A study of the set of probability measures compatible with comparative judgements

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3 month research stay in Oviedo, Spain

expectation

3 month research stay in Oviedo, Spain

reality

Comparative judgements

Consider a possibility space $\mathscr{X}$.

We are given m comparative judgements of the form

$$A_k \succeq B_k \Leftrightarrow P(A_k) \geq P(B_k) \quad \text{with } A_k, B_k \subseteq \mathscr{X} .$$

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This comparative assessment

$$\mathscr{C} := \{(A_k, B_k) \colon k \in \{1, \ldots, m\}\}$$

induces the set of compatible probability mass functions

$$\mathscr{M}_{\mathscr{C}} := \left\{ p \text{ a pmf} \colon (\forall (A, B) \in \mathscr{C}) \sum_{x \in A} p(x) \geq \sum_{y \in B} p(y) \right\}.$$

What has already been done?

Classical works by de Finetti, Koopman, Good or Savage assume that

$$\mathscr{C} \text{ is a total order } \succeq \text{ on } \mathcal{P}(\mathscr{X}).$$

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The work of Miranda & Destercke is most closely related to ours, but they consider comparisons between atoms instead of events.

They show that

- $\mathscr{M}_{\mathscr{C}}$ is *always* non-empty;
- has at most $2^{(n-1)}$ extreme points;
- these extreme points can be easily determined.

When is $\mathscr{M}_{\mathscr{C}}$ non-empty?

Compatibility of a comparative assessment

The comparative assessment $\mathscr{C}$ corresponds to the set of desirable gambles

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From Walley's theory, we know that

$$\begin{aligned}
\mathscr{M}_{\mathscr{C}} \neq \emptyset &\iff \mathcal{K}_{\mathscr{C}} \text{ avoids sure loss} \\
&\iff \max \sum_{\ell=1}^{k} f_\ell \geq 0 \quad \text{for all } k \in \mathbb{N}, f_\ell \in \mathcal{K}_{\mathscr{C}} \\
&\iff \max \sum_{(A,B)\in\mathscr{C}} \lambda_{(A,B)}(\mathbb{I}_A - \mathbb{I}_B) \geq 0 \quad \text{for all } \lambda_{(A,B)} \in \mathbb{Z}_{\geq 0}.
\end{aligned}$$

Redundant and saturated constraints

Assume that $\mathscr{M}_{\mathscr{C}} \neq \emptyset$.

The constraint $(A, B) \in \mathscr{C}$ is *redundant*—that is, $\mathscr{M}_{\mathscr{C}} = \mathscr{M}_{\mathscr{C} \setminus \{(A,B)\}}$—if and only if

$$\mathbb{I}_A - \mathbb{I}_B \in \operatorname{posi}\left(\mathcal{K}_{\mathscr{C} \setminus \{(A,B)\}} \cup \{\mathbb{I}_x : x \in \mathscr{X}\}\right).$$

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If no constraint is redundant, then

$$(\forall (A, B) \in \mathscr{C})(\exists p \in \mathscr{M}_{\mathscr{C}}) \sum_{x \in A} p(x) = \sum_{y \in B} p(y),$$

and vice versa.

Can we say something about the structure of $\mathscr{M}_{\mathscr{C}}$?

$\mathscr{M}_{\mathscr{C}}$ as a convex polytope

We interpret real-valued functions on $\mathscr{X}$ as vectors in $\mathbb{R}^n$.

Then $\mathscr{M}_{\mathscr{C}}$ is the convex polytope defined by $n + m + 1$ half spaces:

$$\begin{aligned} p^{\mathrm{T}} 1 &= 1, & \\ p^{\mathrm{T}} \mathbb{I}_x &\geq 0 & \text{for all } x \in \mathscr{X}, \\ p^{\mathrm{T}} (\mathbb{I}_A - \mathbb{I}_B) &\geq 0 & \text{for all } (A, B) \in \mathscr{C}. \end{aligned}$$

Therefore, it is also uniquely defined by its extreme points $\mathrm{ext}(\mathscr{M}_{\mathscr{C}})$.

Properties of the extreme points

Miranda & Desctercke show that for comparisons between atoms,

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We show that for comparisons between events,

- an extreme point is *not necessarily* a uniform distribution; and
- there can be *more than* 2^n extreme points.

Can we bound the number of extreme points of $\mathscr{M}_{\mathscr{C}}$?

Miranda & Destercke show that for comparisons between atoms,

$$|\mathrm{ext}(\mathscr{M}_{\mathscr{C}})| \leq 2^{(n-1)}.$$

By McMullens's theorem,

$$|\mathrm{ext}(\mathscr{M}_{\mathscr{C}})| \leq \binom{m+1+\lfloor n/2 \rfloor}{m+1} + \binom{m+\lceil n/2 \rceil}{m+1}.$$

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Modifying an argument of Derks & Kuipers and Wallner, we show that

$$|\mathrm{ext}(\mathscr{M}_{\mathscr{C}})| \leq n!\,2^{n}.$$

Can we easily determine the extreme points of $\mathscr{M}_{\mathscr{C}}$?

A helpful graphical representation

Inspired by Miranda & Destercke, we represent the assessment $\mathscr{C}$ as a digraph $\mathcal{G}_{\mathscr{C}}$.

We add one node for every atom x in the possibility space $\mathscr{X}$.

For every judgement $(A, B) \in \mathscr{C}$, we add

- an auxiliary node ξ,
- a directed edge from every x in A to ξ, and
- a directed edge from ξ to every y in B.

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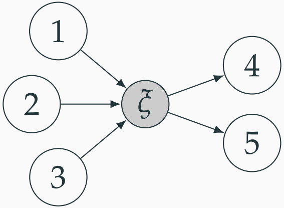
For example, $\{1,2,3\} \succeq \{4,5\}$ yields



Acyclic digraph

- ▷ If the digraph $\mathcal{G}_{\mathscr{C}}$ is *acyclic* and
- ▷ every atom-node has no more than 1 incoming edge,

every extreme point corresponds to an "extreme arborescence" and vice versa,

and these "extreme arborescences" can be easily generated procedurally.

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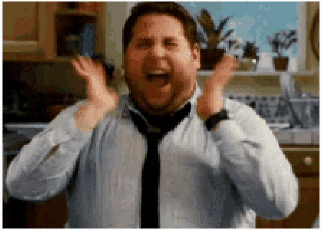

Acyclic digraph

- If the digraph $\mathcal{G}_{\mathscr{C}}$ is *acyclic*,
- every atom-node has no more than 1 incoming edge, and
- every auxiliary-node has precisely 1 incoming edge,

then all the extreme points correspond to uniform distributions.

There is more...

Our results extend easily to a number of more general scenarios:

- when $\mathscr{C}$ is associated with a muti-level partition of $\mathscr{X}$;
- when we consider *strict* probability comparisons;
- by decomposing $\mathcal{G}_{\mathscr{C}}$ in terms of its connected components.

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Our results extend easily to a number of more general scenarios:

- when $\mathscr{C}$ is associated with a muti-level partition of $\mathscr{X}$;
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Still, there are plenty of things to be done:

- Lower the general bound on the number of extreme points.
- Consider other particular cases of acyclic digraphs.
- Check if other graphical representations are more effective.