] = [\overline{\mathrm{T}}_{\Delta_1}\overline{\mathrm{T}}_{\Delta_2} g](z),$$

as required.

For any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and $\Delta \in \mathbb{R}_{\geq 0}$, let $\mathrm{T}^\lambda_\Delta$ map $g \in \mathbb{G}$ to

$$\mathrm{T}^\lambda_\Delta g \colon \mathbb{Z}_{\geq 0} \to \mathbb{R} \colon z \mapsto \overline{\mathrm{E}}_\lambda[g(z+N_\Delta)].$$

Now from Proposition 8, it follows that for all $\Delta \in \mathbb{R}_{\geq 0}$ and $g \in \mathbb{G}$,

$$\mathrm{T}^\lambda_\Delta g \leq \mathrm{S}^\lambda_\Delta g \quad \text{and} \quad -\mathrm{T}^\lambda_\Delta(-g) \geq -\mathrm{S}^\lambda_\Delta(-g) = \mathrm{S}^\lambda_\Delta g,$$

where $\mathrm{S}^\lambda_\bullet$ is the Poisson semigroup with rate $\lambda$. Consequently, $\mathrm{T}^\lambda_\Delta = \mathrm{S}^\lambda_\Delta$ for all $\Delta \in \mathbb{R}_{\geq 0}$ and $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. From this, Propositions 4 and 8, it follows that

$$\mathrm{S}^\lambda_\Delta g = \mathrm{T}^\lambda_\Delta g \leq \overline{\mathrm{T}}_\Delta g \leq \overline{\mathrm{S}}_\Delta g \quad \text{for all } \lambda \in [\underline{\lambda}, \overline{\lambda}], \Delta \in \mathbb{R}_{\geq 0}, g \in \mathbb{G}.$$

Since $\overline{\mathrm{S}}_\bullet$ is equal to Nendel's [9] so called *Nisio semigroup* induced by the family $\{\mathrm{S}^\lambda_\bullet \colon \lambda \in [\underline{\lambda}, \overline{\lambda}]\}$ [3, Proposition 5.2], and this Nisio semigroup is the point-wise smallest semigroup that dominates the family $\{\mathrm{S}^\lambda_\bullet \colon \lambda \in [\underline{\lambda}, \overline{\lambda}]\}$ [9, Remark 5.3], it follows from these inequalities that $\overline{\mathrm{T}}_\Delta = \overline{\mathrm{S}}_\Delta$ for all $\Delta \in \mathbb{R}_{\geq 0}$, which concludes our proof. □

## 6 Conclusion

I am by no means the first to construct an 'imprecise' Poisson process. To the best of my knowledge, Hu and Peng [7, Section 6] were the first to construct a sublinear expectation for $\mathbb{R}^d$-valued Lévy processes, through the viscosity solution to an integro-partial differential equation characterised by a set of Lévy triples; see their Example 43 for the special case of the Poisson process. This inspired Neufeld and Nutz [10] to construct an upper expectation for $\mathbb{R}^d$-valued Lévy processes through the upper envelope of the expectations with respect to the set of probability measures whose 'differential characteristics' take values in the set of triplets; their Example 2.6 treats the Poisson process as a special case.

Unaware of this work, and inspired by work on imprecise finite-state Markov processes [8], Jasper De Bock and I [4] constructed a sublinear expectation for the Poisson process as the upper envelope of the expectations with respect to the set of 'compatible' counting processes—a much less technical condition than the one in [10]. Inspired by [1], I then took an arguably more fruitful approach to the problem in [2, Section 6], constructing a sublinear expectation directly from the sublinear Poisson semigroup.

These earlier approaches have in common with the approach in the present contribution that the upper expectation of $g(N_t)$ can be retrieved as the (viscosity) solution to what is essentially the same differential equation; compare the differential equations in [7, Eq. (9)] and [10, Eq. (2.5)] to the one in [3, Proposition 3.6]. Except for [2], these earlier approaches suffer from the same limitation: they only (practically) determine the upper expectation of finitary variables.

Under the game-theoretic approach, the upper expectation of such (bounded) finitary variables can be determined by means of Theorems 1 and 2 and Corollary 1. However, in Proposition 6 I've also explicitly determined the upper and lower expectation of a non-finitary variable—the time $\rho_\tau - \tau$ until the next increment. Another benefit of the game-theoretic approach over the other approaches is that it involves fewer technicalities—there's no need for measurability, for example, and conditional expectations are defined uniquely and naturally.

Whether the (conditional) upper expectation of other non-finitary variables can also be determined easily is one of the many possible lines of follow-up research I envision. A related future research topic is the continuity of the game-theoretic upper expectation with respect to point-wise convergence of variables. More generally, one could see this work as a first step in the study of imprecise renewal processes, or even imprecise countable-state Markov processes, in the game-theoretic framework.

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## References

1. Denk, R., Kupper, M., Nendel, M.: Kolmogorov-type and general extension results for nonlinear expectations. Banach Journal of Mathematical Analysis **12**(3), 515–540 (2018). `https://doi.org/10.1215/17358787-2017-0024`
2. Erreygers, A.: Convex expectations for countable-state uncertain processes with càdlàg sample paths. International Journal of Approximate Reasoning **175**, 109308 (2024). `https://doi.org/10.1016/j.ijar.2024.109308`
3. Erreygers, A.: Uniformly continuous semigroups of sublinear transition operators. Electronic Journal of Differential Equations **2025**(122) (2025). `https://doi.org/10.58997/ejde.2025.122`

4. Erreygers, A., De Bock, J.: First steps towards an imprecise Poisson process. In: Proceedings of the Eleventh International Symposium on Imprecise Probabilities: Theories and Applications. Proceedings of Machine Learning Research, vol. 103. PMLR (2019). https://doi.org/10.48550/arXiv.1905.05734
5. Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence. Wiley (1986). https://doi.org/10.1002/9780470316658
6. de Finetti, B.: Theory of Probability: A Critical Introductory Treatment. Wiley (2017). https://doi.org/10.1002/9781119286387
7. Hu, M., Peng, S.: $G$-Lévy processes under sublinear expectations. Probability, Uncertainty and Quantitative Risk **6**(1), 1–22 (2021). https://doi.org/10.3934/puqr.2021001
8. Krak, T., De Bock, J., Siebes, A.: Imprecise continuous-time Markov chains. International Journal of Approximate Reasoning **88**, 452–528 (2017). https://doi.org/10.1016/j.ijar.2017.06.012
9. Nendel, M.: On nonlinear expectations and Markov chains under model uncertainty. International Journal of Approximate Reasoning **130**, 226–245 (2021). https://doi.org/10.1016/j.ijar.2020.12.013
10. Neufeld, A., Nutz, M.: Nonlinear Lévy processes and their characteristics. Transactions of the American Mathematical Society **369**(1), 69–95 (2017). https://doi.org/10.1090/tran/6656
11. Shafer, G., Vovk, V.: Probability and Finance: It's Only a Game! Wiley Series in Probability and Statistics, Wiley (2001). https://doi.org/10.1002/0471249696
12. Shafer, G., Vovk, V.: Game-Theoretic Foundations for Probability and Finance. Wiley Series in Probability and Statistics, Wiley (2019). https://doi.org/10.1002/9781118548035
13. Troffaes, M.C.M., De Cooman, G.: Lower Previsions. Wiley (2014). https://doi.org/10.1002/9781118762622
14. Vovk, V.G.: Forecasting point and continuous processes: Prequential analysis. Test (2), 189–217 (1993). https://doi.org/10.1007/bf02562675
15. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall (1991)
16. Watanabe, S.: On discontinuous additive functionals and Lévy measures of a Markov process. Japanese journal of mathematics: transactions and abstracts **34**, 53–70 (1964). https://doi.org/10.4099/jjm1924.34.0_53
17. Williams, P.M.: Notes on conditional previsions. Tech. rep., School of Mathematical and Physical Sciences, University of Sussex (1975), reprinted as [18]
18. Williams, P.M.: Notes on conditional previsions. International Journal of Approximate Reasoning **44**(3), 366–383 (2007). https://doi.org/10.1016/j.ijar.2006.07.019

# A Relegated proofs

## A.1 Additional results and relegated proofs for results in Section 3

*Proof of Proposition 1.* Note that since $\mathfrak{K}$ contains all constant processes and is a cone, the same is true for the set

$$\{\mathcal{K}_\bullet \in \mathfrak{K}\colon\ \inf \mathcal{K}_\bullet > -\infty\},$$

which is non-empty because it contains at least the constant processes. The properties follow almost immediately from this observation and the definition of $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\bullet]$. ☐

One consequence of coherence is that in the definition of the conditional upper expectation $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau](\omega)$, we can limit ourselves to capital processes that are always greater than the infimum of $f$ on $\mathcal{I}_\tau(\omega)$.

**Lemma 1.** *Suppose $\mathfrak{K}$ is coherent. Then for all $\tau \in \mathfrak{T}$, $f \in \overline{\mathbb{V}}_\tau$, $\omega \in \{\tau < +\infty\}$ and $\mathcal{K}_\bullet \in \mathfrak{K}$ such that* $\liminf \mathcal{K}_\bullet \geq_{\mathcal{I}_\tau(\omega)} f$,

$$\mathcal{K}_t|_{\mathcal{I}_\tau(\omega)} \geq \inf f|_{\mathcal{I}_\tau(\omega)} \quad \textit{for all } t \in [\tau(\omega), +\infty[.$$

*Proof.* Follows almost immediately from the definition of coherence. ☐

Lemma 1 comes in handy in our proof for Proposition 2, more specifically when proving (E6).

*Proof of Proposition 2.* For (E6), the upper found follows almost immediately from the definition of $\overline{\mathrm{E}}_{\mathfrak{K}}[\bullet|\tau](\omega)$ and the assumption in the statement that $\mathfrak{K}$ contains all constants, while the lower bound follows from Lemma 1 for $t = \tau(\omega)$.

For (E7), we need to prove that

$$\underline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) = -\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) \leq \overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega). \tag{8}$$

This inequality holds trivially if $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) + \infty$ or $\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) = +\infty$, so we assume that $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) < +\infty$ and $\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) < +\infty$. Since $f - f \geq 0$, it follows from (E2), (E4) and (E6) that

$$\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) + \overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) \geq \overline{\mathrm{E}}_{\mathfrak{K}}[f - f|\tau](\omega) \geq \overline{\mathrm{E}}_{\mathfrak{K}}[0|\tau](\omega) = 0.$$

Since $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) < +\infty$ and $\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) < +\infty$, it follows from this inequality that $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega) > -\infty$ and $\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega) > -\infty$; that is, $\overline{\mathrm{E}}_{\mathfrak{K}}[f|\tau](\omega)$ and $\overline{\mathrm{E}}_{\mathfrak{K}}[-f|\tau](\omega)$ are both real. Consequently, the inequality above indeed implies the one in (8). ☐

## A.2 Proofs for results in Section 4

*Proof of Proposition 3.* That $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ contains the constants follows from choosing $\tau_1 = 0 = \tau_2$ in the definition of the trading strategy $G$, and that it's closed under multiplication with positive scalars clearly follows immediately from the definition of trading strategies. That $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is closed under pointwise addition takes a bit more work to prove formally, but should be intuitively clear. We'll only prove explicitly that $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is coherent. To this end, we fix some $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$, $t \in \mathbb{R}_{\geq 0}$ and $\omega \in \Omega$, and set out to show that for any $\epsilon \in \mathbb{R}_{>0}$, there is some $\varpi \in \mathcal{I}_t(\omega)$ such that

$$\liminf \mathcal{K}_\bullet^{G,c}(\varpi) < \mathcal{K}_t^{G,c}(\omega) + \epsilon. \tag{9}$$

So fix any such $\epsilon \in \mathbb{R}_{>0}$.

We'll show the existence of the path $\varpi$ by constructing it recursively such that the capital doesn't increase by more than $\epsilon$. To this end, we let $\mathscr{K}$ be the set of indices $k \in \{1, \dots, n\}$ such that $t < \tau_{k+1}(\omega)$. In case this index set $\mathscr{K}$ is empty, $\mathcal{K}_r^{G,c}(\omega) = \mathcal{K}_t^{G,c}(\omega)$ for all $r \in [t, +\infty[$, so the path $\varpi = \omega$ satisfies the inequality (9). If $\mathscr{K} \neq \emptyset$, we let $k \coloneqq \min \mathscr{K}$ and distinguish two subcases.

If $\tau_k(\omega) > t$, let $\varpi_k \coloneqq \omega$. Otherwise, we set out to construct some counting path $\varpi_k \in \mathcal{I}_t(\omega)$ for which the 'currently active' bets with stakes $\overline{h}_k(\omega)$ and $\underline{h}_k(\omega)$ don't increase the capital after $t$ by more than $\epsilon/n$. Choose some $\delta_k \in \mathbb{R}_{>0}$ such that $\omega(t+\delta_k) = \omega(t)$ [this is possible because $\omega$ is continuous from the right], $-\overline{h}_k(\omega)\underline{\lambda}\delta_k < \epsilon/n$, $\delta_k < \tau_{k+1}(\omega) - t$ and $\underline{\lambda}\delta_k \leq 1$. It follows from Eq. (4) that the counting path $\varpi_k$ given by

$$r \mapsto \begin{cases} \omega(r) & \text{if } r < t + \delta_k, \\ \omega(t) & \text{if } r \geq t + \delta_k \text{ and } \overline{h}_{k-1}(\omega) \geq 0 \\ \omega(t) + \lceil \underline{\lambda}(r-t) \rceil & \text{if } r \geq t + \delta_k \text{ and } \overline{h}_{k-1}(\omega) < 0 \end{cases}$$

does exactly that. Note that $\tau_{k+1}(\varpi_k) \geq t + \delta_k$—if this were not the case, then since $\varpi_k|_{[0,\tau_{k+1}(\varpi_k)]} = \omega|_{[0,\tau_{k+1}(\varpi_k)]}$ by definition and $\tau_{k+1}$ is a stopping time, it must be that $\tau_{k+1}(\omega) = \tau_{k+1}(\varpi_k) < t + \delta_k$, which contradicts our requirement that $\delta_k < \tau_{k+1}(\omega) - t$. Similarly, note that our construction also ensures that $\overline{h}_k(\varpi_k) = \overline{h}_k(\omega)$ and $\underline{h}_k(\varpi_k) = \underline{h}_k(\omega)$, as the stakes $\overline{h}_k$ and $\underline{h}_k$ are $\tau_k$-measurable and $\varpi_k|_{[0,\tau_k(\omega)]} = \omega|_{[0,\tau_k(\omega)]}$.

If $\tau_{k+1}(\varpi_k) = +\infty$ or $k = n$, our construction ensures that $\mathcal{K}_r^{G,c}(\varpi_k) < \mathcal{K}_t^{G,c}(\omega) + \epsilon/n$ for all $r \in [t, +\infty[$, which implies the inequality in (9). If on the other hand $k < n$ and $\tau_{k+1}(\varpi_k) < +\infty$, we let $\ell$ be the smallest index in $\{k+1, \dots, n\}$ such that $\tau_\ell(\varpi_k) < \tau_{\ell+1}(\varpi_k)$. We repeat the same argument as before, but modifying $\varpi_k$ from $\tau_\ell(\varpi_k)$ onwards rather than $\omega$ from $t$ onwards. After at most $n - k$ repetitions of the same argument, we'll end up with a counting path in $\mathcal{I}_t(\omega)$ that satisfies the inequality in (9). □

### A.3 Additional results and relegated proofs for results in Section 5

*Proof of Proposition 4.* It suffices to make the following observation. Fix any bounded below capital process $\mathcal{K}_\bullet^{G_\mathrm{o},c_\mathrm{o}} \in \mathfrak{K}_{[\underline{\lambda}_\mathrm{o},\overline{\lambda}_\mathrm{o}]}$ and any path $\omega \in \{\tau < +\infty\}$. Then with $c_\mathrm{i} \coloneqq \mathcal{K}_\tau^{G_\mathrm{o},c_\mathrm{o}}(\omega)$ and $G_\mathrm{i}$ a copy of the trading strategy $G_\mathrm{o}$ but with stopping times $\tau_k^\mathrm{i} \coloneqq \tau_k^\mathrm{o} \vee \tau$ instead of $\tau_k^\mathrm{o}$, the capital process $\mathcal{K}_\bullet^{G_\mathrm{i},c_\mathrm{i}} \in \mathfrak{K}_{[\underline{\lambda}_\mathrm{i},\overline{\lambda}_\mathrm{i}]}$ has capital $c_\mathrm{i}$ in $\tau$ for $\omega$ and is constructed in such a way that for all $t \in \,]\tau(\omega), +\infty[$ and $\varpi \in \mathcal{I}_\tau(\omega)$,

$$\begin{aligned}\mathcal{K}_t^{G_\mathrm{i},c_\mathrm{i}}(\varpi) - \mathcal{K}_t^{G_\mathrm{o},c_\mathrm{o}}(\varpi)\\ &= \sum_{k=1}^{n} (\tau_{k+1}^\mathrm{i}(\varpi) \wedge t - \tau_k^\mathrm{i}(\varpi) \wedge t)\big(\overline{h}_k(\varpi)(\overline{\lambda}_\mathrm{o} - \overline{\lambda}_\mathrm{i}) + \underline{h}_k(\varpi)(\underline{\lambda}_\mathrm{i} - \underline{\lambda}_\mathrm{o})\big) \geq 0,\end{aligned}$$

where the inequality follows from the assumptions in the statement. So $\mathcal{K}_\bullet^{G_\mathrm{i},c_\mathrm{i}}$ is uniformly bounded below and dominates $\mathcal{K}_\bullet^{G_\mathrm{o},c_\mathrm{o}}$ on $\mathcal{I}_\tau(\omega)$. □

The following corollary of Proposition 5 will be useful in the proof of Theorem 1 and Proposition 8—through Lemma 3—further on.

**Corollary 2.** *For any two time points $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$ and $m \in \mathbb{N}$,*

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{\{N_t - N_s \geq m\}} | s] \leq \frac{\overline{\lambda}(t-s)}{m}.$$

*Proof.* Since $\mathbb{I}_{\{N_t - N_s \geq m\}} \leq (N_t - N_s)/m$, this follows immediately from Proposition 5 and (E4) & (E3) in Proposition 1. □

We defer proving Proposition 6 to Appendix A.4 further on. The next item on our menu of unproven results is Theorem 1. Our proof for this result will simplify quite a bit thanks to the following intermediary result and Corollary 1 to Proposition 7.

**Lemma 2.** *Fix some $s, t \in \mathbb{R}_{\geq 0}$ such that $s \leq t$. Then for all $t$-measurable $f \in \overline{\mathbb{V}}$ and $\omega \in \Omega$,*

$$\begin{aligned}\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|s](\omega) = \inf \Big\{ \mathcal{K}_s^{G,c}(\omega) \colon \mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}, \inf \mathcal{K}_\bullet^{G,c} > -\infty,\\ (\forall \varpi \in \mathcal{I}_s(\omega))\ \mathcal{K}_t^{G,c}(\varpi) \geq f(\varpi) \Big\}.\end{aligned}$$

*Proof.* Since $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is coherent, it follows immediately from Lemma 1 and the $t$-measurability of $f$ that for any bounded below capital process $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that superhedges $f$ on $\mathcal{I}_s(\omega)$,

$$\mathcal{K}_t^{G,c}(\varpi) \geq \inf f|_{\mathcal{I}_t(\varpi)} = f(\varpi) \quad \text{for all } \varpi \in \mathcal{I}_s(\omega).$$

Conversely, suppose there is some bounded one-sided capital process $\mathcal{K}_\bullet^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\mathcal{K}_t^{G,c}(\varpi) \geq f(\varpi)$ for all $\varpi \in \mathcal{I}_s(\omega)$. Let $G'$ be the betting strategy that copies $G$ but replaces each stopping time $\tau_k$ by $\tau'_k \coloneqq \tau_k \wedge t$. Then $G'$ is a valid trading strategy, and $\mathcal{K}_\bullet^{G',c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is bounded below and defined in such a way that

$$\mathcal{K}_t^{G',c}(\varpi) = \mathcal{K}_t^{G,c}(\varpi) \quad \text{for all } \varpi \in \Omega.$$

Consequently,

$$\liminf \mathcal{K}_{\bullet}^{G',c}(\varpi) = \mathcal{K}_{t}^{G',c}(\varpi) = \mathcal{K}_{t}^{G',c}(\varpi) \geq f(\varpi) \quad \text{for all } \varpi \in \mathcal{I}_s(\omega).$$

This concludes our proof for the equality in the statement. □

*Proof of Corollary 1.* This follows immediately from Proposition 7 once one realises that for all $\varpi \in \Omega$,

$$\big[g(N_{t_1}, \dots, N_{t_{k+1}})\big](\omega \oplus_{t_k} \varpi) = g(\omega(t_1), \dots, \omega(t_k), \omega(t_k) + \varpi(t_{k+1} - t_k)).$$

□

*Proof of Theorem 1.* Fix any $\omega \in \Omega$ and $\epsilon \in \mathbb{R}_{>0}$, and let $f \coloneqq g(N_{t_1}, \dots, N_{t_{k+1}})$. Since $f = (f - \inf f) + \inf f$ and $\inf(f - \inf f) \geq 0$, thanks to (E5) we may assume without loss of generality that $\inf f \geq 0$. Furthermore, since $f$ is positive and bounded, it follows from (E6) that so are $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1]$ and $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]$.

Since $f$ is $t_{k+1}$-measurable, it follows from Lemma 2 that there is some capital process $\mathcal{K}_{\bullet}^{G_1,c_1} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ that is bounded below with $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) \leq \mathcal{K}_{t_1}^{G_1,c_1}(\omega) < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) + \epsilon$ and $\mathcal{K}_{t_{k+1}}^{G_1,c_1} \geq_{\mathcal{I}_{t_1}(\omega)} f$. Consequently, it must be that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k](\varpi) \leq \mathcal{K}_{t_k}^{G_1,c_1}(\varpi)$ for all $\varpi \in \mathcal{I}_{t_k}(\omega)$; thanks to (E4) and Lemma 2, we infer from this that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]\Big|t_1\Big](\omega) \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\mathcal{K}_{t_k}^{G_1,c_1}\Big|t_1\Big](\omega).$$

Clearly, the $t_k$-measurable varibale $\mathcal{K}_{t_k}^{G_1,c_1}$ is hedged on $\mathcal{I}_{t_1}(\omega)$ (and at $t_k$) by $\mathcal{K}_{\bullet}^{G_1,c_1}$; consequently,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]\Big|t_1\Big](\omega) \leq \mathcal{K}_{t_1}^{G_1,c_1}(\omega) < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) + \epsilon. \tag{10}$$

For the converse inequality, recall from Section 2 that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]$ is $t_k$-measurable. Hence, it follows from Lemma 2 that there is some bounded below capital process $\mathcal{K}_{\bullet}^{G,c} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]\Big|t_1\Big](\omega) \leq \mathcal{K}_{t_1}^{G,c}(\omega) < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\Big[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]\Big|t_1\Big](\omega) + \frac{\epsilon}{3}$$

and $\mathcal{K}_{t_k}^{G,c} \geq_{\mathcal{I}_{t_1}(\omega)} \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]$. Let $\Delta \coloneqq t_{k+1} - t_k$, and fix some natural number $m$ such that $3(\sup f)\overline{\lambda}\Delta < m\epsilon$. Then by Corollary 2, there is some positive capital process $\mathcal{K}_{\bullet}^{G_0,c_0} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ with $\mathcal{K}_{t_1}^{G_0,c_0}(\omega) \leq \overline{\lambda}\Delta/m + \epsilon/(3\sup f)$ such that $\mathcal{K}_{t_k}^{G_0,c_0} \geq 1$ for all $\varpi \in \mathcal{I}_{t_1}(\omega)$ with $\varpi(t_k) \geq \varpi(t_1) + m$. On the other hand, for all $z = z_{1:k} \in (\mathbb{Z}_{\geq 0})^k$ with $\omega(t_1) = z_1 \leq z_2 \leq \cdots \leq z_k < \omega(t_1) + m$, thanks to Lemma 2 there is some positive capital process $\mathcal{K}_{\bullet}^{G_z,c_z} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z_1, \dots, z_k, z_k + N_\Delta)] \leq c_z < \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(z_1, \dots, z_k, z_k + N_\Delta)] + \frac{\epsilon}{3}$$

and

$$\mathcal{K}_{\Delta}^{G_z,c_z}(\varpi) \geq g(z_1,\dots,z_k,z_k+\varpi(\Delta)) \quad \text{for all } \varpi \in \Omega.$$

Consider now the trading strategy $G'$ that trades according to $G$ until $t_k$ and, for $\varpi \in \mathcal{I}_{t_1}(\omega)$ with $\varpi(t_k) < \omega(t_1)+m$, from $t_k$ onwards trades according to $G_{(\varpi(t_1)\dots,\varpi(t_k))}$ shifted by $t_k$—it shouldn't take too much effort from the reader to understand that this is still a valid trading strategy. Let $c' \coloneqq c+\epsilon/3$.

By construction, and thanks to Corollary 1, $\mathcal{K}_{t_1}^{G',c'}(\omega) = \mathcal{K}_{t_1}^{G,c}(\omega)+\epsilon/3$, and $\mathcal{K}_{t_{k+1}}^{G',c'}(\varpi) \geq f(\varpi)$ for all $\varpi \in \mathcal{I}_{t_1}(\omega)$ such that $\varpi(t_k) < \omega(t_1)+m$. Additionally, for all $\varpi \in \mathcal{I}_{t_1}(\omega)$—so in particular for those with $\varpi(t_k) \geq \omega(t_1)+m$—$(\sup f)\mathcal{K}_{t_k}^{G_0,c_0}(\varpi) \geq (\sup f)\mathbb{I}_{\{N_{t_k}-N_{t_1}\geq m\}}(\varpi)$. Since furthermore $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is closed under positive linear combinations, it follows that

$$\begin{aligned}
\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_1](\omega) &\leq \mathcal{K}_{t_1}^{G',c'}(\omega) + (\sup f)\mathcal{K}_{t_1}^{G_0,c_0}(\omega) \\
&\leq \mathcal{K}_{t_1}^{G,c}(\omega) + \frac{\epsilon}{3} + (\sup f)\frac{\overline{\lambda}\Delta}{m} + \frac{\epsilon}{3} \\
&< \mathcal{K}_{t_1}^{G,c}(\omega) + \epsilon \\
&< \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}\left[\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[f|t_k]\middle|t_1\right](\omega) + \epsilon. 
\end{aligned} \tag{11}$$

Since $\epsilon$ can be made arbitrarily small, the equality in the statement follows from the inequalities (10) and (11). □

In our proof for Proposition 8, we'll need a bound on the upper probability of having more than one jump in one of a sequence of consecutive intervals of the same length.

**Lemma 3.** *Fix some $s,t \in \mathbb{R}_{\geq 0}$ such that $s < t$. For all $n \in \mathbb{N}$, we define $\Delta_n \coloneqq (t-s)/n$, $t_k^n \coloneqq s+(k-1)\Delta_n$ for all $k \in \{1,\dots,n+1\}$ and*

$$A_n \coloneqq \{\omega \in \Omega \colon (\forall k \in \{1,\dots,n\})\ \omega(t_{k+1}^n) \leq \omega(t_k^n)+1\}.$$

*Then*

$$(\forall \epsilon \in \mathbb{R}_{>0})(\exists n_\epsilon \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_\epsilon)\ \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A_n^c}|s] < \epsilon$$

*Proof.* Fix any $\omega \in \Omega$, $\epsilon \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}$ such that $m > 3\overline{\lambda}(t-s)/\epsilon$. Then by Corollary 2 and Lemma 2, there is some capital process $\mathcal{K}_{\bullet}^{G_m,c_m} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\mathcal{K}_s^{G_m,c_m}(\omega) < \overline{\lambda}(t-s)/m+\epsilon/3$ and $\mathcal{K}_t^{G_m,c_m}(\varpi) \geq \mathbb{I}_{\{N_t-N_s\geq m\}}(\varpi)$ for all $\varpi \in \mathcal{I}_s(\omega)$.

Now consider the elementary betting strategy $G$ that bets with unit stake from the moment there is some jump in the interval $[t_k^n, t_{k+1}^n]$ and stops at the end of this interval, and stops trading altogether once the path has increased by more than $m$; more formally, we consider the stopping times $\tau_1 \coloneqq \tau_1' \wedge \sigma$, …, $\tau_{2m+1} \coloneqq \tau_{2m+1}' \wedge \sigma$ with

$$\sigma \colon \Omega \to \overline{\mathbb{R}}_{\geq 0} \colon \varpi \mapsto \inf\{r \in \mathbb{R}_{\geq 0} \colon \varpi(r) \geq \varpi(s)+m\}$$

and with $\tau'_1, \ldots, \tau'_{2m+1}$ defined recursively by $\tau'_1 \coloneqq t^n_1 = s$ and, for all $j \in \{1, \ldots, m\}$, by

$$\tau'_{2j}\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}\colon \varpi \mapsto \inf\left\{r \in \mathbb{R}_{\geq 0}\colon \tau_{2j-1}(\varpi) < r \leq t, \varpi(r) > \lim_{r_- \nearrow r} \varpi(r_-)\right\}$$

and

$$\tau'_{2j+1}\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}\colon \varpi \mapsto \inf\{t^n_k\colon k \in \{1, \ldots, n+1\}, t^n_k \geq \tau_{2j}(\varpi)\},$$

and the stakes $\overline{h}_{2j} \coloneqq 1$ , $\overline{h}_{2j-1} \coloneqq 0$ and $\underline{h}_{2j} \coloneqq 0 \eqqcolon \underline{h}_{2j-1}$. Then with $c \coloneqq \epsilon/3$, our construction ensures that $\mathcal{K}^{G,c}_t(\varpi) \geq c - m\overline{\lambda}\Delta_n$ for all $\varpi \in \Omega$, and in particular that

$$\mathcal{K}^{G,c}_t(\varpi) \geq c - m\overline{\lambda}\Delta_n + 1 \quad \text{for all } \varpi \in \{N_t - N_s < m\} \cap A^{\mathrm{c}}_n.$$

Let $n_\epsilon \coloneqq 3m\overline{\lambda}(t-s)/\epsilon$; then for all $n \in \mathbb{N}$ such that $n \geq n_\epsilon$,

$$m\overline{\lambda}\Delta_n = m\overline{\lambda}(t-s)/n \leq \frac{\epsilon}{3} = c.$$

Consequently, for all such $n \geq n_\epsilon$, $\mathbb{I}_{A^{\mathrm{c}}_n}$ is superhedged (in $t$) on $\mathcal{I}_s(\omega)$ by $\mathcal{K}^{G,c}_\bullet + \mathcal{K}^{G_m,c_m}_\bullet \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$. Due to Lemma 2, we infer from all this that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A^{\mathrm{c}}_n}|s](\omega) \leq \mathcal{K}^{G,c}_s(\omega) + \mathcal{K}^{G_m,c_m}_s(\omega) < c + \frac{\overline{\lambda}(t-s)}{m} + \frac{\epsilon}{3} < \epsilon,$$

which is what we needed to prove. □

*Proof of Proposition 8.* In the degenerate case $s = t$, the equality in the statement is immediate because (i) $\overline{\mathrm{S}}_0 = \mathrm{I}$; and (ii) $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|s]$ maps the $s$-measurable variable $g(N_s)$ to itself due to (E6). Henceforth, we therefore assume that $s < t$.

We fix any $\omega \in \Omega$ and $g \in \mathbb{G}$, and set out to prove that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s](\omega) \leq \left[\overline{\mathrm{S}}_{t-s}g\right](\omega(s)).$$

For all $n \in \mathbb{N}$, let $\Delta_n$, $t^n_1, \ldots, t^n_{n+1}$ and $A_n$ be defined as in Lemma 3, and let $g^n_k \coloneqq (\mathrm{I} + \Delta_n\overline{\mathrm{G}})^{n+1-k}g$ for all $k \in \{1, \ldots, n+1\}$. Note that $g^n_k = (\mathrm{I} + \Delta_n\overline{\mathrm{G}})g^n_{k+1}$ and that the stopping time

$$\sigma_n\colon \Omega \to \overline{\mathbb{R}}_{\geq 0}\colon \varpi \mapsto \inf \bigcup_{k=1}^{n} \{r \in [t^n_k, t^n_{k+1}]\colon \varpi(r) \geq \varpi(t^n_k) + 2\}$$

is equal to $+\infty$ on the event $A_n$.

Fix any $\epsilon \in \mathbb{R}_{>0}$. Then by Lemma 3 and [3, Theorem 3.1], there is some $n \in \mathbb{N}$ such that $\delta \coloneqq (\sup g - \inf g)\underline{\lambda}\Delta_n < \epsilon$, $0 \leq \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A^{\mathrm{c}}_n}|s](\omega) < \epsilon/2(\sup g - \inf g)$ and $|[\overline{\mathrm{S}}_{t-s}g](\omega(s)) - g^n_1(\omega(s))| < \epsilon$. For this $n$, we consider the initial capital $c \coloneqq g^n_1(\omega(s)) + \delta$ in combination with the elementary betting strategy $G$ with stopping times $\tau_1 \coloneqq t_1^n \wedge \sigma_n, \ldots, \tau_{n+1} \coloneqq t_{n+1}^n \wedge \sigma_n$ and stakes defined for all $k \in \{1, \ldots, n\}$ and $\varpi \in \Omega$ by

$$\overline{h}_k(\varpi) \coloneqq \Big(+g_{k+1}^n\big(\varpi(\tau_k)+1\big) - g_{k+1}^n\big(\varpi(\tau_k)\big)\Big) \vee 0$$

and

$$\underline{h}_k(\varpi) \coloneqq \Big(-g_{k+1}^n\big(\varpi(\tau_k)+1\big) + g_{k+1}^n\big(\varpi(\tau_k)\big)\Big) \vee 0.$$

This way, for all $\varpi \in \mathcal{I}_s(\omega)$, $k \in \{1, \ldots, n\}$ with $\tau_k(\varpi) < \sigma_n(\varpi)$ and $r \in [\tau_k(\varpi), \tau_{k+1}(\varpi)]$ such that $r < \sigma_n(\varpi)$,

$$\begin{aligned}\mathcal{K}_r^{G,c}(\varpi) &- \mathcal{K}_{\tau_k}^{G,c}(\varpi) \\ &= g_{k+1}^n(\varpi(r)) - g_k^n(\varpi(\tau_k)) + \big(\overline{h}_k(\varpi)\overline{\lambda} - \underline{h}_k(\varpi)\underline{\lambda}\big)(t_{k+1}^n - r).\end{aligned} \tag{12}$$

To verify this equality, observe that by construction of the capital process $\mathcal{K}_\bullet^{G,c}$,

$$\begin{aligned}\mathcal{K}_r^{G,c}(\varpi) &- \mathcal{K}_{\tau_k}^{G,c}(\varpi) \\ &= \overline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \overline{\lambda}(r - \tau_k(\varpi))\big) \\ &\quad - \underline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \underline{\lambda}(r - \tau_k(\varpi))\big) \\ &= \overline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \overline{\lambda}\Delta_n\big) + \overline{h}_k(\varpi)\overline{\lambda}(t_{k+1}^n - r) \\ &\quad - \underline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \underline{\lambda}\Delta_n\big) - \underline{h}_k\underline{\lambda}(\varpi)(t_{k+1}^n - r).\end{aligned}$$

Recall that $g_k^n = (\mathrm{I} + \Delta_n\overline{\mathrm{G}})g_{k+1}^n = g_{k+1}^n + \Delta_n\overline{\mathrm{G}}g_{k+1}^n$ by definition. Hence, if $\overline{h}_k(\varpi) > 0$ (and therefore $\underline{h}_k(\varpi) = 0$),

$$\begin{aligned}g_{k+1}^n&(\varpi(r)) - g_k^n(\varpi(\tau_k)) \\ &= g_{k+1}^n\big(\varpi(r)\big) - g_{k+1}^n\big(\varpi(\tau_k)\big) - \overline{\lambda}\Delta_n\Big(g_{k+1}^n\big(\varpi(\tau_k)+1\big) - g_{k+1}^n\big(\varpi(\tau_k)\big)\Big) \\ &= g_{k+1}^n\big(\varpi(r)\big) - g_{k+1}^n\big(\varpi(\tau_k)\big) - \overline{\lambda}\Delta_n\overline{h}_k(\varpi);\end{aligned}$$

a similar equality holds if $\underline{h}_k(\varpi) > 0$ (and therefore $\overline{h}_k(\varpi) = 0$) with $\underline{\lambda}$ in place of $\underline{\lambda}$. Because $r < \sigma_n(\varpi)$ by assumption, it's guaranteed that $0 \le \varpi(r) - \varpi(\tau_k) \le 1$; it's therefore straightforward to verify that if $\overline{h}_k(\varpi) > 0$,

$$\overline{h}_k(\varpi)\big(\varpi(r) - \varpi(\tau_k) - \overline{\lambda}\Delta_n\big) = g_{k+1}^n(\varpi(r)) - g_k^n(\varpi(\tau_k)),$$

and similarly for $\underline{h}_k(\omega) > 0$; this verifies Eq. (12). In the particular case that $\tau_{k+1}(\omega) < \sigma_n(\omega)$, Eq. (12) for $r = \tau_{k+1}(\omega)$ reduces to

$$\mathcal{K}_{\tau_{k+1}}^{G,c}(\omega) - \mathcal{K}_{\tau_k}^{G,c}(\omega) = g_{k+1}^n(\omega(\tau_{k+1})) - g_k^n(\omega(\tau_{k+1})). \tag{13}$$

Since $\mathcal{K}_{\tau_1}^{G,c}(\varpi) = g_1^n(\varpi(s)) + \delta$ for all $\varpi \in \mathcal{I}_s(\omega)$, it follows from Eq. (13) that for all $k \in \{1, \ldots, n\}$ such that $\tau_{k+1}(\varpi) < \sigma_n(\varpi)$,

$$\mathcal{K}_{\tau_k}^{G,c}(\varpi) = g_k^n\big(\varpi(\tau_k)\big) + \delta. \tag{14}$$

Now recall that $\sigma_n(\omega) = +\infty$ for any $\varpi \in A_n \cap \mathcal{I}_{t_1}(\omega)$, so it follows from the preceding equality for $k = n+1$ that

$$\mathcal{K}_t^{G,c}(\varpi) = \mathcal{K}_{\tau_{n+1}}^{G,c}(\varpi) = g_{n+1}^n\big(\omega(\tau_{n+1})\big) + \delta = g\big(\varpi(t)\big) + \delta.$$

Next, observe that for $\varpi \in A_n^{\mathrm{c}} \cap \mathcal{I}_s(\omega)$, there is some largest $k \in \{1, \dots, n\}$ such that $\tau_k(\varpi) < \sigma_n(\varpi)$ (and of course $\tau_{k+1}(\varpi) = \sigma_n(\varpi)$); then since $\varpi$ has unit jumps,

$$\lim_{r \nearrow \sigma_n(\varpi)} \mathcal{K}_{\sigma_n}^{G,c}(\varpi) - \mathcal{K}_r^{G,c}(\varpi) = \lim_{r \nearrow \sigma_n(\varpi)} \overline{h}_k(\varpi)(\varpi(\sigma_n) - \varpi(r)) = \overline{h}_k(\omega).$$

It follows from this, Eqs. (12) and (14) that

$$\begin{aligned}\mathcal{K}_{\sigma_n}^{G,c}(\varpi) &= \lim_{r \nearrow \sigma_n(\varpi)} \delta + g_{k+1}^n(\varpi(r)) + \big(\overline{h}_k(\varpi)\overline{\lambda} - \underline{h}_k(\varpi)\underline{\lambda}\big)(t_{k+1}^n - r) + \overline{h}_k(\varpi) \\ &\geq \delta + \inf g - (\sup g - \inf g)\underline{\lambda}\Delta_n - (\sup g - \inf g) \\ &= 2\inf g - \sup g.\end{aligned}$$

Recall now that $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{A_n^{\mathrm{c}}}|\tau](\omega) < \epsilon/2(\sup g - \inf g)$, so there is some positive capital process $\mathcal{K}_\bullet^{G',c'} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ such that $\mathcal{K}_s^{G',c'}(\omega) < \epsilon/2(\sup g - \inf g)$ and $\mathcal{K}_t^{G',c'}(\varpi) \geq \mathbb{I}_{A_n^{\mathrm{c}}}(\varpi)$ for all $\varpi \in \mathcal{I}_s(\omega)$. Since $\mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$ is a cone and $g(N_t)$ is $t$-measurable, we conclude from this and Lemma 2 that

$$\begin{aligned}\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[g(N_t)|s](\omega) &\leq \mathcal{K}_s^{G,c}(\omega) + 2(\sup g - \inf g)\mathcal{K}_s^{G',c'}(\omega) \\ &< g_1^n(\omega(s)) + \delta + \epsilon \\ &< [\overline{\mathrm{S}}_{t-s} g](\omega(s)) + 3\epsilon.\end{aligned}$$

Since $\omega \in \Omega$, $g \in \mathbb{G}$ and $\epsilon \in \mathbb{R}_{\geq 0}$ were arbitrary, this proves the inequality in the statement. □

### A.4 Proof of Proposition 6

Finally, we set out to prove Proposition 6, and we'll do so with the help of the following result.

**Lemma 4.** *For any* $t, \Delta \in \mathbb{R}_{\geq 0}$,

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{\{N_{t+\Delta}=N_t\}}|t] = e^{-\Delta\underline{\lambda}} \quad \textit{with } \{N_{t+\Delta} = N_t\} \coloneqq \{\omega \in \Omega \colon \omega(t+\Delta) = \omega(t)\}.$$

The reader will have no difficulty in verifying that while our proof invokes results that come after Proposition 6—in particular Theorem 2—these don't rely on this particular result.

*Proof.* Fix any $\omega \in \Omega$, and let $x \coloneqq \omega(t)$. Then it follows from the definition of $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|t](\omega)$ that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{\{N_{t+\Delta}=N_t\}}|t](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_x(N_{t+\Delta})|t](\omega).$$

From this, Theorem 2 and Lemma 53 in [4], it follows that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\mathbb{I}_{\{N_{t+\Delta}=N_t\}}|t](\omega) = [\overline{\mathrm{S}}_\Delta \mathbb{I}_x](x) = [\overline{\mathrm{S}}_\Delta \mathbb{I}_0](0) = \lim_{k\to+\infty}\left(\mathrm{I}+\frac{\Delta}{k}\overline{\mathrm{G}}\right)^k \mathbb{I}_0(0).$$

To obtain the equality in the statement, it therefore suffices to observe that (i) for any $k \in \mathbb{N}$ such that $1 - \underline{\lambda}\Delta/k \geq 0$,

$$\left(\mathrm{I}+\frac{\Delta}{k}\overline{\mathrm{G}}\right)^\ell \mathbb{I}_0 = \left(1-\frac{\Delta\underline{\lambda}}{k}\right)^\ell \mathbb{I}_0 \quad \text{for all } \ell \in \{0,\dots,k\};$$

and (ii) $\lim_{k\to+\infty}(1-\Delta\underline{\lambda}/k)^k = e^{-\Delta\underline{\lambda}}$. □

*Proof of Proposition 6.* Due to a similar argument as the one at the end of our proof for Proposition 5, it suffices to prove that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau] \leq \frac{1}{\underline{\lambda}} \quad \text{and} \quad \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(\rho_\tau - \tau)|\tau] \leq -\frac{1}{\overline{\lambda}}. \tag{15}$$

If $\underline{\lambda} = 0$, the first inequality in (15) is trivial. If $\underline{\lambda} > 0$, we let $c \coloneqq 1/\underline{\lambda}$ and consider the trading strategy $G$ with stopping times $\tau_1 \coloneqq \tau$ and $\tau_2 \coloneqq \rho_\tau$ and stakes $\overline{h}_1 \coloneqq 0$ and $\underline{h}_1 \coloneqq 1/\underline{\lambda}$. Then for all $\varpi \in \Omega$ and $r \in \mathbb{R}_{\geq 0}$,

$$\mathcal{K}_r^{G,c}(\varpi) = \begin{cases} \frac{1}{\underline{\lambda}} & \text{if } r \leq \tau(\varpi) \\ \frac{1}{\underline{\lambda}} + (r - \tau(\varpi)) & \text{if } \tau(\varpi) < r < \rho_\tau(\varpi) \\ \rho_\tau(\varpi) - \tau(\varpi) & \text{if } r \geq \rho_\tau(\varpi). \end{cases}$$

From this equality, we see that $\mathcal{K}_\bullet^{G,c}$ is bounded below (by $1/\underline{\lambda}$) and superhedges $\rho_\tau - \tau$. This implies the first inequality in (15).

Establishing the second inequality is a bit more involved. Let $t \coloneqq \tau(\omega)$, and note that (i) $\mathcal{I}_\tau(\omega) = \mathcal{I}_t(\omega)$; and (ii) $\rho_\tau(\varpi) - \tau(\varpi) = \rho_t(\varpi) - t$ for all $\varpi \in \mathcal{I}_\tau(\omega)$. It follows from this and the definition of $\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\bullet|\tau](\omega)$ that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(\rho_\tau - \tau)|\tau](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[-(\rho_t - t)|t](\omega). \tag{16}$$

Let $c$ be any strictly positive real number such that $c < 1/\overline{\lambda}$, fix some $\Delta \in \mathbb{R}_{\geq 0}$ and consider the trading strategy $G_{c,\Delta}$ with stopping times $\tau_1 \coloneqq t$ and $\tau_2 \coloneqq \rho_t \wedge (t+\Delta)$ and stakes $\overline{h}_1 \coloneqq c$ and $\underline{h}_1 \coloneqq 0$. Then for all $\varpi \in \Omega$ and $r \in \mathbb{R}_{\geq 0}$,

$$\mathcal{K}_r^{G_{c,\Delta},-c}(\varpi) = \begin{cases} -c & \text{if } r \leq t \\ -c - c\overline{\lambda}(r-t) & \text{if } t < r < \rho_t(\varpi) \wedge (t+\Delta) \\ -c\overline{\lambda}(\rho_t(\varpi) - t) & \text{if } r \geq \rho_t(\varpi) \leq t+\Delta \\ -c - c\overline{\lambda}\Delta & \text{if } r \geq t+\Delta < \rho_t(\varpi). \end{cases}$$

From this equality, we see that $\mathcal{K}_\bullet^{G_{c,\Delta},-c}$ is bounded below and superhedges $-(\rho_t - t)$ on $\{\varpi \in \Omega \colon \rho_t(\varpi) \leq t + \Delta\}$. It remains for us to counteract the additional $-c$ term on the remaining set $S \coloneqq \{\varpi \in \Omega\colon \rho_t(\varpi) > t + \Delta\} = \{\varpi \in \Omega\colon \varpi(t+\Delta) = \varpi(t)\}$. By Lemmas 1 and 4, for all $\epsilon \in \mathbb{R}_{>0}$ there is some positive capital process $\mathcal{K}_\bullet^{G'_{\epsilon,\Delta},c_\epsilon}$ that superhedges $\mathbb{I}_S$ on $\mathcal{I}_t(\omega)$ with $\mathcal{K}_t^{G'_{\epsilon,\Delta},c_\epsilon}(\omega) < e^{-\Delta\underline{\lambda}} + \epsilon$. Consequently, the capital process

$$\mathcal{K}_\bullet^{G_{c,\Delta},-c} + c\mathcal{K}_\bullet^{G_{\epsilon,\Delta},c_\epsilon} \in \mathfrak{K}_{[\underline{\lambda},\overline{\lambda}]}$$

is bounded below and superhedges $-(\rho_t - t)$ on $\mathcal{I}_t(\omega) = \mathcal{I}_\tau(\omega)$, and therefore

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_t - t|t](\omega) \leq \mathcal{K}_t^{G_{c,\Delta},-c}(\omega) + c\mathcal{K}_t^{G_{\epsilon,\Delta},c_\epsilon}(\omega) < -c + c(e^{-\Delta\underline{\lambda}} + \epsilon).$$

Since this inequality holds for arbitrary $c < 1/\overline{\lambda}$, $\Delta \in \mathbb{R}_{\geq 0}$ and $\epsilon \in \mathbb{R}_{>0}$, it follows [taking $\epsilon$ arbitrarily small, $\Delta$ arbitrarily large and $c$ as large as possible] that

$$\overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_\tau - \tau|\tau](\omega) = \overline{\mathrm{E}}_{[\underline{\lambda},\overline{\lambda}]}[\rho_t - t|t](\omega) \leq -\frac{1}{\overline{\lambda}},$$

where the first equality is Eq. (16). This proves the second inequality in (15). $\square$