Decide quicker with Total Choice functions

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Setting & Objective

We consider a vector space $\mathscr{V}$, which we also call the option space—things we can choose. The set $\mathscr{Q}$ contains all *finite* sets of options. A choice function $C\colon \mathscr{Q} \to \mathscr{Q}$ is a function that chooses by mapping sets to their (not strict) subsets. With some values of the choice function $C(A_1), C(A_2), \ldots$ as information, we want to choose from a given set A in a way that is coherent with the previous choices.

Totality

A coherent choice function C is called total if $0 \notin C(\{0,u,-u\})$ for any $u \in \mathscr{V} \setminus \{0\}$ or equivalently if $\{u,-u\} \in K$ for it's corresponding SODOS K.

Mixingness

A coherent SODOS K is called mixing if for any $A \in K$, it follows from $B \subseteq A \subset \mathrm{posi}(B)$ that $B \in K$.

E-admissibility

If a choice function C chooses under E-admissibility, then there exists a set of precise linear previsions $\mathscr{P}$ such that

$$C(A) = \{u \in A\colon (\exists P \in \mathscr{P})(\forall v \in A) P(u) \geq P(v)\}.$$

Coherence of choice functions

We call a choice function C coherent if and only if we have for all $A, B \in \mathscr{Q}$:

C_1. $A \neq \emptyset \Rightarrow C(A) \neq \emptyset$;

C_2. $C(\{0,v\}) = \{v\}$ for all $v \in \mathscr{V}_{>0}$;

C_3. a. If $A \subseteq B$, then $C(B) \cap A \subseteq C(A)$;
b. If $C(A) \subseteq B \subseteq A$, then $C(A) = C(B)$;

C_4. a. If $0 \in C(A \cup \{u\})$, then $0 \in C(A \cup \{\lambda u\})$;
b. $C(A+B) \subseteq C(A) + C(B)$.

Minkowski addition: $A + B := \{a + b\colon a \in A, b \in B\}$ for all $A, B \in \mathscr{Q}$.

E-admissibility$\not\Rightarrow$Totality

With E-admissibility, we can have that $\mathscr{P} = \{P\}$ and $P(u) = P(-u) = P(0)$ for some $u \in \mathscr{V} \setminus \{0\}$, such that $C(\{0,u,-u\}) = \{0,u,-u\}$.

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Archimedeanity

For any $A \in \mathscr{Q}$, if $A \in \mathrm{Ex}\{B - \varepsilon(B)\colon B \in K\}$ for all $\varepsilon \in \mathbb{R}^{K}_{>0}$, then also $A \in K$.

Sets of desirable option sets

Every coherent choice function C has a corresponding coherent SODOS K and vice versa. They are linked by the equivalence $u \notin C(A) \Leftrightarrow A - u \in K$ for any $u \in A$ and $A \in \mathscr{Q}$. A SODOS $K \subseteq \mathscr{Q}$ is coherent if it satisfies the following axioms for all $A, B \in \mathscr{Q}$

K_1. $\emptyset \notin K$;

K_2. $\{v\} \in K$ for all $v \in \mathscr{V}_{>0}$;

K_3. a. If $A \in K$ and $A \subseteq B$, then $B \in K$;
b. If $A \in K$, then $A \setminus \{0\} \in K$;

K_4. a. If $A \in K$ and $\lambda \in \mathbb{R}_{>0}$, then $(A \setminus \{u\}) \cup \{\lambda u\} \in K$;
b. If $A, B \in K$, $u \in A$ and $v \in B$, then $(A \setminus \{u\}) \cup (B \setminus \{v\}) \cup \{u + v\} \in K$.

Natural vs. total extension

The idea is to take the information—the assessment—we have and transform it into an assessment $\mathcal{A} = \{B_1, \ldots, B_n\} \subseteq \mathscr{Q}$ in terms of SODOS's. Then we can take the natural extension Ex and the total extension Ex_{tot} of $\mathscr{A}$ and go back using the equivalence.

Without totality: $S \in \mathrm{Ex}(\mathcal{A}) \Leftrightarrow S \cap \mathscr{V}_{>0} \neq \emptyset$ or for every $(u_1, \ldots, u_n) \in \times_{k=1}^{n} B_k$ there is some $s \in S \cup \{0\}$ and $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$, not all zero, for which $\lambda_1 u_1 + \ldots + \lambda_n u_n \leq s$. Worstcase, we will have $|S|$ linear programs per $(u_1, \ldots, u_n)$.

With totality: $S \setminus \{0\} = \{v_1, \ldots, v_m\}$ $S \in \mathrm{Ex}_{tot}(\mathcal{A}) \Leftrightarrow$ for every $(u_1, \ldots, u_n) \in \times_{k=1}^{n} B_k$ there is some $\lambda_1 \geq 0, \ldots, \lambda_{n+m} \geq 0$, not all zero, for which $\lambda_1 u_1 + \ldots + \lambda_n u_n \leq \lambda_{n+1} v_1 + \ldots + \lambda_{n+m} v_m$. Only one linear program per $(u_1, \ldots, u_n)$.