Algebras of Partial Functions

Brett McLean

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Department of Computer Science

University College London

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I, Brett McLean, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

This thesis collects together four sets of results, produced by investigating modifications, in four distinct directions, of the following. Some set-theoretic operations on partial functions are chosen—composition and intersection are examples—and the class of algebras isomorphic to a collection of partial functions, equipped with those operations, is studied. Typical questions asked are whether the class is axiomatisable, or indeed finitely axiomatisable, in any fragment of first-order logic, what computational complexity classes its equational/quasiequational/first-order theories lie in, and whether it is decidable if a finite algebra is in the class.

The first modification to the basic picture asks that the isomorphisms turn any existing suprema into unions and/or infima into intersections, and examines the class so obtained. For composition, intersection, and antidomain together, we show that the suprema and infima conditions are equivalent. We show the resulting class is axiomatisable by a universal-existential-universal sentence, but not axiomatisable by any existential-universal-existential theory.

The second contribution concerns what happens when we demand partial functions on some finite base set. The finite representation property is essentially the assertion that this restriction that the base set be finite does not restrict the algebras themselves. For composition, intersection, domain, and range, plus many supersignatures, we prove the finite representation property. It follows that it is decidable whether a finite algebra is a member of the relevant class.

The third set of results generalises from unary to 'multiplace' functions. For the signatures investigated, finite equational or quasiequational axiomatisations are obtained; similarly when the functions are constrained to be injective. The finite representation property follows. The equational theories are shown to be coNP-complete.

In the last section we consider operations that may only be partial. For most signatures the relevant class is found to be recursively, but not finitely, axiomatisable. For others, finite axiomatisations are provided.

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Acknowledgement

I wish to put on record what an excellent supervisor Robin Hirsch is. His style is to provide as much support as is needed—he is always generous with his time—whilst simultaneously granting complete intellectual freedom. I believe this unpressured atmosphere has been central to making my PhD journey an unexpectedly smooth ride. Though Robin made it clear the selection of topic was entirely my own, his recommendation proved remarkably well chosen, since there are many interesting questions in this area amenable to attack by a novice researcher and they sit well together as a cohesive whole. Robin's week-to-week direction, ideas and suggestions have also been tremendously effective. Thank you Robin.

Chapter 1

Introduction

Across mathematics as a whole, functions have a much more pervasive presence than relations, their more general cousins. However, with regard to the metamathematics of reasoning about these entities, the situation has, historically, been reversed.

Investigations into the laws obeyed by binary relations began in the latter half of the 19th century, starting with De Morgan [82], followed by Peirce [84], Schröder [94], and Russell [88]. For a brief account of this early period, see [85]. Following four decades of almost total neglect, the subject was revived by Tarski's 1941 article *On the calculus of relations* [102], and since then, binary and higher-order relations have been continuously active topics of research.

Initially, the purpose of this work was algebraic logic in the strict sense. That is to say, the relations were providing the semantics for *logical formulas*.[1] This is perhaps the explanation as to why, until recently, the corresponding theory of functions was relatively less developed, since the semantics of formulas is not a role so naturally suited to functions. However, increasingly in the history of relations, computer science has become a source of motivation, with binary relations providing the semantics for, in particular, *(nondeterministic) computer programs*. In this view, the relation relates states of the machine before the program is executed to possible states after it is executed.

Despite the ubiquity of functions in mathematics, prior to the turn of this century the only sustained period of activity on reasoning with (possibly partial) functions was the 1960s, when the semigroup theorist Boris Schein and associates were active in this area.[2,3] In the last fifteen years however, interest has rekindled, and a regular stream of papers has been appearing, with computer science considerations a prime motivation.

The contemporary framework for studying one of these types of entities—either relations or some specialisation thereof—is to first select some operations of interest acting on the entities. For example composition and union are both binary operations we can perform on binary relations; the precise set we choose we call the *signature*. Then any collection of our entities closed under the chosen operations forms an algebraic structure. The object of study is then the isomorphic closure of the class of all these 'concrete' algebras, and we call this class the *representation class*.

[1] A particular interest of Tarski's was for these formulas to be those of the language of set theory [105].

[2] Schein's survey article [91] records the results obtained during this time.

[3] Later, there was a line of work in the category-theoretic tradition [5, 87, 27], but that is outside the interests of this thesis.

Aspirations for reasoning about programs by means of binary relations extend beyond reasoning about program behaviour in the abstract, to the possibility of practical real-world applications to automated program verification [7, 50, 28, 6]. However, a recurring theme in work on relations is the discovery of their poor logical and computational behaviour. To give some explicit examples of what is meant by this, often the representation class is found not to have a simple (for example finite) axiomatisation in first-order logic [80, 46], its validities in fragments of first-order logic are found to have high computational complexity [35] and, for finite structures, membership of the representation class is found to be undecidable [40].

The uninitiated may be surprised to learn that the observed logical and computational unruliness seems to be an intrinsic feature of relations, rather than stemming from any provision for expressing 'unbounded iteration' and the well-known associated complications this presents with regard to computability. Indeed the predisposition towards negative results exists even when only rather simple operations are used [3].

An alternative approach beckons. Though the nondeterministic paradigm certainly has its uses, in practice many programs are actually deterministic. This suggests that algebras of partial functions might provide a useful framework for reasoning about programs, avoiding the use of troublesome relations.

Encouragingly, investigations into classes of algebras of partial functions have tended to indicate they are much better behaved than classes of algebras of binary relations. For many signatures considered, the representation class has been shown to be finitely axiomatisable by quasiequations or even by equations [23, 55, 57], whereas, as we mentioned, for representation by binary relations, usually no finite axiomatisation is possible. Finite axiomatisability immediately gives decidability of membership of the representation class for finite structures, again contrasting with results about relations. The complexity of deciding the validity of equations is also comparatively low, compared to algebras of relations [44].

The work presented in this thesis consists of four further investigations into partial functions. The first considers *complete representations*, which are representations (that is, isomorphisms to concrete algebras) preserving existing infima or suprema. The second concerns satisfaction of the so-called *finite representation property*—the property that a finite algebra, if representable, must be representable using partial functions on a finite set. The third component is an investigation of *multiplace partial functions*—functions taking multiple arguments. The last is about collections of partial functions closed under certain *partial operations*.

The purpose of this work is general rather than specific. It is to advance the understanding of partial functions and reasoning about partial functions, not to accomplish any objective tailored to a precise reasoning task with a particular application. Though providing semantics for programs is now an established motivation, other applications for reasoning with partial functions may emerge. Indeed our final contribution gives one possibility. There, the motivation comes from separation logic, where partial functions model part of the *memory state* of a computer.

At a technical level, this thesis presents many original results.[4] Axiomatisations are found, non-axiomatisability results are proven, the finite representation property is proven to hold (or to fail), and the complexities of equational theories are identified. Overall, these results fall very much on the side of support of the existing evidence that functions are better behaved than relations. Hence at a high level this thesis contributes not just to an understanding of partial functions but also to a cementing of our understanding of the difference between functions and relations, from the point of view of their logical and computational properties.

Structure of document. We now provide a chapter-by-chapter summary of the remainder of this thesis.

Chapter 2 comprises necessary mathematical background—including definitions, notation, and fundamental theorems—in logic, algebra, and computability theory. None of the results in this chapter are original.

In Chapter 3, in order to provide more detailed context for the work in this thesis, we summarise the existing literature on binary and higher-order relations and on partial functions. Here the reader will be able to see the contrast between the metamathematical properties of relations and of partial functions. Again, none of the results in this chapter are original.

Chapters 4–7 contain the original contributions. Each of these chapters is self-contained and so each can be read independently of the others. As a consequence, there is a small amount of duplication of the most essential definitions.

Chapter 4 is an investigation into *complete representation*. We have been able to show that, for a certain signature, the complete representation class is finitely axiomatisable and to determine the precise amount of quantifier alternation needed in the axioms. This contrasts with the case for binary relations, where complete representation classes have been shown not to be axiomatisable by any first-order theory. This is a novel dimension to the difference between relations and functions. Finite axiomatisability occurring for functions in a setting where nonfinite axiomatisability is typical for relations has been observed previously, the setting being ordinary representability. But finite axiomatisability for functions where *nonelementarity* is typical for relations is new. This chapter is entirely the author's own work. All results, unless clearly indicated, are original. The contents of this chapter match, with only extremely minor differences, the published article [73]. In particular, all original results and their order of presentation remain the same, with the exception of Corollary 4.6.5, which is new.

Chapter 5 is about the *finite representation property*. In particular it solves a problem posed in [44], to determine if the finite representation property holds for the signature of composition, intersection, and two unary operations called *domain* and *range*. We settle this in the affirmative, and the result extends to various supersignatures of this signature. This fills in the most significant outstanding cases needed to conclude that for any combination of a quite large set of operations, the finite representation property holds for partial functions. Again, this is in opposition to the position for relations, where the finite representation property can be falsified with very limited and natural signatures. This chapter is joint work with Szabolcs Mikulás. The contributions of each author are summarised at the end of the chapter

[4] Though all already published prior to the assembling of this thesis.

introduction. All results, unless clearly indicated, are original. The contents of this chapter match, with only very minor differences, the published article [75]. In particular, all original results and their order of presentation remain the same. Example 5.2.4 is new.

Chapter 6 contains work on *multiplace partial functions*. Consideration of multiplace functions is not new, but in this chapter we generalise to the multiplace setting methods and results for signatures in which this has not previously been done—namely, signatures containing the unary *antidomain* operation. By doing this, we obtain either finite equational or finite quasiequational axiomatisations for each of the representation classes we consider. We also generalise to multiplace functions the result that the equational theory is coNP-complete, for a large group of signatures. Both these sets of results would be atypical if we were working with relations. This chapter is entirely the author's own work. All results, unless clearly indicated, are original. The contents of this chapter match, with only very minor differences, the published article [72]. In particular, all original results and their order of presentation remain the same.

In Chapter 7 we work with *partial operations* on partial functions, in particular the partial operation of domain-disjoint union, which we define. As operations may be partial, we are examining classes of partial algebras. This chapter differs from those preceding it in its underlying motive. In this case the impetus comes from separation logic, where partial functions model not programs dynamically altering the state but the (static) state of the memory itself. Interestingly, we see an attendant difference in outcomes. In the previous chapters, results are overwhelmingly positive, but this chapter provides a modest counterpoint. Although we obtain some finite axiomatisations, our primary results are refutations of finite axiomatisability. This chapter is joint work with Robin Hirsch. All results, unless clearly indicated, are original. For the most part, the contents of this chapter match the published article [45]. In particular, all original results, with the exception of Corollary 7.7.3, which is new, and Theorem 7.7.4, which has been corrected in this version, remain the same. The argumentation of Section 7.3 has been significantly simplified, thanks to Ian Hodkinson.

In the last chapter, Chapter 8, we discuss briefly where the research program on reasoning with partial functions currently stands and suggest some future directions.

Chapter 2

Mathematical background

In this chapter we give the basic definitions and results we need from logic, algebra, and computability theory. The general schema of algebraic logic is also illustrated by the example of Boolean algebras and the Birkhoff–Stone representation theorem.

Typical undergraduate-level mathematical knowledge is assumed. The set $\mathbb{N}$ of natural numbers starts from zero. The power set of a set X is denoted by $\wp(X)$.

2.1 First-order logic

Here we give a quick summary of the syntax and semantics of first-order logic. We do not need any proof theory. Definitions of the concepts in this section can be found in any sufficiently comprehensive introductory book on logic. Definitions vary slightly from source to source; a book whose treatment is fairly similar to that given here is [60].[1]

2.1.1 Syntax

Definition 2.1.1. A **signature** is a triple (F, R, α), where F and R are disjoint sets, whose elements we call **function symbols** and **relation** or **predicate symbols** respectively, and α is a function assigning a natural number to each function symbol and to each relation symbol. If σ is a symbol with $\alpha(\sigma) = n$ then we say σ has **arity** n or that σ is n**-ary**. Usually we just write a set of symbols and call it a 'signature', and then either the function/relation division and the arities should be specified, or this information should be implicit. A **subsignature** (F', R', α') of (F, R, α) has $F' \subseteq F$ and $R' \subseteq R$, and the arities given by α' agree with those given by α. If Σ' is a subsignature of Σ then Σ is a **supersignature** of Σ'.

Nullary function symbols are called **constants**. We call a signature a **functional signature**, or an **algebraic signature**, if all its symbols are function symbols and a **relational signature** if all its symbols are relation symbols. All the signatures we will be concerned with will be **finite**, that is, having only a finite number of symbols.

The following definitions require a countably infinite set V, whose elements we call **variables**. Although formally our choice of V affects the sets we define, in practice, so long as V is disjoint from the other sets of symbols we use, the choice is immaterial. Hence we leave the identity of V unspecified.

[1] Particularly with regards to semantics on empty structures.

Unlike elsewhere in mathematics, whenever we write two elements of V differently within a single statement, we are excluding the possibility that those variables are equal. Thus if we refer to variables x and y in a statement, then x and y are different elements of V.

Definition 2.1.2. Let Σ be a signature. The set of **Σ-terms** is the smallest set of strings satisfying the following closure properties.

- For any variable x, the single-character string x is in the set.
- If f is a function symbol in Σ of arity n and $t_1, \dots, t_n$ are in the set, then $ft_1 \cdots t_n$ is in the set.

Definition 2.1.3. Let Σ be a signature. The set of **atomic Σ-formulas** is the smallest set of strings satisfying the following closure properties.

- If π is a predicate symbol in Σ of arity n and $t_1, \dots, t_n$ are Σ-terms, then $\pi t_1 \cdots t_n$ is in the set.
- If s and t are Σ-terms, then $(s = t)$ is in the set.[2]
- The single-character string $\bot$ is in the set.

Definition 2.1.4. Let Σ be a signature. The set of **Σ-formulas** is the smallest set of strings satisfying the following closure properties.

- All atomic Σ-formulas are in the set.
- If φ and ψ are in the set, then $(\varphi \to \psi)$ is in the set.
- If φ is in the set and x is a variable, then $\forall x\varphi$ is in the set.

The following abbreviations may be used: $\neg\varphi$ for $(\varphi \to \bot)$, $(\varphi \vee \psi)$ for $(\neg\varphi \to \psi)$, $(\varphi \wedge \psi)$ for $\neg(\neg\varphi \vee \neg\psi)$, $\exists x\varphi$ for $\neg\forall x\neg\varphi$, and $Qx_1 \cdots x_n$ for $Qx_1 \cdots Qx_n$, where Q is either $\forall$ or $\exists$. Brackets may be omitted when this presents no risk of ambiguity.

Definition 2.1.5. An **equation** is a formula of the form $s = t$. A **quasiequation** is a formula of the form $e_1 \wedge \cdots \wedge e_n \to e$, where e and $e_1, \dots, e_n$ are equations. Every equation is considered to be a quasiequation with empty antecedent.

Definitions concerning terms or formulas are always made inductively, by accounting for each possible way that instances are formed from smaller instances. Note that in order to be sure these definitions are well-defined, we should prove that terms and formulas exhibit **unique readability**, that is, have unique constructions from atomic instances. Such a proof is not hard but can be a little tedious, so we shall not give one.

Definition 2.1.6. A **universal formula** is one built from quantifier-free formulas using only $\vee$, $\wedge$, and $\forall$. An **existential formula** is one built from quantifier-free formulas using only $\vee$, $\wedge$, and $\exists$. A **universal-existential formula** is one built from *existential formulas* using only $\vee$, $\wedge$, and $\forall$. This terminology

[2] As with all symbols we introduce, '(', ')' and '=' are assumed to be distinct from the symbols used for function/relation symbols and for variables.

continues in the obvious way. We can do without notation for these fragments, for we encounter only small degrees of such **quantifier alternation**.

A **positive formula** is one built using only atomic formulas and $\vee$, $\wedge$, $\forall$, and $\exists$.

A formula is in **prenex normal form** if it is of the form $Q_1 x_1 \cdots Q_n x_n \varphi$, where each Q_i is either $\forall$ or $\exists$, and φ is quantifier free.

Definition 2.1.7. Whether or not an instance of a variable in a formula φ is **bound** is defined inductively as follows.

- If φ is atomic, then none of the instances of variables in φ are bound.
- If φ is $(\psi \to \chi)$, then instances of variables are bound in φ precisely when they are bound in the subformula ψ or χ that they appear in.
- If φ is $\forall x \psi$, then all instances of x are bound in φ and instances of variables other than x are bound in φ precisely when they are bound in ψ.

If an instance of a variable in a formula is not bound then it is **free**. A variable x is a **free variable** of a formula φ if there is a free instance of x in φ.

Definition 2.1.8. Let Σ be a signature. A Σ**-sentence** is a formula with no free variables. A Σ**-theory** is a set of Σ-sentences.

2.1.2 Semantics

Definition 2.1.9. Let Σ be a signature. A Σ**-structure** $\mathfrak{S}$ consists of a set S, called the **domain** of $\mathfrak{S}$, together with, for each function symbol f of arity n, a function $f_{\mathfrak{S}} \colon S^n \to S$ and, for each relation symbol r of arity n, a subset $r_{\mathfrak{S}}$ of S^n. Note that S^0 is always a set containing a single element. For a function or relation symbol σ, we call $\sigma_{\mathfrak{S}}$ the **interpretation** of σ. We will often write σ in place of $\sigma_{\mathfrak{S}}$ when safe to do so. Two structures are **similar** if they have the same signature.[3]

Note that we *do not* follow the common convention that domains should be nonempty, based on a conviction that the facility to make formal first-order statements about empty structures should be available if desired, and that the semantics of these statements should match those of their metalanguage equivalents. However, empty structures are usually more trouble than they are worth, and we make little use of them—the only time can be found in Theorem 3.1.6. Indeed, we make the following definition.

Definition 2.1.10. Let Σ be a *functional* signature. A Σ**-algebra** is a Σ-structure with nonempty domain.

Definition 2.1.11. Let Σ' be a subsignature of Σ and let $\mathfrak{S}$ be a Σ-structure. The **reduct** of $\mathfrak{S}$ to Σ' is the Σ'-structure interpreting each symbol σ of Σ' as $\sigma_{\mathfrak{S}}$. If $\mathfrak{S}'$ is a reduct of $\mathfrak{S}$ then $\mathfrak{S}$ is an **expansion** of $\mathfrak{S}'$.

[3] In Chapter 7 we take a slightly different view. There, structures do not have an inherent signature, just an ordinal-indexed sequence of relations/operations/partial operations. Correspondingly, signatures are also sequences, to facilitate binding them to structures.

Definition 2.1.12. Let $\mathfrak{S}$ be a Σ-structure. An **assignment** (to $\mathfrak{S}$) is a partial function from the set of variables to the domain of $\mathfrak{S}$.

Each Σ-structure with assignment is a context in which Σ-terms and Σ-formulas can be interpreted. Whenever a structure (or class of similar structures) is present, Σ is fixed, so we can refer unambiguously to terms, formulas, theories, and so on. Otherwise, we must be careful that the signature is clear, for formulas such as $\exists x(x = x)$ are formulas both of signatures without constants and signatures with.

Let $\mathfrak{S}$ be a structure and μ be an assignment to $\mathfrak{S}$. Then there is a unique extension $\bar{\mu}$ of μ to terms all of whose free variables are in the domain of μ, such that $\bar{\mu}$ satisfies the following inductive definition:

- $\bar{\mu}$ and μ agree on variables,
- $\bar{\mu}(ft_1 \cdots t_n) = f_{\mathfrak{S}}(\bar{\mu}(t_1), \ldots, \bar{\mu}(t_n))$ for any function symbol f in the signature and appropriate number $t_1, \ldots, t_n$ of terms.

Definition 2.1.13. Let μ be an assignment to a structure $\mathfrak{S}$ and let x be a variable. Let V' be the domain of μ. An x**-overwrite** ν of μ is an assignment to $\mathfrak{S}$ such that

- ν is defined and in agreement with μ on $V' \setminus \{x\}$,
- ν is defined on x.

Definition 2.1.14. Let $\mathfrak{S}$ be a structure and μ be an assignment to $\mathfrak{S}$. Then there is a unique **valuation** function v_μ with range $\{\text{true}, \text{false}\}$, defined on formulas all of whose free variables are in the domain of μ, such that v_μ satisfies the following inductive definition:

- $v_\mu(\bot) = \text{false}$,
- $v_\mu(t = s) = \text{true} \iff \bar{\mu}(t) = \bar{\mu}(s)$,
- $v_\mu(rt_1 \cdots t_n) = \text{true} \iff (\bar{\mu}(t_1), \ldots, \bar{\mu}(t_n)) \in r_{\mathfrak{S}}$,
- $v_\mu(\varphi \to \psi) = \text{true} \iff v_\mu(\varphi) = \text{false}$ or $v_\mu(\psi) = \text{true}$,
- $v_\mu(\forall x\varphi) = \text{true} \iff$ for all x-overwrites ν of μ we have $v_\nu(\varphi) = \text{true}$.

If $v_\mu(\varphi) = \text{true}$ then we write $\mathfrak{S}, \mu \models \varphi$ and say that $\mathfrak{S}$ **satisfies** φ under assignment μ.

One can show that satisfaction of a given formula φ in a given structure $\mathfrak{S}$ depends only on the values the assignment takes on the free variables of φ. In particular, if φ is a sentence, then for any assignment μ we have that $v_\mu(\varphi)$ is defined and that $\mathfrak{S}, \mu \models \varphi \iff \mathfrak{S}, \emptyset \models \varphi$, viewing $\emptyset$ as the nowhere-defined assignment. If φ is a sentence then we write $\mathfrak{S} \models \varphi$ or $\mathfrak{S} \not\models \varphi$ as appropriate, depending on the unique value that $v_\mu(\varphi)$ takes.

If φ is a quasiequation then we write $\mathfrak{S} \models \varphi$ if $\mathfrak{S} \models \forall x_1 \cdots x_n \varphi$ and $\mathfrak{S} \not\models \varphi$ otherwise, where $\{x_1, \ldots, x_n\}$ is any superset of the free variables of φ. This definition is well-defined for quasiequations, though not for formulas in general.[4] The notation $\mathfrak{S}, (a_1, \ldots, a_n) \models \varphi$ signifies that $\mathfrak{S}, \mu \models \varphi$ for any

[4] If the signature contains no constants, then $\mathfrak{S}$ could be empty, and the truth of $\forall x_1 \cdots x_n \bot$ on an empty structure depends on whether or not $\{x_1, \ldots, x_n\}$ is empty.

assignment μ that assigns the value a_i to variable x_i for each $1 \leq i \leq n$, where $\{x_1, \ldots, x_n\}$ is some superset of the free variables of φ.

2.2 Universal algebra

In this section we present the definitions we need from universal algebra. For a general reference on universal algebra, see [33].

Definition 2.2.1. A **homomorphism** is a map $\theta\colon \mathfrak{S} \to \mathfrak{T}$, between similar structures, satisfying

(i) for every function symbol f in the signature,

$$\theta(f_{\mathfrak{S}}(a_1, \ldots, a_n)) = f_{\mathfrak{T}}(\theta(a_1), \ldots, \theta(a_n)),$$

where n is the arity of f.

(ii) for every relation symbol r in the signature,

$$(a_1, \ldots, a_n) \in r_{\mathfrak{S}} \implies (\theta(a_1), \ldots, \theta(a_n)) \in r_{\mathfrak{T}},$$

where n is the arity of r.

An **embedding** is an injective homomorphism where (ii) is strengthened to a biconditional. An **isomorphism** is a surjective embedding. If there exists a surjective homomorphism $\theta\colon \mathfrak{S} \to \mathfrak{T}$, then $\mathfrak{T}$ is a **homomorphic image** of $\mathfrak{S}$.

If the domain of $\mathfrak{S}$ is a subset of the domain of $\mathfrak{T}$ and the inclusion map is an embedding, then $\mathfrak{S}$ is a **substructure** of $\mathfrak{T}$, and $\mathfrak{T}$ is an **extension** of $\mathfrak{S}$, written $\mathfrak{S} \subseteq \mathfrak{T}$. In the special case that $\mathfrak{S}$ is an algebra it is then a **subalgebra** of $\mathfrak{T}$.

Definition 2.2.2. Let $\mathfrak{A}$ be an algebra. A **congruence** on $\mathfrak{A}$ is an equivalence relation $\sim$ on (the domain of) $\mathfrak{A}$ such that for every function symbol f in the signature,

$$a_1 \sim b_1, \ldots, a_n \sim b_n \implies f_{\mathfrak{A}}(a_1, \ldots, a_n) \sim f_{\mathfrak{A}}(b_1, \ldots, b_n),$$

where n is the arity of f.

Definition 2.2.3. Let $\mathfrak{A}$ be an algebra and $\sim$ be a congruence on $\mathfrak{A}$. The **quotient** structure $\mathfrak{A}/{\sim}$ has as its domain the equivalence classes of $\sim$ and has interpretations defined in the following way: for every function symbol f in the signature,

$$f_{\mathfrak{A}/\sim}([a_1], \ldots, [a_n]) = [f_{\mathfrak{A}}(a_1, \ldots, a_n)],$$

where n is the arity of f. The condition that $\sim$ is a congruence ensures these definitions are well-defined.

Definition 2.2.4. Let Σ be a signature and let $\mathfrak{S}_i$ be a Σ-structure, for each $i \in I$. The **(direct) product** $\prod_i \mathfrak{S}_i$ of the $\mathfrak{S}_i$'s has $\prod_i \mathrm{dom}(\mathfrak{S}_i)$ as its domain and interpretations defined in the following way:

- for every function symbol f in the signature,

$$f_{\prod_i \mathfrak{S}_i}(a_1, \ldots, a_n)(i) = f_{\mathfrak{S}_i}(a_1(i), \ldots, a_n(i)), \text{ for all } i \in I,$$

  where n is the arity of f,

- for every relation symbol r in the signature,

$$(a_1, \ldots, a_n) \in r_{\prod_i \mathfrak{S}_i} \iff (a_1(i), \ldots, a_n(i)) \in r_{\mathfrak{S}_i} \text{ for all } i \in I,$$

  where n is the arity of r.

Definition 2.2.5. An **ultrafilter** on a set I is a subset γ of $\wp(I)$ with the following properties:

proper $\emptyset \notin \gamma$,

upward closed $J \in \gamma$ and $J \subseteq K \implies K \in \gamma$,

downward directed $J, K \in \gamma \implies J \cap K \in \gamma$,

prime $J \subseteq I \implies J \in \gamma$ or $I \setminus J \in \gamma$.

An ultrafilter is **principal** if, for some $i \in I$, it is precisely the set of subsets that contain i.

Definition 2.2.6. Let Σ be a signature, $\mathfrak{S}_i$ be a *nonempty* Σ-structure, for each $i \in I$, and γ be an ultrafilter on I. Define the binary relation $\sim_\gamma$ on the domain of $\prod_i \mathfrak{S}_i$ by $a \sim_\gamma b \iff \{i \in I \mid a(i) = b(i)\} \in \gamma$. The defining properties of ultrafilters ensure $\sim_\gamma$ is an equivalence relation. We write $\prod_\gamma \mathfrak{S}_i$ for the structure whose domain is the set of $\sim_\gamma$-equivalence classes and whose interpretations are given as follows:

- for every function symbol f in the signature,

$$f_{\prod_\gamma \mathfrak{S}_i}([a_1], \ldots, [a_n]) = [f_{\prod_i \mathfrak{S}_i}(a_1, \ldots, a_n)],$$

  where n is the arity of f,

- for every relation symbol r in the signature,

$$([a_1], \ldots, [a_n]) \in r_{\prod_\gamma \mathfrak{S}_i} \iff \{i \in I \mid (a_1(i), \ldots, a_n(i)) \in r_{\mathfrak{S}_i}\} \in \gamma,$$

  where n is the arity of r.

The defining properties of ultrafilters ensure these definitions are well-defined. For any ultrafilter γ, we call $\prod_\gamma \mathfrak{S}_i$ an **ultraproduct** of the $\mathfrak{S}_i$. An ultraproduct of identical structures is called an **ultrapower** and $\mathfrak{S}$ is an **ultraroot** of $\mathfrak{T}$ if $\mathfrak{T}$ is an ultrapower of $\mathfrak{S}$.

Definition 2.2.7. A structure $\mathfrak{S}$ is a **directed union** of a *nonempty* set S of similar structures if

- each structure in S is a substructure of $\mathfrak{S}$,
- for every $\mathfrak{S}_1, \mathfrak{S}_2 \in S$ there exists $\mathfrak{T}$ such that both $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are substructures of $\mathfrak{T}$,
- the domain of $\mathfrak{S}$ is the union of the domains of the structures in S.

2.3 Model theory

We use a few basic notions and results from model theory, many of which are so fundamental as to predate the emergence of model theory as a distinct field of study. The standard general references for model theory are [47] and [18].

Definition 2.3.1. If all sentences of a theory T are satisfied for all members of a class $\mathbf{K}$ of similar structures, then we write $\mathbf{K} \models T$ and say that T is **valid** or **sound** for $\mathbf{K}$ and that $\mathbf{K}$ **models** T. We also use this notation and terminology with a single structure on the left-hand side and/or a single sentence on the right-hand side.

Theorem 2.3.2 (Compactness theorem for first-order logic)**.** *Let T be a theory. If every finite subset of T has a model, then T has a model.*

One method of proof of the compactness theorem is as an immediate consequence of Gödel's completeness theorem (the strong-completeness version). An alternative proof makes (direct) use of ultraproducts.

Definition 2.3.3. Let T and T' be Σ-theories. We write $T \models_\Sigma T'$ and say that T' is a Σ-**semantic consequence** of T if all Σ-structures that are models for T are models for T'.

Usually it is safe to omit the signature—it is only necessary to know whether empty structures are allowed. Again, we may write a sentence as shorthand for a singleton theory.

Definition 2.3.4. Two sentences φ and ψ are **logically equivalent** if $\varphi \models \psi$ and $\psi \models \varphi$.

It is common to define logical equivalence not only for sentences but for formulas in general. However, such definitions usually are not quite equivalence relations, when empty structures are permitted.[5] On the rare occasions we need an equivalence for formulas we can make do with the following definition, which agrees with Definition 2.3.4 when restricted to sentences.

Definition 2.3.5. Two formulas φ and ψ are **strongly logically equivalent** if

- either both have free variables or neither do,
- for any structure $\mathfrak{S}$ and assignment μ to the free variables of both formulas, $\mathfrak{S}, \mu \models \varphi \iff \mathfrak{S}, \mu \models \psi$.

Note that when empty structures are disallowed—in particular when we work with *algebras*—every formula is strongly equivalent to one in prenex normal form.

Definition 2.3.6. A theory T is **complete** for a class $\mathbf{K}$ if for all φ we have $\mathbf{K} \models \varphi \implies T \models \varphi$.[6]

[5] The reader who believes their favoured definition gives an equivalence relation should ponder how it partitions the set $\{x = x, \exists x(x = x), \forall x(x = x)\}$.

[6] This meaning of *complete* is common when studying first-order theories of classes. For example, it is used in this way in [39] and [79].

Definition 2.3.7. If a class $\mathbf{K}$ of similar structures is, for some theory T, precisely the class of models of T, then we say that T **characterises**, or is an **axiomatisation** of, $\mathbf{K}$.

Definition 2.3.8. An **elementary class** is a class of structures that can be axiomatised by a set of first-order formulas. A **basic elementary class** is a class of structures that can by axiomatised by a finite set of first-order sentences or equivalently by a single sentence.

It is worth noting the following well-known and easy-to-prove relationship between axiomatisations and sound and complete theories.

Theorem 2.3.9. *If T axiomatises $\mathbf{K}$, then T is sound and complete for $\mathbf{K}$. If T is sound and complete for $\mathbf{K}$ and $\mathbf{K}$ is elementary, then T axiomatises $\mathbf{K}$.*

Definition 2.3.10. If $\mathbf{K}$ is a class of similar structures the **first-order theory** of $\mathbf{K}$ is the set of all sentences that are true for all members of $\mathbf{K}$. Two similar structures are **elementarily equivalent** if their first-order theories are equal.

We are also interested in certain fragments of first-order logic.

Definition 2.3.11. A **variety** is a class of *algebras* that can be axiomatised by a set of equations, and it is a **finitely based variety** if it can be axiomatised by a finite set of equations. A **quasivariety** is a class of algebras that can be axiomatised by a set of quasiequations, and it is a **proper quasivariety** if it is not also a variety. A **universal class** is any class that can be axiomatised by a set of universal sentences.

Definition 2.3.12. If $\mathbf{K}$ is a class of similar structures, the **equational theory** of $\mathbf{K}$ is the set of all *equations* that are true for all members of $\mathbf{K}$, the **quasiequational theory** the set of all quasiequations, and the **universal theory** the set of all universal sentences.

For algebraic signatures, there is a smallest variety containing $\mathbf{K}$: the class of algebras validating the equational theory of $\mathbf{K}$. This is the **variety generated by** $\mathbf{K}$. The **quasivariety generated by** $\mathbf{K}$ is defined similarly.

It is clear that if a universal sentence is valid on a structure $\mathfrak{S}$, then it is valid on any substructure of $\mathfrak{S}$—the proof is by structural induction on universal formulas. Similarly, if a positive formula is valid on $\mathfrak{S}$, it is valid on any homomorphic image of $\mathfrak{S}$, if a quasiequation is valid on every structure in a sequence of similar structures, it is valid on the direct product of that sequence, and if a universal-existential sentence is valid on a set of similar structures, it is valid on their directed union, if it exists. Hence varieties are closed under substructures, direct products and homomorphic images. The converse is a celebrated theorem of Birkhoff.

Theorem 2.3.13 (Birkhoff's HSP theorem [10]). *A class of similar algebras is a variety if and only if it is closed under the operations of taking products, subalgebras, and homomorphic images.*

There is a similar algebraic characterisation of universality, but it requires as a hypothesis that the class be elementary. It is an immediate corollary of the following result.

Theorem 2.3.14 (Łoś–Tarski preservation theorem)**.** *We say a set Φ of formulas is* ***preserved under substructures*** *if whenever $\mathfrak{S} \subseteq \mathfrak{T}$ and μ is an assignment of elements of $\mathfrak{S}$ to the free variables of Φ, we have*

$$\mathfrak{T}, \mu \models \Phi \implies \mathfrak{S}, \mu \models \Phi.$$

A set Φ of formulas is strongly equivalent to a set of universal formulas if and only if Φ is preserved under substructures.

Corollary 2.3.15. *An elementary class of similar algebras is universal if and only if it is closed under subalgebras.*

There are various possible characterisations of quasivarieties. Here is a useful mixed syntactic/algebraic characterisation. See for example [19, Chapter VI, Corollary 4.4].

Theorem 2.3.16. *A class of similar algebras is a quasivariety if and only if it is universally axiomatisable and closed under direct products.*

We occasionally need the countable case of the downward Löwenheim–Skolem theorem, for which we first need to define elementary substructures.

Definition 2.3.17. An **elementary substructure** of $\mathfrak{T}$ is a substructure $\mathfrak{S}$ of $\mathfrak{T}$ such that for any formula φ and assignment μ of elements in $\mathfrak{S}$,

$$\mathfrak{S}, \mu \models \varphi \implies \mathfrak{T}, \mu \models \varphi.$$

An **elementary embedding** of a structure $\mathfrak{S}$ into a structure $\mathfrak{T}$ is a map $\theta\colon \mathfrak{S} \to \mathfrak{T}$ such that for any formula φ and assignment μ of elements in $\mathfrak{S}$,

$$\mathfrak{S}, \mu \models \varphi \implies \mathfrak{T}, \theta \circ \mu \models \varphi.$$

(This entails that θ is indeed an embedding.)

Theorem 2.3.18 (Downward Löwenheim–Skolem)**.** *Let Σ be a countable signature. Then every Σ-structure has a countable elementary substructure. In particular, if a Σ-theory has a model, then it has a countable model.*

Note that (assuming the axiom of choice, which we always do) ultraproducts will necessarily be nonempty. This is crucial to the proof of the following.

Theorem 2.3.19 (Łoś's theorem [68])**.** *Let $\mathfrak{U} := \prod_\gamma \mathfrak{S}_i$ be an ultraproduct and let φ be a formula. Then $\mathfrak{U}, ([a_1], \dots, [a_n]) \models \varphi$ if and only if $\{i \in I \mid \mathfrak{S}_i, (a_1(i), \dots, a_n(i)) \models \varphi\} \in \gamma$.*

It follows immediately from Łoś's theorem that any class closed under elementary equivalence must be closed under ultrapowers, and that any elementary class must be closed under ultraproducts. Another immediate consequence is the following.

Corollary 2.3.20. *Let $\mathfrak{U} := \prod_\gamma \mathfrak{S}$ be an ultrapower. Let $\theta\colon \mathfrak{S} \to \mathfrak{U}$ be the diagonal map that sends each $a \in \mathfrak{S}$ to equivalence class of the constant sequence with constant value a. Then θ is an elementary embedding, the **diagonal embedding** of $\mathfrak{S}$ into $\mathfrak{U}$.*

Lastly, we will find the notion of pseudoelementarity useful. There are various possible equivalent definitions of this, some purely in terms of the ordinary unsorted first-order logic presented in Section 2.1. However, we find a two-sorted definition more convenient to use, if not to state. It can be found, for example, as [41, Definition 9.1].[7] Briefly, in **two-sorted** first-order logic there are two fixed 'sorts', supplies of variables of each sort, and for each function/relation symbol, a specification of which sort must appear in each position of its argument, and for function symbols, which sort its values take. The syntax of well-formed formulas is restricted in the obvious way. Structures have (disjoint) domains for each sort and the interpretations of symbols must respect their specification. Evaluation of terms and formulas is in the obvious way.

Definition 2.3.21. Given an unsorted signature Σ, a class $\mathbf{K}$ of Σ-structures is **pseudoelementary** if there exist

- a two-sorted signature Σ', with sorts a and b, containing a-sorted copies of all symbols of Σ,
- a Σ'-theory T,

such that $\mathbf{K}$ is the class of all structures $\mathfrak{S}^a{\restriction_\Sigma}$ such that $\mathfrak{S} \models T$, where $\mathfrak{S}^a{\restriction_\Sigma}$ is the reduct to Σ of the a-sorted part of $\mathfrak{S}$.

Intuitively, each structure in a pseudoelementary class is what remains after 'forgetting' one of the sorts. In Chapter 7, Section 7.3 we use the following property of pseudoelementary classes, in order to prove axiomatisability results.

Theorem 2.3.22. *Pseudoelementary classes are closed under ultraproducts.*

A proof of this theorem can be found in [26], where the result is Corollary 4.4.

2.4 Computability theory

Many important questions arising in studies on reasoning with relations or functions involve the language and concepts of computability theory. So in this section we give the necessary definitions. There are many introductory books on computability and computational complexity, for example [97] and [83].

We use nondeterministic Turing machines as our primary model of computation and also introduce deterministic Turing machines.

Definition 2.4.1. A **nondeterministic Turing machine** is a quadruple (Σ, S, s_0, δ) where

- Σ is a nonempty finite set of symbols, the **alphabet**, not containing $\llcorner\!\lrcorner$ (a fixed **blank symbol**),
- S is a finite set of **states**,

[7] The different possible definitions of pseudoelementarity, and the equivalences between them, are discussed in [41, Section 9.4].

- $s_0 \in S$ and is called the **start state**,
- δ, the **transition relation**, is a binary relation between $S \times (\Sigma \cup \{\text{␣}\})$ and $S \times (\Sigma \cup \{\text{␣}\}) \times \{\leftarrow, \rightarrow\}$.

A **deterministic Turing machine** is a nondeterministic Turing machine whose transition relation is a partial function.

Given an alphabet Σ, the set of all finite strings of symbols from Σ is denoted Σ^*, and a **language** over Σ is a subset of Σ^*.

Turing machines operate as follows. An **input** string from Σ^*, of length n say, is placed in **cells** $0, \dots, n-1$ of a $\mathbb{Z}$-indexed sequence of cells, called the **tape**, with the rest of the tape cells filled with blanks. Computation takes place in a sequence of discrete steps. At each step, the configuration is specified by a state $s \in S$, a **head** position—a cell of the tape—and the contents of the tape. Initially, the state is the start state, s_0, and the head position is the zeroth cell. At each step, the possible configurations at the immediately successive step are determined by the transition relation. If the current state is s, the head position is i, the symbol in cell i is σ, and δ relates (s, σ) to (s', σ', d), then a possible next configuration is given by overwriting the contents of cell i with σ, moving the head position one cell in direction d, and changing the state to s'. If there are no possible next configurations, then the machine **halts** and gives as **output** the string in cells 0 (inclusive) up to the next blank cell (exclusive). A **computation path** is any sequence of configurations generated in this way, starting from the initial configuration and either continuing indefinitely or ending with a halt.

Definition 2.4.2. Let f be a partial function from Σ^* to Σ^*. A nondeterministic Turing machine T **computes** f if its alphabet is Σ and for all $x \in \Sigma^*$,

- if $f(x)$ is defined, then on input x all computation paths of T either halt with output $f(x)$ or do not halt, and at least one path halts,
- if $f(x)$ is undefined, then T does not halt on input x.

The partial function f is **computable** or **partial recursive** if there exists a nondeterministic Turing machine T that computes f.

If a partial function is computable, then it can be computed by a deterministic machine, as deterministic machines can simulate nondeterministic ones. In fact, it is well known that by the measure of which partial functions are computable, both types of Turing machine are equivalent to all other proposed models of what constitutes an 'effective procedure', including various types of register machine and the untyped lambda calculus.

In the following, 0 and 1 are two fixed strings, distinct from each other; their precise identity is immaterial.

Definition 2.4.3. A language $\mathcal{L}$ is **semidecided** by a Turing machine T if T computes the partial function $\mathbb{S}_{\mathcal{L}}$ defined by

$$\mathbb{S}_{\mathcal{L}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{L} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The language $\mathcal{L}$ is **recursively enumerable** if it is semidecided by some Turing machine.

A language $\mathcal{L}$ is **decided** by a Turing machine T if T computes the characteristic function of $\mathcal{L}$. The language $\mathcal{L}$ is **recursive** if its decided by some Turing machine. This is easily seen to be equivalent to both $\mathcal{L}$ and the **complement** $\Sigma^* \setminus \mathcal{L}$ of $\mathcal{L}$ being recursively enumerable.

We now define various **computational complexity classes**—a term for which we will not give a formal definition. Informally, a computational complexity class is a set of languages defined by their ability to be decided/semidecided within some limit on the 'computational resources', such as time and space, used.

Definition 2.4.4. Let f be a partial function from Σ^* to Σ^* and $g\colon \mathbb{N} \to \mathbb{N}$ be a (total) function. A nondeterministic Turing machine computes f in **time** $g(n)$ if it computes f and whenever $f(x)$ is defined, at least one of the halting paths is at most $g(n)$ computational steps long, where n is the length of the string x.

A language $\mathcal{L}$ is in $\mathsf{NTIME}(g(n))$ if there is a nondeterministic Turing machine T that semidecides $\mathcal{L}$ in time $\mathcal{O}(g(n))$. The language $\mathcal{L}$ is in $\mathsf{DTIME}(g(n))$ if this can be done with a deterministic machine. If g itself is computable in time $\mathcal{O}(g(n))$,[8] as is usually the case, then this definition of $\mathsf{DTIME}(g(n))$ is equivalent to asking that there be a deterministic machine that decides $\mathcal{L}$ in time $\mathcal{O}(g(n))$.

When defining space complexities, there should be no cost attributed to the space used for either the input or output. One way this can be achieved is by partitioning the tape into an input region that cannot be written to, an output region that cannot be read from, and a free-use 'working' region. Of course the output is then taken to be the string written on the output region when the machine halts. We do not give formal details, but it is such a model of computation that is assumed in the following definition.

Definition 2.4.5. Let f be a partial function from Σ^* to Σ^* and $g\colon \mathbb{N} \to \mathbb{N}$ be a (total) function. A nondeterministic Turing machine computes f in **space** $g(n)$ if it computes f, and whenever $f(x)$ is defined at least one of the halting paths only writes on the first $g(n)$ cells of the working region of the tape, where n is the length of x.

A language $\mathcal{L}$ is in $\mathsf{NSPACE}(g(n))$ if there is a nondeterministic Turing machine T that semidecides $\mathcal{L}$ in space $\mathcal{O}(g(n))$. The language $\mathcal{L}$ is in $\mathsf{DSPACE}(g(n))$ if this can be done with a deterministic machine. If g itself is computable in space $\mathcal{O}(g(n))$, this definition of $\mathsf{DSPACE}(g(n))$ is equivalent to asking that there be a deterministic machine that decides $\mathcal{L}$ in space $\mathcal{O}(g(n))$.

The following terms are used for the asymptotics of functions.

linear	$\mathcal{O}(n)$	**polynomial**	$\mathcal{O}(n^k)$	for some $k \in \mathbb{N}$
quadratic	$\mathcal{O}(n^2)$	**exponential**	$\mathcal{O}(2^{n^k})$	for some $k \in \mathbb{N}$
cubic	$\mathcal{O}(n^3)$	**double exponential**	$\mathcal{O}(2^{2^{n^k}})$	for some $k \in \mathbb{N}$
quartic	$\mathcal{O}(n^4)$			

We say a language is in

[8] Or, rather, some suitable encoding of g as a function $\Sigma^* \to \Sigma^*$ is computable in time $\mathcal{O}(g(n))$.

- P if it is semidecidable (equivalently, decidable) in polynomial time by a deterministic machine,
- NP if it is semidecidable in polynomial time by a nondeterministic machine,
- EXPTIME if it is semidecidable (equivalently, decidable) in exponential time by a deterministic machine,
- NEXPTIME if it is semidecidable in exponential time by a nondeterministic machine,
- PSPACE if it is semidecidable (equivalently, decidable) in polynomial space by a deterministic (equivalently, nondeterministic) machine.[9]

The sets of polynomial, exponential, and double-exponential functions each has the valuable properties of being closed under pointwise products of members.. We mentioned that common models of computation are equivalent with respect to computability, but importantly, more is true: each can guarantee it will compute no more than a polynomial factor slower than any other, measured by number of computational steps, and using no more than a polynomial factor more space. These facts together ensure that the classes P, NP, EXPTIME, NEXPTIME, and PSPACE are all independent of which model of computation is assumed. The fact that these classes are also closed under pointwise addition of members and under pre-composition by a polynomial function grants importance to the following concepts.

Definition 2.4.6. A **many-one reduction** of the language $\mathcal{L}_1$ over Σ_1 to the language $\mathcal{L}_2$ over Σ_2 is a total computable function $f\colon \Sigma_1^* \to \Sigma_2^*$ such that for all $x \in \Sigma_1^*$, we have $x \in \mathcal{L}_1 \iff f(x) \in \mathcal{L}_2$.[10] The function f is a **polynomial-time many-one reduction** of $\mathcal{L}_1$ to $\mathcal{L}_2$ if in addition f is computable in polynomial time by a deterministic machine.

Definition 2.4.7. Let C be one of the complexity classes NP, EXPTIME, NEXPTIME, or PSPACE (not P). A language $\mathcal{L}_2$ is C**-hard** if for any $\mathcal{L}_1 \in$ C there is a polynomial-time many-one reduction from $\mathcal{L}_1$ to $\mathcal{L}_2$. The language $\mathcal{L}_2$ is C**-complete** if in addition $\mathcal{L}_2$ is in C.

We need one more way of generating definitions of complexity classes.

Definition 2.4.8. If C is a complexity class then the language $\mathcal{L}$ (over Σ) is in coC, the **complement class** of C, if the complement of $\mathcal{L}$ is in C.

If we act responsibly, then what we say about languages we can say about any decision problem. A **decision problem** is a class, whose members we call **instances**, partitioned into a class of *yes instances* and a class of *no instances*. Using an encoding of instances as finite strings over some alphabet Σ^*, a decision problem can be viewed as a language—the language of strings encoding yes instances—whose decidability/complexity can be analysed. Our responsibility is to ensure that the encoding used is a sensible one. For instance the interpretation of an α-ary relation symbol on a structure of size n could be encoded as a string of n^α binary digits, and a structure as a concatenation of such strings. In

[9]The second equivalence is nontrivial; it is *Savitch's theorem* [90].

[10]Here, the definition of Turing machines must be relaxed to allow differing input and output alphabets; details would be unenlightening.

certain cases, it is possible to give a precise definition of a sensible encoding: see, for example, [32, Section 3.1.5]. For decidability, and for the polynomially robust complexity classes we have discussed, all sensible encodings will be equivalent. Further, it will be legitimate to conflate, for example, the size of the encoding of a structure with the size of its domain, or the size of the encoding of a formula with the length of the formula itself.

The following is a basic but important complexity result. It tells us that finite axiomatisability of a class implies that membership of the class, for finite structures, can be decided in polynomial time. We use this repeatedly in this thesis: in Chapter 4, to obtain Corollary 4.6.5, in Chapter 6, for Remark 6.2.6, and in Chapter 7, for Corollary 7.7.3.

Theorem 2.4.9. *Let Λ be a signature. Given a fixed Λ-sentence φ, the problem of deciding if φ holds on a Λ-structure $\mathfrak{S}$, with domain of size n, can be checked in time polynomial in n.*

Proof. We prove the more general statement that given any Λ-formula ψ and assignment $(a_1, \dots a_k)$ to its free variables, we can check whether $\mathfrak{S}, (a_1, \dots, a_k) \models \psi$ in time polynomial in n. The proof is by induction on the structure of ψ.

We may assume that the result of application of the interpretation of any function symbol can be computed in polynomial time, given the values of its arguments. Then the value assigned to any term under any assignment can be computed in polynomial time, by induction on term structure.

Next we may assume any atomic formula can be checked in polynomial time, given the values of the terms in it. Hence any atomic formula can be checked in polynomial time under any assignment.

For the inductive cases, if ψ is of the form $\chi_1 \to \chi_2$, the result is obvious. If ψ is of the form $\forall x \chi$, let μ be an assignment to the free variables of ψ. Then there are n possible x-overwrites of μ, and for each overwrite ν, it can be decided whether $\mathfrak{S}, \nu \models \chi$ in polynomial time. Hence ψ can be checked in polynomial time by checking each of these in turn. □

2.5 Algebraic logic

In this section we start with definitions of ordered structures of increasing specificity before giving the foundational example within algebraic logic—the identification, up to isomorphism, of fields of sets with Boolean algebras. We then use this as an illustrative example as we explain the general methodology of algebraic logic. The majority of the concepts and results in the section can be found in [20].

2.5.1 Ordered structures

Definition 2.5.1. A **poset** is a set equipped with a binary relation $\leq$ that validates the following laws.

$$a \leq a \qquad (\textbf{reflexive})$$

$$a \leq b \ \wedge \ b \leq c \ \rightarrow \ a \leq c \qquad (\textbf{transitive})$$

$$a \leq b \ \wedge \ b \leq a \ \rightarrow \ a = b \qquad (\textbf{antisymmetric})$$

A poset is **bounded** if it has both a minimum element and a maximum element.

Definition 2.5.2. A **semilattice** is a *nonempty* set equipped with a binary operation $*$ that is commutative, associative, and idempotent.

There are two ways to view a semilattice as a poset: either $a \leq b$ if and only if $a * b = a$, or the reverse ordering. If we take the first view, we call the semilattice a **meet semilattice**, and if we take the second, a **join semilattice**. And it is indeed true that in the first case $a * b$ will be the **meet** of a and b (that is, their infimum) and in the second the **join** (their supremum).

Definition 2.5.3. A **lattice** is a *nonempty* set equipped with binary operations $\cdot$ and $+$ such that using $\cdot$ it forms a meet semilattice and using $+$ it forms a join semilattice *with the same ordering*. This can be expressed algebraically by adding the **absorption** laws

$$a \cdot (a + b) = a$$
$$a + (a \cdot b) = a$$

to the semilattice conditions for $\cdot$ and $+$ (the idempotency conditions become redundant). A lattice is **distributive** if it validates the law

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Definition 2.5.4. Let X be a set. A **ring of sets** over X is a subset $\mathcal{F}$ of $\wp(X)$ such that $\mathcal{F}$ is closed under the set-theoretic operations of binary intersection and binary union.

Every ring of sets over a set X is a distributive lattice, interpreting $\cdot$ as intersection and $+$ as union. The next theorem tells us that (modulo isomorphisms) the converse is also true.

Theorem 2.5.5 (Birkhoff [9])**.** *Every distributive lattice is isomorphic to a ring of sets.*

Definition 2.5.6. A lattice is **complemented** if it is bounded and for all a there exists b such that $a \cdot b = 0$ and $a + b = 1$, where 0 is the minimum element and 1 the maximum.

For distributive lattices, complements are necessarily unique.

Definition 2.5.7. Consider the functional signature $\{0, 1, +, ^-\}$, where 0 and 1 are constants, $^-$ is unary, and $+$ is binary. A **Boolean algebra** is any $\{0, 1, +, ^-\}$-algebra validating the equational theory of the following algebra $\mathfrak{B}$. The domain of $\mathfrak{B}$ is just 0 and 1 (which are distinct) the operation $+$ is given by $a + b = 1$ unless both a and b are 0, and $^-$ swaps 0 and 1. It is not hard to show that the Boolean algebras are precisely the $\{0, 1, +, ^-\}$-algebras validating the following equations [20], where $a \cdot b := \overline{\overline{a} + \overline{b}}$.

- $+$ and $\cdot$ are associative
- $+$ and $\cdot$ are commutative
- $+$ distributes over $\cdot$
- $\cdot$ distributes over $+$
- 0 is an identity for $+$
- 1 is an identity for $\cdot$
- $a + \overline{a} = 1$
- $a \cdot \overline{a} = 0$

The absorption laws are derivable from these equations. Hence the Boolean algebras are (in a sense) precisely the complemented distributive lattices (though the differing signatures affect notions such as homomorphism, subalgebra, and so on).

Definition 2.5.8. Let X be a set. A **field of sets** over X is a nonempty subset $\mathcal{F}$ of $\wp(X)$ such that $\mathcal{F}$ is closed under the set-theoretic operations of binary union and of complementation-relative-to-X. Note this means that $\mathcal{F}$ necessarily contains both $\emptyset$ and X.

Every field of sets over a set X is a Boolean algebra, interpreting

- 0 as $\emptyset$,
- 1 as X,
- $+$ as union,
- $^-$ as complementation.

Again, the converse is also true.

Theorem 2.5.9 (Birkhoff–Stone representation theorem [98][11]). *Every Boolean algebra is isomorphic to a field of sets.*

Definition 2.5.10. A **filter** on a Boolean algebra $\mathfrak{B}$ is a subset γ of $\mathfrak{B}$ with the following properties:

upward closed $a \in \gamma$ and $a \leq b \implies b \in \gamma$,

downward directed $a, b \in \gamma \implies a \cdot b \in \gamma$.

It is an **ultrafilter** if it is also

proper $0 \notin \gamma$,

prime $a \in \mathfrak{B} \implies a \in \gamma$ or $\overline{a} \in \gamma$.

The most direct proof of the Birkhoff–Stone representation theorem is to represent each element a of the Boolean algebra by the set of ultrafilters containing a, and then to check that this really is an isomorphism.

2.5.2 Representation

In algebraic logic, theorems such as Theorem 2.5.9 are of fundamental importance. Whenever we specify some type of entity, and some set-theoretic operations on those entities, we naturally obtain a class of rather 'concrete' algebras. Fields of sets are obtained in this way by specifying unary relations (that is, subsets of X) and the operations $\emptyset, X, \cup$, and $^\mathsf{c}$ (complementation). Now suppose we wish to study these concrete algebras from the perspective of first-order logic. We cannot hope to find an axiomatisation of precisely the concrete algebras, as this class will not be closed under isomorphisms—isomorphisms being blind to the actual identities of elements of algebras. We are therefore led to study the isomorphic closure of the class of concrete algebras.

[11] A version closer to the full duality-of-categories result was proved by Stone and appeared later, in [99]. (He did not consider morphisms.) For distributive lattices, the equivalent version is also due to Stone [100].

The most immediate question to ask is whether this isomorphic closure is axiomatisable, and if it is, we will wish to exhibit an axiomatisation. Exhibiting an axiomatisation usually involves three tasks: identifying a suspected axiomatisation, showing it is sound, and then giving an explicit construction of an isomorphism from an arbitrary algebra satisfying the axioms to a concrete algebra. We refer to an isomorphism from an abstract to a concrete algebra as a **representation**, and accordingly, the isomorphic closure of the class of concrete algebras as the **representation class**, and the proof that the axioms are necessary and sufficient for representability as a **representation theorem**.[12] In the case of fields of sets, the Birkhoff–Stone representation theorem proves that the representation class is the class of Boolean algebras. The representations used later in this thesis to prove Proposition 4.5.1 and Theorem 6.4.5—the key representation theorems of Chapters 4 and 6 respectively—are both descendents in some way of the Birkhoff–Stone representation.

Concrete algebras usually have an associated base.[13] For example, the base of a field of sets over X is the set X. Given a fixed class of concrete algebras, we say that the **finite representation property** holds if whenever a finite algebra is representable as a concrete algebra, it is representable as a concrete algebra *on a finite base*. Typically the class of concrete algebras is identified by a signature and the type of entities comprising the concrete algebras. So, for example, we may say 'the finite representation property holds for the signature Σ for representation by injective partial functions'.

The finite representation property is interesting in its own right, but it is also worth briefly examining its relationships with problems of decidability of representability. Most immediately, if the finite representation property holds, then the problems of deciding if a finite algebra is representable and of deciding if a finite algebra is representable on a finite base, become one and the same. If in addition the representation class has a recursively enumerable first-order axiomatisation, then both the *representable* finite algebras are recursively enumerable (by searching through all concrete algebras on finite bases, ordered by increasing base size) and the *non-representable* finite algebras are recursively enumerable (by checking for violation of the axioms). Hence the finite representation property plus recursively enumerable axiomatisability is sufficient for representability of finite algebras to be decidable. Alternatively, we can do without the axiomatisation if the size of base necessary to represent a finite and representable algebra is bounded by a computable function of the size of the algebra, for then the search through concrete algebras can at the appropriate point be established to have failed. It is the norm for proofs of the finite representation property to provide an explicit bound, so for this second alternative to apply.

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[12] One additional piece of relevant terminology is the term **faithful**, which, applied to a putative representation, means injective.

[13] Or at least a unique minimal base.

Chapter 3

Related work

This chapter is divided into two sections. In the first we give a summary of the substantial body of work that has been carried out in investigating algebras of binary relations. Typically, questions that are asked about the logical or computational properties of classes of such algebras yield negative answers.

The second section is an account of the somewhat smaller body of work that exists relating to algebras of partial functions, that is, of single-valued relations. Here the situation in regard to logical and computational properties is markedly more positive.

3.1 Algebras of relations

3.1.1 Relation algebras

In 1941, Tarski [102] gave a set of axioms that characterises a certain class of algebras that have since come to be known as relation algebras. The axiomatisation given in the following definition is taken from Hirsch and Hodkinson's monograph [41]. It is equivalent to Tarski's original set of axioms.

Definition 3.1.1. Consider the functional signature $\{0, 1, +, ^-, 1', ^\smile, ;\}$, where 0, 1, and 1' are constants, $^-$ and $^\smile$ are unary, and $+$ and ; are binary.[1] A **relation algebra** is a $\{0, 1, +, ^-, 1', ^\smile, ;\}$-algebra such that the reduct to the Boolean algebra signature $\{0, 1, +, ^-\}$ is a Boolean algebra and the following seven additional equations are validated:

- ; is associative,
- $(a + b) ; c = a ; c + b ; c$,
- $a ; 1' = a$,
- $a^{\smile\smile} = a$ ($^\smile$ is **involutive**),
- $(a + b)^\smile = a^\smile + b^\smile$,
- $(a ; b)^\smile = b^\smile ; a^\smile$,
- $(a^\smile ; \overline{a ; b}) + \overline{b} = \overline{b}$.

We write $\mathbf{RA}$ for the class of all relation algebras. By (our) definition, $\mathbf{RA}$ is a variety.

The motivation for relation algebras is as an algebraic abstraction of algebras of binary relations.

[1] Here and elsewhere, the symbol ; binds more tightly than all other binary operation symbols, and unary symbols bind more tightly than binary ones.

Definition 3.1.2. We take the view that binary relations on a set X are subsets of $X \times X$. A **concrete relation algebra** $\mathcal{R}$ is a set of binary relations on some set X, which we call the **base**, such that

(i) all the relations in $\mathcal{R}$ are subsets of some fixed equivalence relation E on X,

(ii) the set $\mathcal{R}$ of relations is closed under the following operations:

- the constants $\emptyset$, $\Delta := \{(x, x) \in X^2\}$, and E,
- the unary operation of complementation-relative-to-E, written $^{\mathsf{c}}$,
- the unary operation of **converse**, written $^{-1}$, defined by

$$xR^{-1}y \iff yRx,$$

- binary union, $\cup$,
- the binary operation of **composition**, written $|$, defined by

$$xR|Sy \iff \exists z(xRz \wedge zSy). \tag{3.1}$$

Using the correspondence

$$\begin{array}{llll} 0 \longleftrightarrow \emptyset & + \longleftrightarrow \cup & 1\text{'} \longleftrightarrow \Delta & ; \longleftrightarrow | \\ 1 \longleftrightarrow E & ^{-} \longleftrightarrow {}^{\mathsf{c}} & ^{\smile} \longleftrightarrow {}^{-1} & \end{array}$$

a concrete relation algebra becomes an algebra of the relation algebra signature in which all the relation algebra axioms hold. Hence, viewed in this way, every concrete relation algebra is a relation algebra. A concrete relation algebra is **square** if the equivalence relation E is the universal relation $X \times X$.[2]

The relation algebra signature and its subsignatures and supersignatures are the most extensively studied signatures for binary relations. So much so, that we will not even mention some topics here—relativised representations, weakly representable algebras, and so on—instead sticking to those issues relevant to later chapters. Note that it is sometimes convenient to view the derived operation $a \cdot b := \overline{(\overline{a} + \overline{b})}$, which is intersection in concrete algebras, as being part of the relation algebra signature.

3.1.2 Representable relation algebras

Definition 3.1.3. A relation algebra (or indeed any $\{0, 1, +, ^{-}, 1\text{'}, ^{\smile}, ;\}$-algebra) is **representable** if it is isomorphic to a concrete relation algebra. The class of representable relation algebras is denoted $\mathbf{RRA}$.

Since all the relation algebra axioms hold on concrete relation algebras, $\mathbf{RRA}$ is a subclass of $\mathbf{RA}$. In 1950, Lyndon showed that this inclusion is proper, by describing a relation algebra with 2^{56} elements and proving it is not representable [69].

In 1955, Tarski proved that $\mathbf{RRA}$ is a variety [104]. However, in 1964, Monk, using the so-called *Lyndon algebras* [70], showed that $\mathbf{RRA}$ is not finitely axiomatisable by first-order logic [80]. Further negative results followed. In 1991, Jónsson, again exploiting Lyndon algebras, showed that $\mathbf{RRA}$ cannot

[2]The reason for the relativisation to E, in general, is so the representation class is closed under direct products.

be axiomatised by a set of equations that uses only finitely many variables [61]. In other words, any equational axiomatisation of $\mathbf{RRA}$ contains equations with arbitrarily large numbers of variables. For full first-order logic, this question of axiomatisability with finitely many variables appears to still be open.

Problem 3.1.4 ([41], Problem 17.14)**.** Can $\mathbf{RRA}$ be axiomatised by a set of first-order sentences that uses only finitely many variables?

A further negative result about the axiomatisability of $\mathbf{RRA}$ is Venema's finding [108], building on a result of Hodkinson [48], that $\mathbf{RRA}$ is not axiomatisable by Sahlqvist equations, but it would be too much of a diversion to describe Sahlqvist equations and their significance here.

Tarski showed that the equational theory of $\mathbf{RRA}$ is undecidable [103]. Hirsch and Hodkinson showed that representability of finite relation algebras is also undecidable [40]. They note that this implies $\mathbf{RRA}$ is not finitely axiomatisable in nth-order logic for any n, for on a finite structure, the truth of any nth-order formula can be determined by brute force.

Simple examples show that $\mathbf{RRA}$ does not satisfy the finite representation property. Tarski's point algebra, which describes a dense linear order without end points, is perhaps the most natural. Hence *finite* representation of finite relation algebras is a distinct decision problem to that examined by Hirsch and Hodkinson, leaving us with the following open question.

Problem 3.1.5 (Maddux)**.** Is finite representability of finite relation algebras decidable?

3.1.3 Subsignatures and supersignatures of the relation algebra signature

Finite axiomatisability of representation classes for signatures that are subsignatures of the relation algebra signature has also been extensively studied. Generally, results are negative, except for rather inexpressive signatures. With just composition and intersection, the representable algebras are axiomatised by the quasiequations defining semilattice-ordered semigroups [13]. But, for example, any subsignature of the relation algebra signature containing union, intersection, and composition has a representation class that is not finitely axiomatisable [3]. Even the signature $\{;, +\}$ proscribes finite axiomatisations [2].

Similarly for supersignatures of the relation algebra signature many rather general negative results about finite axiomatisability have been found. A brief overview can be found in Section 6.4.2 of [41].

With regard to equational theories there are some positive results. Andréka and Bredikhin have shown that for the signature $\{;, ^\smile, +, 1'\}$ the equational theory of the representable algebras is decidable [4], and Bredikhin did the same for the signature $\{;, ^\smile, +, \cdot\}$, in [14].

In [43], Hirsch and Jackson showed that for any subsignature of the relation algebra signature containing $;, +, \cdot$, and 1', representability of finite algebras is undecidable. For all of those same signatures not containing converse, they showed that finite representability of finite algebras is also undecidable.

Mikulás and Maddux have observed, independently, that the reduct of Tarski's point algebra to the signature $\{;, \cdot, 1'\}$ can still only be represented on an infinite base, so the finite representation property still fails to hold on this rather inexpressive signature. The same is true of $\{;, ^\smile, 1'\}$.

3.1.4 The fundamental theorem of relation algebras

Before introducing further variants of the representability problem, we note that for a quite general family of such problems the representation class is guaranteed to be at least *universally* axiomatisable. This result, called the fundamental theorem of relation algebras, is due to Schein [91]. Schein states it for binary relations but notes that it applies equally well to relations of any finite arity; we shall do the same. We also very slightly reduce the generality given by Schein for reasons of simplicity. (Schein takes five pages to articulate the theorem.) When reading the theorem, it may be helpful to have a representation class in mind. One should think of Σ as the signature of some class of concrete algebras, and the Λ'-formulas as making the assertion that a given Σ-structure is a concrete algebra.

Theorem 3.1.6 (Fundamental theorem of relation algebras, Schein [91])**.** *Let Λ be a signature, Λ' be the signature obtained by the addition to Λ of a countably infinite set $\{R_1, R_2, \dots\}$ of binary relation symbols, and T be a Λ'-theory.*

Let Σ also be a signature. For each function symbol f of arity n in Σ, let φ_f be a Λ'-formula using only Λ plus the relation symbols $\{R_1, \dots, R_n\}$, and whose only free variables are x_1 and x_2. For each relation symbol r of arity n in Σ, let φ_r be a Λ'-sentence using only Λ plus the relation symbols $\{R_1, \dots, R_n\}$.

Let $\mathbf{K}$ be the class of all Σ-structures $\mathfrak{S}$ of the following form:

- *the domain of $\mathfrak{S}$ is a subset of all the binary relations on some Λ-structure $\mathfrak{X}$,*[3]

and, writing $\mathfrak{X}(a_1, a_2, \dots)$ for the expansion of $\mathfrak{X}$ given by interpreting each R_i as a_i,

- *for each $\varphi \in T$ and $a_1, a_2, \dots \in \mathfrak{S}$, it holds that $\mathfrak{X}(a_1, a_2, \dots) \models \varphi$,*
- *for each function symbol f in Σ, the interpretation of f in $\mathfrak{S}$ is the operation on binary relations given by*

$$f(a_1, \dots, a_n) \coloneqq \{(p_1, p_2) \mid \mathfrak{X}(a_1, \dots, a_n), (p_1, p_2) \models \varphi_f\},$$

- *for each relation symbol r in Σ, the interpretation of r in $\mathfrak{S}$ is the predicate on binary relations given by*

$$r(a_1, \dots, a_n) \iff \mathfrak{X}(a_1, \dots, a_n) \models \varphi_r.$$

Then the isomorphic closure of $\mathbf{K}$ is universally axiomatisable by Σ-sentences.[4] *If T and the sets of all φ_f's and of all φ_r's are recursive, then the universal axiomatisation obtained will be recursive.*

The theory T can perform two roles: placing conditions on the structure $\mathfrak{X}$, and restricting the allowable binary relations in $\mathfrak{S}$. An example of a sentence performing the first role, when there is a binary relation symbol E in Λ, is the statement that E is an equivalence relation on $\mathfrak{X}$. An example of a sentence restricting the allowable binary relations is the sentence

$$\forall x \exists y\ xRy$$

[3]The structure $\mathfrak{X}$ may be empty. This is the only point in this thesis that empty structures are used.

[4]Strictly, this is only true if we define Σ-structures as necessarily nonempty. But we will only mention the fundamental theorem in the context of classes of *algebras*, where we have already made this stipulation.

expressing that the relations are all **total**. An example of a φ_f is the formula defining composition in (3.1). An example of a φ_r is the sentence $\forall xy(xRy \rightarrow xSy)$ defining what it means for one relation to be a subset of another.

Though we will not give a proof of the fundamental theorem, we note that the theorem can be factored into two separate results. The first is that the hypotheses of the theorem represent sufficient conditions for a class to have a *pseudouniversal* axiomatisation, a particular form of the type of axiomatisation delineated in the definition of pseudoelementarity, Definition 2.3.21. The second is that having a pseudouniversal axiomatisation is equivalent to having a universal first-order axiomatisation.

Often a representation class to which the fundamental theorem applies will also be closed under direct products. In this case Theorem 2.3.16 can be applied to deduce that the class is a quasivariety. This is the case for the representable relation algebras. Indeed, as we indicated earlier, the reason that concrete relation algebras are relativised to some equivalence relation, rather than requiring all concrete relation algebras be square, is that this allows the representation class to be closed under direct products.

3.1.5 Kleene algebras

Kleene algebras are members of a certain abstractly defined class of algebras, and emerged from investigations into deductive reasoning about regular languages initiated by Kleene [64]. Since then, they have found applications in many other areas, including relational reasoning.

As with relation algebras, the literature is extensive. We will only state a few of the most important results, choosing particularly from those pertaining to algebras of relations.[5]

Definition 3.1.7. Let as usual ; and $+$ be binary function symbols and 0 and 1' be constants. Let * be a unary function symbol. A **Kleene algebra** is an algebra of the signature $\{;, +, 0, 1', ^*\}$ validating the following (quasi)equations:

- $+$ is commutative, associative, and idempotent,
- 0 is an identity for $+$,
- ; is associative,
- 1' is a (two-sided) identity for ;,
- ; distributes over $+$,
- $0 ; a = a ; 0 = 0$ (0 is an **annihilator** for ;),

and, defining $a \leq b \iff a + b = b$,

- $1' + a ; a^* \leq a^*$,
- $1' + a^* ; a \leq a^*$,
- $a ; b \leq b \rightarrow a^* ; b \leq b$,
- $b ; a \leq b \rightarrow b ; a^* \leq b$.

[5] And we say nothing about the two-sorted Kleene algebra with tests [66], or about Kleene algebra with domain [21].

When the language of Kleene algebra is used to talk about *regular languages*, $+$ is interpreted as union, ; as concatenation, 0 as the empty language, 1' as the language containing only the empty string, and * as the Kleene star operation. The defining quasiequations of Kleene algebras are easily seen to be sound for regular languages. The justification for fixating on the particular quasivariety given in Definition 3.1.7 is the following result.

Theorem 3.1.8 (Kozen [65]). *Kleene algebras are* ***equationally complete*** *for regular languages, that is, any equation that is valid for regular languages is valid for the class of all Kleene algebras (and thus is a semantic consequence of the quasiequational axiomatisation of Kleene algebras).*[6]

When Kleene algebra is used in the context of *binary relations*, ;, $+$, 0, and 1' are interpreted in the usual way for binary relations, and * is the reflexive transitive closure operation:

$$xR^*y \iff \exists x_0 \dots x_n(x_0 = x) \wedge (x_n = y) \wedge x_0Rx_1 \wedge \dots \wedge x_{n-1}Rx_n \text{ for some } n \in \mathbb{N}. \tag{3.2}$$

For these relational semantics, the Kleene algebra axioms are again sound. In fact, it is a folk theorem that the equational theories for language and relational semantics coincide, so happily, we again have equational completeness.

Theorem 3.1.9. *Kleene algebras are equationally complete for* $\{;, +, 0, 1', ^*\}$*-algebras of binary relations.*

The equational theory shared by Kleene algebras, algebras of regular languages and $\{;, +, 0, 1', ^*\}$-algebras of binary relations is PSPACE-complete [78].

Having $*$ available when working with binary relations is very valuable, as it allows expression of a form of unbounded iteration. However, this expressibility comes at a cost. Observe that (3.2) is not in an appropriate form for the fundamental theorem of relation algebras to be applicable. So, unusually, we cannot deduce that the algebras representable as collections of binary relations form a universal class. Indeed, it is easy to show that the representation class is not closed under ultrapowers and hence not under elementary equivalence, so is not first-order axiomatisable at all.

The penalty imposed by the presence of * is also evident in decidability of validity, for we have the following result.

Theorem 3.1.10 (Hardin and Kozen [35]). *The quasiequational theory of the class of* $\{;, +, 0, 1', ^*\}$*-algebras of binary relations is not recursively enumerable.*

Once we move beyond the level of equations then, the logic of Kleene algebras of binary relations has undesirable computational properties, at least as far as automated reasoning is concerned. As we have seen, binary relations can behave problematically even in the absence of unbounded iteration. It is hardly surprising then that considerable obstacles are encountered when * is added to this already-rocky foundation.

[6]Descriptions of the equational theory of regular languages were discovered prior to Kozen's result, by Salomaa [89], and at around the same time as Kozen, by Bloom and Ésik [11], and by Krob [67].

3.1.6 Higher-order relations

Beyond binary relations, there has also been a great deal of work done on algebras of higher-order relations. There are a few different approaches, each of which arose for the purpose of giving an algebraic semantics to first-order logic. Again, we only give a brief summary. The important point to note is that for the properties of interest to us, results mirror those for binary relations closely.

In work on algebras of higher-order relations, the arity of the relations is a fixed ordinal, referred to as the **dimension**. Although in general any ordinal is permitted, here we will assume it is finite. This both simplifies the presentation and matches the picture for algebras of higher-order *functions*, which we introduce later. An n**-ary relation** then, on a set X, is a subset of X^n. We write an n-tuple as $\boldsymbol{x}$ and its value at i as x_i.

Before we define operations, we must explain an important technicality. For concrete relation algebras, the pairs in relations are constrained to have components that are equivalent under some fixed equivalence relation (E in Definition 3.1.2). Put another way, the components must lie in the same block of the partition defined by the equivalence relation. Similarly, for higher-order relations it is required that there be some partition P of the base X and that all relations must only contain tuples whose components all lie in the same block of P. As with relation algebras, the reason for this is that it improves the algebraic properties of the representation classes: they become closed under direct products.

Definition 3.1.11. Let X be a set and P a partition of X. The following operations on n-ary relations are important in the literature:

- the constant $0 \coloneqq \emptyset$,
- the constant 1, given by

$$1 \coloneqq \{\boldsymbol{x} \in X^n \mid \exists B \in P : \forall i \in \{1, \ldots, n\}\ x_i \in B\},$$

- binary union, $+$,
- complement, $\overline{}$, given by

$$\overline{R} \coloneqq \{\boldsymbol{x} \in 1 \mid \boldsymbol{x} \notin R\},$$

- for each $i \in \{1, \ldots, n\}$, the ith **cylindrification** operation, c_i, given by

$$c_i(R) \coloneqq \{\boldsymbol{y} \in 1 \mid \exists \boldsymbol{x} \in R : \forall j \in \{1, \ldots, n\}\ (j \neq i \rightarrow y_j = x_i)\},$$

- for each $i, j \in \{1, \ldots, n\}$, the ijth **diagonal** constant, d_{ij}, given by

$$d_{ij} \coloneqq \{\boldsymbol{x} \in 1 \mid x_i = x_j\},$$

- for each function $\tau \colon \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$, the τ-**substitution** operation, s_τ, given by

$$s_\tau(R) \coloneqq \{\boldsymbol{x} \in 1 \mid (x_{\tau(1)}, \ldots, x_{\tau(n)}) \in R\}.$$

We mention four classes of concrete algebras. For each, there is a similarly named but equationally defined abstract class approximating, but not coinciding with, the representation class. However, they are not relevant here.

Definition 3.1.12. A Σ-algebra of n-ary relations on some set X relativised to some partition P of X is a

- **generalised cylindric set algebra** if Σ consists of the Boolean operations, all cylindrifications, and all diagonals,
- **generalised diagonal-free algebra** if Σ consists of the Boolean operations and all cylindrifications,
- **generalised polyadic set algebra** if Σ consists of the Boolean operations, all cylindrifications, and all substitutions,
- **generalised polyadic equality set algebra** if Σ consists of the Boolean operations, all cylindrifications, all diagonals, and all substitutions.[7]

Cylindric algebras were introduced by Tarski in [102]; extensive coverage can be found in [36] and [37]. Polyadic algebras are due to Halmos, and [34] is a collection of his writings on this subject.

For all four classes of algebras, the fundamental theorem of relation algebras can be applied to see that the representation class has a recursive universal axiomatisation. As the representation classes are also closed under direct products, they are quasivarieties, by Theorem 2.3.16.

When $n \geq 3$, for the classes that can express substitutions (all but diagonal-free set algebras), there is a natural way to obtain a representable relation algebra from each algebra, and every representable relation algebra can be obtained in this way. In most cases this is the explanation behind the similarity between results we are about to recount and those of Section 3.1.2.

In the same paper that Tarski proved the representable relation algebras form a variety, [104], he also showed that the representation class for generalised cylindric set algebras is a variety for any n. For $n < 3$ the representation classes have finite equational axiomatisations, but Monk showed that for $n \geq 3$, the representation class is not finitely axiomatisable in first-order logic [81]. Also for $n \geq 3$, the equational theory is undecidable, following from the equivalent result for relation algebras.

For diagonal-free set algebras, when $n \geq 3$, the representation class is not finitely axiomatisable, as shown by Johnson [59]. And again, for $n \geq 3$, the equational theory is undecidable. The polyadic and polyadic equality algebras were introduced by Halmos. Johnson also showed in [59] that in both cases the representation class is not finitely axiomatisable for $n \geq 3$.

For all four classes of algebra, decidability of representability for finite algebras has been shown undecidable for $n \geq 3$, utilising the analogous result for relation algebras. This was proven for diagonal-free set algebras in [42], and for the other three types by Hodkinson in [49]. As the representation

[7] The non-'generalised' versions of these algebras are those with trivial partition P. The generalised algebras can alternatively (modulo isomorphisms) be defined as the subdirect products of these.

classes have recursive axiomatisations, but representability of finite algebras is undecidable, we know immediately that the finite representation property fails in all four cases. Alternatively, this can be seen by utilising the impossibility of finitely representing Tarski's point algebra, together with arguments connecting relation algebras and n-ary algebras.

3.1.7 Complete representations

Many signatures can express the inclusion-ordering relation $\subseteq$. For example for any signature containing $+$, the relation defined by $a \leq b \iff a + b = b$ is interpreted as $\subseteq$ in any representation and so partially orders any representable algebra. Similarly for any signature containing $\cdot$ and the definition $a \leq b \iff a \cdot b = a$. Whenever this happens we can make the following definitions of specialised types of representations.

Definition 3.1.13. A representation θ of a poset $\mathfrak{P}$ is **meet complete** if, for every nonempty subset S of $\mathfrak{P}$, if $\prod S$ (the infimum of S) exists, then

$$\theta(\prod S) = \bigcap\{\theta(s) \mid s \in S\}.$$

Definition 3.1.14. A representation θ of a poset $\mathfrak{P}$ is **join complete** if, for every subset S of $\mathfrak{P}$, if $\sum S$ (the supremum of S) exists, then

$$\theta(\sum S) = \bigcup\{\theta(s) \mid s \in S\}.$$

Note how S is required to be nonempty in Definition 3.1.13 but not in Definition 3.1.14, for there is not always a sensible way to define the empty intersection. For representations of Boolean algebras, relation algebras, and cylindric algebras, the notions of meet complete and join complete are equivalent, that is, a representation is meet complete if and only if it is join complete. So in cases such as these, we may simply use the adjective **complete**.

Complete representations have been investigated by Hirsch and Hodkinson, who have shown that neither the class of completely representable relation algebras nor the class of completely representable cylindric algebras (for dimension at least three) is elementary [38]. We say more about complete representations in Chapter 4, where we obtain a contrasting result for partial functions.

3.2 Algebras of partial functions

We have seen, in Section 3.1, various classes of algebras of relations. Algebras of partial functions are algebras of **functional** relations, which for a binary relation R means

$$xRy \wedge xRy' \rightarrow y = y'$$

for all x, y and y'. Hence algebras of partial functions are simply yet further variants of algebras of relations, and the methodology we use is exactly the same: each choice of set-theoretic operations gives a notion of representability for abstract algebras, and we can then study the representation class and related issues such as finite or complete representability.

3.2.1 Unary functions

The basic and most common case is to consider **unary partial functions**, that is, functional binary relations on some base set X.

We first give a non-exhaustive list of operations that have appeared in work on algebras of unary partial functions. So great a proportion of the operations have only a single symbol for both set-theoretic and the abstract operations that in this section we only use one symbol set. We have already met

- function composition: $;$ (a special case of relation composition),[8]
- intersection: $\cdot$
- empty function: 0
- identity function: 1' (defined on the specified base),

and there is also

- **domain**: D a unary operation—$\mathrm{D}(f)$ is the identity function restricted to the domain of f,
- **antidomain**: A a unary operation—$\mathrm{A}(f)$ is the identity function restricted to those points in the base where f is *not* defined,
- **range**: R a unary operation—$\mathrm{R}(f)$ is the identity function restricted to the range of f,
- **fixset**: F a unary operation—$\mathrm{F}(f)$ is the identity function restricted to the fixed points of f,
- **preferential union**: $\sqcup$ a binary operation—the preferential union of f and g takes the value of f where f is defined and the value of g where f is not defined and g is,
- **relative complement**: $\setminus$ the usual binary relative complement operation on sets,
- **maximum iterate**: $^\uparrow$ a unary operation—$f^\uparrow(x)$ is defined if only a finite number of iterations of f are defined on x and takes the value $f^n(x)$ for the maximum value of n that this is defined. So
$$f^\uparrow(x) = \bigcup_{n\in\mathbb{N}} (f^n \,;\, \mathrm{A}(f)).$$

The reader will note there are operations featuring heavily in the section on binary relations but absent in the above list. If an operation on partial functions does not *in general* yield a function, then it is not terribly useful to be able to reason about algebras of partial functions with that operation in the signature. Firstly, if we ever want to apply a validity of such algebras to any specific functions, we are burdened with proving that those functions can coexist in an algebra of that signature and not generate a non-function. Secondly, such algebras are often so restricted as to not be interesting. Take collections of partial functions closed under unions: there cannot be even one point that can map to more than one

[8] Note that compared to the usual mathematical notation $\circ$ for composition, the ; notation reverses the order the functions are applied. That is, $f \,;\, g = g \circ f$.

place. Signatures containing complement are even worse: there could not be more than two points in the base.

The restriction to a single fixed base set X is for many signatures not important, as we can reduce to the single base case by taking a union of bases. This is not true of signatures containing antidomain though, because the antidomain operation is corrupted by expanding the base. Nevertheless, throughout this thesis we only concern ourselves with the single-base-set setup.

Having indicated a correspondence between operations on partial functions and symbols, as we have done above, we get a definition of representation by partial functions for any signature containing any combination of symbols in the correspondence. Let Σ be such a signature and $\mathfrak{A}$ a Σ-algebra. A **representation of $\mathfrak{A}$ by partial functions** is an isomorphism from $\mathfrak{A}$ to an algebra whose elements are partial functions and whose interpretations are the indicated operations.

The most basic example of a representation theorem is instructive.

Theorem 3.2.1 (Suschkewitsch [101])**.** *The class of* $\{;\}$*-algebras representable by partial functions is axiomatised by the associativity law.*

Proof. First we must note that function composition is associative, so every $\{;\}$-algebra representable by partial functions must validate the associativity law.

Conversely, let $\mathfrak{A}$ be an algebra of the signature $\{;\}$ that validates the associativity law. Let A be the domain of $\mathfrak{A}$ and write 1 for $\{A\}$ (so $1 \notin A$). For each $a \in \mathfrak{A}$, define the function $\theta(a) \colon A \cup \{1\} \to A \cup \{1\}$ by

$$\theta(a)(b) = b \,;\, a \quad \text{for } b \neq 1,$$
$$\theta(a)(1) = a.$$

The associativity law is precisely what is needed to make the composition of functions $\theta(a_1) \,;\, \theta(a_2)$ equal to the function $\theta(a_1 \,;\, a_2)$. The addition of 1 to the base ensures distinct elements of $\mathfrak{A}$ are mapped to distinct functions. Hence θ defines a representation of $\mathfrak{A}$ by partial (in fact, total) functions. □

Algebras with a single binary operation that is associative are called **semigroups**, and in semigroup theory a representation would be called a *faithful action* of the semigroup on its base.

The proof of Theorem 3.2.1 is clearly a simplification of Cayley's famous proof that every group is isomorphic to a group of permutations (a representation theorem of its own, for bijective functions). For this reason, it is common to invoke Cayley's name when describing representations constructed in similar ways. Cayley's representation and that used in the proof of the Birkhoff–Stone representation theorem are the two prototypical examples of representations. Often representation theorems are obtained by combining the Cayley idea with the Birkhoff–Stone idea of using ultrafilters, or some other type of distinguished filters on the algebra.

Axiomatisability. We will list axiomatisability results for prominent signatures in chronological order of their discovery. In each case, either or both of 0 and 1' can be added to the signature as desired, by supplementing the axiomatisation by the equations $0 \,;\, a = a \,;\, 0 = 0$ and/or $1' \,;\, a = a \,;\, 1' = a$ as appropriate.

In 1970, Schein axiomatised the representation class for the signature $\{;, \mathrm{D}, \mathrm{R}\}$ using a finite number of quasiequations [92]. This cannot be improved to equations—the representation class is a proper quasivariety. In 1971, Garvac'kiĭ showed that the representation class for $\{;, \cdot\}$ is a finitely based variety [29]. For the signature $\{;, \mathrm{D}\}$, the representation class is again axiomatisable by a finite number of equations. Such sets of equations have been discovered and rediscovered a number of times, but this seems to have been done first by Trokhimenko [106], in 1973. For the signature $\{;, \setminus\}$, Schein gave a finite equational axiomatisation in [93].

A special case of a result by Dudek and Trokhimenko [23] gives a finite equational axiomatisation of the representation class for the signature $\{;, \cdot, \mathrm{D}\}$. In [55], Jackson and Stokes gave a finite equational axiomatisation for the signature $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$. In [57], Jackson and Stokes consider various signatures containing ; and A. For $\{;, \mathrm{A}\}$ they show that the representation class is a proper quasivariety and give a finite quasiequational axiomatisation. For each of the signatures $\{;, \mathrm{A}, \sqcup\}$, $\{;, \cdot, \mathrm{A}\}$, and $\{;, \cdot, \mathrm{A}, \sqcup\}$ they give finite equational axiomatisations. Note that any signature containing antidomain can express domain, as a double application of antidomain. In [44], Hirsch, Jackson, and Mikulás give a finite equational axiomatisation of the representation class for the signature $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$.

Unbounded iteration. A modest amount of work has been done with operations that express some sort of unbounded iteration. For maximum iterate, in [57], for the signature $\{;, \mathrm{A}, \sqcup, {}^{\uparrow}\}$ a finite set of equations is given that, if we restrict attention to finite algebras, axiomatises the representable ones. They obtain the same result for $\{;, \cdot, \mathrm{A}, \sqcup, {}^{\uparrow}\}$ by the addition of a few equations. In [44], the same type of result (again using finite sets of equations) is proven when we add range to these two signatures, that is, for the signatures $\{;, \mathrm{A}, \mathrm{R}, \sqcup, {}^{\uparrow}\}$ and $\{;, \cdot, \mathrm{A}, \mathrm{R}, \sqcup, {}^{\uparrow}\}$. It follows from a result by Goldblatt and Jackson [30] that when the signature contains $\{;, \cdot\, \mathrm{A}, {}^{\uparrow}\}$, the equational theory of the class of *all* (not just the finite) representable algebras is not recursively enumerable, so we cannot hope to find a recursively enumerable full axiomatisation.

In [56], Jackson and Stokes consider the representation class for various types of two-sorted algebras, including those of partial functions and unary relations ("tests") with Boolean operations on tests, composition of functions, `if-then-else`, `while` and the test-valued operation asking if a given test would be true after application of a given function. For this case they find that no finite axiomatisation is possible. In [58], Jackson and Stokes consider a quite similar signature that omits the mixed-type test-valued operation, instead providing some of its functionality within an 'extended' `if-then-else` and an 'extended' `while`. For this signature they provide a finite set of equations that, over finite algebras, axiomatises the representable ones.[9]

Equational theories. In [44], it is proved that for any combination of operations from $\{;, \cdot, \mathrm{D}, \mathrm{A}, \mathrm{R}, \mathrm{F}, \sqcup, 0, 1'\}$, the equational theory of the representation class is in coNP, and provided both composition and antidomain are present it is coNP-complete. As we mentioned, the presence of $;, \cdot, \mathrm{A}$, and ${}^{\uparrow}$ causes the equational theory to be undecidable. In fact, Goldblatt and Jackson's result implies this for the less expressive combination of $;, \cdot, \mathrm{F}$, and ${}^{\uparrow}$.

[9] They actually prove the stronger statement that this is an axiomatisation over algebras for which ; is *periodic*.

Decidability of representability. Recall that checking any first-order sentence on a finite structure can be done in time polynomial in the size of the structure. Hence whenever a class has a finite first-order axiomatisation, as all the representation classes mentioned do, decidability of membership, for finite structures, is in P.[10]

So, by the measures of axiomatisability, complexity of equational theories, and decidability of representability, algebras of partial functions score rather positively, based on results obtained so far. One dissenting result is that of Gould and Kambites, who showed that the representability of finite $\{;, \mathrm{D}, \mathrm{R}\}$ algebras by *injective* partial functions is undecidable [31].

3.2.2 The finite representation property

For partial functions it is fairly common for the finite representation property to follow immediately from representation theorems. For example this is true for the standard Cayley-style representation theorems for the signatures of composition plus any subset of $\{0, 1'\}$. The representations for $\{;, \cdot, \mathrm{D}\}$ in [23] and for signatures including antidomain in [57] also have this property.

There is a very simple argument proving the finite representation property for a large class of signatures (including, incidentally, those just mentioned), which in [44], Hirsch, Jackson, and Mikulás use for all combinations from $\{;, \cdot, \mathrm{D}, \mathrm{A}, \sqcup, \mathrm{F}, {}^{\uparrow}, 0, 1'\}$ that include composition (note, no range). This argument will work whenever the signature contains composition plus only other 'forward looking' operations. We will not attempt a completely precise definition of when an operation f is 'forward looking', but roughly, whether $f(R_1, \ldots, R_n)$ holds on (x, y) should depend only on where $R_1, \ldots, R_n$ hold on the points reachable from x.

In [44], they prove the result mentioned in the previous paragraph and then state that 'Similarly, the finite representation property is easy to establish for signatures that cannot express d' (their notation for domain). Hence they focus on those composition-containing signatures that *do* contain both domain and range. They prove the finite representation property for all signatures between $\{;, \mathrm{D}, \mathrm{R}\}$ and $\{;, \mathrm{D}, \mathrm{R}, \mathrm{A}, \mathrm{F}, 0, 1'\}$ inclusive and for all between $\{;, \mathrm{D}, \mathrm{R}, \mathrm{A}, \sqcup\}$ and $\{;, \mathrm{D}, \mathrm{R}, \mathrm{A}, \sqcup, \mathrm{F}, {}^{\uparrow}, 0, 1'\}$ inclusive. Note that none of these contain intersection. They pose the problem of determining if the finite representation property holds for $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ and supersignatures, and it is this problem we provide a solution to in Chapter 5.

For the signatures not including range, the size of base necessary is bounded by the cube of the size of the algebra. It follows (by considering a brute-force algorithm) that decidability of representability of finite algebras is in NP. For the signatures not including intersection, the bound is exponential in the size of the algebra, and hence decidability of representability of finite algebras is in NEXPTIME. Note however, that as finite first-order axiomatisations are known for many of the signatures concerned, the corresponding decision problem is already known to be in P.

Problem 3.2.2. For signatures between $\{;, \mathrm{D}, \mathrm{R}\}$ and $\{;, \mathrm{D}, \mathrm{R}, \mathrm{A}, \mathrm{F}, 0, 1'\}$, and between $\{;, \mathrm{D}, \mathrm{R}, \mathrm{A}, \sqcup\}$ and $\{;, \mathrm{D}, \mathrm{R}, \mathrm{A}, \sqcup, \mathrm{F}, {}^{\uparrow}, 0, 1'\}$, is the exponential bound on the size of the base tight?

[10] In fact it will be in AC^0, which is strictly contained in P.

3.2.3 Multiplace functions

There also exists work investigating algebras of partial functions from X^n to X—so-called **multiplace functions**—for some fixed set X. This was initiated by Menger's 1944 article *Algebra of analysis* [76], where in Section III he pursues an algebraic abstraction of differentiable functions of many variables, and although there Menger allows the arity n to be variable, it has subsequently become expected that it be fixed.

In [77], Menger defined the key operation, the multiplace analogue of composition. The $(n+1)$-ary **superposition** operation $\langle\ \rangle;$ on n-ary functions is defined by

$$(\langle f_1, \ldots, f_n\rangle\ ;g)(\boldsymbol{x}) \coloneqq g(f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})),$$

where $\boldsymbol{x}$ is an arbitrary n-tuple of base elements.

In [77], Menger wrote down an n-ary analogue of associativity, which he termed 'superassociativity', and noted it was sound for n-ary partial functions. Later, Dicker provided the Cayley-style representation theorem proving this single equation is sufficient to axiomatise the representation class when superposition is the sole operation [22].

Other operations on unary functions (range excepted) are either immediately applicable to multiplace functions, or generalise to a set of n indexed operations. For example, one can consider n domain operations, with $\mathrm{D}_i(f)$ the restriction of the ith projection to those points where f is defined.

The study of multiplace functions can in particular be associated with Trokhimenko, who took up the subject as a Ph.D. student under Schein in the late 1960s and has been publishing on this topic ever since. So in fact the theory of multiplace functions continued to be developed during the period that its unary counterpart was dormant. For various signatures, finite equational axiomatisations have been found for the representation classes. Some references can be found in Chapter 6, where we investigate signatures containing indexed antidomain operations.

3.2.4 Tabular summary

Table 3.1 gives a partial summary of known results concerning algebras of partial functions, with a focus on questions and signatures for which this thesis makes a contribution. Bold entries indicate contributions of this thesis, with numbers indicating where in the thesis the result can be found. Blank cells indicate the problem has not been studied. The (partial) operations $\dot{\smile}$ and $\dot{\backslash}$ are introduced and studied in Chapter 7. The meanings of each column are as follows.

Axiomatisability Axiomatisability of the class of algebras representable by unary partial functions

Multiplace functions Axiomatisability of the class of algebras representable by multiplace partial functions (for the multiplace analogue of the given signature)

Equational theory Complexity of the equational theory of the class of algebras representable by unary partial functions.

FRP Whether the finite representation property holds for representation by unary partial functions. These also hold for the multiplace case, for applicable signatures (those not containing range).

Complete rep. Axiomatisability of the class of algebras completely representable by unary partial functions

Decidability of representability of finite algebras holds whenever either there is a finite axiomatisation (by Theorem 2.4.9) or the finite representation property holds (in all cases there is a computable bound on the size of base necessary).

Signature	Axiomatisability	Multiplace functions	Equational theory	FRP	Complete rep.
$;$	finite eq. [101]	finite eq. [22]	P	yes	N/A[1]
$;\ \cdot$	finite eq. [29]	equational [107]	coNP[2] [44]	yes	
$;\ \mathrm{D}$	finite eq. [106]	equational [106]	coNP[2] [44]	yes	
$;\ \cdot\ \mathrm{D}$	finite eq. [23]	finite eq. [23]	coNP[2] [44]	yes	
$;\ \mathrm{A}$	finite quasieq. [57]	**finite quasieq.** 6.4.5	coNP-complete[2] [44]	yes	
$;\ \cdot\ \mathrm{A}$	finite eq. [57]	**finite eq.** 6.6.5	coNP-complete[2] [44]	yes	**finite 1st-order**[3] 4.6.4
$;\ \mathrm{A}\ \sqcup$	finite eq. [57]	**finite eq.** 6.7.3	coNP-complete[2] [44]	yes	
$;\ \cdot\ \mathrm{A}\ \sqcup$	finite eq. [57]	**finite eq.** 6.7.6	coNP-complete[2] [44]	yes	
$;\ \mathrm{A}\ \mathrm{F}$	finite quasieq. [57]	**finite quasieq.** 6.8.3	coNP-complete[2] [44]	yes	
$;\ \mathrm{A}\ \mathrm{F}\ \sqcup$	finite quasieq. [57]	**finite quasieq.** 6.8.6	coNP-complete[2] [44]	yes	
$;\ \mathrm{D}\ \mathrm{R}$	finite quasieq. [92]	N/A[4]	coNP [44]	yes [44]	
$;\ \cdot\ \mathrm{D}\ \mathrm{R}$	finite eq. [55]	N/A[4]	coNP [44]	**yes** 5.4.3	
$;\ \cdot\ \mathrm{A}\ \mathrm{R}$	finite eq. [44]	N/A[4]	coNP-complete [44]	**yes** 5.4.3	
$\dot{\smile}$	**1st-order** 7.3.6 **not finitely** 7.5.6	**1st-order**[5] **not finitely**[5]	**P** 7.7.4	**yes** 7.4.3	**nonelementary**[6] 7.5.8
$\dot{\setminus}$	**1st-order** 7.3.6 **not finitely** 7.5.14	**1st-order**[5] **not finitely**[5]		**yes** 7.4.3	**nonelementary**[6] 7.5.14
$\dot{\smile}\ \dot{\setminus}$	**1st-order** 7.3.6 **not finitely** 7.5.14	**1st-order**[5] **not finitely**[5]		**yes** 7.4.3	**nonelementary**[6] 7.5.14

Table 3.1: Summary of representability results for partial functions

[1] This signature cannot express the subset relation.

[2] In Chapter 6, we prove the multiplace version (Theorem 6.9.1).

[3] Not simpler than an existential-universal-existential theory.

[4] No n-ary generalisation of R has been given.

[5] These results are not stated in Chapter 7, but should be clear to the reader after reading that chapter (in particular Proposition 7.2.9(1) and its proof).

[6] For $\lesssim$-complete representability/$\lesssim'$-complete representability, as appropriate. See Chapter 7 for definitions of these notions.

Chapter 4

Complete representation by partial functions for composition, intersection, and antidomain

The work in this chapter has been published as: Brett McLean, *Complete representation by partial functions for composition, intersection and anti-domain*, Journal of Logic and Computation **27** (2017), no. 4, 1143–1156.[1]

ABSTRACT. For representation by partial functions in the signature with composition, intersection, and antidomain, we show that a representation is meet complete if and only if it is join complete. We show that a representation is complete if and only if it is atomic, but not all atomic representable algebras are completely representable. We show that the class of completely representable algebras is not axiomatisable by any existential-universal-existential first-order theory. By giving an explicit representation, we show that the completely representable algebras form a basic elementary class, axiomatisable by a universal-existential-universal sentence.

4.1 Introduction

Whenever we have a concrete class of algebras whose operations are set-theoretically defined, we have a notion of a representation: an isomorphism from an abstract algebra to a concrete algebra. Then the representation class—the class of representable algebras—becomes an object of interest itself.

One possibility—the focus of this thesis—is for the concrete algebras to be algebras of partial functions, and for this scenario various signatures have been considered. Often, the representation classes have turned out to be finitely axiomatisable varieties or quasivarieties [92, 23, 55, 57]; more details were given in Section 3.2.1.

Extra conditions we can impose on a representation are to require that it be meet complete or to require that it be join complete. A representation is meet complete if it turns any existing infima into intersections and join complete if it turns any existing suprema into unions. Hence we can define meet-complete representation classes and join-complete representation classes. In many important cases these two classes coincide. Bounded distributive lattices represented as rings of sets is an example where they do not [25].

In [38], Hirsch and Hodkinson showed that when the representation class is elementary, the complete representation class may (as is the case for Boolean algebras represented as fields of sets) or may not (relation algebras by binary relations) also be elementary.

In this chapter we investigate complete representation by partial functions for the signature $\{;, \cdot, \mathrm{A}\}$ of composition, intersection, and antidomain. In Section 4.2 we see that for this particular signature the algebras behave in many ways like Boolean algebras. We show that, as one consequence of this similarity to Boolean algebras, a representation by partial functions is meet complete if and only if it is join complete.

In Section 4.3 we show that a representation is complete if and only if it is atomic. We use the requirement that completely representable algebras be atomic to prove that the class of completely representable algebras is not closed under subalgebras, directed unions or homomorphic images and is not axiomatisable by any existential-universal-existential first-order theory.

In Section 4.4 we investigate the validity of various distributive laws with respect to the classes of representable and completely representable $\{;, \cdot, \mathrm{A}\}$-algebras. This enables us to give an example of an algebra that is representable and atomic, but not completely representable.

In Section 4.5 we present an explicit representation, which we use, in Section 4.6, to prove our main result: the class of completely representable algebras is a basic elementary class, axiomatisable by a universal-existential-universal first-order sentence.

4.2 Representations and complete representations

In this section we give preliminary definitions and then proceed to show that for the signature $\{;, \cdot, \mathrm{A}\}$, a representation by partial functions is meet complete if and only if it is join complete.

Given an algebra $\mathfrak{A}$, when we write $a \in \mathfrak{A}$ or say that a is an element of $\mathfrak{A}$, we mean that a is an element of the domain of $\mathfrak{A}$. Similarly for the notation $S \subseteq \mathfrak{A}$ or saying that S is a subset of $\mathfrak{A}$. The notation $|\mathfrak{A}|$ denotes the cardinality of the domain of $\mathfrak{A}$. We follow the convention that algebras are always nonempty. If S is a subset of the domain of a map θ then $\theta[S]$ denotes the set $\{\theta(s) \mid s \in S\}$. If S_1 and S_2 are subsets of the domain of a binary operation $*$ then $S_1 * S_2$ denotes the set $\{s_1 * s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}$. In a poset $\mathfrak{P}$ (whose identity should be clear) the notation $\downarrow a$ signifies the down-set $\{b \in \mathfrak{P} \mid b \leq a\}$.

Definition 4.2.1. Let σ be an algebraic signature whose symbols are a subset of $\{;, \cdot, 0, 1', \mathrm{D}, \mathrm{R}, \mathrm{A}\}$. An **algebra of partial functions** of the signature σ is an algebra of the signature σ whose elements are partial functions and with operations given by the set-theoretic operations on those partial functions described in the following.

Let X be the union of the domains and ranges of all the partial functions. We call X the **base**. In an algebra of partial functions

- the binary operation ; is composition of partial functions:

$$f ; g = \{(x, z) \in X^2 \mid \exists y \in X : (x, y) \in f \text{ and } (y, z) \in g\},$$

- the binary operation $\cdot$ is intersection:

$$f \cdot g = \{(x, y) \in X^2 \mid (x, y) \in f \text{ and } (x, y) \in g\},$$

- the constant 0 is the nowhere-defined function:

$$0 = \emptyset,$$

- the constant 1' is the identity function on X:

$$1\text{'} = \{(x, x) \in X^2\},$$

- the unary operation D is the operation of taking the diagonal of the domain of a function:

$$\mathrm{D}(f) = \{(x, x) \in X^2 \mid \exists y \in X : (x, y) \in f\},$$

- the unary operation R is the operation of taking the diagonal of the range of a function:

$$\mathrm{R}(f) = \{(y, y) \in X^2 \mid \exists x \in X : (x, y) \in f\},$$

- the unary operation A is the operation of taking the diagonal of the antidomain of a function—those points of X where the function is not defined:

$$\mathrm{A}(f) = \{(x, x) \in X^2 \mid \nexists y \in X : (x, y) \in f\}.$$

The list of operations in Definition 4.2.1 does not exhaust those that have been considered for partial functions but does include the most commonly appearing operations.

Definition 4.2.2. Let $\mathfrak{A}$ be an algebra of one of the signatures specified by Definition 4.2.1. A **representation of $\mathfrak{A}$ by partial functions** is an isomorphism from $\mathfrak{A}$ to an algebra of partial functions of the same signature. If $\mathfrak{A}$ has a representation then we say it is **representable**.

Theorem 4.2.3 (Jackson and Stokes [57])**.** *The class of $\{;, \cdot, \mathrm{A}\}$-algebras representable by partial functions is a finitely based variety.*

In fact in [57] a finite equational axiomatisation of the representation class is given, implicitly. So there exist known examples of such axiomatisations.

If an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ is representable by partial functions, then it forms a $\cdot$-semilattice. Whenever we treat such an algebra as a poset, we are using the order induced by this semilattice.

The next two definitions apply to any situation where the concept of a representation has been defined. So in particular, these definitions apply to representations as fields of sets as well as to representations by partial functions.

Definition 4.2.4. A representation θ of a poset $\mathfrak{P}$ over the base X is **meet complete** if, for every nonempty subset S of $\mathfrak{P}$, if $\prod S$ exists, then

$$\theta(\prod S) = \bigcap \theta[S].$$

Definition 4.2.5. A representation θ of a poset $\mathfrak{P}$ over the base X is **join complete** if, for every subset S of $\mathfrak{P}$, if $\sum S$ exists, then

$$\theta(\sum S) = \bigcup \theta[S].$$

Note that S is required to be nonempty in Definition 4.2.4 but not in Definition 4.2.5. For representations of Boolean algebras as fields of sets, the notions of meet complete and join complete are equivalent, so in this case we may simply use the adjective **complete**.

Note that if $\mathfrak{A}$ is an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ and $\mathfrak{A}$ is representable by partial functions, then $\mathfrak{A}$ must have a least element, 0, given by $\mathrm{A}(a) \mathbin{;} a$ for any $a \in \mathfrak{A}$ and any representation must represent 0 with the empty set. Similarly $\mathrm{D} := \mathrm{A}^2$ must be represented by the set-theoretic domain operation.

The following lemma demonstrates the utility of the particular signature $\{;, \cdot, \mathrm{A}\}$. The similarity of representable $\{;, \cdot, \mathrm{A}\}$-algebras to Boolean algebras allows results from the theory of Boolean algebras to be imported into the setting of $\{;, \cdot, \mathrm{A}\}$-algebras.

Lemma 4.2.6. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$. If $\mathfrak{A}$ is representable by partial functions, then for every $a \in \mathfrak{A}$, the set $\downarrow a$, with least element 0, greatest element a, meet given by $\cdot$, and complementation given by $\overline{b} := \mathrm{A}(b) \mathbin{;} a$ is a Boolean algebra. Any representation θ of $\mathfrak{A}$ by partial functions restricts to a representation of $\downarrow a$ as a field of sets over $\theta(a)$. If θ is a meet-complete or join-complete representation, then the representation of $\downarrow a$ is complete.*

Proof. If θ is a representation of $\mathfrak{A}$ by partial functions, then $b \leq a \implies \theta(b) \subseteq \theta(a)$, so θ does indeed map elements of $\downarrow a$ to subsets of $\theta(a)$. We have $b, c \in \downarrow a \implies b \cdot c \in \downarrow a$ and $\theta(b \cdot c) = \theta(b) \cap \theta(c)$ is always true by the definition of functional representability. For $b \leq a$

$$\theta(\overline{b}) = \theta(\mathrm{A}(b) \mathbin{;} a) = \mathrm{A}(\theta(b)) \mathbin{;} \theta(a) = \theta(a) \setminus \theta(b),$$

so $\overline{b} \in \downarrow a$ and $\theta(\overline{b}) = \theta(b)^{\mathsf{c}}$, where the set complement is taken relative to $\theta(a)$. Hence the restriction of θ to $\downarrow a$ is a representation of $(\downarrow a, 0, a, \cdot, ^-)$ as a field of sets over $\theta(a)$ (from which it follows that $\downarrow a$ is a Boolean algebra).

Suppose θ is meet complete. If S is a nonempty subset of $\downarrow a$, then all lower bounds for S in $\mathfrak{A}$ are also in $\downarrow a$. Hence if $\prod_{\downarrow a} S$ exists then it equals $\prod_{\mathfrak{A}} S$, and so $\theta(\prod_{\downarrow a} S) = \bigcap \theta[S]$. So the representation of $\downarrow a$ is complete.

Suppose that θ is join complete, $S \subseteq \downarrow a$, and $\sum_{\downarrow a} S$ exists. If $c \in \mathfrak{A}$ and c is an upper bound for S, then $c \geq c \cdot a \geq \sum_{\downarrow a} S$. Hence $\sum_{\downarrow a} S = \sum_{\mathfrak{A}} S$, giving $\theta(\sum_{\downarrow a} S) = \theta(\sum_{\mathfrak{A}} S) = \bigcup \theta[S]$. So the representation of $\downarrow a$ is complete. □

Corollary 4.2.7. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ and θ be a representation of $\mathfrak{A}$ by partial functions. If θ is meet complete, then it is join complete.*

Proof. Suppose that θ is meet complete. Let S be a subset of $\mathfrak{A}$ and suppose that $\sum_{\mathfrak{A}} S$ exists. Let $a = \sum_{\mathfrak{A}} S$. Then

$$\theta(\sum_{\mathfrak{A}} S) = \theta(\sum_{\downarrow a} S) = \bigcup \theta[S].$$ □

Corollary 4.2.8. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ and θ be a representation of $\mathfrak{A}$ by partial functions. If θ is join complete, then it is meet complete.*

Proof. Suppose that θ is join complete. Let S be a nonempty subset of $\mathfrak{A}$ and suppose that $\prod_{\mathfrak{A}} S$ exists. As S is nonempty, we can find $s \in S$. Then

$$\theta(\prod_{\mathfrak{A}} S) = \theta(\prod_{\mathfrak{A}}(S \cdot \{s\})) = \theta(\prod_{\downarrow s}(S \cdot \{s\})) = \bigcap \theta[S \cdot \{s\}] = \bigcap \theta[S]. \qquad \square$$

Corollaries 4.2.7 and 4.2.8 tell us that, just as for representations of Boolean algebras, we can describe representations of $\{;, \cdot, \mathrm{A}\}$-algebras by partial functions as **complete**, without any risk of confusion about whether we mean meet complete or join complete.

4.3 Atomicity

We begin our investigation of the complete representation class by considering the property of being atomic, both for algebras and for representations.

Definition 4.3.1. Let $\mathfrak{P}$ be a poset with a least element, 0. An **atom** of $\mathfrak{P}$ is a minimal nonzero element of $\mathfrak{P}$. We say that $\mathfrak{P}$ is **atomic** if every nonzero element is greater than or equal to an atom.

If $\mathfrak{P}$ is a poset, then $\mathrm{At}(\mathfrak{P})$ denotes the set of atoms of $\mathfrak{P}$.

We noted in the proof of Lemma 4.2.6 that representations of $\{;, \cdot, \mathrm{A}\}$-algebras necessarily represent the partial order by set inclusion. The following definition is meaningful for any notion of representation where this is the case.

Definition 4.3.2. Let $\mathfrak{P}$ be a poset with a least element and let θ be a representation of $\mathfrak{P}$. Then θ is **atomic** if $x \in \theta(a)$ for some $a \in \mathfrak{P}$ implies $x \in \theta(b)$ for some atom b of $\mathfrak{P}$.

We will need the following theorem.

Theorem 4.3.3 (Hirsch and Hodkinson [38])**.** *Let $\mathfrak{B}$ be a Boolean algebra. A representation of $\mathfrak{B}$ as a field of sets is atomic if and only if it is complete.*

Note that being completely representable does not imply a Boolean algebra is complete, but having an atomic representation *does* imply a Boolean algebra is atomic. Hence the existence of Boolean algebras that are atomic but not complete, for example, the finite-cofinite algebra on any infinite set.

Proposition 4.3.4. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ and θ be a representation of $\mathfrak{A}$ by partial functions. Then θ is atomic if and only if it is complete.*

Proof. Suppose that θ is atomic, S is a nonempty subset of $\mathfrak{A}$, and $\prod S$ exists. It is always true that $\theta(\prod S) \subseteq \bigcap \theta[S]$, regardless of whether or not θ is atomic. For the reverse inclusion, we have

$$\begin{aligned}
& (x, y) \in \bigcap \theta[S] & \\
\implies\ & (x, y) \in \theta(s) & \text{for all } s \in S \\
\implies\ & (x, y) \in \theta(a) & \text{for some atom } a \text{ such that } (\forall s \in S)\ a \leq s \\
\implies\ & (x, y) \in \theta(a) & \text{for some atom } a \text{ such that } a \leq \textstyle\prod S \\
\implies\ & (x, y) \in \theta(\textstyle\prod S). &
\end{aligned}$$

The third line follows from the second because, taking an a with $(x, y) \in \theta(a)$—which exists by the second line, since $S \neq \emptyset$—we have $(x, y) \in \theta(a \cdot s)$ for any $s \in S$. So for all $s \in S$, the element $a \cdot s$ is nonzero, so equals a, by atomicity of a, giving $a \leq s$.

Conversely, suppose that θ is complete. Let (x, y) be a pair contained in $\theta(a)$ for some $a \in \mathfrak{A}$. By Lemma 4.2.6, the map θ restricts to a complete representation of $\downarrow a$ as a field of sets. Hence, by Theorem 4.3.3, $(x, y) \in \theta(b)$ for some atom b of the Boolean algebra $\downarrow a$. Since an atom of $\downarrow a$ is clearly an atom of $\mathfrak{A}$, the representation θ is atomic. □

Corollary 4.3.5. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$. If $\mathfrak{A}$ is completely representable by partial functions then $\mathfrak{A}$ is atomic.*

Proof. Let a be a nonzero element of $\mathfrak{A}$. Let θ be any complete representation of $\mathfrak{A}$. Then $\emptyset = \theta(0) \neq \theta(a)$, so there exists $(x, y) \in \theta(a)$. By Proposition 4.3.4, the map θ is atomic, so $(x, y) \in \theta(b)$ for some atom b in $\mathfrak{A}$. Then $(x, y) \in \theta(a \cdot b)$, so $a \cdot b > 0$, from which we may conclude that the atom b satisfies $b \leq a$. □

So far we have exploited the Boolean algebras that are contained in any representable $\{;, \cdot, \mathrm{A}\}$-algebra. But we can also travel in the opposite direction and interpret any Boolean algebra as an algebra of the signature $\{;, \cdot, \mathrm{A}\}$, by using the Boolean meet for both the composition and meet operations, and Boolean complement for antidomain. Again this enables us to easily prove results about $\{;, \cdot, \mathrm{A}\}$-algebras using results about Boolean algebras.

We know by the following argument that a Boolean algebra, $\mathfrak{B}$, viewed as an algebra of the signature $\{;, \cdot, \mathrm{A}\}$, is representable by partial functions. By the Birkhoff–Stone representation theorem we may assume that $\mathfrak{B}$ is a field of sets. Then the set of all identity functions on elements of $\mathfrak{B}$ forms a representation of $\mathfrak{B}$ by partial functions. Using the same argument, it is easy to see that a Boolean algebra is completely representable as a field of sets if and only if it is completely representable by partial functions.

Hirsch and Hodkinson used Theorem 4.3.3 to identify those Boolean algebras completely representable as fields of sets as precisely the atomic Boolean algebras.[2] Hence a Boolean algebra is completely representable by *partial functions* if and only if it is atomic. The following proposition uses this fact to prove various negative results about the axiomatisability of the class of completely representable $\{;, \cdot, \mathrm{A}\}$-algebras.

Proposition 4.3.6. *The class of $\{;, \cdot, \mathrm{A}\}$-algebras that are completely representable by partial functions is not closed with respect to the operations shown in the following table and so is not axiomatisable by first-order theories of the indicated corresponding form.*

[2]This result, that a Boolean algebra is completely representable if and only if it is an atomic algebra, had also been discovered previously by Abian [1].

	Operation	*Axiomatisation*
(i)	*subalgebra*	*universal*
(ii)	*directed union*	*universal-existential*
(iii)	*homomorphism*	*positive*

Proof. In each case we use the fact, which we noted previously, that a Boolean algebra is completely representable by partial functions if and only if it is atomic.

(i) We show that the class is not closed under subalgebras. It follows that the class cannot be axiomatised by any universal first-order theory. Let $\mathfrak{B}$ be any non-atomic Boolean algebra, for example the countable atomless Boolean algebra, which is unique up to isomorphism.[3] By the Birkhoff–Stone representation theorem we may assume that $\mathfrak{B}$ is a field of sets, with base X say. Then $\mathfrak{B}$ is a subalgebra of $\wp(X)$, and $\wp(X)$ is atomic, but $\mathfrak{B}$ is not.

(ii) We show that the class is not closed under directed unions. It follows that the class cannot be axiomatised by any universal-existential first-order theory. Again, let $\mathfrak{B}$ be any non-atomic Boolean algebra. Then $\mathfrak{B}$ is the union of its finitely generated subalgebras, which form a directed set of algebras. The finitely generated subalgebras, being Boolean algebras, are finite and hence atomic. So we have, as required, a directed set of atomic Boolean algebras whose union is not atomic.

(iii) We show that the class is not closed under homomorphic images. It follows that the class cannot be axiomatised by any positive first-order theory. Let X be any infinite set and I the ideal of $\wp(X)$ consisting of finite subsets of X. Then $\wp(X)$ is atomic, but the quotient $\wp(X)/I$ is atomless and nontrivial and so is not atomic. □

Since we have mentioned the subalgebra and homomorphism operations, we note that the class of completely representable $\{;, \cdot, \mathrm{A}\}$-algebras *is* closed under direct products. Indeed, it is routine to verify that given complete representations of each factor in a product we can form a complete representation of the product using disjoint unions in the obvious way.

Proposition 4.3.7. *The class of* $\{;, \cdot, \mathrm{A}\}$*-algebras that are completely representable by partial functions is not axiomatisable by any existential-universal-existential first-order theory.*

Proof. Let $\mathfrak{B}$ be any atomic Boolean algebra with an infinite number of atoms and $\mathfrak{B}'$ be any Boolean algebra that is not atomic but also has an infinite number of atoms. We will show that $\mathfrak{B}'$ satisfies any existential-universal-existential sentence satisfied by $\mathfrak{B}$. Since $\mathfrak{B}$ is completely representable by partial functions and $\mathfrak{B}'$ is not, this shows that the complete representation class cannot be axiomatised by any existential-universal-existential theory.

We will show that for certain Ehrenfeucht–Fraïssé games, duplicator has a winning strategy. For an overview of Ehrenfeucht–Fraïssé games see, for example, [47, Chapter 3]. Briefly, two players, spoiler and duplicator, take turns to choose elements from two algebras. Duplicator wins if the two sequences of choices determine an isomorphism between the subalgebras generated by all the elements chosen.

[3] This can be realised as the periodic elements of $2^{\mathbb{N}}$.

Consider the game in which spoiler must in the first round choose n_1 elements of $\mathfrak{B}$, in the second round n_2 elements of $\mathfrak{B}'$, and in the third and final round n_3 elements of $\mathfrak{B}$. Each round, duplicator responds with corresponding choices from the other algebra. Let φ be any sentence in prenex normal form whose quantifiers are, starting from the outermost, n_1 universals, then n_2 existentials, and finally n_3 universals. It is not hard to convince oneself that if duplicator has a winning strategy for the game then $\mathfrak{B}' \models \varphi \implies \mathfrak{B} \models \varphi$. Hence if duplicator has a winning strategy for all games of this form—where spoiler chooses finite numbers of elements from $\mathfrak{B}$ then $\mathfrak{B}'$ then $\mathfrak{B}$—then all universal-existential-universal sentences satisfied by $\mathfrak{B}'$ are satisfied by $\mathfrak{B}$. Equivalently, $\mathfrak{B}'$ satisfies any existential-universal-existential sentence satisfied by $\mathfrak{B}$, which is what we are aiming to show.

Since our algebras are Boolean algebras, a choice of a finite number of elements from one of the algebras generates a finite subalgebra, with a finite number of atoms. The atoms form a partition, that is, a sequence $(a_1, \ldots, a_n)$ of nonzero elements with $\sum_i a_i = 1$ and $a_i \cdot a_j = 0$ for all $i \neq j$. As the game progresses and more elements are chosen, the partition is refined—the elements of the partition are (finitely) further subdivided. The elements the two players have actually chosen are all uniquely expressible as a join of some subset of the partition.

Suppose that, throughout the game, duplicator is able to maintain a correspondence between the partitions on the two algebras. That is, if spoiler subdivides an element a of the existing partition into $(a_1, \ldots, a_n)$ then the element corresponding to a should be partitioned into a corresponding $(a'_1, \ldots, a'_n)$. Then clearly this determines a winning sequence of moves for duplicator: each of spoiler's choices is the join of some subset of one partition and duplicator's choice should be the join of the corresponding elements of the other partition. At the end of the game there will exist an isomorphism between the generated subalgebras that sends each element chosen during the game to the corresponding choice from the other algebra. Hence a strategy for maintaining a correspondence between the two partitions provides a winning strategy for duplicator.

For an element a of $\mathfrak{B}$ or $\mathfrak{B}'$ we will say that a is of size n, for finite n, if a is the join of n distinct atoms, otherwise a is of infinite size. Duplicator can maintain a correspondence by playing as follows.

Round 1 (Spoiler plays on atomic algebra, duplicator on non-atomic) Duplicator should simply provide a partition with matching sizes.

Round 2 (Spoiler non-atomic, duplicator atomic) For subdivisions of elements of finite size, duplicator can provide a subdivision with matching sizes. For subdivisions of elements of infinite size, there is necessarily at least one element in the subdivision of infinite size—duplicator should select one such, match everything else with distinct single atoms and match this infinite size element with what remains on the atomic side.

Round 3 (Spoiler atomic, duplicator non-atomic) At the start of this round every element of the partition of the atomic algebra is matched with something of greater or equal size on the non-atomic side. Hence duplicator can easily provide matching subdivisions. □

4.4 Distributivity

We now turn our attention to the validity of various distributive laws with respect to the classes of representable and completely representable $\{;, \cdot, \mathrm{A}\}$-algebras. We give the first definition that we will use. Other distributive properties that we refer to later are defined similarly. For distributive properties 'over meets' it should be assumed that definitions only require that the relevant equation holds when *nonempty* subsets are used.

Definition 4.4.1. Let $\mathfrak{P}$ be a poset and $*$ be a binary operation on $\mathfrak{P}$. We say that $*$ is **completely right-distributive over joins** if, for any subset S of $\mathfrak{P}$ and any $a \in \mathfrak{P}$, if $\sum S$ exists, then

$$\sum S * a = \sum(S * \{a\}).$$

Proposition 4.4.2. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ that is representable by partial functions. Then composition is completely right-distributive over joins.*

Proof. As $\mathfrak{A}$ is representable, we may assume the elements of $\mathfrak{A}$ are partial functions. Let S be a subset of $\mathfrak{A}$ such that $\sum S$ exists and let $a \in \mathfrak{A}$.

Firstly, for all $s \in S$ we have $\sum S \,;\, a \geq s \,;\, a$ and so $\sum S \,;\, a$ is an upper bound for $S \,;\, \{a\}$.

Now suppose that for all $s \in S$, the element $b \in \mathfrak{A}$ satisfies $b \geq s \,;\, a$. For $s \in S$, suppose s is defined on x and let $s(x) = y$. If a is defined on y, then $s \,;\, a$ is defined on x, so, since $b \geq s \,;\, a$ and $\sum S \,;\, a \geq s \,;\, a$, in this case $b \cdot (\sum S \,;\, a)$ is defined on x. If a is *not* defined on y then, as $(\sum S)(x) = y$, in this case $\sum S \,;\, a$ is not defined on x. Hence the sub-identity function $\mathrm{D}(b \cdot (\sum S \,;\, a)) + \mathrm{A}(\sum S \,;\, a)$ is defined on the entire domain of s. Therefore

$$(\mathrm{D}(b \cdot (\sum S \,;\, a)) + \mathrm{A}(\sum S \,;\, a)) \,;\, \sum S \geq s.$$

Since s was an arbitrary element of S, we have

$$(\mathrm{D}(b \cdot (\sum S \,;\, a)) + \mathrm{A}(\sum S \,;\, a)) \,;\, \sum S \geq \sum S$$

and so

$$(\mathrm{D}(b \cdot (\sum S \,;\, a)) + \mathrm{A}(\sum S \,;\, a)) \,;\, \sum S = \sum S.$$

Therefore

$$\begin{aligned} \mathrm{D}(b \cdot (\sum S \,;\, a)) \,;\, \sum S \,;\, a &= (\mathrm{D}(b \cdot (\sum S \,;\, a)) + \mathrm{A}(\sum S \,;\, a)) \,;\, \sum S \,;\, a \\ &= \sum S \,;\, a, \end{aligned}$$

which says that wherever the function $\sum S \,;\, a$ is defined, it agrees with the function b, that is to say $b \geq \sum S \,;\, a$. So $\sum S \,;\, a$ is the least upper bound for $\sum(S \,;\, \{a\})$. □

Remark 4.4.3. For $\{;, \cdot, \mathrm{A}\}$-algebras representable by partial functions it is easy to see that the following two laws hold.

(i) For finite S, if $\sum S$ exists, then

$$a \,;\, \sum S = \sum(\{a\} \,;\, S). \qquad \text{(composition is left-distributive over joins)}$$

(ii) For finite, nonempty S,

$$a \mathbin{;} \prod S = \prod(\{a\} \mathbin{;} S). \qquad \text{(composition is left-distributive over meets)}$$

We now give an example that shows that the these distributive laws cannot, in general, be extended to arbitrary joins and meets. We will use this example to show that there exist $\{;, \cdot, \mathrm{A}\}$-algebras that are representable as partial functions, and atomic, but have no atomic representation.

Example 4.4.4. Consider the following concrete algebra of partial functions, $\mathfrak{F}$. Its base is the disjoint union of a one element set, $\{p\}$, and $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$. Let $\mathcal{S}$ be all the subsets of $\mathbb{N}_\infty$ that are either finite and do not contain ∞, or cofinite and contain ∞. The elements of $\mathfrak{F}$ are precisely the following functions.

1. Restrictions of the identity to $A \cup B$ where $A \subseteq \{p\}$ and $B \in \mathcal{S}$.

2. The function f, defined only on p and taking p to ∞.

One can check that $\mathfrak{F}$ is closed under the operations of composition, intersection, and antidomain, that $\mathfrak{F}$ is atomic and that f is an atom.

For $i \in \mathbb{N}$, let g_i be the restriction of the identity to $\{0, \dots, i\}$. Then $\sum_i g_i$ exists and is equal to the identity restricted to $\mathbb{N}_\infty$. So

$$f \mathbin{;} \sum_{i \in \mathbb{N}} g_i = f \neq \emptyset = \sum_{i \in \mathbb{N}} (f \mathbin{;} g_i).$$

For $i \in \mathbb{N}$, let h_i be the restriction of the identity to $\{i, \dots\} \cup \{\infty\}$. Then $\prod_i h_i$ exists and is equal to the nowhere-defined function. So

$$f \mathbin{;} \prod_{i \in \mathbb{N}} h_i = \emptyset \neq f = \prod_{i \in \mathbb{N}} (f \mathbin{;} h_i).$$

Lemma 4.4.5. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ that is completely representable by partial functions. Then composition in $\mathfrak{A}$ is completely left-distributive over joins and completely left-distributive over meets.*

Proof. First we prove that composition is completely left-distributive over joins. Let S be a subset of $\mathfrak{A}$ such that $\sum S$ exists and let $a \in \mathfrak{A}$. Let θ be any complete representation of $\mathfrak{A}$. Suppose that for all $s \in S$ the element $b \in \mathfrak{A}$ satisfies $b \geq a \mathbin{;} s$. Then for all $s \in S$ we have $\theta(b) \supseteq \theta(a \mathbin{;} s)$. Hence

$$\begin{aligned}
\theta(b) &\supseteq \bigcup \theta[\{a\} \mathbin{;} S] \\
&= \bigcup(\{\theta(a)\} \mathbin{;} \theta[S]) \\
&= \theta(a) \mathbin{;} \bigcup \theta[S] \\
&= \theta(a \mathbin{;} \sum S).
\end{aligned}$$

The second equality is a true property of any collection of functions, indeed of any collection of relations. We conclude that $b \geq a \mathbin{;} \sum S$ and hence $a \mathbin{;} \sum S$ is the least upper bound for $\{a\} \mathbin{;} S$.

The proof that composition is completely left-distributive over meets is similar. Let S be a nonempty subset of $\mathfrak{A}$ such that $\prod S$ exists and let $a \in \mathfrak{A}$. Let θ be any complete representation of $\mathfrak{A}$. Suppose that for all $s \in S$, the element $b \in \mathfrak{A}$ satisfies $b \leq a \,;\, s$. Then for all $s \in S$, we have $\theta(b) \subseteq \theta(a \,;\, s)$. Hence

$$\begin{aligned}\theta(b) &\subseteq \bigcap \theta[\{a\} \,;\, S] \\ &= \bigcap(\{\theta(a)\} \,;\, \theta[S]) \\ &= \theta(a) \,;\, \bigcap \theta[S] \\ &= \theta(a \,;\, \prod S).\end{aligned}$$

This time the second equality holds only because we are working with functions. It is not, in general, a true property of relations. We conclude from the above that $b \geq a \,;\, \prod S$ and hence $a \,;\, \prod S$ is the greatest lower bound for $\{a\} \,;\, S$. □

Proposition 4.4.6. *There exist $\{;, \cdot, \mathrm{A}\}$-algebras that are representable by partial functions, and atomic, but have no atomic representation.*

Proof. Let $\mathfrak{F}$ be the algebra of Example 4.4.4. Since $\mathfrak{F}$ *is* an algebra of partial functions, it is certainly representable by partial functions. We have already mentioned that $\mathfrak{F}$ is atomic. We have demonstrated that composition in $\mathfrak{F}$ is neither completely left-distributive over joins nor over meets. Hence, by Lemma 4.4.5, $\mathfrak{F}$ has no complete representation. So, by Proposition 4.3.4, $\mathfrak{F}$ has no atomic representation. □

To make the discussion of distributive laws comprehensive we finish by mentioning right-distributivity of composition over meets. Here the weakest possible result, that the finite version of the law is valid for completely representable algebras, does not hold for representation by partial functions. In the algebra of partial functions shown in Figure 4.1, where sub-identity elements are omitted, we have

$$(f_1 \cdot f_2) \,;\, g = 0 \,;\, g = 0 \neq h = h \cdot h = (f_1 \,;\, g) \cdot (f_2 \,;\, g).$$

The algebra is completely representable because it is already an algebra of partial functions and it is finite.

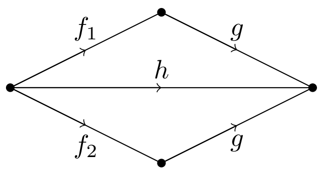

Figure 4.1: An algebra refuting right-distributivity over meets

4.5 A representation

We have seen that for an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ to be completely representable by partial functions it is necessary for it to be representable by partial functions and atomic and for composition to be completely left-distributive over joins. Next we show that these conditions are also sufficient. The representation used for the proof is a Cayley-style representation but also has a certain similarity to the Birkhoff–Stone representation, for our representation uses atoms for its base, and atoms correspond to principal ultrafilters in Boolean algebras.

Proposition 4.5.1. *Let* $\mathfrak{A}$ *be an algebra of the signature* $\{;, \cdot, \mathrm{A}\}$. *Suppose* $\mathfrak{A}$ *is representable by partial functions and atomic, and that composition is completely left-distributive over joins. For each* $a \in \mathfrak{A}$, *let* $\theta(a)$ *be the following partial function on* $\mathrm{At}(\mathfrak{A})$.

$$\theta(a)(x) = \begin{cases} x\,;a & \text{if } x\,;a \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then θ *is a complete representation of* $\mathfrak{A}$ *by partial functions, with base* $\mathrm{At}(\mathfrak{A})$.

Proof. We first need to show that, for each $a \in \mathfrak{A}$, the partial function $\theta(a)$ maps into $\mathrm{At}(\mathfrak{A})$. Let x be an atom and suppose that $x\,;a$ is nonzero. Let $b \in \mathfrak{A}$ and suppose $b \leq x\,;a$. Then $\mathrm{D}(b) \leq \mathrm{D}(x\,;a) \leq \mathrm{D}(x)$. Hence if $\mathrm{D}(b)\,;x = 0$ then $b = 0$. If $\mathrm{D}(b)\,;x > 0$, then we must have $\mathrm{D}(b)\,;x = x$ and hence $\mathrm{D}(b) = \mathrm{D}(x\,;a) = \mathrm{D}(x)$. Therefore $b = x\,;a$. So $x\,;a$ is an atom.

To show that θ represents composition correctly, let $a, b \in \mathfrak{A}$ and $x \in \mathrm{At}(\mathfrak{A})$. Then clearly $\theta(a\,;b)(x) = \theta(a)\,;\theta(b)(x)$ if both sides are defined. The left-hand side is defined precisely when $x\,;a\,;b$ is nonzero and the right-hand side when $x\,;a$ and $x\,;a\,;b$ are both nonzero. Since $x\,;a\,;b \neq 0$ implies $x\,;a \neq 0$, the domains of definition are the same.

To show that θ represents binary meet correctly, let $a, b \in \mathfrak{A}$ and $x, y \in \mathrm{At}(\mathfrak{A})$. Then

$$\begin{aligned} & (x,y) \in \theta(a \cdot b) \\ \implies\ & (x,y) \in \theta(a) \text{ and } (x,y) \in \theta(b) && \text{as } a, b \geq a \cdot b \\ \implies\ & (x,y) \in \theta(a) \cap \theta(b) \end{aligned}$$

and

$$\begin{aligned} & (x,y) \in \theta(a) \cap \theta(b) \\ \implies\ & x\,;a = y \text{ and } x\,;b = y \\ \implies\ & (x\,;a) \cdot (x\,;b) = y \\ \implies\ & x\,;(a \cdot b) = y && \text{by Remark 4.4.3} \\ \implies\ & (x,y) \in \theta(a \cdot b). \end{aligned}$$

To show that antidomain is represented correctly, let $a \in \mathfrak{A}$ and $x \in \mathrm{At}(\mathfrak{A})$. Then $0 < \theta(\mathrm{A}(a))(x) = x\,;\mathrm{A}(a) \leq x$ if $\theta(\mathrm{A}(a))(x)$ is defined. Since x is an atom we have, in this case, $\theta(\mathrm{A}(a))(x) = x$. The partial function $\mathrm{A}(\theta(a))$ is also a restriction of the identity function. The domains of $\theta(\mathrm{A}(a))$ and $\mathrm{A}(\theta(a))$ are the same, since we have seen that $\theta(\mathrm{A}(a))(x)$ is defined precisely when $x\,;\mathrm{A}(a) = x$, which is when $x\,;a = 0$, which is precisely when $\mathrm{A}(\theta(a))(x)$ is defined.

To show that θ is injective, let a and b be distinct elements of $\mathfrak{A}$. Then without loss of generality, $a \not\leq b$. This implies $\mathrm{A}(a \cdot b) \cdot \mathrm{D}(a) \neq 0$ in any representable algebra, such as $\mathfrak{A}$ is. Take an atom x with

$x \leq \mathrm{A}(a \cdot b) \cdot \mathrm{D}(a)$. Then $x ; a$ is nonzero and $x ; b \neq x ; a$, again by inferences valid on any representable algebra. So $\theta(a)(x)$ equals $x ; a$ and $\theta(b)(x)$ does not equal $x ; a$ (it may be undefined). Hence θ maps a and b to distinct partial functions. This completes the proof that θ is a representation of $\mathfrak{A}$ by partial functions.

Finally, we show that the representation θ is complete. Let S be a subset of $\mathfrak{A}$ such that $\sum S$ exists. Let $x, y \in \mathrm{At}(\mathfrak{A})$. Then

$$\begin{aligned} & (x, y) \in \bigcup \theta[S] & \\ \implies \quad & (x, y) \in \theta(s) & \text{for some } s \in S \\ \implies \quad & (x, y) \in \theta(\textstyle\sum S) & \text{as } \textstyle\sum S \geq s \end{aligned}$$

and

$$\begin{aligned} & (x, y) \in \theta(\textstyle\sum S) & \\ \implies \quad & x ; \textstyle\sum S = y & \\ \implies \quad & \textstyle\sum(\{x\} ; S) = y & \text{as } ; \text{ is completely left-distributive over joins} \\ \implies \quad & x ; s = y & \text{for some } s \in S\text{, since } y \text{ is an atom} \\ \implies \quad & (x, y) \in \theta(s) & \text{for some } s \in S \\ \implies \quad & (x, y) \in \bigcup \theta[S]. & \end{aligned}$$

Hence $\theta(\sum S) = \bigcup \theta[S]$. □

4.6 Axiomatising the class

In this final section, we use the conditions for complete representability that we have uncovered to obtain a finite first-order axiomatisation of the complete-representation class.

Definition 4.6.1. A poset $\mathfrak{P}$ is **atomistic** if its atoms are join dense in $\mathfrak{P}$. That is to say that every element of $\mathfrak{P}$ is the join of the atoms less than or equal to it.

Clearly any atomistic poset is atomic. For $\{;, \cdot, \mathrm{A}\}$-algebras representable by partial functions, the converse is also true.

Lemma 4.6.2. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ that is representable by partial functions. If $\mathfrak{A}$ is atomic, then it is atomistic.*

Proof. Suppose $\mathfrak{A}$ is atomic and let $a \in \mathfrak{A}$. By Lemma 4.2.6, the algebra $\downarrow a$ is a Boolean algebra and clearly it is atomic. It is well known that atomic Boolean algebras are atomistic. So we have

$$a = \sum_{\downarrow a} \{x \in \mathrm{At}(\downarrow a) \mid x \leq a\} = \sum_{\mathfrak{A}} \{x \in \mathrm{At}(\downarrow a) \mid x \leq a\} = \sum_{\mathfrak{A}} \{x \in \mathrm{At}(\mathfrak{A}) \mid x \leq a\}.$$

The second equality holds because any upper bound $c \in \mathfrak{A}$ for $\{x \in \mathrm{At}(\downarrow a) \mid x \leq a\}$ is above an upper bound in $\downarrow a$, for example $c \cdot a$. Hence the least upper bound in $\downarrow a$ is least in $\mathfrak{A}$ also. □

Lemma 4.6.3. *Let $\mathfrak{A}$ be an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ that is representable by partial functions and atomic. Let φ be the first-order sentence asserting that for any a, b, c, if $c \geq a ; x$ for all atoms x less than or equal to b, then $c \geq a ; b$. Then composition is completely left-distributive over joins if and only if $\mathfrak{A} \models \varphi$.*

Proof. Suppose first that composition is completely left-distributive over joins. As $\mathfrak{A}$ is atomic it is atomistic. So for any $a, b \in \mathfrak{A}$ we have

$$a ; b = a ; \sum\{x \in \mathrm{At}(\mathfrak{A}) \mid x \leq b\} = \sum(\{a\} ; \{x \in \mathrm{At}(\mathfrak{A}) \mid x \leq b\})$$

and so φ holds.

Now suppose that $\mathfrak{A} \models \varphi$. Let $a \in \mathfrak{A}$ and let S be a subset of $\mathfrak{A}$ such that $\sum S$ exists. Then certainly $a ; \sum S$ is an upper bound for $\{a\} ; S$. To show it is the least upper bound, let c be an arbitrary upper bound for $\{a\} ; S$. Then

$$\begin{array}{lll} & \text{for all } s \in S & c \geq a ; s \\ \Longrightarrow & \text{for all } s \in S \text{ and } x \in \mathrm{At}(\downarrow\sum S) \text{ with } x \leq s & c \geq a ; x \\ \Longrightarrow & \text{for all } x \in \mathrm{At}(\downarrow\sum S) & c \geq a ; x \\ \Longrightarrow & \text{for all } x \in \mathrm{At}(\mathfrak{A}) \text{ with } x \leq \sum S & c \geq a ; x \\ \Longrightarrow & & c \geq a ; \sum S. \end{array}$$

The third line follows from the second because $x \in \mathrm{At}(\downarrow\sum S)$ implies $x \leq s$ for some $s \in S$. To see this, consider the Boolean algebra $\downarrow\sum S$. When x is an atom, $x \nleq s$ if and only if $x \cdot s = 0$, which is equivalent to $\overline{x} \geq s$. So if $x \nleq s$ for all $s \in S$ then $\overline{x} \geq \sum S$, forcing x to be zero—a contradiction. The fifth line can be seen to follow from the fourth by first writing $\sum S$ as the join of the atoms below it and then using φ. □

We now have everything we need to prove our main result.

Theorem 4.6.4. *The class of $\{;, \cdot, \mathrm{A}\}$-algebras that are completely representable by partial functions is a basic elementary class.*

Proof. By Corollary 4.3.5, Lemma 4.4.5 and Proposition 4.5.1, an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ is completely representable by partial functions if and only if it is representable by partial functions, atomic, and composition is completely left-distributive over joins. By Theorem 4.2.3, the property of being representable by partial functions is characterised by a finite set of first-order sentences. The property of being atomic is easily written as a first-order sentence. By Lemma 4.6.3, in the presence of the axioms for the first two properties, the property that composition is completely left-distributive over joins can be written as a first-order sentence. □

We immediately obtain the following corollary (by Theorem 2.4.9).

Corollary 4.6.5. *The problem of determining whether an algebra of the signature $\{;, \cdot, \mathrm{A}\}$ is completely representable by partial functions is decidable in polynomial time (as a function of $|\mathfrak{A}|$).*

Any attempt at writing down our axioms will readily reveal that each can be expressed in a universal-existential-universal form. We know from Proposition 4.3.7 that no existential-universal-existential axiomatisation is possible, hence we have determined the precise amount of quantifier alternation necessary to axiomatise the class.

Note that if range had been included in our signature then the function θ in Proposition 4.5.1 would not be a representation, as it would not represent range correctly. Figure 4.2 shows how this can happen. The atom f satisfies $f \mathbin{;} \mathrm{R}(g) = f$ and so $(f, f) \in \theta(\mathrm{R}(g))$, but there is no h such that $h \mathbin{;} g = f$ and so $(f, f) \notin \mathrm{R}(\theta(g))$. Hence questions about the axiomatisability of the complete representation class for the signature $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$ remain open. Equally for the less expressive signature $\{;, \cdot, \mathrm{D}\}$, where the meet-complete and join-complete representations do not coincide.

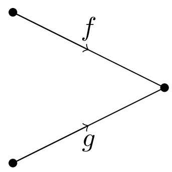

Figure 4.2: Algebra for which θ does not represent range correctly

Chapter 5

The finite representation property for composition, intersection, domain, and range

The work in this chapter has been published as: Brett McLean and Szabolcs Mikulás, *The finite representation property for composition, intersection, domain and range*, International Journal of Algebra and Computation **26** (2016), no. 5, 1199–1216.[1]

ABSTRACT. We prove that the finite representation property holds for representation by partial functions for the signature consisting of composition, intersection, domain, and range and for any expansion of this signature by the antidomain, fixset, preferential union, maximum iterate, and opposite operations. The proof shows that, for all these signatures, the size of base required is bounded by a double-exponential function of the size of the algebra. This establishes that representability of finite algebras is decidable for all these signatures. We also give an example of a signature for which the finite representation property fails to hold for representation by partial functions.

5.1 Introduction

The investigation of the abstract algebraic properties of partial functions involves studying the isomorphism class of algebras whose elements are partial functions and whose operations are some specified set of operations on partial functions—operations such as composition or intersection, for example. We refer to an algebra isomorphic to an algebra of partial functions as representable.

As we have indicated in previous chapters, one of the primary aims is to determine how simply the class of representable algebras can be axiomatised and to find such an axiomatisation. Often, the representation classes have turned out to be axiomatisable by finitely many equations or quasi-equations [92, 52, 53, 55, 57, 44]; we detailed this earlier, in Section 3.2.1.

Another question to ask is whether every finite representable algebra can be represented by partial functions on some finite set. Interest in this so-called finite representation property originates from its potential to help prove decidability of representability, which in turn can help give decidability of the equational or universal theories of the representation class.

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Journal URL: http://www.worldscientific.com/worldscinet/ijac

Recently, Hirsch, Jackson, and Mikulás established the finite representation property for many signatures, but they leave the case for signatures containing the intersection, domain, and range operations together open [44].

In this chapter we prove the finite representation property for the most significant group of outstanding signatures, which includes a signature containing all the most commonly considered operations on partial functions. From our proof we obtain a double-exponential bound on the size of base set required for a representation. It follows as a corollary that representability of finite algebras is decidable for all these signatures. As an additional observation, we give an example showing that there are signatures for which the finite representation property does not hold for representation by partial functions.

The results presented here originate with McLean [71]. The contribution of the second author is to translate the original proof of the finite representation property into a semantical setting, so that the presence of antidomain is not necessary.

5.2 Algebras of partial functions

In this section we give the fundamental definitions that are needed in order to state the results contained in this chapter.

Given an algebra $\mathfrak{A}$, when we write $a \in \mathfrak{A}$ or say that a is an element of $\mathfrak{A}$, we mean that a is an element of the domain of $\mathfrak{A}$. We follow the convention that algebras are always nonempty.

Definition 5.2.1. Let σ be an algebraic signature whose symbols are a subset of $\{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, \uparrow, {}^{-1}\}$. An **algebra of partial functions** of the signature σ is an algebra of the signature σ whose elements are partial functions and with operations given by the set-theoretic operations on those partial functions described in the following.

Let X be the union of the domains and ranges of all the partial functions occurring in an algebra $\mathfrak{A}$. We call X the **base** of $\mathfrak{A}$. The interpretations of the operations in σ are given as follows:

- the binary operation ; is **composition** of partial functions:

$$f \,;\, g = \{(x, z) \in X^2 \mid \exists y \in X : (x, y) \in f \text{ and } (y, z) \in g\},$$

  that is, $(f \,;\, g)(x) = g(f(x))$,

- the binary operation $\cdot$ is **intersection**:

$$f \cdot g = \{(x, y) \in X^2 \mid (x, y) \in f \text{ and } (x, y) \in g\},$$

- the unary operation D is the operation of taking the diagonal of the **domain** of a function:

$$\mathrm{D}(f) = \{(x, x) \in X^2 \mid \exists y \in X : (x, y) \in f\},$$

- the unary operation R is the operation of taking the diagonal of the **range** of a function:

$$\mathrm{R}(f) = \{(y, y) \in X^2 \mid \exists x \in X : (x, y) \in f\},$$

- the constant 0 is the nowhere-defined **empty function**:

$$0 = \emptyset,$$

- the constant 1' is the **identity function** on X:

$$1\text{'} = \{(x, x) \in X^2\},$$

- the unary operation A is the operation of taking the diagonal of the **antidomain** of a function—those points of X where the function is not defined:

$$\mathrm{A}(f) = \{(x, x) \in X^2 \mid \nexists y \in X : (x, y) \in f\},$$

- the unary operation F is **fixset**, the operation of taking the diagonal of the fixed points of a function:

$$\mathrm{F}(f) = \{(x, x) \in X^2 \mid (x, x) \in f\},$$

- the binary operation $\sqcup$ is **preferential union**:

$$(f \sqcup g)(x) = \begin{cases} f(x) & \text{if } f(x) \text{ defined} \\ g(x) & \text{if } f(x) \text{ undefined, but } g(x) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

- the unary operation $^{\uparrow}$ is the **maximum iterate**:

$$f^{\uparrow} = \bigcup_{n \in \mathbb{N}} (f^n \mathbin{;} \mathrm{A}(f)),$$

  where $f^0 := 1\text{'}$ and $f^{n+1} := f \mathbin{;} f^n$,

- the unary operation $^{-1}$ is an operation we call **opposite**:

$$f^{-1} = \{(y, x) \in X^2 \mid (x, y) \in f \text{ and } ((x', y) \in f \implies x = x')\}.$$

The list of operations in Definition 5.2.1 does not exhaust those that have been considered for partial functions but does include the most commonly appearing operations.

Definition 5.2.2. Let $\mathfrak{A}$ be an algebra of one of the signatures permitted by Definition 5.2.1. A **representation of $\mathfrak{A}$ by partial functions** is an isomorphism from $\mathfrak{A}$ to an algebra of partial functions of the same signature. If $\mathfrak{A}$ has a representation then we say it is **representable**.

In [55], Jackson and Stokes give a finite equational axiomatisation of the representation class for the signature $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ and similarly for any expansion of this signature by operations in $\{0, 1\text{'}, \mathrm{F}\}$.

In [44], Hirsch, Jackson, and Mikulás give a finite equational axiomatisation of the representation class for the signature $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$ and similarly for any expansion of this signature by operations in $\{0, 1\text{'}, \mathrm{D}, \mathrm{F}, \sqcup\}$. For expanded signatures containing the maximum iterate operation they give finite sets of axioms that, if we restrict attention to finite algebras, axiomatise the representable ones.

The operation that we call opposite is described in [77], where Menger calls the concrete operation 'bilateral inverse' and uses 'opposite' to refer to an abstract operation intended to model this bilateral inverse. The opposite operation appears again in Schweizer and Sklar's [95] and [96] but thereafter does not appear to have received any further attention. In particular, for signatures containing opposite, axiomatisations of the representation classes remain to be found.

Definition 5.2.3. Let σ be a signature. We say that σ has the **finite representation property** (for representation by partial functions) if whenever a finite algebra of the signature σ is representable by partial functions, it is representable on a finite base.[2]

In [44], Hirsch, Jackson, and Mikulás establish the finite representation property for many signatures that are subsets of $\{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, ^\uparrow\}$. Assuming composition is in the signature, they prove the finite representation property holds for any such signature that cannot express domain, any not containing range, and almost all that do not contain intersection. This leaves one significant group of cases, which they highlight as an open problem: signatures containing $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$.

In this chapter we prove that $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ and any expansion of $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ by operations that we have mentioned (including opposite) all have the finite representation property. The following example may give some intuition about this problem and its solution.

Example 5.2.4. Let $\mathfrak{F}_1$ be the algebra of partial functions, of the signature $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ and with base $\mathbb{Z} \times 2$, consisting of the following five elements.

- 0, the empty function,
- d, the identity function on $\mathbb{Z} \times \{0\}$,
- r, the identity function on $\mathbb{Z} \times \{1\}$,
- f, the function with domain d and range r sending each $(n, 0)$ to $(n, 1)$,
- g, the function with domain d and range r sending each $(n, 0)$ to $(n + 1, 1)$.

See Figure 5.1.

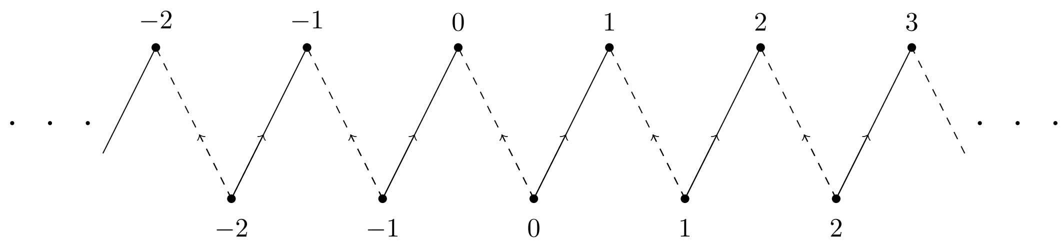

Figure 5.1: The algebra $\mathfrak{F}_1$. Dashed lines for f, solid lines for g

The algebra $\mathfrak{F}_1$, being an actual algebra of partial functions, is trivially representable. However, this representation uses an infinite base. To try to reduce the infinite representation to a finite one, we may

[2]This property has also been called the *finite algebra on finite base property*.

observe that there are two types of base points: those mapped to themself by d (the points in $\mathbb{Z} \times \{0\}$), and those mapped to themself by r (the points in $\mathbb{Z} \times \{1\}$). We may then attempt to identify all base points of the same type. But doing this produces the structure pictured on the left of Figure 5.2. This does not yield a representation of $\mathfrak{F}_1$, for the 'representations' of f and g are not disjoint, conflicting with the fact that $f \cdot g = 0$ in $\mathfrak{F}_1$.

We cannot then necessarily construct a representation using only one copy of each type of base point. However, the structure pictured on the right of Figure 5.2 uses two copies of each type, and *does* yield a representation of $\mathfrak{F}_1$. Hence, in this case, with enough copies, a finite representation can be given. The proof of the main theorem of this chapter, Theorem 5.4.3, shows it is always possible to gather 'enough' copies of parts of infinite representations to be able to construct a finite representation, and describes that representation.

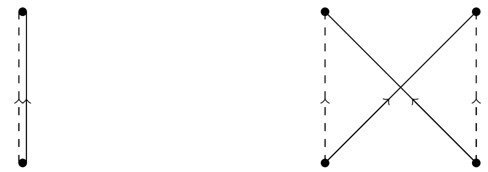

Figure 5.2: Left: a non-representation of $\mathfrak{F}_1$. Right: a representation of $\mathfrak{F}_1$

5.3 Uniqueness of presents and futures

In this section we derive some results about representations of finite algebras, which we will use in the following section to prove the finite representation property holds.

Throughout this section, let σ be a signature with $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, {}^{\uparrow}, {}^{-1}\}$ and let $\mathfrak{A}$ be a finite representable σ-algebra. Since $\mathfrak{A}$ is representable, we may freely make use of basic properties of algebras of partial functions in the process of our deductions.

First note that the algebra $\mathfrak{A}$ is a meet-semilattice, with meet given by $\cdot$. Whenever we treat $\mathfrak{A}$ as a poset, we are using the order induced by this semilattice. The set $\mathrm{D}(\mathfrak{A}) := \{\mathrm{D}(a) \mid a \in \mathfrak{A}\}$ of **domain elements** forms a subsemilattice of $\mathfrak{A}$. We will usually use Greek letters to denote domain elements.

In any representation θ of $\mathfrak{A}$ we have that $(x, y) \in \theta(a)$ if and only if a is greater than or equal to the meet of the finite set $\{b \in \mathfrak{A} \mid (x, y) \in \theta(b)\}$. Hence we may identify each representation of $\mathfrak{A}$ with a particular edge-labelled directed graph (with reflexive edges). The label of an edge (x, y) is the least element of $\{b \in \mathfrak{A} \mid (x, y) \in \theta(b)\}$. Since we take the base of a representation to be the union of the domains and ranges of all the partial functions, every vertex participates in some edge. Given that the domain and range operations are in the signature, this means that all vertices will have a reflexive edge.

The previous paragraph motivates our interest in the following type of object, of which representations of $\mathfrak{A}$ are a special case.

Definition 5.3.1. A **network** over $\mathfrak{A}$ will be an edge-labelled directed graph, with labels drawn from $\mathfrak{A}$ and with a reflexive edge on every vertex.

Given a network N, we will follow the usual convention of writing $x \in N$ to mean x is a vertex of

N and we will denote the label of an edge (x, y) by $N(x, y)$. We will speak of an element $b \in \mathfrak{A}$ holding on an edge (x, y) when $N(x, y) \leq b$. We will call a vertex x an **α-vertex** if the reflexive edge at that vertex is labelled α, that is if $N(x, x) = \alpha$.

Note that if x is an α-vertex of a representation then α is necessarily a domain element. Indeed if α holds on (x, x) it follows that $\mathrm{D}(\alpha)$ holds on (x, x). Since α is the least such element, we have $\alpha \leq \mathrm{D}(\alpha)$. It follows that $\alpha = \mathrm{D}(\alpha)$, by a property of partial functions.

An isomorphism of networks is just an isomorphism of labelled graphs, that is, a graph isomorphism that preserves labels.

Definition 5.3.2. Let N be a network (over $\mathfrak{A}$) and let W be a subset of the vertices of N. We define the **future** of W to be the subnetwork induced by the vertices reachable via an edge starting in W. We define the future of a vertex x to be the future of the singleton set containing x.

Since $\mathfrak{A}$ is finite and we are representing by partial functions, in a *representation*, the future of a vertex must be a finite network.

Note also that in a representation, the taking of futures is a closure operator. Indeed, each $x \in W$ is reachable from W via the reflexive edge at x. If there is an edge from $x \in W$ to y, labelled a, and from y to z, labelled b, then z is reachable via an edge starting in W, labelled $a \mathbin{;} b$.

Definition 5.3.3. Let N be a network (over $\mathfrak{A}$). The **present** of a vertex x of N is the set of all vertices y such that y is in the future of x and x is in the future of y.

We are interested in presents and futures because in Section 5.4 we will describe how to use the presents and futures extant in representations in order to construct a representation on a finite base.

Definition 5.3.4. Let $a \in \mathfrak{A}$. If there exists a representation of $\mathfrak{A}$ in which a labels an edge, then we will call a **realisable**.

Proposition 5.3.5. *For any realisable domain element $\alpha \in \mathfrak{A}$, any two α-vertices x and x' from any two representations have isomorphic futures, and the isomorphism can be chosen so that x maps to x'.*

Proof. Let x be an α-vertex of the network N obtained from a representation, that is, $\alpha = N(x, x)$. We claim that for every $a \in \mathfrak{A}$, there is an edge starting at x labelled with a if and only if $\mathrm{D}(a) = \alpha$.

Suppose first that $\mathrm{D}(a) = \alpha$. Then, by the definition of the domain operation, there must be an edge starting at x labelled with some $b \leq a$. Then $\mathrm{D}(b)$ must hold on (x, x) and so $\mathrm{D}(a) \leq \mathrm{D}(b)$. From $b \leq a$ and $\mathrm{D}(a) \leq \mathrm{D}(b)$ it follows that $a = b$, by a property of partial functions. Hence there is an a-labelled edge starting at x.

Conversely, suppose there is an edge labelled with a starting at x, ending at y say. Then $\alpha \mathbin{;} a$ holds on (x, y) and so $a \leq \alpha \mathbin{;} a$. Since α is a domain element, this implies $\mathrm{D}(a) \leq \alpha$, by a property of partial functions. But $\mathrm{D}(a)$ holds on (x, x) and so $\alpha \leq \mathrm{D}(a)$, since $\alpha = N(x, x)$. We conclude that $\mathrm{D}(a) = \alpha$.

Note that as the elements of $\mathfrak{A}$ are represented by partial functions, there cannot be multiple edges starting at x on which the same element holds. In particular there cannot be multiple edges with the same label. We therefore now know that for any α-vertex x, the edges starting at x are precisely a single edge labelled a for every a with $\mathrm{D}(a) = \alpha$ (the α-labelled edge being the reflexive one). So we have an obvious candidate for an isomorphism between the futures of α-vertices: given two α-vertices x and x' in networks N and N' respectively, we let $y \mapsto y'$ if and only if $N(x, y) = N'(x', y')$. This entails $x \mapsto x'$.

For a and b with $\mathrm{D}(a) = \mathrm{D}(b) = \alpha$, let (x, y) and (x, z) be the two edges starting at x and labelled by a and b respectively. To show that the future of x is uniquely determined up to isomorphism, we only need show that the set of elements of $\mathfrak{A}$ holding on (y, z) is uniquely determined. We claim that

$$\text{if } N(x, y) = a \text{ and } N(x, z) = b, \text{ then } c \text{ holds on } (y, z) \text{ if and only if } a \,;\, c = b, \tag{5.1}$$

which gives us what we want.

Suppose first that $a \,;\, c = b$. Then as (x, y) is the *unique* edge starting at x on which a holds, c must hold on (y, z) in order that composition be represented correctly. Conversely, suppose that c holds on (y, z). Then $a \,;\, c$ holds on (x, z) and so $b \leq a \,;\, c$. But $\mathrm{D}(a \,;\, c) \leq \mathrm{D}(a)$ is valid in all representable algebras and $\mathrm{D}(a) = \mathrm{D}(b)$. From $b \leq a \,;\, c$ and $\mathrm{D}(a \,;\, c) \leq \mathrm{D}(b)$ we may conclude $a \,;\, c = b$, by a property of partial functions. □

For realisable domain elements α and β, write $\alpha \lesssim \beta$ if in a (or every) representation of $\mathfrak{A}$ there is a β-vertex in the future of every α-vertex, or equivalently if there exists an $a \in \mathfrak{A}$ with $\mathrm{D}(a) = \alpha$ and $\mathrm{R}(a) = \beta$. Then $\lesssim$ is easily seen to be a preorder on the realisable domain elements.

Proposition 5.3.6. *For any realisable domain elements $\alpha, \beta \in \mathfrak{A}$, if α and β are $\lesssim$-equivalent, then any α-vertex and any β-vertex from any two representations have isomorphic futures.*

Proof. Let x be an α-vertex, from some representation of $\mathfrak{A}$. As $\alpha \lesssim \beta$, in the same representation there is a β-vertex, y say, in the future of x. As $\beta \lesssim \alpha$ there is an α-vertex, z say, in the future of y. Hence z is in the future of x, meaning that the future of z is a subnetwork of the future of x. But these are finite isomorphic objects and therefore equal. So x is in the future of z and therefore x is in the future of y. Hence the futures of the α-vertex x and the β-vertex y are equal in this representation. By Proposition 5.3.5, we conclude the required result. □

In a representation, the present of an α-vertex x is always the initial, strongly connected component of x's future—the one that can 'see' the entire future of x. So we get the following immediate corollary of Proposition 5.3.6.

Corollary 5.3.7. *For any realisable domain elements $\alpha, \beta \in \mathfrak{A}$, if α and β are $\lesssim$-equivalent, then any α-vertex and any β-vertex from any two representations have isomorphic presents.*

Given a $\lesssim$-equivalence class E, we will speak of 'the future of E' to mean the unique isomorphism class of futures of α-vertices in representations, for any $\alpha \in E$. Similarly for 'the present of E'.

5.4 The finite representation property

In this section, we prove our main result: the finite representation property holds for all the signatures we are interested in. We then use our proof to calculate an upper bound on the size of base required.

To construct a representation over a finite base we will use the realisable domain elements and the preorder $\lesssim$ on them defined in Section 5.3. Recall that the realisable domain elements are the domain elements appearing as reflexive-edge labels in some representation.

We start with a lemma that is little more than a translation of the definition of a representation into the language of graphs, but which gives us an opportunity to state exactly what is needed in order for a network to be a representation.

Lemma 5.4.1. *Let $\mathfrak{A}$ be a finite representable algebra of a signature σ with $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, {}^\uparrow, {}^{-1}\}$ and let N be a network (over $\mathfrak{A}$). Then N is a representation of $\mathfrak{A}$ if and only if the following conditions are satisfied.*

(i) ***(Relations are functions)*** *For any vertex x of N and any $a \in \mathfrak{A}$ there is at most one edge starting at x on which a holds.*

(ii) ***(Operations represented correctly)*** *Let $*$ be an operation in the signature (excluding $\cdot$). Then (assuming for simplicity of presentation that $*$ is a binary operation) if we apply the appropriate set-theoretic operation to the set of edges where a holds and the set of edges where b holds then we get precisely the set of edges where $a * b$ holds:*

$$\{(x, y) \mid N(x, y) \leq a\} * \{(x', y') \mid N(x', y') \leq b\} = \{(x'', y'') \mid N(x'', y'') \leq a * b\}.$$

(iii) ***(Faithful)*** *For every $a, b \in \mathfrak{A}$ with $a \not\leq b$, there is an edge of N on which a holds, but b does not.*

Proof. Routine. □

We also require the following definition.

Definition 5.4.2. Let $\mathfrak{A}$ be a finite representable algebra of a signature σ with $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, {}^\uparrow, {}^{-1}\}$. From the relation $\lesssim$ defined in Section 5.3, form the partial order of $\lesssim$-equivalence classes (of realisable domain elements of $\mathfrak{A}$). The **depth** of a $\lesssim$-equivalence class E will be the length of the longest increasing chain in this partial order, starting at E. (We take the length of a chain to be one fewer than the number of elements it contains, so a maximal $\lesssim$-equivalence class has depth zero.) Since $\mathfrak{A}$ is finite, the depth of every $\lesssim$-equivalence class is finite and bounded by the size of $\mathfrak{A}$. The depth of a realisable domain element will be the depth of its $\lesssim$-equivalence class.

We are now ready to prove our main result, but note that the following theorem does not cover signatures containing opposite.

Theorem 5.4.3. *The finite representation property holds for representation by partial functions for any signature σ with $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, {}^\uparrow\}$.*

Proof. Let $\mathfrak{A}$ be a finite representable algebra of one of the signatures under consideration. We construct a finite network N step by step, by adding copies of the present of $\lesssim$-equivalence classes of increasing depths. Then we argue that the resulting network is a representation of $\mathfrak{A}$. The idea of the proof is to always 'add everything we can, $|\mathfrak{A}|$ times'.

Assume inductively that we have carried out steps $0, \ldots, n-1$ of our construction, giving us the network N_{n-1}. We form N_n as follows. (For the base case of this induction, we let N_{-1} be the empty network.) Let E be a $\lesssim$-equivalence class of depth n, and P a copy of the present of E. A choice of edges from P to N_{n-1} labelled by elements of $\mathfrak{A}$ is **allowable** if adding P and these edges to N_{n-1} would make the future of P in the extended network isomorphic to the future of E. For every allowable choice, to N_{n-1} we add $|\mathfrak{A}|$ copies of both P and the edges from P specified by the choice. The network N_n is the network we have once we have done this for every $\lesssim$-equivalence class of depth n.

Note that the order that $\lesssim$-equivalence classes of a given depth n are processed is immaterial, since no allowable choice could have an edge ending at a vertex that had not been in N_{n-1}. By induction, each N_n is finite: assume that N_{n-1} is finite; then as each P is also finite and $\mathfrak{A}$ is finite we see that the number of allowable choices is finite, so N_n is finite. We take N to be N_M, where M is the maximum depth of any $\lesssim$-equivalence class of $\mathfrak{A}$.

For any vertex x of N, the future of x during the various stages of the construction of N is unaltered once x has been added to the construction. So the future of x in N is isomorphic to the future of some vertex in a representation of $\mathfrak{A}$, since this is true at the moment that x is added to the construction, by the definition of an allowable choice.

The next lemma will ensure that allowable choices always exist. Let N' be the underlying network for a representation of $\mathfrak{A}$. Fix a vertex $x \in N'$ and write F for the future of x in N'. Define F_n to be the subnetwork of F induced by vertices y such that the depth of $N'(y, y)$ is at most n.

Lemma 5.4.4. *For every $n \geq -1$, there is an embedding $f_n \colon F_n \hookrightarrow N_n$. Moreover, if G is any future-closed subset of F and $g \colon G \hookrightarrow N$ is an embedding, then f_n can be chosen so that it agrees with g wherever f_n and g are both defined.*

Proof. We use induction on n. As before, the base case for the induction is depth -1, so we define f_{-1} to be the empty map.

For $n > -1$, suppose we have an embedding $f_{n-1} \colon F_{n-1} \hookrightarrow N_{n-1}$, a future-closed subset G of F, and an embedding $g \colon G \hookrightarrow N$ such that f_{n-1} and g agree where both are defined. We can form f_n, an extension of f_{n-1}, as follows. First use g to define an intermediate extension f'_{n-1} of f_{n-1} to those vertices in $(F_n \setminus F_{n-1}) \cap G$, that is,

$$f'_{n-1}(y) = \begin{cases} f_{n-1}(y) & \text{if } y \in F_{n-1} \\ g(y) & \text{if } y \in (F_n \setminus F_{n-1}) \cap G. \end{cases}$$

Using the assumption that G is future closed, we will show that this intermediate extension f'_{n-1} is still an embedding. Observe that for any $y \in (F_n \setminus F_{n-1}) \cap G$, the future of $g(y)$ in N is isomorphic to the future of y in N', since the future of any vertex of N is isomorphic to the future, in any representation, of any vertex with the same reflexive-edge label. Hence f'_{n-1} maps elements of $(F_n \setminus F_{n-1}) \cap G$ to elements of $N_n \setminus N_{n-1}$, from which we see that f'_{n-1} is injective. Now for f'_{n-1} to be an embedding of $F_{n-1} \cup (F_n \cap G)$, we need to show that for arbitrary $y, z \in F_{n-1} \cup (F_n \cap G)$ and $a \in \mathfrak{A}$, there is an edge labelled a from y to z if and only if there is an edge labelled a from $f'_{n-1}(y)$ to $f'_{n-1}(z)$. If $y, z \in F_{n-1}$

then we use that f_{n-1} is an embedding. Similarly, if $y, z \in (F_n \setminus F_{n-1}) \cap G$ then we use that g is an embedding. If $y \in F_{n-1}$ and $z \in (F_n \setminus F_{n-1}) \cap G$ then there is no edge (y, z), because there are no edges from F_{n-1} to $F_n \setminus F_{n-1}$, nor is there an edge $(f'_{n-1}(y), f'_{n-1}(z))$, because there are no edges from N_{n-1} to $N_n \setminus N_{n-1}$. It remains to consider the case when $y \in (F_n \setminus F_{n-1}) \cap G$ and $z \in F_{n-1}$.

First assume there is an edge from y to z labelled a. Then $z \in G$, since G is future closed. Hence $f'_{n-1}(z) = f_{n-1}(z) = g(z)$, by the inductive hypothesis, and the edge $(g(y), g(z))$ is labelled a in N, since g is an embedding.

Conversely, assume there is an edge from $g(y)$ to $f'_{n-1}(z)$ labelled a. Then $f'_{n-1}(z)$ is in the future of $g(y)$ in N. Since the future of $g(y)$ in N is isomorphic to the future of y in the representation N', we see firstly that $f'_{n-1}(z)$ is the unique vertex of N being the end of an a-labelled edge starting at $g(y)$. Secondly, in N' there is a unique a-labelled edge, (y, z') say, starting at y. Then $z' \in G \cap F_n$, since G and F_n are future closed, and $g(z') = f'_{n-1}(z)$, since g is an embedding. By injectivity of f'_{n-1}, we have $z' = z$ and so $N'(y, z) = a$, as desired. This completes the proof that f'_{n-1} is an embedding.

The remaining vertices we need to extend to are partitioned into copies of the present of various $\lesssim$-equivalence classes of depth n. Fix one copy P in F_n of one of these equivalence classes. Then P and f_{n-1} (being an embedding of F_{n-1} into N_{n-1}) together specify an allowable choice of edges from P to N_{n-1}. Since every allowable choice has been replicated $|\mathfrak{A}|$ times during each step of the construction of N_n, this provides not just one but $|\mathfrak{A}|$ possible ways to extend f_{n-1} to P and to the edges starting in P. The number of edges starting at x in F is bounded by the number of elements of $\mathfrak{A}$. So F, and therefore F_n, certainly contain no more than $|\mathfrak{A}|$ copies of the present of any $\lesssim$-equivalence class of depth n (including any copies in G). Hence there exists a way to extend f'_{n-1} to all these copies simultaneously. □

It remains to show that N is a representation of $\mathfrak{A}$, so we need to show that N satisfies the conditions of Lemma 5.4.1. It is easy to see that the relations are functions. From the fact that the future of any vertex of N is isomorphic to the future of some vertex in a representation of $\mathfrak{A}$, it follows that there is at most one edge starting at x on which any given $a \in \mathfrak{A}$ holds.

Next we need to show that the operations are represented correctly by N. With the exception of range, all the operations are straightforward and similar to show. We again rely on the fact that for any vertex x in N, the future of x is isomorphic to the future of some vertex in a representation of $\mathfrak{A}$.

To see that composition is represented correctly, suppose first that a holds on (x, y) and b holds on (y, z). Then as the future of x matches the future of some vertex in a representation, $a\,;b$ holds on (x, z). Conversely, suppose that $a;b$ holds on (x, z). Then again by matching x with a vertex in a representation, we know there is a y such that a holds on (x, y) and b holds on (y, z).

To see that domain is represented correctly, suppose first that $\mathrm{D}(a)$ holds on (x, y). Then by matching x with a vertex in a representation, we know both that $x = y$ and that there is an edge starting at x on which a holds. Conversely, suppose that a holds on an edge (x, y). Then by matching x, we see that $\mathrm{D}(a)$ must hold on (x, x).

If 0 is in the signature, then no edge in any representation of $\mathfrak{A}$ is labelled with 0. Hence 0 does not hold on any edge in N and so N represents 0 correctly. If 1' is in the signature then in any representation, 1' holds on all reflexive edges and no others. Hence the same is true of N and so N represents 1' correctly.

To see that antidomain is represented correctly if it is in the signature, suppose first that $\mathrm{A}(a)$ holds on (x, y). Then by matching x with a vertex in a representation, we know both that $x = y$ and that there is no edge starting at x on which a holds. Conversely, suppose there is no edge starting at x on which a holds. Then by matching x, we see that $\mathrm{A}(a)$ must hold on (x, x).

To see that fixset is represented correctly if it is in the signature, suppose first that $\mathrm{F}(a)$ holds on (x, y). Then by matching x with a vertex in a representation, we know both that $x = y$ and that a holds on (x, x). Conversely, suppose a holds on (x, x). Then by matching x, we see that $\mathrm{F}(a)$ must hold on (x, x).

To see that preferential union is represented correctly if it is in the signature, suppose first that $a \sqcup b$ holds on (x, y). Then by matching x with a vertex in a representation, we know that on (x, y) either a holds, or a does not hold and b does. Conversely, suppose that on (x, y) either a holds, or a does not hold and b does. Then by matching x, we see that $a \sqcup b$ must hold on (x, y).

To see that maximum iterate is represented correctly if it is in the signature, suppose first that $a^{\uparrow}$ holds on (x_0, x_n). Then by matching x_0 with a vertex in a representation, we know there exist $x_0, x_1, \dots, x_n$ such that a holds on each x_i, x_{i+1} and there is no edge starting at x_n on which a holds. Conversely, suppose there exist $x_0, x_1, \dots, x_n$ such that a holds on each (x_i, x_{i+1}) and there is no edge starting at x_n on which a holds. Then by matching x_0, we see that $a^{\uparrow}$ must hold on (x_0, x_{i+1}).

One direction of range being represented correctly is clear: if N has an edge from x to y on which a holds, then $\mathrm{R}(a)$ will hold on the reflexive edge at y. For the other direction, suppose that $\mathrm{R}(a)$ holds on a vertex y in N and let β be the label of y. Then we know that we can find a β-vertex, y' say, in a representation and that $\mathrm{R}(a)$ will hold on y'. So there is an edge (x', y') in this representation on which a holds. Since the future of y' is isomorphic to the future of y via an isomorphism sending y' to y, there is an embedding of the future of y' into N sending y' to y. Then Lemma 5.4.4 ensures we can embed the future of x' into N in such a way that y' is mapped to y and so there is an a-labelled edge ending at y.

For the condition that N be faithful, consider any $a, b \in \mathfrak{A}$ with $a \not\leq b$. Then as $\mathfrak{A}$ is representable, there certainly exists some realisable $\alpha \in \mathfrak{A}$ having an edge in its future on which a holds, but b does not. Since the futures of α-vertices in N are isomorphic to the futures of α-vertices in representations, it suffices to show that for every $\lesssim$-equivalence class E, a nonzero number of copies of the present of E are added at the appropriate stage of the construction. But this is obvious, by Lemma 5.4.4. $\square$

With a little more work we can expand the list of operations to include opposite.

Theorem 5.4.5. *The finite representation property holds for representation by partial functions for any signature σ with* $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, {}^{\uparrow}, {}^{-1}\}$.

Proof. Let $\mathfrak{A}$ be a finite representable algebra of one of the specified signatures. We may assume that $\mathfrak{A}$ has more than one element, as the one-element algebra is representable using the empty set as a base.

We will argue that if the signature contains opposite, then the network N described in the proof of Theorem 5.4.3 represents opposite correctly.

Suppose first that a^{-1} holds on (y, x). We want to show that (x, y) is the unique edge ending at y on which a holds. As the future of y matches the future of some vertex in a representation, we know that a holds on (x, y). To show that (x, y) is the unique such edge, suppose a holds on (x', y). Then x, x', and y are all in the future of x'. So in the future of x' we have a^{-1} holding on (y, x), and a holding on (x, y) and (x', y). As the future of x' matches the future of some vertex in a representation, it follows that $x = x'$, as required.

Conversely, suppose that (x, y) is the unique edge ending at y on which a holds. We want to show that a^{-1} holds on (y, x). Let α be the label of the reflexive edge at x, let β be the label of the reflexive edge at y, and let b be the label of (x, y). First note that if α were in a deeper $\lesssim$-equivalence class than β, then, because of the way N is constructed, there would be at least $|\mathfrak{A}|$ edges ending at y on which a holds. Hence α and β are in the same $\lesssim$-equivalence class.

Now the present of x is isomorphic to the present of any α-vertex in any representation of $\mathfrak{A}$. So it suffices to show that for an α-vertex x' in a representation of $\mathfrak{A}$, if (x', y') is the b-labelled edge from x' to a β-vertex, then a^{-1} holds on (y', x'). Being situated in a representation, we can show this by proving that (x', y') is the unique edge ending at y' on which a holds.

Suppose then that a holds on (z', y') and let γ be the label of the reflexive edge at z'. Suppose $\gamma \neq \alpha$. We saw, in proving that range is represented correctly, that we can embed the future of z' into N in such a way that y' is mapped to y. So there is an edge starting at a γ-vertex and ending at y on which a holds. But this is a contradiction, as the edge (x, y), starting at an α-vertex, is supposed to be the unique edge ending at y on which a holds. We conclude that $\gamma = \alpha$ and hence z' is in the present of y', since α and β are in the same $\lesssim$-equivalence class. We must now have $x' = z'$, for otherwise the present of x' would feature two distinct edges ending at y' on which a holds. We know this not to be the case, by comparison with y, in the present of x. Hence (x', y') is the unique edge ending at y' on which a holds, as required. $\square$

Given some representation θ of an algebra $\mathfrak{A}$, we could give an alternate definition of the realisable elements of $\mathfrak{A}$ as those appearing as edge labels in *the particular representation* θ, rather than just in any representation. Then our proofs of Theorem 5.4.3 and Theorem 5.4.5 would work equally well. However, with the definition we gave, the constructed representation is in a sense the richest possible, in that if it is possible for an element to appear as a label in a representation, then it appears as a label in the constructed representation.

It is clear that from the proof of Theorem 5.4.3 we can extract a bound on the size required for the base.

Proposition 5.4.6. *For any signature* σ *with* $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1\text{'}, \mathrm{A}, \mathrm{F}, \sqcup, {}^{\uparrow}, {}^{-1}\}$ *every* *finite σ-algebra $\mathfrak{A}$ is representable over a base of size*

$$|\mathfrak{A}|^{|\mathfrak{A}|^{\mathcal{O}(|\mathfrak{A}|^{\frac{1}{2}})}}.$$

Proof. We may assume $|\mathfrak{A}| \geq 2$. Let N and $(N_n)_{n\geq -1}$ be as in the proofs of Theorem 5.4.3 and Theorem 5.4.5. Let E be a $\lesssim$-equivalence class of depth n and let P be a copy of the present of E. An allowable choice from P to N_{n-1} is determined by the labelled edges from a single vertex of P, since it follows from claim (5.1) in the proof of Proposition 5.3.5 that if a is the label of an edge (x, y) and b is the label of an edge (y, z) then $a \mathbin{;} b$ is the label of the edge (x, z). There are at most $|\mathfrak{A}|$ labels, so at most $|N_{n-1}|^{|\mathfrak{A}|}$ allowable choices (unless E is of depth 0, in which case there is a single allowable choice). When N_n is constructed from N_{n-1}, for each allowable choice, $|\mathfrak{A}|$ copies of P are added, so $|\mathfrak{A}||P|$ vertices are added. The sum, over all $\lesssim$-equivalence classes of depth n, of the number of vertices in the present of each class, is at most $|\mathfrak{A}|$. Hence at most $|\mathfrak{A}|^2|N_{n-1}|^{|\mathfrak{A}|}$ vertices are added when N_n is constructed from N_{n-1}. We obtain

$$|N_0| \leq |\mathfrak{A}|^2,$$
$$|N_n| \leq |N_{n-1}| + |\mathfrak{A}|^2|N_{n-1}|^{|\mathfrak{A}|} \text{ for } n \geq 1,$$

from which it is provable by induction that

$$|N_n| \leq |\mathfrak{A}|^{2(n+1)|\mathfrak{A}|^n}.$$

Suppose there is a $\lesssim$-equivalence class of depth n. Then there is a chain $\alpha_0 \lnsim \cdots \lnsim \alpha_n$ of distinct realisable domain elements. For each i there is an $a \in \mathfrak{A}$ with $\mathrm{D}(a) = \alpha_i$ and $\mathrm{R}(a) = \alpha_{i+1}$. Then since composition is in the signature, for each $i < j$ there is an $a \in \mathfrak{A}$ with $\mathrm{D}(a) = \alpha_i$ and $\mathrm{R}(a) = \alpha_j$, and these are all necessarily distinct. Hence at least $n(n + 1)/2$ distinct elements of $\mathfrak{A}$ are required in order for there to be a $\lesssim$-equivalence class of depth n. So the construction of N is completed by a depth that is $\mathcal{O}(|\mathfrak{A}|^{\frac{1}{2}})$. Hence

$$|N| = |\mathfrak{A}|^{2(\mathcal{O}(|\mathfrak{A}|^{\frac{1}{2}})+1)|\mathfrak{A}|^{\mathcal{O}(|\mathfrak{A}|^{\frac{1}{2}})}} = |\mathfrak{A}|^{|\mathfrak{A}|^{\mathcal{O}(|\mathfrak{A}|^{\frac{1}{2}})}}. \qquad \square$$

For comparison, note that in [44], whenever a signature is shown to have the finite representation property, a bound on the size required for the base is derived that has either polynomial or exponential asymptotic growth.

Problem 5.4.7. Could Proposition 5.4.6 still be true if the double-exponential bound were replaced with some exponential bound?

We mentioned in the introduction that proving the finite representation property can help show that representability of finite algebras is decidable. The most direct way this can happen is by finding a (computable) bound on the size required for a representation. Then the representability of a finite algebra can be decided by searching for an isomorph amongst the concrete algebras with bases no larger than the bound.

For most of the signatures that we have considered, decidability has already been established, because finite equational or quasiequational axiomatisations of the representation classes (or at least the finite representable algebras) are known. However, this is not the case for some of our signatures. Specifically, the antidomain-free expansions of $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ by $^\uparrow$ and/or $\sqcup$ and also any of the signatures containing opposite. So it is worth stating the following corollary of Proposition 5.4.6.

Corollary 5.4.8. *Representability of finite algebras by partial functions is decidable for any signature σ with $\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subseteq \sigma \subseteq \{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \sqcup, ^\uparrow, ^{-1}\}$.*

5.5 Entirely algebraic constructions

Most of the construction detailed in Section 5.4 can be carried out based only on direct inspection of the algebra under consideration. However, we noted that the construction does depend in one respect on information contained in representations of the algebra: the representations determine which are the realisable domain elements. We also noted that our construction works equally well if our realisable elements are those appearing as edge labels in one particular representation. So if we were to give an algebraic characterisation of the elements appearing as reflexive-edge labels in a particular representation, we would have a method of constructing a representation on a finite base using only algebraic properties of the algebra. Giving such characterisations, for certain signatures, is precisely what we do in this section.

We first mention the signature $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ and expansions of this signature by operations in $\{0, 1', \mathrm{F}\}$. The representation that Jackson and Stokes give in [55] for these signatures uses for the base of the representation Schein's 'permissible sequences', as originally described in [92]. A permissible sequence, is a sequence $(a_1, b_1, \dots, a_n, b_n, a_{n+1})$ with $\mathrm{R}(a_i) = \mathrm{R}(b_i)$ and $\mathrm{D}(b_i) = \mathrm{D}(a_{i+1})$ for each i (and 0 cannot participate in a sequence if it is in the signature). There is an edge on which c holds, starting at such a sequence, if and only if $\mathrm{D}(c) \geq \mathrm{R}(a_{n+1})$. Hence for Schein's representation we can identify the elements labelling reflexive edges quite easily: they are those of the form $\mathrm{R}(a)$, for some a (excluding 0 if it is in the signature).

Now we examine the signature $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$. An arbitrary representable $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$-algebra, $\mathfrak{A}$, has a least element, 0, given by $\mathrm{A}(a) ; a$ for any $a \in \mathfrak{A}$ and any representation of $\mathfrak{A}$ must represent 0 by the empty set. We can define $\mathrm{D} \coloneqq \mathrm{A}^2$ and in any representation this must be represented by the domain operation.

The down-set $\downarrow a$ of any element $a \in \mathfrak{A}$ forms a Boolean algebra using the meet operation of $\mathfrak{A}$ and with complementation given by $\overline{b} \coloneqq \mathrm{A}(b) ; a$. Any representation θ of $\mathfrak{A}$ by partial functions restricts to a representation of each $\downarrow a$ as a field of sets over $\theta(a)$. From this we see that θ turns any finite joins in $\mathfrak{A}$ into unions.

Definition 5.5.1. Let $\mathfrak{P}$ be a poset with a least element, 0. An **atom** of $\mathfrak{P}$ is a minimal nonzero element of $\mathfrak{P}$. We say that $\mathfrak{P}$ is **atomic** if every nonzero element is greater than or equal to an atom.

A *finite* representable $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$-algebra, $\mathfrak{A}$, is necessarily atomic. Any $a \in \mathfrak{A}$ can be expressed as a finite join of atoms of $\mathfrak{A}$, since, given $0 < a < b$, we can split b as $b = a + (\mathrm{A}(a) \,;\, b)$.

From the preceeding discussion, we see that in any representation of a $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$-algebra, for any edge a finite sum of atoms holds, so at least one of the atoms holds, as finite joins are represented by unions. We know that at most one atom holds, since the meet of two distinct atoms is 0, which can never hold on an edge. Hence a unique atom holds on each edge and necessarily labels the edge. In every representation every atom must appear as a label, otherwise it is not separated from 0. We conclude that in any representation the elements appearing as edge labels are precisely the atoms and so the elements labelling reflexive edges are precisely the atomic domain elements. Hence for the signature $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$ the realisable domain elements are the atomic domain elements. This also applies to any expansion of this signature by operations we have mentioned.

The purpose of the next example is simply to illustrate that, unlike Boolean algebras for example, the set of atoms in a finite representable $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$-algebra can be almost as large as the algebra itself. Hence applying the knowledge that the number of labels is at most the number of atoms to the calculation in Proposition 5.4.6, does not improve the bound.

Example 5.5.2. Let G be any finite group. We can make $G \cup \{0\}$ into an algebra of the signature $\{;, \cdot, \mathrm{A}, \mathrm{R}\}$ by using the group operation for composition (and $g \,;\, 0 = 0 \,;\, g = 0$ for all g) and defining $g \cdot h = 0$ unless $g = h$, every antidomain of a nonzero element to be 0 (and $\mathrm{A}(0) = e$, the group identity), and every range of a nonzero element to be e (and $\mathrm{R}(0) = 0$). Then every nonzero element of $G \cup \{0\}$ is an atom. Augmenting the Cayley representation of G (the representation $\theta(g)(h) = hg$) by setting $\theta(0) = \emptyset$ demonstrates that $G \cup \{0\}$ is representable.

5.6 Failure of the finite representation property

Finally, one might reasonably wonder if it is possible for the finite representation property *not* to hold for algebras of partial functions. After all, for every signature for which it has been settled, the finite representation property has been shown to hold. We finish with a simple example showing that we can indeed force a finite representable algebra of partial functions to fail to have representations over finite bases.

Example 5.6.1. Let U be the unary operation on partial functions given by

$$\mathrm{U}(f) = \{(y, y) \in X^2 \mid \exists! x \in X : (x, y) \in f\}.$$

Let $\mathfrak{F}_2$ be the algebra of partial functions, of the signature $\{;, \cdot, \mathrm{D}, \mathrm{R}, \mathrm{U}\}$ and with base $\mathbb{N} \times 2$, consisting of the following five elements.

- 0, the empty function,
- d, the identity function on $\mathbb{N} \times \{0\}$,

- r, the identity function on $\mathbb{N} \times \{1\}$,
- f, the function with domain d and range r sending each $(n, 0)$ to $(n, 1)$,
- g, a function with domain d and range r such that each $(n, 1) \in \mathbb{N} \times \{1\}$ has precisely two g-preimages: the least two elements of $\mathbb{N} \times \{0\}$ that are neither the f-preimage $(n, 0)$ nor g-preimages of $(m, 1)$ for $m < n$. See Figure 5.3.

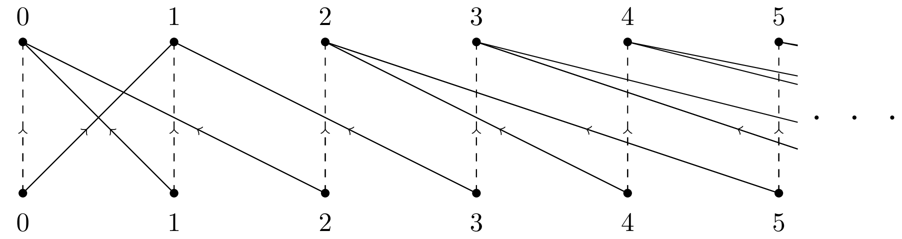

Figure 5.3: The algebra $\mathfrak{F}_2$. Dashed lines for f, solid lines for g

Since $\mathfrak{F}_2$ *is* an algebra of partial functions, it is certainly representable by partial functions. It is easy to see that $\mathfrak{F}_2$ cannot be represented over a finite base. Indeed, $\mathrm{R}(f) = \mathrm{U}(f)$, so in any representation f is a bijection from its domain, the d-vertices, to its range, the r-vertices. On the other hand, $\mathrm{R}(g) \neq \mathrm{U}(g)$ so g maps the d-vertices onto the r-vertices but not injectively. Hence these sets of vertices cannot have finite cardinality.

By including the operation U in less expressive signatures, it is possible to give slightly simpler examples than Example 5.6.1. However, we chose a supersignature of the signature $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$ in order to contrast with the other supersignatures that are the subject of this chapter, for which we have seen that the finite representation property does hold.

Note that our example allows us to observe the finite representation property behaving non monotonically as a function of expressivity. Indeed U is expressible in terms of domain and opposite, $\mathrm{U}(f) = \mathrm{D}(f^{-1})$, and so we have

$$\{;, \cdot, \mathrm{D}, \mathrm{R}\} \subset \{;, \cdot, \mathrm{D}, \mathrm{R}, \mathrm{U}\} \subset \{;, \cdot, \mathrm{D}, \mathrm{R}, ^{-1}, \mathrm{U}\}$$

with the finite representation property holding for the outer two signatures but failing in the middle.

Chapter 6

Algebras of multiplace functions for signatures containing antidomain

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ABSTRACT. We define antidomain operations for algebras of multiplace partial functions. For all signatures containing composition, the antidomain operations, and any subset of intersection, preferential union and fixset, we give finite equational or quasiequational axiomatisations for the representation class. We do the same for the question of representability by injective multiplace partial functions. For all our representation theorems, it is an immediate corollary of our proof that the finite representation property holds for the representation class. We show that for a large set of signatures, the representation classes have equational theories that are coNP-complete.

6.1 Introduction

The scheme for investigating the abstract algebraic properties of functions takes the following form. First choose some sort of functions of interest, for example partial functions or injective functions. Second, specify some set-theoretically-defined operations possible on such functions, for example function composition or set intersection. Finally, study the isomorphism class of algebras that consist of some such functions together with the specified set-theoretic operations. We have discussed the basic case—unary functions—extensively in previous chapters, particularly in Section 3.2.1.

The study of algebras of so-called multiplace functions started with Menger [76]. Here the objects in the concrete algebras are (usually partial) functions from X^n to X for some fixed X and n. Since then, representation theorems—axiomatisations of isomorphism classes via explicit representations—have been given for various cases [22, 107, 106, 23, 24].

For unary functions, the antidomain operation yields the identity function restricted to the complement of a function's domain. In the setting of partial functions, this operation seems first to have been described in [54], where it is referred to as domain complement.[2] Some recent work has been direc-

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[2] Though for an earlier appearance in the setting of *binary relations*, see [51], where Hollenberg calls the operation *dynamic negation*.

ted towards providing representation theorems in the case of unary functions for signatures including antidomain [57, 44].

In this chapter we define, for n-ary multiplace functions, n indexed antidomain operations by simultaneous analogy with the indexed domain operations studied on multiplace functions and the antidomain operation studied on unary functions. This definition together with other fundamental definitions we need comprise Section 6.2.

The majority of this chapter, Sections 6.3–6.8, consists of representation theorems for multiplace functions for signatures containing composition and the antidomain operations. Much of this is a straightforward translation of [57], where the same is done for unary functions.

In Sections 6.3 and 6.4 we work over the signature containing composition and the antidomain operations. We show that for multiplace partial functions the representation class cannot form a variety and we state and prove the correctness of a finite quasiequational axiomatisation of the class. It follows, as it does for our later representation theorems, that the representation class has the finite representation property.

In Section 6.5 we use a single quasiequation to extend the axiomatisation of Section 6.3 to a finite quasiequational axiomatisation for the case of injective multiplace partial functions.

In Section 6.6 we add intersection to our signature and for both partial multiplace functions and injective partial multiplace functions are able to give finite equational axiomatisations of the representation class.

In Sections 6.7 and 6.8 we consider all our previous representation questions with the preferential union and fixset operations, respectively, added to the signature. In all cases we give either finite equational or finite quasiequational axiomatisations of the representation class.

In Section 6.9 we switch our focus to equational theories. We prove that for any signature containing operations that we mention, the equational theory of the representation class of multiplace partial functions lies in coNP. If the signature contains the antidomain operations and either composition or intersection then the equational theory is coNP-complete.

6.2 Algebras of multiplace functions

In this section we give the fundamental definitions of algebras of multiplace functions and of the various operations that may be included.

Given an algebra $\mathfrak{A}$, when we write $a \in \mathfrak{A}$ or say that a is an element of $\mathfrak{A}$, we mean that a is an element of the domain of $\mathfrak{A}$. We follow the convention that algebras are always nonempty. We use n to denote an arbitrary nonzero natural number. A bold symbol, $\boldsymbol{a}$ say, is either simply shorthand for $\langle a_1, \ldots, a_n\rangle$ in a term of the form $\langle a_1, \ldots, a_n\rangle$; b or denotes an actual n-tuple $\langle a_1, \ldots, a_n\rangle$. We may abuse notation, when convenient, by writing $(\boldsymbol{x}, y)$ for the $(n+1)$-tuple $\langle x_1, \ldots, x_n, y\rangle$. If $\mathrm{A}_1, \ldots, \mathrm{A}_n$ are unary operation symbols, the notation $\langle \mathrm{A}_1^n a\rangle$ is shorthand for $\langle \mathrm{A}_1 a, \ldots, \mathrm{A}_n a\rangle$. When a function f acts on an n-tuple $\langle a_1, \ldots, a_n\rangle$ we omit the angle brackets and write $f(a_1, \ldots, a_n)$. If i is an index, then 'for all i' or 'for every i' means for all $i \in \{1, \ldots, n\}$.

First we make clear what we mean by a multiplace function.

Definition 6.2.1. An n**-ary relation** is a subset of a set of the form $X_1 \times \cdots \times X_n$. Without loss of generality we may assume all the X_i's are equal. In the context of a given value of n, a **multiplace partial function** is an $(n+1)$-ary relation f validating

$$\langle x_1, \dots, x_n, y\rangle \in f \quad \wedge \quad \langle x_1, \dots, x_n, z\rangle \in f \quad \rightarrow \quad y = z. \tag{6.1}$$

We may also use the terminology n**-ary partial function** for the same concept. We import all the usual notation and terminology for partial functions, for instance if $(\boldsymbol{x}, y) \in f$ then we may write $f(\boldsymbol{x}) = y$, say '$f(\boldsymbol{x})$ is defined', and so on.

Henceforth, we will use the epithet 'n-ary' in favour of 'multiplace' in order to make the arity of the functions in question explicit.

Definition 6.2.2. Let σ be an algebraic signature whose symbols are a subset of $\{\langle\ \rangle;, \cdot, 0, \pi_i, \mathrm{D}_i, \mathrm{A}_i, \mathrm{F}_i, \rtimes_i, \sqcup\}$, where we write, for example, A_i to indicate that $\mathrm{A}_1, \dots, \mathrm{A}_n \in \sigma$ for some fixed n. An **algebra of** n**-ary partial functions** of the signature σ is an algebra, $\mathfrak{A}$, of the signature σ whose elements are n-ary partial functions and that has the following properties.

(i) There is a set X, the **base**, and a partition P of X with the following property. For all $f \in \mathfrak{A}$ and all $\langle x_1, \dots, x_{n+1}\rangle \in f$, we have that $x_1, \dots, x_{n+1}$ lie in a single block of P.

(ii) The operations are given by the set-theoretic operations on partial functions described in the following.

In an algebra of n-ary partial functions

- the $(n+1)$-ary operation $\langle\ \rangle;$ is **composition**, given by

$$\boldsymbol{f} \,;\, g = \{(\boldsymbol{x}, z) \in X^{n+1} \mid \exists \boldsymbol{y} \in X^n : (\boldsymbol{x}, y_i) \in f_i \text{ for each } i \text{ and } (\boldsymbol{y}, z) \in g\},$$

- the binary operation $\cdot$ is intersection:

$$f \cdot g = \{(\boldsymbol{x}, y) \in X^{n+1} \mid (\boldsymbol{x}, y) \in f \text{ and } (\boldsymbol{x}, y) \in g\},$$

- the constant 0 is the nowhere-defined function:

$$0 = \emptyset = \{(\boldsymbol{x}, y) \in X^{n+1} \mid \bot\},$$

- for each i the constant π_i is the ith projection on the set of all n-tuples whose components lie in a single block of P:

$$\pi_i = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid \exists B \in P :\ x_1, \dots, x_n \in B\},$$

- for each i the unary operation D_i is the operation of taking the ith projection restricted to the domain of a function:

$$\mathrm{D}_i(f) = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid \exists y \in X : (\boldsymbol{x}, y) \in f\},$$

- for each i, the unary operation A_i is the operation of taking the ith projection restricted to the **antidomain** of a function—those n-tuples with components in a single block of P where the function is not defined on the n-tuple:

$$\mathrm{A}_i(f) = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid \exists B \in P : \ x_1, \dots, x_n \in B \text{ and } \nexists y \in X : (\boldsymbol{x}, y) \in f\},$$

- for each i, the unary operation F_i, the ith **fixset** operation, is the ith projection function intersected with the function itself:

$$\mathrm{F}_i(f) = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid (\boldsymbol{x}, x_i) \in f\},$$

- for each i, the binary operation $\bowtie_i$, the ith **tie** operation, is the ith projection function restricted to n-tuples whose components lie in a single block of P and where the two arguments do not disagree on the n-tuple, that is, either neither is defined or they are both defined and are equal:

$$f \bowtie_i g = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid (\boldsymbol{x}, x_i) \in \mathrm{A}_i f \cap \mathrm{A}_i g \text{ or } \exists y \in X : (\boldsymbol{x}, y) \in f \cap g\},$$

- the binary operation $\sqcup$ is **preferential union**:

$$(f \sqcup g)(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \text{if } f(\boldsymbol{x}) \text{ defined} \\ g(\boldsymbol{x}) & \text{if } f(\boldsymbol{x}) \text{ undefined, but } g(\boldsymbol{x}) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

If the partition P is the trivial partition $\{X\}$ of X, then we say that the algebra is **square**.

The indexed domain operations $\mathrm{D}_1, \dots, \mathrm{D}_n$ have been studied for many years. See, for example, [106, 23]. Multiplace versions of the antidomain, fixset, and tie operations do not appear to have been defined before. Their definitions are made by generalising their unary versions (appearing, for example, in [57]) by analogy with the generalisation of unary domain to its indexed multiplace incarnations.

Definition 6.2.3. Let $\mathfrak{A}$ be an algebra of one of the signatures specified by Definition 6.2.2. A **representation of $\mathfrak{A}$ by n-ary partial functions** is an isomorphism from $\mathfrak{A}$ to an algebra of n-ary partial functions of the same signature. If $\mathfrak{A}$ has a representation then we say it is **representable**.

As we have signified, in this chapter the focus is on isomorphs of algebras of n-ary partial functions in general, rather than the square ones in particular. However, now is an opportune moment for a brief discussion of the merits of each of these concepts and the relationship between them.

The square algebras of n-ary functions have the advantage of being the simpler and more natural concept. However, for certain signatures they are not as algebraically well behaved, failing to be closed under direct products. Indeed there are simple examples of pairs of algebras that are each representable as square algebras of functions but whose product is not. The presence of the antidomain operations in the signature will always cause this problem, as the example we now give demonstrates.

Example 6.2.4. Assume $n \geq 2$ and work over any one of the signatures specified by Definition 6.2.2 containing the n indexed antidomain operations $\mathrm{A}_1, \ldots, \mathrm{A}_n$. Consider the two-element algebra $\mathfrak{A}$ consisting of both of the n-ary partial functions on some base of size one. As $\mathfrak{A}$ *is* a square algebra of partial functions it is trivially representable as a square algebra of partial functions. We argue that $\mathfrak{A} \times \mathfrak{A}$ is *not* representable as a square algebra of functions.

Suppose, for contradiction, that θ is a square representation of $\mathfrak{A} \times \mathfrak{A}$ with base X. Since $|\mathfrak{A} \times \mathfrak{A}| = 4$, we know X must contain at least two distinct points, in order that θ distinguishes all the elements of $\mathfrak{A} \times \mathfrak{A}$. Let $\boldsymbol{x}$ be any n-tuple from X^n not lying on the diagonal. Denote the two elements of $\mathfrak{A}$ by a and b. Then $\mathrm{A}_i a = b$ and $\mathrm{A}_i b = a$ for every i. So $\mathrm{A}_1(a, a) = (b, b)$, and hence the domains of the partial functions $\theta(a, a)$ and $\theta(b, b)$ must partition X^n. Without loss of generality we may assume $\boldsymbol{x}$ is not in the domain of $\theta(a, a)$. But then $\theta(\mathrm{A}_i(a, a))(\boldsymbol{x}) = x_i$ for every i. As every $\theta(\mathrm{A}_i(a, a))$ is the same function—namely, $\theta(b, b)$—all components of $\boldsymbol{x}$ are equal, contradicting the assumption that $\boldsymbol{x}$ is not on the diagonal. We conclude that $\mathfrak{A} \times \mathfrak{A}$ cannot be represented as a square algebra of partial functions.

An immediate consequence of not being closed under direct products is that the class of algebras having a square representation cannot be a quasivariety. We note however that these classes always possess recursive universal axiomatisations in first-order logic, for any of the signatures covered by Definition 6.2.2. This can be seen by appealing to the $(n+1)$-ary form of Schein's fundamental theorem of relation algebra. (We gave the binary form in Theorem 3.1.6.) There are two conditions of Schein's theorem that need to be checked. The first is that n-ary partial functions can be defined as those $(n+1)$-ary relations satisfying a recursive set of sentences in the first-order language with equality and a countable supply of $(n+1)$-ary relation symbols, which is precisely what we did in Definition 6.2.1 by using (6.1). The second is that, using the same first-order language, the operations we are considering can each be defined using a formula with $n+1$ free variables. Definitions of the operations for square algebras can be formed from the more general definitions we gave in Definition 6.2.2 by removing any stipulations that components lie in the same block of P. The resulting definitions are of the required form.

The purpose of relativising operations to P in Definition 6.2.2 is to ensure that the class of algebras representable by n-ary partial functions *is* closed under direct products. A direct product of representable algebras can be represented using a 'disjoint union' of representations of the factors.

Definition 6.2.5. Let $(\mathfrak{A}_i)_{i \in I}$ be a family of algebras all of the same signature and $(\theta_i \colon \mathfrak{A}_i \to \mathfrak{F}_i)_{i \in I}$ be a corresponding family of homomorphisms to algebras of n-ary partial functions, with $\mathfrak{F}_i$ having base X_i and partition P_i of X_i.

A **disjoint union** of $(\theta_i)_{i \in I}$ is any homomorphism θ out of $\prod_{i \in I} \mathfrak{A}_i$ formed by the following process. First rename the elements of the X_i's in such a way that the X_i's are pairwise disjoint. Then the codomain of θ will be an algebra $\mathfrak{F}$ consisting of all n-ary partial functions of the form $\bigcup_{i \in I} \theta_i(a_i)$ for some element $(a_i)_{i \in I}$. The base of $\mathfrak{F}$ will be $X \coloneqq \bigcup_{i \in I} X_i$ and the partition of X will be $P \coloneqq \bigcup_{i \in I} P_i$. The operations on $\mathfrak{F}$ will be given by the concrete operations described in Definition 6.2.2. Define $\theta((a_i)_{i \in I}) = \bigcup_{i \in I} \theta_i(a_i)$ for each element $(a_i)_{i \in I}$ of $\prod_{i \in I} \mathfrak{A}_i$. The map θ is straightforwardly a homomorphism.

A disjoint union of injective homomorphisms will be injective and that is why we remarked that a product of representable algebras can be represented by a disjoint union of representations of the factors. If our definition of algebras of n-ary partial functions were restricted to square algebras only, then we could not guarantee that a disjoint union of representations would be a representation, since the disjoint union of two trivial partitions is not trivial.

Schein's fundamental theorem of relation algebras shows that the representable algebras form a universal class. To see this, use a binary relation symbol E and the first-order $\{E\}$-sentence asserting that E is an equivalence relation. Then replace every occurrence of '$\exists B \in P : \ x_1, \dots, x_n \in B$' in Definition 6.2.1 by '$x_1 E x_2 \wedge \cdots \wedge x_{n-1} E x_n$', to obtain an axiomatisation of the required form. Thus, for any one of the signatures specified in Definition 6.2.1, the class of representable algebras is both universal and closed under direct products, and hence, by Theorem 2.3.16, is a quasivariety.

Our final remark about square algebras of partial functions is that it is easily seen that every algebra representable by n-ary partial functions is a subalgebra of a product of algebras each having a square representation. Hence the general representation class is contained in the quasivariety generated by the square representation class. Since the general representation class *is* a quasivariety, it is precisely the quasivariety generated by the square representation class.

For algebras of n-ary functions, the first representation theorem was provided by Dicker in [22], showing that the equation that has come to be known as the superassociativity law axiomatises the representation class (for total functions, although the equation is valid for partial functions) in the signature consisting only of composition. Trokhimenko gave equational axiomatisations for the signatures of composition and intersection, in [107], and composition and domain, in [106]. In [23], Dudek and Trokhimenko gave a finite equational axiomatisation for the signature of composition, intersection, and domain.

The subject of this chapter is signatures containing composition and antidomain. Note that $0, \pi_i$, and D_i are all definable using composition and antidomain, using $0 := \langle \mathrm{A}_1^n a \rangle ; a$, for any a, and then $\pi_i := \mathrm{A}_i 0$ and using $\mathrm{D}_i := \mathrm{A}_i^2$ (that is, a double application of A_i). Further, in the presence of composition and antidomain, the tie operations and intersection are interdefinable. The tie operations are definable as $a \bowtie_i b := \mathrm{D}_i(a \cdot b) +_i \langle \mathrm{A}_1^n a \rangle ; \mathrm{A}_i b$, where $\alpha +_i \beta := \mathrm{A}_i(\langle \mathrm{A}_1^n \alpha \rangle ; \mathrm{A}_i \beta)$. Intersection is definable as $a \cdot b := \langle a \bowtie_1^n b \rangle ; a$. This leaves $\cdot$, F_i and $\sqcup$ as the only interesting additional operations among those we have mentioned. When intersection is present, the fixset operations are definable as $\mathrm{F}_i f := \pi_i \cdot f$.

We include here, for ease of reference, a summary of the results about representation classes contained in this chapter. All classes have finite axiomatisations of the relevant form.

Signature	**Partial functions**	**Injective partial functions**
$\langle\ \rangle;, \mathrm{A}_i$	proper quasivariety	quasivariety
$\langle\ \rangle;, \mathrm{A}_i, \cdot$	variety	variety
$\langle\ \rangle;, \mathrm{A}_i, \sqcup$	variety	quasivariety
$\langle\ \rangle;, \mathrm{A}_i, \cdot, \sqcup$	variety	variety
$\langle\ \rangle;, \mathrm{A}_i, \mathrm{F}_i$	quasivariety	quasivariety
$\langle\ \rangle;, \mathrm{A}_i, \mathrm{F}_i, \sqcup$	quasivariety	quasivariety

Table 6.1: Summary of representation classes for n-ary functions

Remark 6.2.6. Note (as a special case of Theorem 2.4.9) that whenever a representation class has a finite quasiequational axiomatisation the decision problem of representability of finite algebras is solvable in polynomial time, and if we know such an axiomatisation then we know such an algorithm. Hence it follows from our results that representability of finite algebras is solvable in polynomial time for all the representation classes presented in Table 6.1.

Where an entry in Table 6.1 is just ‘quasivariety’, we have not investigated whether the class is a *proper* quasivariety, so these questions are left open.

Beyond representability, we may also be interested in representability on a finite base. Our final fundamental definition can be invoked in any circumstance where there is a notion of representability.

Definition 6.2.7. The **finite representation property** holds if any finite representable algebra is representable on a finite base.

6.3 Composition and antidomain

First we examine the signature $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n\}$ consisting of composition and the antidomain operations. After presenting some equations and one quasiequation that are valid for algebras of n-ary partial functions, we deduce some consequences of these (quasi)equations that we use in Section 6.4 to prove that our (quasi)equations axiomatise the representation class.

In [57], Jackson and Stokes give a finite quasiequational axiomatisation of the representation class of unary partial functions for the signature of composition, antidomain. (Actually, their signature also contains the constants 0 and 1’, but these are definable from composition and antidomain.) They call algebras validating their laws modal restriction semigroups.

Definition 6.3.1. A **modal restriction semigroup** [57] is an algebra of the signature $\{;, \mathrm{A}\}$ validating the equations

$$(a\,;b)\,;c = a\,;(b\,;c)$$
$$1'\,;a = a$$
$$\mathrm{A}(a)\,;a = 0$$
$$0\,;a = 0$$
$$a\,;0 = 0$$
$$a\,;\mathrm{A}(b) = \mathrm{A}(a\,;b)\,;a \qquad \text{(the twisted law for antidomain)}$$

and the quasiequation

$$\mathrm{D}(a)\,;b = \mathrm{D}(a)\,;c \quad \wedge \quad \mathrm{A}(a)\,;b = \mathrm{A}(a)\,;c \quad \rightarrow \quad b = c$$

where $0 \coloneqq \mathrm{A}(b)\,;b$ for any b (and the third equation says this is a well-defined constant), $1' \coloneqq \mathrm{A}(0)$ and $\mathrm{D} \coloneqq \mathrm{A}^2$.

Note that the definition of modal restriction semigroups given by Jackson and Stokes states they should be monoids, so 1' should also be a right identity. But this is a consequence of the equations we gave in Definition 6.3.1, for

$$a\,;1' = a\,;\mathrm{A}(0) = \mathrm{A}(a\,;0)\,;a = \mathrm{A}(0)\,;a = 1'\,;a = a$$

using the twisted law for the second equality.

For n-ary functions, working over the signature $\{\langle\ \rangle ;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$, we can try to write down valid n-ary versions of the (quasi)equations appearing in Definition 6.3.1. This is easy in every case except that of the twisted law for antidomain, which needs more care.

This is a good point at which to note that we do not need to bracket expressions like $\boldsymbol{a}\,;\boldsymbol{b}\,;c$, since this can only mean $\boldsymbol{a}\,;(\boldsymbol{b}\,;c)$. When we do write the brackets, we do so only for emphasis.

Proposition 6.3.2. *The following equations and quasiequations are valid for the class of* $\{\langle\ \rangle ;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$*-algebras representable by* n*-ary partial functions.*

$$\langle \boldsymbol{a}\,;b_1, \dots, \boldsymbol{a}\,;b_n\rangle\,;c = \boldsymbol{a}\,;(\boldsymbol{b}\,;c) \tag{6.2}$$

(**superassociativity**)

$$\boldsymbol{\pi}\,;a = a \tag{6.3}$$

$$\langle \mathrm{A}_1^n a\rangle\,;a = 0 \tag{6.4}$$

$$\langle a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n\rangle\,;b = 0 \quad \textit{for every } i \tag{6.5}$$

$$\boldsymbol{a}\,;0 = 0 \tag{6.6}$$

$$\boldsymbol{a}\,;\mathrm{A}_i b = \langle \mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle \mathrm{D}_1^n a_1\rangle\,;\dots\,;\langle \mathrm{D}_1^n a_n\rangle\,;a_i \quad \textit{for every } i \tag{6.7}$$

(the **twisted laws for antidomain**)

$$\langle \mathrm{D}_1^n a\rangle\,;b = \langle \mathrm{D}_1^n a\rangle\,;c \quad\wedge\quad \langle \mathrm{A}_1^n a\rangle\,;b = \langle \mathrm{A}_1^n a\rangle\,;c \quad\rightarrow\quad b = c \tag{6.8}$$

where $0 \coloneqq \langle \mathrm{A}_1^n b\rangle\,;b$ *for any* b *(and* (6.4)*, which is really* $\langle \mathrm{A}_1^n a\rangle\,;a = \langle \mathrm{A}_1^n b\rangle\,;b$*, says this is a well-defined constant),* $\pi_i \coloneqq \mathrm{A}_i 0$ *and* $\mathrm{D}_i \coloneqq \mathrm{A}_i^2$ *(a double application of* A_i*).*

Proof. We noted in the previous section that every algebra representable by n-ary partial functions is isomorphic to a subalgebra of a product of algebras having a square representation. As the validity of quasiequations is preserved by taking products and subalgebras, it suffices to prove validity only for algebras having square representations. Further, since representations are themselves isomorphisms, it is sufficient to prove validity for an arbitrary square algebra of n-ary partial functions. So suppose we have such an algebra, with base X.

The validity of the superassociative law has been recognised since Menger noted it in [76]. We turn next to (6.4). Given an n-ary partial function a, if $\langle \mathrm{A}_1^n a\rangle\,;a$ is to be defined at an n-tuple $\boldsymbol{x}$ then there should be a $\boldsymbol{y}$ with $\mathrm{A}_i(a)(\boldsymbol{x}) = y_i$ for each i and with a defined at $\boldsymbol{y}$. Since each $\mathrm{A}_i a$ is a restriction of the ith projection, $\boldsymbol{y}$ can only be $\boldsymbol{x}$. But if $\mathrm{A}_1 a$ is defined at $\boldsymbol{x}$ then a cannot be. Hence $\langle \mathrm{A}_1^n a\rangle\,;a$ is the nowhere-defined function. So 0 is well-defined, that is, the value of $\langle \mathrm{A}_1^n a\rangle\,;a$ does not depend on the choice of a, and so (6.4) is valid. The validity of (6.5) and the validity of (6.6) are now both clear.

Now $\pi_i \coloneqq \mathrm{A}_i 0$ is the ith projection restricted to those n-tuples in X^n where 0 is not defined. So π_i is, as the notation indicates, the ith projection on the set of *all* n-tuples in X^n. The validity of (6.3) is now clear.

For the twisted laws for antidomain, first suppose that $\boldsymbol{a}\,;\mathrm{A}_i b$ is defined at $\boldsymbol{x}$. Then we know that $a_1, \dots, a_n$ are all defined at $\boldsymbol{x}$ and that b is not defined at $\langle a_1(\boldsymbol{x}), \dots, a_n(\boldsymbol{x})\rangle$. Hence $\mathrm{D}_j a_k$ is defined at $\boldsymbol{x}$ for every j, k and $\boldsymbol{a}\,;b$ is not defined at $\boldsymbol{x}$. It follows that $\mathrm{A}_j(\boldsymbol{a}\,;b)$ *is* defined at $\boldsymbol{x}$ for every j. It is now apparent that $\langle \mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle \mathrm{D}_1^n a_1\rangle\,;\dots;\langle \mathrm{D}_1^n a_n\rangle\,;a_i$ is defined at $\boldsymbol{x}$ with value $a_i(\boldsymbol{x})$—the same value as $\boldsymbol{a}\,;\mathrm{A}_i b$.

If $\boldsymbol{a}\,;\mathrm{A}_i b$ is *not* defined at an n-tuple $\boldsymbol{x}$, then this is either because a_j is undefined at $\boldsymbol{x}$ for some j or all a_j *are* defined at $\boldsymbol{x}$, but $\mathrm{A}_i b$ is not defined at $\langle a_1(\boldsymbol{x}), \dots, a_n(\boldsymbol{x})\rangle$. If a_j is undefined at $\boldsymbol{x}$ then it is clear that $\langle \mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle \mathrm{D}_1^n a_1\rangle\,;\dots;\langle \mathrm{D}_1^n a_n\rangle\,;a_i$ cannot be defined at $\boldsymbol{x}$. In the second case, b must be defined at $\langle a_1(\boldsymbol{x}), \dots, a_n(\boldsymbol{x})\rangle$ and so $\boldsymbol{a}\,;b$ is defined at $\boldsymbol{x}$. Again it is clear that $\langle \mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle \mathrm{D}_1^n a_1\rangle\,;\dots;\langle \mathrm{D}_1^n a_n\rangle\,;a_i$ cannot be defined at $\boldsymbol{x}$.

For (6.8), suppose the antecedent of the implication is true. Let $\boldsymbol{x}$ be an n-tuple in X^n. If a is defined on $\boldsymbol{x}$ then $\mathrm{D}_i a$ is defined at $\boldsymbol{x}$ for each i and accordingly $\langle \mathrm{D}_1^n a\rangle\,;b = \langle \mathrm{D}_1^n a\rangle\,;c$ says that either $b(\boldsymbol{x}) = c(\boldsymbol{x})$ or both b and c are undefined at $\boldsymbol{x}$. If a is undefined at $\boldsymbol{x}$ then $\mathrm{A}_i a$ is defined at $\boldsymbol{x}$ for each i and this time $\langle \mathrm{A}_1^n a\rangle\,;b = \langle \mathrm{A}_1^n a\rangle\,;c$ says that either $b(\boldsymbol{x}) = c(\boldsymbol{x})$ or both b and c are undefined at $\boldsymbol{x}$. Hence regardless of whether or not a is defined on $\boldsymbol{x}$, either $b(\boldsymbol{x}) = c(\boldsymbol{x})$ or both b and c are undefined at $\boldsymbol{x}$. As $\boldsymbol{x}$ was arbitrary, we conclude the two partial functions b and c are equal, that is $b = c$. □

Note that the naive n-ary versions of the twisted law for antidomain—namely, $\boldsymbol{a}\,;\mathrm{A}_i b = \langle \mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;a_i$, for every i—are not valid (except in the unary case). Indeed if at an n-tuple, a_i is defined, but a_j is undefined for some j different to i, then $\boldsymbol{a}\,;\mathrm{A}_i b$ is undefined, but $\langle \mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;a_i$ will be defined.

To compensate for the complication with the twisted laws, we introduce as an axiom the equation

$$\langle \mathrm{D}_1^n a\rangle \,;\, a = a \tag{6.9}$$

whose validity is clear and has been noted before; see for example [23, Equation 10].

In addition we will need one extra indexed set of equations (trivial in the unary case)—namely,

$$\mathrm{A}_i\mathrm{A}_j a = \mathrm{A}_i\mathrm{A}_k a \qquad \text{for every } i, j, k \tag{6.10}$$

—whose validity we now prove.

Proposition 6.3.3. *The indexed equations of* (6.10) *are valid for the class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-*algebras representable by* n*-ary partial functions.*

Proof. As before it is sufficient to prove validity for an arbitrary square algebra of n-ary partial functions. So suppose we have such an algebra, with base X.

Suppose that $\mathrm{A}_i\mathrm{A}_j a$ is defined on an n-tuple $\boldsymbol{x}$, necessarily with value x_i. Then $\mathrm{A}_j a$ is not defined on $\boldsymbol{x}$. Hence a *is* defined on $\boldsymbol{x}$. It follows that $\mathrm{A}_k a$ is not defined on $\boldsymbol{x}$ and from there we deduce that $\mathrm{A}_i\mathrm{A}_k a$ is defined on $\boldsymbol{x}$, necessarily with value x_i. Hence the function $\mathrm{A}_i\mathrm{A}_j a$ is a restriction of $\mathrm{A}_i\mathrm{A}_k a$. By symmetry the reverse is true and the two functions are equal. □

We are going to prove that (6.2)–(6.10) axiomatise the class of $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-algebras that are representable by n-ary partial functions. But before we do that, we show that the representation class is not a variety.

Proposition 6.3.4. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-*algebras that are representable by* n*-ary partial functions is not closed under quotients and hence is not a variety.*

Proof. We adapt an example given in [57] to describe an algebra of n-ary partial functions having a quotient that does not validate (6.8) and so is not representable by partial functions.[3]

We describe an algebra $\mathfrak{F}$ of n-ary partial functions, with base $\{1, 2, 3\}$. The partition of the base to which the antidomain operations are relativised has blocks $\{1\}$ and $\{2, 3\}$. The elements of $\mathfrak{F}$ are the following $2(n + 3)$ elements.

- the empty function
- the ith projection on $\{2, 3\}^n$, for each i
- the function with domain $\{2, 3\}^n$ that is constantly 2
- the function with domain $\{2, 3\}^n$ that is constantly 3
- each of the aforementioned $n + 3$ functions with the pair $(\mathbf{1}, 1)$ adjoined

[3] Hollenberg proves the equivalent result for *binary relations* in [51].

It is clear that $\mathfrak{F}$ is closed under the n antidomain operations. Checking that $\mathfrak{F}$ is closed under composition is also straightforward.

It is easy to check, directly, that identifying all the elements with domain $\{2, 3\}^n$ produces a quotient of $\mathfrak{A}$. Let a be any element with domain $\{2, 3\}^n$, let b be the element sending $\mathbf{1}$ to 1 and constantly 2 elsewhere, and let c be the element sending $\mathbf{1}$ to 1 and constantly 3 elsewhere. Then in the quotient

$$\langle \mathrm{D}^n_1[a]\rangle \,;\, [b] = \langle \mathrm{D}^n_1[a]\rangle \,;\, [c]$$

and

$$\langle \mathrm{A}^n_1[a]\rangle \,;\, [b] = \langle \mathrm{A}^n_1[a]\rangle \,;\, [c],$$

but $[b]$ and $[c]$ are not equal. Hence (6.8) is refuted in the quotient. □

Next comes the work of deducing the various consequences of (6.2)–(6.10) that are needed to prove their sufficiency for representability.

We noted earlier that the equation $\boldsymbol{a} \,;\, \mathrm{A}_i b = \langle \mathrm{A}^n_1(\boldsymbol{a} \,;\, b)\rangle \,;\, a_i$ is not valid. However, it is valid whenever $a_1, \dots, a_n$ all have the same domain. Hence there is a valid equational version in the special case that $\boldsymbol{a}$ is of the form $\langle \mathrm{A}^n_1 a'\rangle$ for some a', and as we now show, we can obtain this as a consequence of our axioms.

Lemma 6.3.5. *The indexed equations*

$$\langle \mathrm{A}^n_1 a\rangle \,;\, \mathrm{A}_i b = \langle \mathrm{A}^n_1(\langle \mathrm{A}^n_1 a\rangle \,;\, b)\rangle \,;\, \mathrm{A}_i a \qquad \textit{for every } i \tag{6.11}$$

are consequences of axioms (6.2)–(6.10).

Proof. We have

$$\begin{aligned}\langle \mathrm{A}^n_1 a\rangle \,;\, \mathrm{A}_i b &= \langle \mathrm{A}^n_1(\langle \mathrm{A}^n_1 a\rangle \,;\, b)\rangle \,;\, \langle \mathrm{D}^n_1 \mathrm{A}_1 a\rangle \,;\, \dots \,;\, \langle \mathrm{D}^n_1 \mathrm{A}_n a\rangle \,;\, \mathrm{A}_i a\\ &= \langle \mathrm{A}^n_1(\langle \mathrm{A}^n_1 a\rangle \,;\, b)\rangle \,;\, \langle \mathrm{D}^n_1 \mathrm{A}_i a\rangle \,;\, \dots \,;\, \langle \mathrm{D}^n_1 \mathrm{A}_i a\rangle \,;\, \mathrm{A}_i a\\ &= \langle \mathrm{A}^n_1(\langle \mathrm{A}^n_1 a\rangle \,;\, b)\rangle \,;\, \mathrm{A}_i a\end{aligned}$$

by first applying the ith twisted law for antidomain, then applying (6.10) and then repeatedly applying (6.9). □

We will give (6.11) the full title 'the **restricted twisted laws for antidomain**', but since these are the twisted laws we apply most frequently, when we refer simply to 'the ith twisted law' we will mean the i-indexed version of (6.11).

In the following lemma and in later proofs an 's' above an equality sign indicates an appeal to superassociativity, a 't' an appeal to the twisted laws and any number an appeal to the corresponding equation.

Lemma 6.3.6. *The following equations are consequences of axioms* (6.2)–(6.10).

$$\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i a = \mathrm{A}_i a \qquad \textit{for every } i \tag{6.12}$$

$$\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b = \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_i a \qquad \textit{for every } i \tag{6.13}$$

$$\langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} c = \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} c \tag{6.14}$$

$$\mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) = \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b \qquad \textit{for every } i, j \tag{6.15}$$

Proof. We have

$$\begin{aligned}
\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i a &= \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} a)\rangle \mathbin{;} \mathrm{A}_i a && \text{by the } i\text{th twisted law}\\
&= \langle \mathrm{A}_1^n 0\rangle \mathbin{;} \mathrm{A}_i a && \text{by (6.4)}\\
&= \boldsymbol{\pi} \mathbin{;} \mathrm{A}_i a && \text{by the definition of } \boldsymbol{\pi}\\
&= \mathrm{A}_i a && \text{by (6.3)}
\end{aligned}$$

proving (6.12).

Before proceeding with (6.13)–(6.15), we note the following useful consequences of (6.2)–(6.10). By (6.10) then (6.4) we see that

$$\langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_i a = \langle \mathrm{A}_1^n \mathrm{A}_i a\rangle \mathbin{;} \mathrm{A}_i a = 0 \tag{6.16}$$

and by first applying superassociativity and then (6.16) to $\langle \mathrm{D}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} b$ we obtain

$$\langle \mathrm{D}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} b = 0. \tag{6.17}$$

We will use (6.8) to prove (6.13). Firstly

$$\begin{aligned}
\langle \mathrm{A}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) &\overset{\text{s}}{=} \langle \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_1 a, \dots, \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_n a\rangle \mathbin{;} \mathrm{A}_i b\\
&\overset{6.12}{=} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathrm{A}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_i a) &\overset{\text{s}}{=} \langle \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_1 b, \dots, \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_n b\rangle \mathbin{;} \mathrm{A}_i a\\
&\overset{\text{t}}{=} \langle \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{A}_1 a, \dots, \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{A}_n a\rangle \mathbin{;} \mathrm{A}_i a\\
&\overset{\text{s}}{=} \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i a\\
&\overset{6.12}{=} \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{A}_i a\\
&\overset{\text{t}}{=} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b
\end{aligned}$$

so we see that $\langle \mathrm{A}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b)$ and $\langle \mathrm{A}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_i a)$ coincide. We also have

$$\langle \mathrm{D}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) \overset{6.17}{=} 0$$

and

$$\begin{aligned}\langle \mathrm{D}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_i a) &\stackrel{\mathrm{s}}{=} \langle \langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_1 b, \ldots, \langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_n b\rangle \mathbin{;} \mathrm{A}_i a\\ &\stackrel{\mathrm{t}}{=} \langle \langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{D}_1 a, \ldots, \langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{D}_n a\rangle \mathbin{;} \mathrm{A}_i a\\ &\stackrel{\mathrm{s}}{=} \langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} \langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_i a\\ &\stackrel{6.16}{=} \langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} b)\rangle \mathbin{;} 0\\ &\stackrel{6.6}{=} 0\end{aligned}$$

and so $\langle \mathrm{D}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b)$ and $\langle \mathrm{D}_1^n a\rangle \mathbin{;} (\langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_i a)$ coincide, completing the proof of (6.13).

Equation (6.14) is a simple, but useful, consequence of (6.13). We have

$$\begin{aligned}\langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} c &= \langle \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_1 b, \ldots, \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_n b\rangle \mathbin{;} c && \text{by superassociativity}\\ &= \langle \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_1 a, \ldots, \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_n a\rangle \mathbin{;} c && \text{by (6.13)}\\ &= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} c && \text{by superassociativity}\end{aligned}$$

as required.

To prove (6.15) we prove that

$$\mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) = \mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) \qquad \text{for every } i, j \tag{6.18}$$

and that

$$\mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) = \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b \qquad \text{for every } j \tag{6.19}$$

are consequences of (6.2)–(6.10).

For (6.18) we have

$$\begin{aligned}\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) &= \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b)\rangle \mathbin{;} \mathrm{A}_j a && \text{by (6.13)}\\ &= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j \mathrm{A}_i b && j\text{th twisted law}\\ &= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j \mathrm{A}_j b && \text{by (6.10)}\end{aligned}$$

and in the same way

$$\begin{aligned}\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) &= \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b)\rangle \mathbin{;} \mathrm{A}_j a && \text{by (6.13)}\\ &= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j \mathrm{A}_j b && j\text{th twisted law}\end{aligned}$$

and we have

$$\begin{aligned}\langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) &= \langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b)\rangle \mathbin{;} \mathrm{D}_j a && j\text{th twisted law}\\ &= \langle \mathrm{A}_1^n 0\rangle \mathbin{;} \mathrm{D}_j a && \text{by (6.17)}\\ &= \boldsymbol{\pi} \mathbin{;} \mathrm{D}_j a && \text{by definition of } \boldsymbol{\pi}\\ &= \mathrm{D}_j a && \text{by (6.3)}\end{aligned}$$

and similarly

$$\langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) = \mathrm{D}_j a$$

and so from an application of (6.8) we deduce the required equation.

Equation (6.19) can be deduced with two applications of (6.8), composing on the left with $\langle \mathrm{A}_1^n a\rangle$ and $\langle \mathrm{D}_1^n a\rangle$ and with $\langle \mathrm{A}_1^n b\rangle$ and $\langle \mathrm{D}_1^n b\rangle$. One can show that any of the compositions with $\langle \mathrm{D}_1^n a\rangle$ or $\langle \mathrm{D}_1^n b\rangle$ evaluate to 0, for example

$$\begin{aligned}
&\langle \mathrm{D}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) && \text{by (6.14)} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_j\mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) && \text{by definition of } \mathrm{D}_j \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} \mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b))\rangle \mathbin{;} \mathrm{D}_j a && \text{by } j\text{th twisted law} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b)\rangle \mathbin{;} \mathrm{D}_j a)\rangle \mathbin{;} \mathrm{D}_j a && \text{by } j\text{th twisted law} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n 0\rangle \mathbin{;} \mathrm{D}_j a)\rangle \mathbin{;} \mathrm{D}_j a && \text{by (6.17)} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n(\boldsymbol{\pi} \mathbin{;} \mathrm{D}_j a)\rangle \mathbin{;} \mathrm{D}_j a && \text{by definition of } \boldsymbol{\pi} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n \mathrm{D}_j a\rangle \mathbin{;} \mathrm{D}_j a && \text{by (6.3)} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} 0 && \text{by (6.4)} \\
&= 0 && \text{by (6.6)}
\end{aligned}$$

and the others are similar. The compositions with $\langle \mathrm{A}_1^n a\rangle$ and $\langle \mathrm{A}_1^n b\rangle$ both equal $\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b$. Observe

$$\begin{aligned}
\langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) &= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{D}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b)\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.13)} \\
&= \langle \mathrm{D}_1^n(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b)\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.14)} \\
&= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.9)}
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} (\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) &= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.14)} \\
&= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.12)} \\
&= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_j a && \text{by (6.13)} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j a && \text{by (6.14)} \\
&= \langle \mathrm{A}_1^n b\rangle \mathbin{;} \mathrm{A}_j a && \text{by (6.12)} \\
&= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.13)}
\end{aligned}$$

as claimed.

The equations of (6.15) now follow easily, for

$$\begin{aligned}
\mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) &= \mathrm{A}_j\mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b) && \text{by definition} \\
&= \mathrm{A}_j\mathrm{A}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) && \text{by (6.18)} \\
&= \mathrm{D}_j(\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b) && \text{by definition} \\
&= \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_j b && \text{by (6.19)}
\end{aligned}$$

as required. □

We will refer to elements of the form $\mathrm{A}_i a$, for any a, as A_i-elements. For each i define a product on A_i-elements by $\mathrm{A}_i a \bullet_i \mathrm{A}_i b \coloneqq \langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b$. We will omit the subscript and write $\bullet$ where possible. To prove these are well-defined we need to show

$$\mathrm{A}_i a = \mathrm{A}_i b \to \mathrm{A}_j a = \mathrm{A}_j b \qquad \text{for every } i, j \tag{6.20}$$

all hold. But by (6.15), with $a = 0$, we know that

$$\mathrm{D}_j \mathrm{A}_i c = \mathrm{A}_j c \tag{6.21}$$

is a consequence of our axioms for all i and j. Then assuming $\mathrm{A}_i a = \mathrm{A}_i b$, we have $\mathrm{A}_j a = \mathrm{D}_j \mathrm{A}_i a = \mathrm{D}_j \mathrm{A}_i b = \mathrm{A}_j b$. Note also that, by (6.15), every product of A_i-elements is an A_i-element.

Lemma 6.3.7. *It follows from* (6.2)–(6.10) *that the* A_i*-elements with the operation* $\bullet$ *form a semilattice.*

Proof. Equations (6.12) and (6.13) state that $\bullet$ is idempotent and commutative respectively.

For associativity we have

$$\begin{aligned}
&\mathrm{A}_i a \bullet (\mathrm{A}_i b \bullet \mathrm{A}_i c) && \\
&= \langle \mathrm{A}_1^n a \rangle \mathbin{;} (\langle \mathrm{A}_1^n b \rangle \mathbin{;} \mathrm{A}_i c) && \text{by the definition of } \bullet \\
&= \langle \langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_1 b, \dots, \langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_n b \rangle \mathbin{;} \mathrm{A}_i c && \text{by superassociativity} \\
&= \langle \mathrm{D}_1(\langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b), \dots, \mathrm{D}_n(\langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b) \rangle \mathbin{;} \mathrm{A}_i c && \text{by (6.15)} \\
&= \langle \mathrm{A}_1^n \mathrm{A}_i(\langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b) \rangle \mathbin{;} \mathrm{A}_i c && \text{by (6.10)} \\
&= \mathrm{A}_i \mathrm{A}_i(\langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b) \bullet \mathrm{A}_i c && \text{by the definition of } \bullet \\
&= \mathrm{D}_i(\langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b) \bullet \mathrm{A}_i c && \text{by the definition of } \mathrm{D}_i \\
&= (\langle \mathrm{A}_1^n a \rangle \mathbin{;} \mathrm{A}_i b) \bullet \mathrm{A}_i c && \text{by (6.15)} \\
&= (\mathrm{A}_i a \bullet \mathrm{A}_i b) \bullet \mathrm{A}_i c && \text{by the definition of } \bullet
\end{aligned}$$

as required. □

Lemma 6.3.8. *It follows from* (6.2)–(6.10) *that for every i the* A_i*-elements, with product* $\bullet$ *and complement given by* A_i*, form a Boolean algebra with top element* π_i *and bottom element* 0*.*

Proof. We already know, by Lemma 6.3.7, that the A_i-elements form a semilattice. Equation (6.3) says that π_i is the top element of the semilattice. We want to show that 0 is an A_i-element, then both (6.5) and (6.6) independently say that 0 is the bottom element of the semilattice. This is easy: $\mathrm{A}_i \pi_i = \mathrm{A}_i \pi_i \bullet \pi_i = \langle \mathrm{A}_1^n \pi_i \rangle \mathbin{;} \pi_i = 0$.

To complete the proof that we have a Boolean algebra we use the dual of the axiomatisation of Boolean algebras given, for example, in [41, Definition 2.3]. Let $\alpha + \beta$ abbreviate $\mathrm{A}_i(\mathrm{A}_i \alpha \bullet \mathrm{A}_i \beta)$. We need complement axioms:

$$\mathrm{A}_i\mathrm{A}_i\alpha = \alpha \tag{6.22}$$

$$\mathrm{A}_i\alpha \bullet \alpha = 0 \tag{6.23}$$

$$\mathrm{A}_i 0 = \pi_i \tag{6.24}$$

and distributivity:

$$\alpha + \beta \bullet \gamma = (\alpha + \beta) \bullet (\alpha + \gamma) \tag{6.25}$$

where Greek letters denote arbitrary A_i-elements.

The first complement axiom follows from (6.15), the second is (6.4), and the third is true by definition. To prove distributivity, we first need an auxiliary equation:

$$\mathrm{A}_i a \bullet \mathrm{A}_i\mathrm{A}_i b = \mathrm{A}_i(\mathrm{A}_i a \bullet \mathrm{A}_i b) \bullet \mathrm{A}_i a. \tag{6.26}$$

We have

$$\begin{aligned}
\mathrm{A}_i a \bullet \mathrm{A}_i\mathrm{A}_i b &= \langle \mathrm{A}_1^n a\rangle \,;\, \mathrm{A}_i\mathrm{A}_i b && \text{by the definition of } \bullet\\
&= \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle \,;\, \mathrm{A}_i b)\rangle \,;\, \mathrm{A}_i a && \text{by the } i\text{th twisted law}\\
&= \langle \mathrm{A}_1^n(\mathrm{A}_i a \bullet \mathrm{A}_i b)\rangle \,;\, \mathrm{A}_i a && \text{by the definition of } \bullet\\
&= \mathrm{A}_i(\mathrm{A}_i a \bullet \mathrm{A}_i b) \bullet \mathrm{A}_i a && \text{by the definition of } \bullet
\end{aligned}$$

as required.

Now the distributivity axiom expands to

$$\mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i(\beta \bullet \gamma)) = \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\beta) \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma)$$

and we prove this using (6.8). We have by applying the ith twisted law

$$\begin{aligned}
\langle \mathrm{D}_1^n\alpha\rangle \,;\, \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i(\beta \bullet \gamma)) &= \langle \mathrm{A}_1^n(\mathrm{D}_i\alpha \bullet \mathrm{A}_i\alpha \bullet \mathrm{A}_i(\beta \bullet \gamma))\rangle \,;\, \mathrm{D}_i\alpha\\
&= \langle \mathrm{A}_1^n 0\rangle \,;\, \mathrm{D}_i\alpha\\
&= \boldsymbol{\pi} \,;\, \mathrm{D}_i\alpha\\
&= \mathrm{D}_i\,\alpha\\
&= \alpha
\end{aligned}$$

and again using the ith twisted law

$$\begin{aligned}
\langle \mathrm{D}_1^n\alpha\rangle \,;\, (\mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\beta) \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma)) &= \mathrm{D}_i\alpha \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\beta) \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma)\\
&= \mathrm{A}_i 0 \bullet \mathrm{D}_i\alpha \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma)\\
&= \mathrm{A}_i 0 \bullet \mathrm{A}_i 0 \bullet \mathrm{D}_i\alpha\\
&= \alpha
\end{aligned}$$

and we have

$$\begin{aligned}\langle \mathrm{A}_1^n\alpha\rangle\ ;\ \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i(\beta\bullet\gamma)) &= \mathrm{A}_i\alpha \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i(\beta\bullet\gamma))\\ &= \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i(\beta\bullet\gamma)) \bullet \mathrm{A}_i\alpha\\ &\overset{6.26}{=} \mathrm{A}_i\alpha \bullet \mathrm{A}_i\mathrm{A}_i(\beta\bullet\gamma)\\ &= \mathrm{A}_i\alpha \bullet \mathrm{D}_i(\beta\bullet\gamma)\\ &= \mathrm{A}_i\alpha \bullet \beta\bullet\gamma\end{aligned}$$

and

$$\begin{aligned}\langle \mathrm{A}_1^n\alpha\rangle\ ;\ (\mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\beta) \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma)) &= \mathrm{A}_i\alpha \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\beta) \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma)\\ &= \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma) \bullet \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\beta) \bullet \mathrm{A}_i\alpha\\ &\overset{6.26}{=} \mathrm{A}_i(\mathrm{A}_i\alpha \bullet \mathrm{A}_i\gamma) \bullet \mathrm{A}_i\alpha \bullet \mathrm{A}_i\mathrm{A}_i\beta\\ &\overset{6.26}{=} \mathrm{A}_i\alpha \bullet \mathrm{A}_i\mathrm{A}_i\gamma \bullet \mathrm{A}_i\mathrm{A}_i\beta\\ &= \mathrm{A}_i\alpha \bullet \mathrm{D}_i\gamma \bullet \mathrm{D}_i\beta\\ &= \mathrm{A}_i\alpha \bullet \beta\bullet\gamma\end{aligned}$$

giving the result. □

We know, from (6.20), that the map $\theta_{ji}\colon\ \mathrm{A}_i a \mapsto \mathrm{A}_j a$ is well-defined for every i and j. Hence it is a bijection from the A_i-elements to the A_j-elements. (By (6.21), it is given by the restriction of D_j to A_i-elements.) Then

$$\begin{aligned}\theta_{ji}(\mathrm{A}_i\mathrm{A}_i a) &= \mathrm{A}_j\mathrm{A}_i a && \text{by the definition of } \theta_{ji}\\ &= \mathrm{A}_j\mathrm{A}_j a && \text{by (6.10)}\\ &= \mathrm{A}_j\theta_{ji}(\mathrm{A}_i a) && \text{by the definition of } \theta_{ji}\end{aligned}$$

and

$$\begin{aligned}\theta_{ji}(\mathrm{A}_i a \bullet_i \mathrm{A}_i b) &= \mathrm{D}_j(\mathrm{A}_i a \bullet_i \mathrm{A}_i b) && \text{by (6.21)}\\ &= \mathrm{A}_j a \bullet_j \mathrm{A}_j b && \text{by (6.15)}\\ &= \theta_{ji}(\mathrm{A}_i a) \bullet_j \theta_{ji}(\mathrm{A}_i b) && \text{by the definition of } \theta_{ji}\end{aligned}$$

and so θ_{ji} is an isomorphism of the Boolean algebras.

Notice that the collection (θ_{ji}) of Boolean algebra isomorphisms commute, that is, each θ_{ii} is the identity and $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for all i, j, and k. Hence we may fix a representative of the isomorphism class of these Boolean algebras and fix isomorphisms to the Boolean algebras that commute with the isomorphisms θ_{ji}. For definiteness we will use the A_1-elements as the representative Boolean algebra. Then for each i the isomorphism to the A_i-elements will be θ_{i1}.

We will refer to elements of the representative Boolean algebra as A-elements and use Greek letters to denote arbitrary A-elements. If α is an A-element then $\mathrm{A}\alpha$ is the complement of α within the Boolean algebra of A-elements, $\boldsymbol{\alpha}$ is shorthand for $\langle \alpha_1, \dots, \alpha_n\rangle$, consisting of the images of α in the algebras of A_i-elements and $\overline{\boldsymbol{\alpha}}$ is shorthand for $\langle \mathrm{A}_1\alpha_1, \dots, \mathrm{A}_n\alpha_n\rangle$, consisting of the images of $\mathrm{A}\alpha$.

Lemma 6.3.9. *The following quasiequations are consequences of axioms* (6.2)–(6.10).

$$\langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, b)\rangle \,;\, \mathrm{D}_i a_j = \mathrm{D}_i(\boldsymbol{a} \,;\, b) \qquad \textit{for every } i, j \qquad (6.27)$$

$$\boldsymbol{a} \,;\, \mathrm{D}_i b = \langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, b)\rangle \,;\, a_i \qquad \textit{for every } i \qquad (6.28)$$

(the **twisted laws for domain**)

$$\boldsymbol{a} \,;\, \mathrm{A}_i b = 0 \to \boldsymbol{a} \,;\, \mathrm{A}_j b = 0 \qquad \textit{for every } i, j \qquad (6.29)$$

$$\boldsymbol{\alpha} \,;\, a = \boldsymbol{\alpha} \,;\, b \wedge \boldsymbol{\beta} \,;\, a = \boldsymbol{\beta} \,;\, b \to (\boldsymbol{\alpha} + \boldsymbol{\beta}) \,;\, a = (\boldsymbol{\alpha} + \boldsymbol{\beta}) \,;\, b \qquad (6.30)$$

where $+$ is the Boolean sum and we have extended notation componentwise to sequences.

Proof. Equation (6.27) is the statement that $\mathrm{D}(\boldsymbol{a} \,;\, b) \leq \mathrm{D}a_j$ within the Boolean algebra of A-elements. This is equivalent to $\mathrm{A}(\boldsymbol{a} \,;\, b) \geq \mathrm{A}a_j$, that is $\langle \mathrm{A}_1^n a_j\rangle \,;\, \mathrm{A}_1(\boldsymbol{a} \,;\, b) = \mathrm{A}_1 a_j$. This is true, for

$$\begin{aligned}
&\langle \mathrm{A}_1^n a_j\rangle \,;\, \mathrm{A}_1(\boldsymbol{a} \,;\, b) \\
&\overset{\mathrm{t}}{=} \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a_j\rangle \,;\, \boldsymbol{a} \,;\, b)\rangle \,;\, \mathrm{A}_1 a_j \\
&\overset{\mathrm{s}}{=} \langle \mathrm{A}_1^n(\langle\langle \mathrm{A}_1^n a_j\rangle \,;\, a_1, \dots, \langle \mathrm{A}_1^n a_j\rangle \,;\, a_n\rangle \,;\, b)\rangle \,;\, \mathrm{A}_1 a_j \\
&\overset{6.4}{=} \langle \mathrm{A}_1^n(\langle\langle \mathrm{A}_1^n a_j\rangle \,;\, a_1, \dots, \langle \mathrm{A}_1^n a_j\rangle \,;\, a_{j-1}, 0, \langle \mathrm{A}_1^n a_j\rangle \,;\, a_{j+1}, \dots, \langle \mathrm{A}_1^n a_j\rangle \,;\, a_n\rangle \,;\, b)\rangle \,;\, \mathrm{A}_1 a_j \\
&\overset{6.5}{=} \langle \mathrm{A}_1^n 0\rangle \,;\, \mathrm{A}_1 a_j \\
&= \boldsymbol{\pi} \,;\, \mathrm{A}_1 a_j \\
&\overset{6.3}{=} \mathrm{A}_1 a_j
\end{aligned}$$

and so (6.27) is valid.

In order to prove the twisted laws for domain we first prove

$$\langle \mathrm{D}_1^n c\rangle \,;\, d = \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n c\rangle \,;\, d)\rangle \,;\, d \qquad (6.31)$$

and we do this by an application of (6.8). We have

$$\langle \mathrm{D}_1^n c\rangle \,;\, (\langle \mathrm{D}_1^n c\rangle \,;\, d) = \langle \mathrm{D}_1^n c\rangle \,;\, d$$

and

$$\begin{aligned}
\langle \mathrm{D}_1^n c\rangle \,;\, (\langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n c\rangle \,;\, d)\rangle \,;\, d) &\overset{\mathrm{s}}{=} \langle\langle \mathrm{D}_1^n c\rangle \,;\, \mathrm{A}_1(\langle \mathrm{A}_1^n c\rangle \,;\, d), \dots, \langle \mathrm{D}_1^n c\rangle \,;\, \mathrm{A}_n(\langle \mathrm{A}_1^n c\rangle \,;\, d)\rangle \,;\, d \\
&\overset{\mathrm{t}}{=} \langle\langle \mathrm{A}_1^n(\langle \mathrm{D}_1^n c\rangle \,;\, \langle \mathrm{A}_1^n c\rangle \,;\, d)\rangle \,;\, \mathrm{D}_1 c, \dots\rangle \,;\, d \\
&= \langle\langle \mathrm{A}_1^n 0\rangle \,;\, \mathrm{D}_1 c, \dots, \langle \mathrm{A}_1^n 0\rangle \,;\, \mathrm{D}_n c\rangle \,;\, d \\
&= \langle \boldsymbol{\pi} \,;\, \mathrm{D}_1 c, \dots, \boldsymbol{\pi} \,;\, \mathrm{D}_n c\rangle \,;\, d \\
&\overset{6.3}{=} \langle \mathrm{D}_1^n c\rangle \,;\, d
\end{aligned}$$

and we also have

$$\langle \mathrm{A}_1^n c\rangle \,;\, (\langle \mathrm{D}_1^n c\rangle \,;\, d) = 0$$

and

$$\begin{aligned}
\langle \mathrm{A}_1^n c\rangle \,;\, (\langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n c\rangle \,;\, d)\rangle \,;\, d) &\overset{6.14}{=} \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n c\rangle \,;\, d)\rangle \,;\, \langle \mathrm{A}_1^n c\rangle \,;\, d \\
&\overset{6.4}{=} 0
\end{aligned}$$

giving us what we require to deduce (6.31).

Now to deduce the ith twisted law for domain, firstly $\boldsymbol{a} \,;\, \mathrm{D}_i b = \boldsymbol{a} \,;\, \mathrm{A}_i \mathrm{A}_i b$ by the definition of D_i. Applying the ith twisted law for antidomain to the right-hand side we get

$$\langle \mathrm{A}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_i b)\rangle \,;\, \langle \mathrm{D}_1^n a_1\rangle \,;\, \dots \,;\, \langle \mathrm{D}_1^n a_n\rangle \,;\, a_i$$

then by applying the ith twisted law for antidomain again this equals

$$\langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n(\boldsymbol{a} \,;\, b)\rangle \,;\, \langle \mathrm{D}_1^n a_1\rangle \,;\, \dots \,;\, \langle \mathrm{D}_1^n a_n\rangle \,;\, a_i)\rangle \,;\, \langle \mathrm{D}_1^n a_1\rangle \,;\, \dots \,;\, \langle \mathrm{D}_1^n a_n\rangle \,;\, a_i$$

and by setting $c = \boldsymbol{a} \,;\, b$ and $d = \langle \mathrm{D}_1^n a_1\rangle \,;\, \dots \,;\, \langle \mathrm{D}_1^n a_n\rangle \,;\, a_i$ in (6.31), this is equal to

$$\langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, b)\rangle \,;\, \langle \mathrm{D}_1^n a_1\rangle \,;\, \dots \,;\, \langle \mathrm{D}_1^n a_n\rangle \,;\, a_i$$

and this equals $\langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, b)\rangle \,;\, a_i$ by repeated application of superassociativity and (6.27).

For (6.29), suppose $\boldsymbol{a} \,;\, \mathrm{A}_i b = 0$. Then

$$\begin{aligned}
\langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_j b)\rangle \,;\, a_i &= \boldsymbol{a} \,;\, \mathrm{D}_i \mathrm{A}_j b && \text{by the } i\text{th twisted law for domain}\\
&= \boldsymbol{a} \,;\, \mathrm{A}_i \mathrm{A}_i \mathrm{A}_j b && \text{by the definition of } \mathrm{D}_i\\
&= \boldsymbol{a} \,;\, \mathrm{A}_i \mathrm{A}_i \mathrm{A}_i b && \text{by (6.10)}\\
&= \boldsymbol{a} \,;\, \mathrm{A}_i b && \text{as } \mathrm{A}_i \text{ is complement on the } \mathrm{A}_i\text{-elements}\\
&= 0 && \text{by assumption}
\end{aligned}$$

and so

$$\begin{aligned}
\boldsymbol{a} \,;\, \mathrm{A}_j b &\overset{6.9}{=} \langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_j b)\rangle \,;\, \boldsymbol{a} \,;\, \mathrm{A}_j b\\
&\overset{\mathrm{S}}{=} \langle \langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_j b)\rangle \,;\, a_1, \dots, \langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_j b)\rangle \,;\, a_n\rangle \,;\, \mathrm{A}_j b\\
&= \langle \dots, \langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_j b)\rangle \,;\, a_{i-1}, 0, \langle \mathrm{D}_1^n(\boldsymbol{a} \,;\, \mathrm{A}_j b)\rangle \,;\, a_{i+1}, \dots\rangle \,;\, \mathrm{A}_j b\\
&\overset{6.5}{=} 0
\end{aligned}$$

hence (6.29) holds.

For (6.30), suppose $\boldsymbol{\alpha} \,;\, a = \boldsymbol{\alpha} \,;\, b$ and $\boldsymbol{\beta} \,;\, a = \boldsymbol{\beta} \,;\, b$. Then by Boolean reasoning and the assumptions

$$\begin{aligned}
\boldsymbol{\alpha} \,;\, (\boldsymbol{\alpha} + \boldsymbol{\beta}) \,;\, a &= \boldsymbol{\alpha} \,;\, a\\
&= \boldsymbol{\alpha} \,;\, b\\
&= \boldsymbol{\alpha} \,;\, (\boldsymbol{\alpha} + \boldsymbol{\beta}) \,;\, b
\end{aligned}$$

and

$$\begin{aligned}
\overline{\boldsymbol{\alpha}} \,;\, (\boldsymbol{\alpha} + \boldsymbol{\beta}) \,;\, a &= \overline{\boldsymbol{\alpha}} \,;\, \boldsymbol{\beta} \,;\, a\\
&= \overline{\boldsymbol{\alpha}} \,;\, \boldsymbol{\beta} \,;\, b\\
&= \overline{\boldsymbol{\alpha}} \,;\, (\boldsymbol{\alpha} + \boldsymbol{\beta}) \,;\, b
\end{aligned}$$

so (6.30) follows, by (6.8). □

Write $a \leq b$ to mean $\langle \mathrm{D}^n_1(a)\rangle \mathbin{;} b = a$.

Lemma 6.3.10. *It follows from* (6.2)–(6.10) *that the relation $\leq$ is a partial order and with respect to this order $\langle\ \rangle;$ is order preserving in each of its arguments.*

Proof. Reflexivity is just (6.9). For antisymmetry, suppose that $\langle \mathrm{D}^n_1 a\rangle \mathbin{;} b = a$ and $\langle \mathrm{D}^n_1 b\rangle \mathbin{;} a = b$. Then

$$\begin{aligned}
a &= \langle \mathrm{D}^n_1 a\rangle \mathbin{;} b && \text{by the first assumption}\\
&= \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \langle \mathrm{D}^n_1 b\rangle \mathbin{;} a && \text{by the second assumption}\\
&= \langle \mathrm{A}^n_1 \mathrm{A}_1 a\rangle \mathbin{;} \langle \mathrm{A}^n_1 \mathrm{A}_1 b\rangle \mathbin{;} a && \text{by (6.10)}\\
&= \langle \mathrm{A}^n_1 \mathrm{A}_1 b\rangle \mathbin{;} \langle \mathrm{A}^n_1 \mathrm{A}_1 a\rangle \mathbin{;} a && \text{by (6.14)}\\
&= \langle \mathrm{D}^n_1 b\rangle \mathbin{;} \langle \mathrm{D}^n_1 a\rangle \mathbin{;} a && \text{by (6.10)}\\
&= \langle \mathrm{D}^n_1 b\rangle \mathbin{;} a && \text{by (6.9)}\\
&= b && \text{by the second assumption}
\end{aligned}$$

as required.

To prove transitivity, suppose $\langle \mathrm{D}^n_1 a\rangle \mathbin{;} b = a$ and $\langle \mathrm{D}^n_1 b\rangle \mathbin{;} c = b$. We first claim that

$$\mathrm{D}_i a = \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \mathrm{D}_i b \qquad \text{for every } i \tag{6.32}$$

follows from these assumptions. Observe that

$$\begin{aligned}
&\langle \mathrm{D}^n_1 a\rangle \mathbin{;} b = a\\
\implies\ & \mathrm{D}_j(\langle \mathrm{D}^n_1 a\rangle \mathbin{;} b) = \mathrm{D}_j a && \text{for every } j\\
\implies\ & \langle \mathrm{D}^n_1(\langle \mathrm{D}^n_1 a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{D}_i a = \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \mathrm{D}_i a = \mathrm{D}_i a && \text{for every } i,
\end{aligned}$$

but $\langle \mathrm{D}^n_1(\langle \mathrm{D}^n_1 a\rangle \mathbin{;} b)\rangle \mathbin{;} \mathrm{D}_i a = \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \mathrm{D}_i b$ by the ith twisted law for domain, establishing that $\mathrm{D}_i a = \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \mathrm{D}_i b$.

Now to prove transitivity, we have

$$\begin{aligned}
a &= \langle \mathrm{D}^n_1 a\rangle \mathbin{;} b && \text{by the first assumption}\\
&= \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \langle \mathrm{D}^n_1 b\rangle \mathbin{;} c && \text{by the second assumption}\\
&= \langle \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \mathrm{D}_1 b, \dots, \langle \mathrm{D}^n_1 a\rangle \mathbin{;} \mathrm{D}_n b\rangle \mathbin{;} c && \text{by superassociativity}\\
&= \langle \mathrm{D}^n_1 a\rangle \mathbin{;} c && \text{by (6.32)}
\end{aligned}$$

as required.

To see that $\langle\ \rangle;$ is order preserving in its final argument, suppose that $c \leq d$, that is, $\langle \mathrm{D}^n_1 c\rangle \mathbin{;} d = c$. Then for an arbitrary $\boldsymbol{a}$ we have

$$\begin{aligned}
\langle \mathrm{D}^n_1(\boldsymbol{a} \mathbin{;} c)\rangle \mathbin{;} (\boldsymbol{a} \mathbin{;} d) &\overset{\mathrm{S}}{=} \langle \langle \mathrm{D}^n_1(\boldsymbol{a} \mathbin{;} c)\rangle \mathbin{;} a_1, \dots, \langle \mathrm{D}^n_1(\boldsymbol{a} \mathbin{;} c)\rangle \mathbin{;} a_n\rangle \mathbin{;} d\\
&\overset{6.28}{=} \langle \boldsymbol{a} \mathbin{;} \mathrm{D}_1 c, \dots, \boldsymbol{a} \mathbin{;} \mathrm{D}_n c\rangle \mathbin{;} d\\
&\overset{\mathrm{S}}{=} \boldsymbol{a} \mathbin{;} \langle \mathrm{D}^n_1 c\rangle \mathbin{;} d\\
&= \boldsymbol{a} \mathbin{;} c
\end{aligned}$$

where the last equality holds by the assumption.

To see that $\langle\ \rangle;$ is order preserving in each of its first n arguments, suppose that $a_i \leq b_i$ for every i. That is, $\langle \mathrm{D}_1^n a_i\rangle ; b_i = a_i$ for every i. Then

$$\begin{aligned}
\langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; (\boldsymbol{b} ; c) &\overset{\mathrm{S}}{=} \langle\langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; b_1, \dots, \langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; b_n\rangle ; c \\
&= \langle\langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; \langle \mathrm{D}_1^n a_1\rangle ; b_1, \dots, \langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; \langle \mathrm{D}_1^n a_n\rangle ; b_n\rangle ; c \\
&= \langle\langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; a_1, \dots, \langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; a_n\rangle ; c \\
&\overset{\mathrm{S}}{=} \langle \mathrm{D}_1^n(\boldsymbol{a} ; c)\rangle ; \boldsymbol{a} ; c \\
&\overset{6.9}{=} \boldsymbol{a} ; c
\end{aligned}$$

utilising (6.27) for the second equality and the assumptions for the third. □

An easy application of laws we have so far shows that the partial order on the entire algebra agrees with the partial orders on each of the embedded Boolean algebras.

Note that

$$\langle \mathrm{A}_1^n a\rangle ; b = 0 \to \mathrm{A}_i a \leq \mathrm{A}_i b \qquad \text{for every } i \tag{6.33}$$

all hold, for assuming $\langle \mathrm{A}_1^n a\rangle ; b = 0$ gives

$$\begin{aligned}
\langle \mathrm{A}_1^n a\rangle ; \mathrm{A}_i b &= \langle \mathrm{A}_1^n(\langle \mathrm{A}_1^n a\rangle ; b)\rangle ; \mathrm{A}_i a && \text{by the } i\text{th twisted law} \\
&= \langle \mathrm{A}_1^n 0\rangle ; \mathrm{A}_i a && \text{by the assumption} \\
&= \boldsymbol{\pi} ; \mathrm{A}_i a && \text{by the definition of } \boldsymbol{\pi} \\
&= \mathrm{A}_i a && \text{by (6.3)}
\end{aligned}$$

which says that $\mathrm{A}_i a \bullet \mathrm{A}_i b = \mathrm{A}_i a$.

6.4 The representation

We are now finally ready to start describing our representation. In this section we prove the correctness of our representation for the signature $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$, but the representation is the same one we will use for all the expanded signatures that follow. The representation is a multiplace generalisation of the representation used in [57]; the technique originates with Schein, who calls it the method of *determinative pairs*.[4]

Definition 6.4.1. Let $\mathfrak{A}$ be an algebra of a signature containing composition. A **right congruence** is an equivalence relation $\sim$ on $\mathfrak{A}$ such that if $a_i \sim b_i$ for every i then $\boldsymbol{a} ; c \sim \boldsymbol{b} ; c$ for any $c \in \mathfrak{A}$.

For the remainder of this section, let $\mathfrak{A}$ be an algebra of the signature $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$ validating (6.2)–(6.10). Hence all the consequences deduced in Section 6.3 are true of $\mathfrak{A}$.

For a filter F of A-elements of $\mathfrak{A}$, define the binary relation $\sim_F$ on $\mathfrak{A}$ by $a \sim_F b$ if and only if there exists $\alpha \in F$ such that $\boldsymbol{\alpha} ; a = \boldsymbol{\alpha} ; b$.

[4] The 'pair' here is $(\sim, [0])$, where $\sim$ is a right congruence, and $[0]$ is the $\sim$-equivalence class of 0.

Lemma 6.4.2. *For any filter F of* A*-elements of* $\mathfrak{A}$*, the binary relation* $\sim_F$ *is a right congruence.*

Proof. It is clear that $\sim_F$ is reflexive and symmetric. To see that $\sim_F$ is transitive, first note that for any A-elements α and β and any c we have

$$(\boldsymbol{\alpha} \bullet \boldsymbol{\beta}) ; c = \langle \alpha_1 \bullet_1 \beta_1, \ldots, \alpha_n \bullet_n \beta_n \rangle ; c = \langle \boldsymbol{\alpha} ; \beta_1, \ldots, \boldsymbol{\alpha} ; \beta_n \rangle ; c = \boldsymbol{\alpha} ; (\boldsymbol{\beta} ; c) \tag{6.34}$$

Now suppose that $a \sim_F b$ and $b \sim_F c$ and let $\alpha \in F$ be such that $\boldsymbol{\alpha} ; a = \boldsymbol{\alpha} ; b$ and $\beta \in F$ be such that $\boldsymbol{\beta} ; b = \boldsymbol{\beta} ; c$. Then $\alpha \bullet \beta \in F$, since F is a filter, and (6.34) and commutativity of the Boolean product operations is precisely what is needed to give $(\boldsymbol{\alpha} \bullet \boldsymbol{\beta}) ; a = (\boldsymbol{\alpha} \bullet \boldsymbol{\beta}) ; c$. So $\sim_F$ is transitive.

Suppose now that $a_i \sim_F b_i$ for every i and let c be an arbitrary element of $\mathfrak{A}$. By hypothesis, for each i we can find $\alpha^i \in F$ such that $\boldsymbol{\alpha^i} ; a_i = \boldsymbol{\alpha^i} ; b_i$. Then $\prod_i \alpha^i \in F$ and $(\prod_i \boldsymbol{\alpha^i}) ; (\boldsymbol{a} ; c) = \langle (\prod_i \boldsymbol{\alpha^i}) ; a_1, \ldots, (\prod_i \boldsymbol{\alpha^i}) ; a_n \rangle ; c = \langle (\prod_i \boldsymbol{\alpha^i}) ; b_1, \ldots, (\prod_i \boldsymbol{\alpha^i}) ; b_n \rangle ; c = (\prod_i \boldsymbol{\alpha^i}) ; (\boldsymbol{b} ; c)$. So $\boldsymbol{a} ; c \sim_F \boldsymbol{b} ; c$. □

The next lemma describes a family of Cayley-style homomorphisms from which we will build a faithful representation.

Lemma 6.4.3. *Let U be an ultrafilter of* A*-elements of* $\mathfrak{A}$*. Write $[a]$ for the $\sim_U$-equivalence class of an element $a \in \mathfrak{A}$. Let $X \coloneqq \{[a] \mid a \in \mathfrak{A}\} \setminus \{[0]\}$ and for each $b \in \mathfrak{A}$ let $\theta_U(b)$ be the partial function from X^n to X given by*

$$\theta_U(b) \colon ([a_1], \ldots, [a_n]) \mapsto \begin{cases} [\langle a_1, \ldots, a_n \rangle ; b] & \text{if this is not equal to } [0] \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then the set $\{\theta_U(b) \mid b \in \mathfrak{A}\}$ forms a square algebra of n-ary partial functions, which we will call $\mathfrak{F}$ and $\theta_U \colon \mathfrak{A} \to \mathfrak{F}$ is a (surjective) homomorphism of $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n\}$-algebras. Further, if a is inequivalent to both 0 and b then θ_U separates a from b.

Proof. That $\sim_U$ is a right congruence says that $\theta_U(b)$ is well-defined for every $b \in \mathfrak{A}$. If we show that θ_U satisfies the conditions for being a homomorphism, then it automatically follows that the domain of $\mathfrak{F}$ is closed under the operations and so really is an algebra of n-ary partial functions.

We write $[\boldsymbol{a}]$ for $([a_1], \ldots, [a_n])$. To see that composition is represented correctly we first argue that $\theta_U(\boldsymbol{b} ; c)$ is defined if and only if $\langle \theta_U(b_1), \ldots, \theta_U(b_n) \rangle ; \theta_U(c)$ is defined. If $\langle \theta_U(b_1), \ldots, \theta_U(b_n) \rangle ; \theta_U(c)$ is defined at $[\boldsymbol{a}]$ then in particular $[\langle \boldsymbol{a} ; b_1, \ldots, \boldsymbol{a} ; b_n \rangle ; c]$ must be inequivalent to 0. By superassociativity, this equals $[\boldsymbol{a} ; (\boldsymbol{b} ; c)]$ and hence $\theta_U(\boldsymbol{b} ; c)$ is defined at $\boldsymbol{a}$.

If $\langle \theta_U(b_1), \ldots, \theta_U(b_n) \rangle ; \theta_U(c)$ is *un*defined at $[\boldsymbol{a}]$ then this is either because $\boldsymbol{a} ; (\boldsymbol{b} ; c)$ is equivalent to 0, in which case $\theta_U(\boldsymbol{b} ; c)$ is undefined at $[\boldsymbol{a}]$, or because there is an $\alpha \in U$ such that $\boldsymbol{\alpha} ; (\boldsymbol{a} ; b_i) = 0$ for some i. In the second case

$$\begin{aligned} \boldsymbol{\alpha} ; (\boldsymbol{a} ; (\boldsymbol{b} ; c)) &\overset{\mathrm{S}}{=} \boldsymbol{\alpha} ; \langle \boldsymbol{a} ; b_1, \ldots, \boldsymbol{a} ; b_n \rangle ; c \\ &\overset{\mathrm{S}}{=} \langle \boldsymbol{\alpha} ; \boldsymbol{a} ; b_1, \ldots, \boldsymbol{\alpha} ; \boldsymbol{a} ; b_n \rangle ; c \\ &= \langle \boldsymbol{\alpha} ; \boldsymbol{a} ; b_1, \ldots, \boldsymbol{\alpha} ; \boldsymbol{a} ; b_{i-1}, 0, \boldsymbol{\alpha} ; \boldsymbol{a} ; b_{i+1}, \ldots, \boldsymbol{\alpha} ; \boldsymbol{a} ; b_n \rangle ; c \\ &\overset{6.5}{=} 0 \end{aligned}$$

and so $\theta_U(\boldsymbol{b}\,;c)$ is again undefined at $[\boldsymbol{a}]$.

If $\theta_U(\boldsymbol{b}\,;c)$ and $\langle\theta_U(b_1),\dots,\theta_U(b_n)\rangle\,;\theta_U(c)$ are both defined at $[\boldsymbol{a}]$ then they both equal $[\boldsymbol{a}\,;\boldsymbol{b}\,;c]$. We conclude that composition is represented correctly by θ_U.

We now show that each A_i is represented correctly by θ_U. It is helpful to first note that θ_U represents 0 correctly, as $\boldsymbol{a}\,;0=0$ for any $\boldsymbol{a}$ and so $\theta_U(0)$ is undefined everywhere.

Next we will show that $\theta_U(\mathrm{A}_i b)$ is a restriction of the ith projection, for any $b\in\mathfrak{A}$ and for any i. Suppose that $\theta_U(\mathrm{A}_i b)$ is defined on $[\boldsymbol{a}]$, so that $a_1,\dots,a_n$ and $\boldsymbol{a}\,;\mathrm{A}_i b$ are all inequivalent to 0. We wish to show that $[\boldsymbol{a}\,;\mathrm{A}_i b]=[a_i]$. As $\langle\mathrm{A}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;(\boldsymbol{a}\,;\mathrm{A}_i b)=0=\langle\mathrm{A}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;0$, we know that $\mathrm{A}(\boldsymbol{a}\,;\mathrm{A}_i b)\notin U$ and so $\mathrm{D}(\boldsymbol{a}\,;\mathrm{A}_i b)\in U$, since U is an ultrafilter. Then

$$\begin{aligned}\langle\mathrm{D}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;(\boldsymbol{a}\,;\mathrm{A}_i b)&=\langle\mathrm{D}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;(\boldsymbol{a}\,;\mathrm{D}_i(\mathrm{A}_i b))\\&=\langle\mathrm{D}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;(\langle\mathrm{D}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;a_i)\\&=\langle\mathrm{D}_1^n(\boldsymbol{a}\,;\mathrm{A}_i b)\rangle\,;a_i\end{aligned}$$

where the second equality follows by the ith twisted law for domain. We conclude that $[\boldsymbol{a}\,;\mathrm{A}_i b]=[a_i]$, as desired.

Next we will show that where $\theta_U(b)$ is not defined, $\theta_U(\mathrm{A}_i b)$ *is* defined. Suppose that $a_1,\dots,a_n$ are all inequivalent to 0, so $\mathrm{D}a_1,\dots,\mathrm{D}a_n\in U$, but that $\theta_U(b)$ is undefined at $[\boldsymbol{a}]$, meaning $[\boldsymbol{a}\,;b]=[0]$. So there is an $\alpha\in U$ with $\boldsymbol{\alpha}\,;(\boldsymbol{a}\,;b)=\boldsymbol{\alpha}\,;0=0$. Then (6.33) tells us that $\alpha\leq\mathrm{A}(\boldsymbol{a}\,;b)$ and so U, being an ultrafilter, contains $\mathrm{A}(\boldsymbol{a}\,;b)$. Hence we know $\mathrm{A}(\boldsymbol{a}\,;b)\bullet\mathrm{D}a_1\bullet\cdots\bullet\mathrm{D}a_n\in U$, as U is $\bullet$-closed. Let this element of U be α'. Then by repeated application of (6.34), for any $c\in\mathfrak{A}$, we have $\boldsymbol{\alpha'}\,;c=\langle\mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle\mathrm{D}_1^n a_1\rangle\,;\dots;\langle\mathrm{D}_1^n a_n\rangle\,;c$. Then we can observe that

$$\begin{aligned}&\langle\mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle\mathrm{D}_1^n a_1\rangle\,;\dots;\langle\mathrm{D}_1^n a_n\rangle\,;(\boldsymbol{a}\,;\mathrm{A}_i b)\\&=\langle\langle\mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle\mathrm{D}_1^n a_1\rangle\,;\dots;\langle\mathrm{D}_1^n a_n\rangle\,;a_1,\dots,\\&\quad\ \langle\mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle\mathrm{D}_1^n a_1\rangle\,;\dots;\langle\mathrm{D}_1^n a_n\rangle\,;a_n\rangle\,;\mathrm{A}_i b&&\text{by superassociativity}\\&=\langle\boldsymbol{a}\,;\mathrm{A}_1 b,\dots,\boldsymbol{a}\,;\mathrm{A}_n b\rangle\,;\mathrm{A}_i b&&\text{twisted laws for antidomain}\\&=\boldsymbol{a}\,;\langle\mathrm{A}_1^n b\rangle\,;\mathrm{A}_i b&&\text{by superassociativity}\\&=\boldsymbol{a}\,;\mathrm{A}_i b&&\text{by (6.12)}\\&=\langle\mathrm{A}_1^n(\boldsymbol{a}\,;b)\rangle\,;\langle\mathrm{D}_1^n a_1\rangle\,;\dots;\langle\mathrm{D}_1^n a_n\rangle\,;a_i&&\text{twisted law for antidomain}\end{aligned}$$

and hence $[\boldsymbol{a}\,;\mathrm{A}_i b]=[a_i]$. As $[a_i]\neq[0]$, this means $\theta_U(\mathrm{A}_i b)$ is defined at $[\boldsymbol{a}]$, with value $[a_i]$.

It remains to show that $\theta_U(\mathrm{A}_i b)$ cannot be defined when $\theta_U(b)$ is defined. Suppose for a contradiction that both $\theta_U(b)$ and $\theta_U(\mathrm{A}_i b)$ are defined on an n-tuple $[\boldsymbol{a}]$. Now (6.29) tells us that $\boldsymbol{a}\,;\mathrm{A}_1 b,\dots,\boldsymbol{a}\,;\mathrm{A}_n b$ must be simultaneously equivalent or inequivalent to 0, for if there is an $\alpha\in U$ with $\boldsymbol{\alpha}\,;(\boldsymbol{a}\,;\mathrm{A}_j b)=0$ then by superassociativity $\langle\boldsymbol{\alpha}\,;a_i,\dots,\boldsymbol{\alpha}\,;a_n\rangle\,;\mathrm{A}_j b=0$ and so $\boldsymbol{\alpha}\,;(\boldsymbol{a}\,;\mathrm{A}_k b)=\langle\boldsymbol{\alpha}\,;a_i,\dots,\boldsymbol{\alpha}\,;a_n\rangle\,;\mathrm{A}_k b=0$. Hence $\theta_U(\mathrm{A}_1 b),\dots\theta_U(\mathrm{A}_1 b)$ are all defined on $[\boldsymbol{a}]$, with each $\theta_U(\mathrm{A}_j b)$, being a restriction of the jth projection, having value $[a_j]$. But then

$$\begin{aligned}
\theta_U(b)([\boldsymbol{a}]) &= \theta_U(b)(\theta_U(\mathrm{A}_1 b)([\boldsymbol{a}]), \dots, \theta_U(\mathrm{A}_n b)([\boldsymbol{a}])) \\
&= (\langle \theta_U(\mathrm{A}_1 b), \dots, \theta_U(\mathrm{A}_n b)\rangle \,;\, \theta_U(b))([\boldsymbol{a}]) && \text{by the definition of } \langle\ \rangle; \\
&= \theta_U(\langle \mathrm{A}_1 b, \dots, \mathrm{A}_n b\rangle \,;\, b)([\boldsymbol{a}]) && \text{as } \langle\ \rangle; \text{ represented} \\
&= \theta_U(0)([\boldsymbol{a}]) && \text{by (6.4)}
\end{aligned}$$

contradicting our observation that 0 is represented by θ_U as the empty function. This completes the proof that the antidomain operations are represented correctly by θ_U.

For the last part, if a is inequivalent to both 0 and b, then we know that π_i is inequivalent to 0, for each i, otherwise

$$a = \boldsymbol{\pi} \,;\, a \sim_U \langle \pi_1, \dots, \pi_{i-1}, 0, \pi_{i+1}, \dots, \pi_n\rangle \,;\, a = 0.$$

So $\theta_U(a)([\boldsymbol{\pi}]) = [\boldsymbol{\pi} \,;\, a] = [a]$ and if $\theta_U(b)([\boldsymbol{\pi}])$ is defined then it equals $[b]$, which is distinct from $[a]$. □

The next lemma shows that there are enough ultrafilters to form a faithful representation.

Lemma 6.4.4. *Let $a, b \in \mathfrak{A}$ and suppose that $a \not\leq b$. Then there is an ultrafilter U of* A*-elements for which $a \nsim_U 0$ and $a \nsim_U b$.*

Proof. Let F be the filter of A-elements generated by $\{\alpha \mid \boldsymbol{\alpha} \,;\, a = a\} \cup \{\mathrm{A}(\beta) \mid \boldsymbol{\beta} \,;\, a = \boldsymbol{\beta} \,;\, b\}$. If $0 \in F$ then (employing Equation (6.34)) $0 = \alpha \bullet \mathrm{A}(\beta^1) \bullet \dots \bullet \mathrm{A}(\beta^m)$ for some A-elements $\alpha, \beta^1, \dots, \beta^m$ with $\boldsymbol{\alpha} \,;\, a = a$ and $\boldsymbol{\beta^i} \,;\, a = \boldsymbol{\beta^i} \,;\, b$ for each $i \in \{1, \dots, m\}$. Define $\beta \coloneqq \sum_i \beta^i$. Then $0 = \alpha \bullet \mathrm{A}(\beta)$ and so $\alpha \leq \beta$, giving $\boldsymbol{\alpha} \,;\, a \leq \boldsymbol{\beta} \,;\, a$. From our assumption that $\boldsymbol{\beta^i} \,;\, a = \boldsymbol{\beta^i} \,;\, b$ for each $i \in \{1, \dots, m\}$, repeated application of (6.30) gives $\boldsymbol{\beta} \,;\, a = \boldsymbol{\beta} \,;\, b$, so we have $a = \boldsymbol{\alpha} \,;\, a \leq \boldsymbol{\beta} \,;\, a = \boldsymbol{\beta} \,;\, b \leq \boldsymbol{\pi} \,;\, b = b$ contradicting the assumption that $a \not\leq b$. Hence the filter F is proper and so can be extended to an ultrafilter, U, say.

Suppose that $a \sim_U 0$, in which case there is an $\alpha \in U$ such that $\boldsymbol{\alpha} \,;\, a = \boldsymbol{\alpha} \,;\, 0 = 0$. Now if we compose on the left each of $\langle \mathrm{A}_1\alpha_1, \dots, \mathrm{A}_n\alpha_n\rangle \,;\, a$ and a with $\boldsymbol{\alpha}$ and $\langle \mathrm{A}_1\alpha_1, \dots, \mathrm{A}_n\alpha_n\rangle$ in turn, we obtain, by an application of (6.8), the equation $\langle \mathrm{A}_1\alpha_1, \dots, \mathrm{A}_n\alpha_n\rangle \,;\, a = a$. So, by the definition of U, we get $\mathrm{A}(\alpha) \in U$—a contradiction, as U is a proper filter containing α. Hence $a \nsim_U 0$.

Suppose that $a \sim_U b$ in which case there is a $\beta \in U$ such that $\boldsymbol{\beta} ; a = \boldsymbol{\beta} ; b$. Then $\mathrm{A}(\beta) \in F \subseteq U$—a contradiction, as U is a proper filter containing β. Hence $a \nsim_U b$. □

Theorem 6.4.5. *The class of $\{\langle\ \rangle ;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-algebras that are representable by n-ary partial functions is a proper quasivariety, finitely axiomatised by (quasi)equations* (6.2)–(6.10).

Proof. We continue to let $\mathfrak{A}$ be an arbitrary $\{\langle\ \rangle ;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-algebra validating (6.2)–(6.10). For each $a, b \in \mathfrak{A}$ with $a \not\leq b$, let U_{ab} be a choice of an ultrafilter of A-elements for which $a \nsim_U 0$ and $a \nsim_U b$. Let θ_{ab} be the corresponding homomorphism as described in Lemma 6.4.3, which is guaranteed to separate a from b. Take a disjoint union, in the sense of Definition 6.2.5, of the family $(\theta_{ab})_{a,b\in\mathfrak{A}}$ of homomorphisms and call this φ. So φ is a homomorphism from some power $\mathfrak{A}^S$ of $\mathfrak{A}$ to an algebra of n-ary partial functions. Let Δ be the diagonal embedding of $\mathfrak{A}$ into $\mathfrak{A}^S$. Then the map $\theta \colon \mathfrak{A} \to \mathrm{Im}(\varphi \circ \Delta)$

defined by $\theta(a) = (\varphi \circ \Delta)(a)$ is a surjective homomorphism from $\mathfrak{A}$ to an algebra of n-ary partial functions.

For distinct $a, b \in \mathfrak{A}$, either $a \not\lesssim b$ or $b \not\lesssim a$ and so θ_{ab}, and therefore θ, separates a and b. Hence θ is an isomorphism, so a representation of $\mathfrak{A}$ by n-ary partial functions. □

Note that whilst Lemma 6.4.3 only uses square algebras of functions, in Theorem 6.4.5, by taking a disjoint union of homomorphisms, we require non-square algebras of functions for our representation. (It is linguistically convenient to treat the θ of Theorem 6.4.5 as uniquely specified and then refer to 'our representation' or 'the representation' in defiance of the fact that there is some nonconstructive choice involved in selecting which ultrafilters to use.)

It is clear that if $\mathfrak{A}$ is finite then the representation described in Theorem 6.4.5 has a finite base. More specifically the size of the base is no greater than the cube of the size of the algebra.

Corollary 6.4.6. *The finite representation property holds for the signature* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n\}$ *for representation by* n*-ary partial functions.*

6.5 Injective partial functions

In this section we present an algebraic characterisation of the injective partial functions within algebras of n-ary partial functions. This allows us to extend the axiomatisation of Section 6.3 to an axiomatisation of the class of $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n\}$-algebras representable as injective n-ary partial functions.

The following definition applies to any algebra with composition in the signature and with the domain operations either in the signature or definable via antidomain operations.

Definition 6.5.1. We will call an element a **injective** if it validates the indexed quasiequations

$$\boldsymbol{b} ; a = \boldsymbol{c} ; a \rightarrow \boldsymbol{b} ; \mathrm{D}_i a = \boldsymbol{c} ; \mathrm{D}_i a \qquad \text{for every } i \tag{6.35}$$

Definition 6.5.1 is made by analogy with Jackson and Stokes' definition of injective elements in the unary case, which is those a validating

$$b ; a = c ; a \rightarrow b ; \mathrm{D}(a) = c ; \mathrm{D}(a).$$

This appears as (27) in [57].

Proposition 6.5.2. *The representation described in Theorem 6.4.5 represents as injective functions precisely the injective elements of the algebra.*

Proof. We first argue that in algebras of n-ary partial functions injective functions are injective elements; then if an element of a representable algebra is represented as an injective function it must be an injective element. To this end, suppose a is an injective n-ary partial function and that $\boldsymbol{b} ; a = \boldsymbol{c} ; a$. Suppose further that $(\boldsymbol{x}, z) \in \boldsymbol{b} ; \mathrm{D}_i a$. Then $b_1, \ldots, b_n$ are all defined on $\boldsymbol{x}$, the function a is defined on $\langle b_1(\boldsymbol{x}), \ldots, b_n(\boldsymbol{x})\rangle$ and $z = b_i(\boldsymbol{x})$. The first two of these facts tell us that $\boldsymbol{b} ; a$ is defined on $\boldsymbol{x}$, with value w say. Then by assumption, $\boldsymbol{c} ; a$ is defined on $\boldsymbol{x}$, also with value w. So $c_1, \ldots, c_n$ are all defined on $\boldsymbol{x}$ and $a(b_1(\boldsymbol{x}), \ldots, b_n(\boldsymbol{x})) = w = a(c_1(\boldsymbol{x}), \ldots, c_n(\boldsymbol{x}))$. By injectivity of a, we get $b_j(\boldsymbol{x}) = c_j(\boldsymbol{x})$, for every j. As $\boldsymbol{c} \mathbin{;} a$ is defined on $\boldsymbol{x}$, so is $\boldsymbol{c} \mathbin{;} \mathrm{D}_i a$ and it takes value $c_i(\boldsymbol{x}) = b_i(\boldsymbol{x}) = z$. That is, $(\boldsymbol{x}, z) \in \boldsymbol{c} \mathbin{;} \mathrm{D}_i a$. We conclude that $\boldsymbol{b} \mathbin{;} \mathrm{D}_i a \subseteq \boldsymbol{c} \mathbin{;} \mathrm{D}_i a$. By symmetry, the reverse inclusion also holds. Hence a validates (6.35).

We now prove the converse: that every injective element is represented by our representation as an injective function. We will argue that, for any ultrafilter U of A-elements, the map θ_U described in Lemma 6.4.3 maps injective elements to injective functions. Since a disjoint union of injective functions is injective, the result follows.

Suppose that a is an injective element and that $\theta_U(a)([\boldsymbol{b}]) = \theta_U(a)([\boldsymbol{c}])$. That is, there is an $\alpha \in U$ such that $\boldsymbol{\alpha} \mathbin{;} (\boldsymbol{b} \mathbin{;} a) = \boldsymbol{\alpha} \mathbin{;} (\boldsymbol{c} \mathbin{;} a)$ (and neither $\boldsymbol{b} \mathbin{;} a$ nor $\boldsymbol{c} \mathbin{;} a$ is equivalent to 0). Then

$$\langle \boldsymbol{\alpha} \mathbin{;} b_1, \dots, \boldsymbol{\alpha} \mathbin{;} b_n \rangle \mathbin{;} a = \langle \boldsymbol{\alpha} \mathbin{;} c_1, \dots, \boldsymbol{\alpha} \mathbin{;} c_n \rangle \mathbin{;} a \qquad \text{by superassociativity}$$

so

$$\langle \boldsymbol{\alpha} \mathbin{;} b_1, \dots, \boldsymbol{\alpha} \mathbin{;} b_n \rangle \mathbin{;} \mathrm{D}_i a = \langle \boldsymbol{\alpha} \mathbin{;} c_1, \dots, \boldsymbol{\alpha} \mathbin{;} c_n \rangle \mathbin{;} \mathrm{D}_i a \qquad \text{for every } i\text{, by (6.35)}$$

so

$$\boldsymbol{\alpha} \mathbin{;} \boldsymbol{b} \mathbin{;} \mathrm{D}_i a = \boldsymbol{\alpha} \mathbin{;} \boldsymbol{c} \mathbin{;} \mathrm{D}_i a \qquad \text{by superassociativity}$$

so

$$\boldsymbol{\alpha} \mathbin{;} \langle \mathrm{D}^n_1(\boldsymbol{b} \mathbin{;} a) \rangle \mathbin{;} b_i = \boldsymbol{\alpha} \mathbin{;} \langle \mathrm{D}^n_1(\boldsymbol{c} \mathbin{;} a) \rangle \mathbin{;} c_i \qquad \text{twisted laws for domain}$$

from which we can derive

$$\boldsymbol{\alpha} \mathbin{;} \langle \mathrm{D}^n_1(\boldsymbol{b} \mathbin{;} a) \rangle \mathbin{;} \langle \mathrm{D}^n_1(\boldsymbol{c} \mathbin{;} a) \rangle \mathbin{;} b_i = \boldsymbol{\alpha} \mathbin{;} \langle \mathrm{D}^n_1(\boldsymbol{b} \mathbin{;} a) \rangle \mathbin{;} \langle \mathrm{D}^n_1(\boldsymbol{c} \mathbin{;} a) \rangle \mathbin{;} c_i$$

using superassociativity and the commutativity and idempotency of the $\bullet_i$ operations.

Since $\boldsymbol{b} \mathbin{;} a$ is inequivalent to 0, we know that $\mathrm{A}(\boldsymbol{b} \mathbin{;} a) \notin U$ and so $\mathrm{D}(\boldsymbol{b} \mathbin{;} a) \in U$. Similarly $\mathrm{D}(\boldsymbol{c} \mathbin{;} a) \in U$. As α, $\mathrm{D}(\boldsymbol{b} \mathbin{;} a)$ and $\mathrm{D}(\boldsymbol{c} \mathbin{;} a)$ are all in the ultrafilter U, we conclude, for every i, that $[b_i] = [c_i]$. Hence $\theta_U(a)$ is injective. □

The proof of Proposition 6.5.2 showed that if an element is represented as an injective function by *any* representation (not just the one described in Theorem 6.4.5), then the element is an injective element. Hence the indexed quasiequations of (6.35) are valid for the class of algebras representable by injective n-ary partial functions. So Proposition 6.5.2 yields the following corollary.

Corollary 6.5.3. *Adding* (6.35) *to* (6.2)–(6.10) *gives a finite quasiequational axiomatisation of the class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$*-algebras that are representable by injective* n*-ary partial functions.*

Since Corollary 6.5.3 uses the same representation as Theorem 6.4.5, it again follows as a corollary that the finite representation property holds.

Corollary 6.5.4. *The finite representation property holds for the signature* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$ *for representation by injective* n*-ary partial functions.*

6.6 Intersection

In this section we consider the signature $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \cdot\}$. We could search for extensions to the quasiequational axiomatisations of the previous sections. However, the presence of intersection in the signature allows us to give equational axiomatisations, deducing the quasiequations that we need.

We first present some valid equations involving intersection.

Proposition 6.6.1. *The following equations are valid for the class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \cdot\}$*-algebras representable by* n*-ary partial functions.*

$$a \cdot a = a \tag{6.36}$$

$$a \cdot b = b \cdot a \tag{6.37}$$

$$\boldsymbol{a} \mathbin{;} (b \cdot c) = (\boldsymbol{a} \mathbin{;} b) \cdot (\boldsymbol{a} \mathbin{;} c) \tag{6.38}$$

$$\langle \mathrm{D}_1^n(a \cdot b)\rangle \mathbin{;} a = a \cdot b \tag{6.39}$$

Proof. Equations (6.36) and (6.37) are both well-known properties of intersection. The validity of (6.38) and the validity of (6.39) are both easy to see and are noted in [23], where they appear as Equation (29) and Equation (28) respectively. □

We will include all the equational axioms of Section 6.3 in our axiomatisation, that is (6.2)–(6.7), (6.9), and (6.10), as well as including (6.36)–(6.39). All the consequences of Section 6.3 will follow from our axiomatisation if only we can deduce (6.8). Next we give three more valid equations whose inclusion enables us to do just that. Notice that (6.12) was deduced without (6.8), so is available to us.

We make use of the tie operations. Define $a \bowtie_i b \coloneqq \mathrm{D}_i(a \cdot b) +_i \langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b$, where $\alpha +_i \beta \coloneqq \mathrm{A}_i(\langle \mathrm{A}_1^n \alpha\rangle \mathbin{;} \mathrm{A}_i \beta)$.

Proposition 6.6.2. *The following equations are valid for the class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \cdot\}$*-algebras representable by* n*-ary partial functions.*

$$\langle a \bowtie_1^n b\rangle \mathbin{;} a = \langle a \bowtie_1^n b\rangle \mathbin{;} b \tag{6.40}$$

$$\boldsymbol{\alpha} \mathbin{;} \mathrm{D}_i(\boldsymbol{\alpha} \mathbin{;} (a \cdot b)) +_i \boldsymbol{\alpha} \mathbin{;} \langle \mathrm{A}_1^n(\boldsymbol{\alpha} \mathbin{;} a)\rangle \mathbin{;} \mathrm{A}_i(\boldsymbol{\alpha} \mathbin{;} b) = \boldsymbol{\alpha} \mathbin{;} (a \bowtie_i b) \qquad \textit{for every } i \tag{6.41}$$

$$\langle \mathrm{D}_1^n a\rangle \mathbin{;} (b \bowtie_i c) +_i \langle \mathrm{A}_1^n a\rangle \mathbin{;} (b \bowtie_i c) = b \bowtie_i c \qquad \textit{for every } i \tag{6.42}$$

Proof. We first need to convince ourselves that in an algebra of n-ary functions, $\bowtie_i$, as we have defined it, really does give the ith tie operation on its two arguments. And before we do that we need to see that if α and β are restrictions of the ith projection, then $\alpha +_i \beta$ is the ith projection on the union of the domains of α and β. It suffices to prove these for the square algebras of n-ary functions.

In a square algebra of n-ary partial functions, with base X, the function $\alpha +_i \beta$ is by definition the ith projection restricted to where $\langle \mathrm{A}_1^n \alpha\rangle \mathbin{;} \mathrm{A}_i \beta$ is not defined. Now $\langle \mathrm{A}_1^n \alpha\rangle \mathbin{;} \mathrm{A}_i \beta$ is defined precisely where $\mathrm{A}_1 \alpha$ (or indeed any $\mathrm{A}_j \alpha$) and $\mathrm{A}_i \beta$ are both defined, which is those n-tuples in the domains of neither α nor β. By De Morgan, $\alpha +_i \beta$ is as claimed.

Examining the definition of $a \bowtie_i b$, we note that $\mathrm{D}_i(a \cdot b)$ is the ith projection restricted to where a and b are both defined and are equal, and $\langle \mathrm{A}_1^n a\rangle \mathbin{;} \mathrm{A}_i b$ is the ith projection on those n-tuples where neither a nor b are defined. Hence $a \bowtie_i b$, being defined as the result of applying the $+_i$ operation to these projections, is exactly the ith tie of a and b.

Now for (6.40). Suppose that $\langle a \bowtie_1^n b\rangle \mathbin{;} a$ is defined at an n-tuple $\boldsymbol{x}$, with value z. This means $a \bowtie_1 b, \dots, a \bowtie_n b$ are all defined at $\boldsymbol{x}$ and a is defined at $\langle (a \bowtie_1 b)(\boldsymbol{x}), \dots, (a \bowtie_n b)(\boldsymbol{x})\rangle = \boldsymbol{x}$, with value z. Then as a and $a \bowtie_1 b$ are both defined at $\boldsymbol{x}$, it must be that b is also defined at $\boldsymbol{x}$ with the same value as a—namely, z. It follows that $\langle a \bowtie_1^n b\rangle \mathbin{;} b$ is defined at $\boldsymbol{x}$, with value z. We conclude that $\langle a \bowtie_1^n b\rangle \mathbin{;} a \subseteq \langle a \bowtie_1^n b\rangle \mathbin{;} b$. Similarly (utilising the symmetry of the tie operations on n-ary partial functions) $\langle a \bowtie_1^n b\rangle \mathbin{;} a \supseteq \langle a \bowtie_1^n b\rangle \mathbin{;} b$ and so (6.40) is valid.

We know that in an algebra of n-ary partial functions the A_i-elements, with $\bullet_i$ as in Section 6.3 as product and A_i as complement, form a Boolean algebra. By the definition of $\bullet_i$ and De Morgan, the operation $+_i$ acts as the Boolean sum on the A_i-elements. Then (6.41) is the statement that

$$\alpha_i \bullet_i \mathrm{D}_i(\boldsymbol{\alpha} \mathbin{;} (a \cdot b)) +_i \alpha_i \bullet_i \mathrm{A}_i(\boldsymbol{\alpha} \mathbin{;} a) \bullet_i \mathrm{A}_i(\boldsymbol{\alpha} \mathbin{;} b) = \alpha_i \bullet_i (\mathrm{D}_i(a \cdot b) +_i \mathrm{A}_i a \bullet_i \mathrm{A}_i b) \tag{6.43}$$

holds for every i. Now $\mathrm{D}_i(\boldsymbol{\alpha} \mathbin{;} (a \cdot b))$ is easily seen to be equal to $\boldsymbol{\alpha} \mathbin{;} \mathrm{D}_i(a \cdot b)$, which is the definition of $\alpha_i \bullet_i \mathrm{D}_i(a \cdot b)$. It is similarly easy to see that $\mathrm{A}_i(\boldsymbol{\alpha} \mathbin{;} a) = \mathrm{A}_i\alpha_i +_i \mathrm{A}_i a$ and $\mathrm{A}_i(\boldsymbol{\alpha} \mathbin{;} b) = \mathrm{A}_i\alpha_i +_i \mathrm{A}_i b$. After making these substitutions, (6.43) follows by Boolean reasoning.

Equation (6.42) is the statement that

$$\mathrm{A}_i\mathrm{A}_i a \bullet_i (b \bowtie_i c) +_i \mathrm{A}_i a \bullet_i (b \bowtie_i c) = b \bowtie_i c$$

holds for every i. This follows directly by Boolean reasoning. □

The equations (6.2)–(6.7), (6.9), (6.10), and (6.36)–(6.42) will form our axiomatisation. Equation (6.40) says that the A-element $a \bowtie b$ is an 'equaliser' of a and b. In order to deduce (6.8), we start by showing that $a \bowtie b$ is the greatest such equaliser. (Note though that we have not yet deduced that the sets of A_i-elements, for each i, form isomorphic Boolean algebras nor even that they are partially ordered by the $\bullet_i$ operations of Section 6.3.)

Lemma 6.6.3. *The following indexed quasiequations are consequences of* (6.2)–(6.7)*,* (6.9)*,* (6.10)*, and* (6.36)–(6.42)*.*

$$\boldsymbol{\alpha} \mathbin{;} a = \boldsymbol{\alpha} \mathbin{;} b \to \boldsymbol{\alpha} \mathbin{;} (a \bowtie_i b) = \alpha_i \qquad \textit{for every } i \tag{6.44}$$

Proof. Assume $\boldsymbol{\alpha} ; a = \boldsymbol{\alpha} ; b$. Then we have

$$
\begin{aligned}
&\boldsymbol{\alpha} ; (a \bowtie_i b) \\
&= \boldsymbol{\alpha} ; \mathrm{D}_i(\boldsymbol{\alpha} ; (a \cdot b)) +_i \boldsymbol{\alpha} ; \langle \mathrm{A}_1^n(\boldsymbol{\alpha} ; a)\rangle ; \mathrm{A}_i(\boldsymbol{\alpha} ; b) && \text{by (6.41)} \\
&= \boldsymbol{\alpha} ; \mathrm{D}_i(\boldsymbol{\alpha} ; (a \cdot b)) +_i \boldsymbol{\alpha} ; \langle \mathrm{A}_1^n(\boldsymbol{\alpha} ; a)\rangle ; \mathrm{A}_i(\boldsymbol{\alpha} ; a) && \text{by assumption} \\
&= \boldsymbol{\alpha} ; \mathrm{D}_i((\boldsymbol{\alpha} ; a) \cdot (\boldsymbol{\alpha} ; b)) +_i \boldsymbol{\alpha} ; \langle \mathrm{A}_1^n(\boldsymbol{\alpha} ; a)\rangle ; \mathrm{A}_i(\boldsymbol{\alpha} ; a) && \text{by (6.38)} \\
&= \boldsymbol{\alpha} ; \mathrm{D}_i((\boldsymbol{\alpha} ; a) \cdot (\boldsymbol{\alpha} ; a)) +_i \boldsymbol{\alpha} ; \langle \mathrm{A}_1^n(\boldsymbol{\alpha} ; a)\rangle ; \mathrm{A}_i(\boldsymbol{\alpha} ; a) && \text{by assumption} \\
&= \boldsymbol{\alpha} ; \mathrm{D}_i(\boldsymbol{\alpha} ; (a \cdot a)) +_i \boldsymbol{\alpha} ; \langle \mathrm{A}_1^n(\boldsymbol{\alpha} ; a)\rangle ; \mathrm{A}_i(\boldsymbol{\alpha} ; a) && \text{by (6.38)} \\
&= \boldsymbol{\alpha} ; (a \bowtie_i a) && \text{by (6.41)} \\
&= \boldsymbol{\alpha} ; (\mathrm{D}_i(a \cdot a) +_i \langle \mathrm{A}_1^n a\rangle ; \mathrm{A}_i a) && \text{by definition of } \bowtie_i \\
&= \boldsymbol{\alpha} ; (\mathrm{D}_i(a \cdot a) +_i \mathrm{A}_i a) && \text{by (6.12)} \\
&= \boldsymbol{\alpha} ; (\mathrm{D}_i a +_i \mathrm{A}_i a) && \text{idempotency of } \cdot \\
&= \boldsymbol{\alpha} ; \mathrm{A}_i(\langle \mathrm{A}_1^n \mathrm{D}_i a\rangle ; \mathrm{A}_i \mathrm{A}_i a) && \text{by definition of } +_i \\
&= \boldsymbol{\alpha} ; \mathrm{A}_i(\langle \mathrm{A}_1^n \mathrm{D}_i a\rangle ; \mathrm{D}_i a) && \text{by definition of } \mathrm{D}_i \\
&= \boldsymbol{\alpha} ; \mathrm{A}_i 0 && \text{by (6.4)} \\
&= \langle \mathrm{A}_1^n(\boldsymbol{\alpha} ; 0)\rangle ; \alpha_i && \text{by } i\text{th twisted law} \\
&= \langle \mathrm{A}_1^n 0\rangle ; \alpha_i && \text{by (6.6)} \\
&= \boldsymbol{\pi} ; \alpha_i && \text{by definition of } \boldsymbol{\pi} \\
&= \alpha_i && \text{by (6.3)}
\end{aligned}
$$

which is the required conclusion. □

Now it is straightforward to deduce (6.8).

Lemma 6.6.4. *Equation* (6.8) *is a consequence of* (6.2)–(6.7), (6.9), (6.10), *and* (6.36)–(6.42).

Proof. Suppose that $\langle \mathrm{D}_1^n a\rangle ; b = \langle \mathrm{D}_1^n a\rangle ; c$ and $\langle \mathrm{A}_1^n a\rangle ; b = \langle \mathrm{A}_1^n a\rangle ; c$. Then we have

$$b \bowtie_i c = \pi_i \qquad \text{for every } i \tag{6.45}$$

because

$$
\begin{aligned}
b \bowtie_i c &= \langle \mathrm{D}_1^n a\rangle ; (b \bowtie_i c) +_i \langle \mathrm{A}_1^n a\rangle ; (b \bowtie_i c) && \text{by (6.42)} \\
&= \langle \mathrm{D}_1^n a\rangle ; (b \bowtie_i c) +_i \mathrm{A}_i a && \text{by (6.44)} \\
&= \mathrm{D}_i a +_i \mathrm{A}_i a && \text{by (6.44)} \\
&= \mathrm{A}_i(\langle \mathrm{A}_1^n \mathrm{D}_i a\rangle ; \mathrm{A}_i \mathrm{A}_i a) && \text{by the definition of } +_i \\
&= \mathrm{A}_i(\langle \mathrm{A}_1^n \mathrm{D}_i a\rangle ; \mathrm{D}_i a) && \text{by the definition of } \mathrm{D}_i \\
&= \mathrm{A}_i 0 && \text{by (6.4)} \\
&= \pi_i && \text{by the definition of } \pi_i
\end{aligned}
$$

and so

$$\begin{aligned} b &= \boldsymbol{\pi} \,;\, b && \text{by (6.3)} \\ &= \langle b \bowtie_1^n c \rangle \,;\, b && \text{by (6.45)} \\ &= \langle b \bowtie_1^n c \rangle \,;\, c && \text{by (6.40)} \\ &= \boldsymbol{\pi} \,;\, c && \text{by (6.45)} \\ &= c && \text{by (6.3)} \end{aligned}$$

and hence (6.8) holds. □

We are now in a position to state and prove our representation theorem.

Theorem 6.6.5. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \cdot\}$*-algebras that are representable by* n*-ary partial functions is a variety, finitely axiomatised by equations* (6.2)–(6.7), (6.9)*, and* (6.10)*, together with* (6.36)–(6.42).

Proof. Let $\mathfrak{A}$ be an algebra of the signature $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \cdot\}$ validating the specified equations. We will show that, for any ultrafilter U of A-elements, the map θ_U described in Lemma 6.4.3 represents intersection correctly. The result follows.

We first show that $\theta_U(a) \cap \theta_U(b) \subseteq \theta_U(a \cdot b)$ for all $a, b \in \mathfrak{A}$. Suppose that $([\boldsymbol{c}], [d]) \in \theta_U(a) \cap \theta_U(b)$. Then there is an $\alpha \in U$ with $\boldsymbol{\alpha} \,;\, (\boldsymbol{c} \,;\, a) = \boldsymbol{\alpha} \,;\, d$ and a $\beta \in U$ with $\boldsymbol{\beta} \,;\, (\boldsymbol{c} \,;\, b) = \boldsymbol{\beta} \,;\, d$. As U is an ultrafilter we may assume $\alpha = \beta$. Then

$$\begin{aligned} \boldsymbol{\alpha} \,;\, (\boldsymbol{c} \,;\, (a \cdot b)) &= \boldsymbol{\alpha} \,;\, ((\boldsymbol{c} \,;\, a) \cdot (\boldsymbol{c} \,;\, b)) && \text{by distributivity of } \langle\ \rangle; \text{ over } \cdot \\ &= (\boldsymbol{\alpha} \,;\, (\boldsymbol{c} \,;\, a)) \cdot (\boldsymbol{\alpha} \,;\, (\boldsymbol{c} \,;\, b)) && \text{by distributivity of } \langle\ \rangle; \text{ over } \cdot \\ &= (\boldsymbol{\alpha} \,;\, d) \cdot (\boldsymbol{\alpha} \,;\, d) && \text{by equality of the factors} \\ &= \boldsymbol{\alpha} \,;\, d && \text{by idempotency of } \cdot \end{aligned}$$

and hence $[\boldsymbol{c} \,;\, (a \cdot b)] = [d]$. This says that $([\boldsymbol{c}], [d]) \in \theta_U(a \cdot b)$, since we know that $[d] \neq [0]$. We conclude that $\theta_U(a) \cap \theta_U(b) \subseteq \theta_U(a \cdot b)$.

We now show that the reverse inclusion, $\theta_U(a \cdot b) \subseteq \theta_U(a) \cap \theta_U(b)$, holds. Suppose that $([\boldsymbol{c}], [d]) \in \theta_U(a \cdot b)$. This means that $[\boldsymbol{c} \,;\, (a \cdot b)] \neq [0]$, equivalently $\mathrm{D}(\boldsymbol{c} \,;\, (a \cdot b)) \in U$, and that $[d] = [\boldsymbol{c} \,;\, (a \cdot b)]$. Then

$$\begin{aligned} \langle \mathrm{D}_1^n(\boldsymbol{c} \,;\, (a \cdot b)) \rangle \,;\, (\boldsymbol{c} \,;\, a) &= \langle \mathrm{D}_1^n((\boldsymbol{c} \,;\, a) \cdot (\boldsymbol{c} \,;\, b)) \rangle \,;\, (\boldsymbol{c} \,;\, a) && \text{by (6.38)} \\ &= (\boldsymbol{c} \,;\, a) \cdot (\boldsymbol{c} \,;\, b) && \text{by (6.39)} \\ &= \boldsymbol{c} \,;\, (a \cdot b) && \text{by (6.38)} \\ &= \langle \mathrm{D}_1^n(\boldsymbol{c} \,;\, (a \cdot b)) \rangle \,;\, (\boldsymbol{c} \,;\, (a \cdot b)) && \text{by (6.9)} \end{aligned}$$

and so $[\boldsymbol{c} \,;\, a] = [\boldsymbol{c} \,;\, (a \cdot b)] = [d] \neq [0]$, which tells us $([\boldsymbol{c}], [d]) \in \theta_U(a)$. Similarly and using commutativity of $\cdot$ we get $([\boldsymbol{c}], [d]) \in \theta_U(b)$ and so $([\boldsymbol{c}], [d]) \in \theta_U(a) \cap \theta_U(b)$. We conclude that $\theta_U(a \cdot b) \subseteq \theta_U(a) \cap \theta_U(b)$, completing the proof. □

With the aid of intersection, we can also replace the indexed quasiequations of (6.35) to give an equational axiomatisation for the case of injective n-ary partial functions.

Proposition 6.6.6. *The representation used in the proof of Theorem 6.6.5 represents an element a as an injective function if and only if it validates the following indexed equations.*

$$\langle \mathrm{D}_1^n((\boldsymbol{b}\,;a)\cdot(\boldsymbol{c}\cdot a))\rangle\,;\mathrm{A}_i(b_i \bowtie_i c_i) = 0 \qquad \textit{for all } i \tag{6.46}$$

Proof. We first argue that any injective function a validates (6.46). Then if an element a is represented as an injective function it must validate (6.46). To this end, suppose a is an injective n-ary partial function and that $\langle \mathrm{D}_1^n((\boldsymbol{b}\,;a)\cdot(\boldsymbol{c}\cdot a))\rangle\,;\mathrm{A}_i(b_i \bowtie_i c_i)$ is defined on the n-tuple $\boldsymbol{x}$. Then both $\boldsymbol{b}\,;a$ and $\boldsymbol{c}\,;a$ should be defined on $\boldsymbol{x}$ and take the same value. This means that $\langle b_1(\boldsymbol{x}),\dots,b_n(\boldsymbol{x})\rangle$ and $\langle c_1(\boldsymbol{x}),\dots,c_n(\boldsymbol{x})\rangle$ are both defined and

$$a(b_1(\boldsymbol{x}),\dots,b_n(\boldsymbol{x})) = a(c_1(\boldsymbol{x}),\dots,c_n(\boldsymbol{x})).$$

By injectivity of a, we get $b_j(\boldsymbol{x}) = c_j(\boldsymbol{x})$ for every j. In particular $b_i(\boldsymbol{x}) = c_i(\boldsymbol{x})$ and so $b_i \bowtie_i c_i$ is defined on $\boldsymbol{x}$. Hence $\mathrm{A}_i(b_i \bowtie_i c_i)$ is *not* defined on $\boldsymbol{x}$. This contradicts $\langle \mathrm{D}_1^n((\boldsymbol{b}\,;a)\cdot(\boldsymbol{c}\cdot a))\rangle\,;\mathrm{A}_i(b_i \bowtie_i c_i)$ being defined on $\boldsymbol{x}$ and so $\langle \mathrm{D}_1^n((\boldsymbol{b}\,;a)\cdot(\boldsymbol{c}\cdot a))\rangle\,;\mathrm{A}_i(b_i \bowtie_i c_i)$ must be the empty function.

We now prove the converse: that every a validating (6.46) is represented by our representation as an injective function. We will argue that, for any ultrafilter U of A-elements, the map θ_U described in Lemma 6.4.3 maps elements validating (6.46) to injective functions. Since a disjoint union of injective functions is injective, the result follows.

Suppose a validates (6.46) and suppose for a contradiction that $\theta_U(a)([\boldsymbol{b}]) = \theta_U(a)([\boldsymbol{c}])$ (with both sides defined) and that $[\boldsymbol{b}] \neq [\boldsymbol{c}]$. The second of these statements means that U contains some equaliser of $\boldsymbol{b}\,;a$ and $\boldsymbol{c}\,;a$, so $(\boldsymbol{b}\,;a)\bowtie(\boldsymbol{c}\,;a) \in U$, as this is the greatest such equaliser. Since both $\boldsymbol{b}\,;a$ and $\boldsymbol{c}\,;a$ are inequivalent to 0 we know that $\mathrm{D}(\boldsymbol{b}\,;a) \in U$ and $\mathrm{D}(\boldsymbol{c}\,;a) \in U$. Since $[\boldsymbol{b}] \neq [\boldsymbol{c}]$, we have $[b_i] \neq [c_i]$ for some i. Then $b_i \bowtie c_i \notin U$, so that $\mathrm{A}(b_i \bowtie c_i) \in U$. Marshalling all our elements of U we have

$$((\boldsymbol{b}\,;a)\bowtie(\boldsymbol{c}\,;a))\,\mathrm{D}(\boldsymbol{b}\,;a)\,\mathrm{D}(\boldsymbol{c}\,;a)\,\mathrm{A}(b_i \bowtie c_i) = \mathrm{D}((\boldsymbol{b}\,;a)\cdot(\boldsymbol{c}\,;a))\,\mathrm{A}(b_i \bowtie c_i) \in U$$

where we now use juxtaposition for the Boolean meet. We are told by (6.46) that this element of the ultrafilter U is 0—a contradiction. We conclude that $\theta_U(a)$ is injective. □

Corollary 6.6.7. *The class of $\{\langle\ \rangle;, \mathrm{A}_1,\dots,\mathrm{A}_n,\cdot\}$-algebras that are representable by injective n-ary partial functions is a variety, finitely axiomatised by the equations specified in Theorem 6.6.5 together with* (6.46)*.*

Corollary 6.6.8. *The finite representation property holds for the signature $\{\langle\ \rangle;, \mathrm{A}_1,\dots,\mathrm{A}_n,\cdot\}$ for representation by n-ary partial functions and for representation by injective n-ary partial functions.*

6.7 Preferential union

For signatures including composition and the antidomain operations, there is a simple equational characterisation of preferential union in terms of composition and the antidomain operations.

Proposition 6.7.1. *In an algebra of n-ary partial functions, for signatures containing composition and the antidomain operations, h is the preferential union of f and g if and only if $\langle \mathrm{D}^n_1 f\rangle \,;\, h = f$ and $\langle \mathrm{A}^n_1 f\rangle \,;\, h = \langle \mathrm{A}^n_1 f\rangle \,;\, g$.*

Proof. First suppose that $h = f \sqcup g$. If $\langle \mathrm{D}^n_1 f\rangle \,;\, h$ is defined on an n-tuple $\boldsymbol{x}$ then f is defined on $\boldsymbol{x}$ and so h is defined on $\boldsymbol{x}$ with the same value as f. Hence $(\langle \mathrm{D}^n_1 f\rangle \,;h)(\boldsymbol{x}) = f(\boldsymbol{x})$. Conversely, if f is defined on $\boldsymbol{x}$ then h is too, with the same value. Then $\langle \mathrm{D}^n_1 f\rangle \,;\, h$ is defined on $\boldsymbol{x}$ and $(\langle \mathrm{D}^n_1 f\rangle \,;\, h)(\boldsymbol{x}) = f(\boldsymbol{x})$. This completes the argument that $\langle \mathrm{D}^n_1 f\rangle \,;\, h = f$.

Continuing to suppose that $h = f \sqcup g$, if $\langle \mathrm{A}^n_1 f\rangle \,;\, h$ is defined on an n-tuple $\boldsymbol{x}$ then f is not defined on $\boldsymbol{x}$ and h *is* defined on $\boldsymbol{x}$. As h is the preferential join of f and g, this implies that g is defined on $\boldsymbol{x}$ with the same value as h. So $\langle \mathrm{A}^n_1 f\rangle \,;\, h$ agrees with $\langle \mathrm{A}^n_1 f\rangle \,;\, g$ on $\boldsymbol{x}$. Conversely, if $\langle \mathrm{A}^n_1 f\rangle \,;\, g$ is defined on $\boldsymbol{x}$ then f is not defined on $\boldsymbol{x}$ and g is. This implies that h is defined on $\boldsymbol{x}$ with the same value as g. So again $\langle \mathrm{A}^n_1 f\rangle \,;\, h$ agrees with $\langle \mathrm{A}^n_1 f\rangle \,;\, g$ on $\boldsymbol{x}$. This completes the argument that $\langle \mathrm{A}^n_1 f\rangle \,;\, h = \langle \mathrm{A}^n_1 f\rangle \,;\, g$.

We now show that for any f, g and h satisfying the two equations, h is the preferential join of f and g. Given such an f, g and h, first suppose that h is defined on the n-tuple $\boldsymbol{x}$. If f is also defined on $\boldsymbol{x}$ then $\langle \mathrm{D}^n_1 f\rangle \,;\, h$ is defined on $\boldsymbol{x}$ with the same value as h. In this case we are told by the equation $\langle \mathrm{D}^n_1 f\rangle \,;\, h = f$ that $h(\boldsymbol{x}) = (\langle \mathrm{D}^n_1 f\rangle \,;\, h)(\boldsymbol{x}) = f(\boldsymbol{x}) = (f \sqcup g)(\boldsymbol{x})$. If f is *un*defined at $\boldsymbol{x}$ then $\langle \mathrm{A}^n_1 f\rangle \,;\, h$ is defined on $\boldsymbol{x}$ with the same value as h. Then the equation $\langle \mathrm{A}^n_1 f\rangle \,;\, h = \langle \mathrm{A}^n_1 f\rangle \,;\, g$ tells us that $h(\boldsymbol{x}) = (\langle \mathrm{A}^n_1 f\rangle \,;h)(\boldsymbol{x}) = (\langle \mathrm{A}^n_1 f\rangle \,;g)(\boldsymbol{x})$. So g must be defined at $\boldsymbol{x}$ with the same value as h. But $g(\boldsymbol{x}) = (f \sqcup g)(\boldsymbol{x})$, as f is undefined here. Again we have found $h(\boldsymbol{x}) = (f \sqcup g)(\boldsymbol{x})$. We conclude that $h \subseteq f \sqcup g$.

Conversely, suppose that $f \sqcup g$ is defined on $\boldsymbol{x}$. If f is defined on $\boldsymbol{x}$ then $(f \sqcup g)(\boldsymbol{x}) = f(\boldsymbol{x}) = (\langle \mathrm{D}^n_1 f\rangle \,;\, h)(\boldsymbol{x}) = h(\boldsymbol{x})$, utilising the equation $\langle \mathrm{D}^n_1 f\rangle \,;\, h = f$. If f is *not* defined on $\boldsymbol{x}$ then g must be, since $f \sqcup g$ is defined, and for the same reason $\mathrm{A}_1 f, \dots, \mathrm{A}_n f$ must be defined on $\boldsymbol{x}$. Then $(f \sqcup g)(\boldsymbol{x}) = g(\boldsymbol{x}) = (\langle \mathrm{A}^n_1 f\rangle \,;\, g)(\boldsymbol{x}) = (\langle \mathrm{A}^n_1 f\rangle \,;\, h)(\boldsymbol{x}) = h(\boldsymbol{x})$, utilising the equation $\langle \mathrm{A}^n_1 f\rangle \,;\, h = \langle \mathrm{A}^n_1 f\rangle \,;\, g$. We conclude that $h \supseteq f \sqcup g$, completing the proof that $h = f \sqcup g$. □

The content of Proposition 6.7.1 means we only need add the following two equations in order to extend the axiomatisations of the previous sections so as to include $\sqcup$ in the signature.

$$\langle \mathrm{D}^n_1 a\rangle \,;\, (a \sqcup b) = a \tag{6.47}$$

$$\langle \mathrm{A}^n_1 a\rangle \,;\, (a \sqcup b) = \langle \mathrm{A}^n_1 a\rangle \,;\, b \tag{6.48}$$

For the signature $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \sqcup\}$ this gives us quasiequational axiomatisations. However, it is possible to replace the quasiequation (6.8) with a valid equation that trivially implies it.

Proposition 6.7.2. *For any signature containing composition, the antidomain operations, and preferential union, the following equation is valid for the class of algebras representable by n-ary partial functions.*

$$(\langle \mathrm{D}^n_1 a\rangle \,;\, b) \sqcup (\langle \mathrm{A}^n_1 a\rangle \,;\, b) = b \tag{6.49}$$

Proof. As usual, we prove validity for an arbitrary square algebra of n-ary partial functions. So let a and b be elements of such an algebra, with base X, and let $\boldsymbol{x}$ be an n-tuple in X^n.

If a is defined on $\boldsymbol{x}$ then $\mathrm{D}_1 a, \dots, \mathrm{D}_n a$ are too. Then $(\langle \mathrm{D}_1^n a \rangle \,;\, b) \sqcup (\langle \mathrm{A}_1^n a \rangle \,;\, b)$ and b agree on $\boldsymbol{x}$, since if b is defined on $\boldsymbol{x}$ then $\langle \mathrm{D}_1^n a \rangle \,; b$ is and so $((\langle \mathrm{D}_1^n a \rangle \,; b) \sqcup (\langle \mathrm{A}_1^n a \rangle \,; b))(\boldsymbol{x}) = (\langle \mathrm{D}_1^n a \rangle \,; b)(\boldsymbol{x}) = b(\boldsymbol{x})$ and if b is *not* defined on $\boldsymbol{x}$ then neither $\langle \mathrm{D}_1^n a \rangle \,; b$ nor $\langle \mathrm{A}_1^n a \rangle \,; b$ are and so $(\langle \mathrm{D}_1^n a \rangle \,; b) \sqcup (\langle \mathrm{A}_1^n a \rangle \,; b)$ is also not defined on $\boldsymbol{x}$.

The other case needing consideration is when a is *not* defined on $\boldsymbol{x}$. Then $\langle \mathrm{D}_1^n a \rangle \,; b$ is not defined on $\boldsymbol{x}$ and $\mathrm{A}_1 a, \dots, \mathrm{A}_n a$ are all defined on $\boldsymbol{x}$. Again $(\langle \mathrm{D}_1^n a \rangle \,; b) \sqcup (\langle \mathrm{A}_1^n a \rangle \,; b)$ and b agree on $\boldsymbol{x}$, since if b is defined then $((\langle \mathrm{D}_1^n a \rangle \,; b) \sqcup (\langle \mathrm{A}_1^n a \rangle \,; b))(\boldsymbol{x}) = (\langle \mathrm{A}_1^n a \rangle \,; b)(\boldsymbol{x}) = b(\boldsymbol{x})$ and if b is not defined on $\boldsymbol{x}$ then neither $\langle \mathrm{D}_1^n a \rangle \,; b$ nor $\langle \mathrm{A}_1^n a \rangle \,; b$ are and so $(\langle \mathrm{D}_1^n a \rangle \,; b) \sqcup (\langle \mathrm{A}_1^n a \rangle \,; b)$ also is not. □

We obtain the following results.

Theorem 6.7.3. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \sqcup\}$*-algebras that are representable by* n*-ary partial functions is a variety, finitely axiomatised by equations* (6.2)–(6.7)*,* (6.9)*, and* (6.10)*, together with* (6.47)*,* (6.48)*, and* (6.49)*.*

Theorem 6.7.4. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \sqcup\}$*-algebras that are representable by injective* n*-ary partial functions is a finitely axiomatised by* (6.2)–(6.10) *together with* (6.35)*,* (6.47)*, and* (6.48)*.*

Corollary 6.7.5. *The finite representation property holds for the signature* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \sqcup\}$ *for representation by* n*-ary partial functions and for representation by injective* n*-ary partial functions.*

For the signature $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \cdot, \sqcup\}$ we can simply extend the equational axiomatisations of Section 6.6.

Theorem 6.7.6. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \cdot, \sqcup\}$*-algebras that are representable by* n*-ary partial functions is a variety, finitely axiomatised by the equations specified in Theorem 6.6.5 together with* (6.47) *and* (6.48)*.*

Corollary 6.7.7. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \cdot, \sqcup\}$*-algebras that are representable by injective* n*-ary partial functions is a variety, finitely axiomatised by the equations specified in Theorem 6.6.5 together with* (6.35)*,* (6.47)*, and* (6.48)*.*

Corollary 6.7.8. *The finite representation property holds for the signature* $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n, \cdot, \sqcup\}$ *for representation by* n*-ary partial functions and for representation by injective* n*-ary partial functions.*

6.8 Fixset

As we noted previously, the fixset operations can be expressed using intersection and the antidomain operations as $\mathrm{F}_i f \coloneqq \pi_i \cdot f$. So, having already given axiomatisations for signatures containing intersection, only the signatures without intersection are interesting to us—namely, $\{\langle\ \rangle;, \mathrm{A}_i, \mathrm{F}_i\}$ and $\{\langle\ \rangle;, \mathrm{A}_i, \mathrm{F}_i, \sqcup\}$.

There is a simple equational axiomatisation of *restrictions* of the ith fixset in terms of composition and the domain operations, getting us halfway to axiomatising fixset.

Proposition 6.8.1. *In an algebra of n-ary partial functions, for signatures containing composition and the antidomain operations, g is a restriction of $\mathrm{F}_i f$ if and only if $\mathrm{D}_i g = g$ and $\langle \mathrm{D}^n_1 g\rangle \,;\, f = g$.*[5]

Proof. By definition, $\mathrm{F}_i f = \pi_i \cap f$ and so g is a restriction of $\mathrm{F}_i f$ if and only if g is both a restriction of π_i and a restriction of f. Being a restriction of the ith projection is equivalent to satisfying $\mathrm{D}_i g = g$ and being a restriction of f is equivalent to satisfying $\langle \mathrm{D}^n_1 g\rangle \,;\, f = g$. □

The upshot of Proposition 6.8.1 is that the following equations are valid and ensure that any representation of a $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-reduct represents each $\mathrm{F}_i a$ both as a restriction of the ith projection and as a restriction of the representation of a.

$$\mathrm{D}_i(\mathrm{F}_i a) = \mathrm{F}_i a \qquad \text{for every } i \tag{6.50}$$

$$\langle \mathrm{D}^n_1(\mathrm{F}_i a)\rangle \,;\, a = \mathrm{F}_i a \qquad \text{for every } i \tag{6.51}$$

Hence adding (6.50) and (6.51) as axioms is sufficient to give $\theta(\mathrm{F}_i(a)) \subseteq \mathrm{F}_i(\theta(a))$ in Theorem 6.4.5, for all a and every i. The next proposition presents valid quasiequations that are sufficient for the reverse inclusions to hold.

Proposition 6.8.2. *The following indexed quasiequations are valid for algebras representable by n-ary partial functions for any signature containing composition and the fixset operations.*

$$\boldsymbol{b} \,;\, a = b_i \rightarrow \boldsymbol{b} \,;\, \mathrm{F}_i a = b_i \qquad \textit{for every } i \tag{6.52}$$

Further, let $\mathfrak{A}$ be an algebra of a signature containing composition and the antidomain and fixset operations and suppose the $\{\langle\ \rangle;, \mathrm{A}_1, \dots, \mathrm{A}_n\}$-reduct of $\mathfrak{A}$ is representable by n-ary partial functions. Let θ be the representation of the reduct described in Theorem 6.4.5. If the element $a \in \mathfrak{A}$ validates the i-indexed version of (6.52) *then $\theta(\mathrm{F}_i(a)) \supseteq \mathrm{F}_i(\theta(a))$.*

Proof. For the first part it is sufficient to prove validity for an arbitrary square algebra of n-ary partial functions. So let a and $b_1, \dots, b_n$ be elements of such an algebra, with base X, and suppose $\boldsymbol{b} \,;\, a = b_i$. If $\boldsymbol{b} \,;\, \mathrm{F}_i a$ is defined on $\boldsymbol{x}$, with value z, then $b_1, \dots, b_n$ are all defined on $\boldsymbol{x}$ and $\mathrm{F}_i a$ is defined on $\langle b_1(\boldsymbol{x}), \dots, b_n(\boldsymbol{x})\rangle$, so a is too, with value $b_i(\boldsymbol{x}) = z$. Hence $\boldsymbol{b} \,;\, \mathrm{F}_i a \subseteq b_i$.

Conversely, if b_i is defined on $\boldsymbol{x}$ then, by the assumption, $\boldsymbol{b} \,;\, a$ is defined on $\boldsymbol{x}$, with value $b_i(\boldsymbol{x})$. Then $b_1, \dots, b_n$ are all defined on $\boldsymbol{x}$ and a is defined on $\langle b_1(\boldsymbol{x}), \dots, b_n(\boldsymbol{x})\rangle$, also with value $b_i(\boldsymbol{x})$. This tells us that $\mathrm{F}_i a$ is defined on $\langle b_1(\boldsymbol{x}), \dots, b_n(\boldsymbol{x})\rangle$ and so $\boldsymbol{b} \,;\, \mathrm{F}_i a$ is defined on $\boldsymbol{x}$, necessarily with the same value as b_i. Hence $\boldsymbol{b} \,;\, \mathrm{F}_i a \supseteq b_i$ and we conclude that $\boldsymbol{b} \,;\, \mathrm{F}_i a$ and b_i are equal, so (6.52) is valid.

For the second part it is sufficient to prove that, for any ultrafilter U of A-elements, the homomorphism θ_U, as defined in Lemma 6.4.3, satisfies $\theta_U(\mathrm{F}_i(a)) \supseteq \mathrm{F}_i(\theta_U(a))$. So suppose that $([\boldsymbol{b}], [c]) \in \mathrm{F}_i(\theta_U(a))$. Then $[c] = [b_i] \neq [0]$ and $([\boldsymbol{b}], [b_i]) \in \theta_U(a)$, that is, there is some $\alpha \in U$ such that $\boldsymbol{\alpha} \,;\, (\boldsymbol{b} \,;\, a) = \boldsymbol{\alpha} \,;\, b_i$. Then by superassociativity

$$\langle \boldsymbol{\alpha} \,;\, b_1, \dots, \boldsymbol{\alpha} \,;\, b_n\rangle \,;\, a = \boldsymbol{\alpha} \,;\, b_i$$

[5]Of course, the equation $\langle \mathrm{D}^n_1 g\rangle \,;\, f = g$ expresses $g \leq f$ using the $\leq$ relation of Sections 6.3 and 6.4.

so by (6.52)

$$\langle \boldsymbol{\alpha} ; b_1, \ldots, \boldsymbol{\alpha} ; b_n\rangle ; \mathrm{F}_i a = \boldsymbol{\alpha} ; b_i$$

and then by superassociativity

$$\boldsymbol{\alpha} ; (\boldsymbol{b} ; \mathrm{F}_i a) = \boldsymbol{\alpha} ; b_i$$

and so $[\boldsymbol{b} ; \mathrm{F}_i a] = [b_i]$. Hence $([\boldsymbol{b}], [c]) = ([\boldsymbol{b}], [b_i]) \in \theta_U(\mathrm{F}_i(a))$ and we are done. □

Combining Propositions 6.8.1 and 6.8.2, we obtain quasiequational axiomatisations for signatures containing the fixset operations.

Theorem 6.8.3. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{F}_1, \ldots, \mathrm{F}_n\}$*-algebras that are representable by* n*-ary partial functions is finitely axiomatised by the (quasi)equations specified in Theorem 6.4.5 together with* (6.50)–(6.52).

Corollary 6.8.4. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{F}_1, \ldots, \mathrm{F}_n\}$*-algebras that are representable by injective* n*-ary partial functions is finitely axiomatised by the (quasi)equations specified in Theorem 6.4.5 together with* (6.35) *and* (6.50)–(6.52).

Corollary 6.8.5. *The finite representation property holds for the signature* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{F}_1, \ldots, \mathrm{F}_n\}$ *for representation by* n*-ary partial functions and for representation by injective* n*-ary partial functions.*

Theorem 6.8.6. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{F}_1, \ldots, \mathrm{F}_n, \sqcup\}$*-algebras that are representable by* n*-ary partial functions is finitely axiomatised by the (quasi)equations specified in Theorem 6.4.5 together with* (6.47), (6.48), *and* (6.50)–(6.52).

Corollary 6.8.7. *The class of* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{F}_1, \ldots, \mathrm{F}_n, \sqcup\}$*-algebras that are representable by injective* n*-ary partial functions is finitely axiomatised by the (quasi)equations specified in Theorem 6.4.5 together with* (6.35), (6.47), (6.48), *and* (6.50)–(6.52).

Corollary 6.8.8. *The finite representation property holds for the signature* $\{\langle\ \rangle;, \mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{F}_1, \ldots, \mathrm{F}_n, \sqcup\}$ *for representation by* n*-ary partial functions and for representation by injective* n*-ary partial functions.*

6.9 Equational theories

We conclude with an examination of the computational complexity of equational theories. The following theorem and proof are straightforward adaptations to the n-ary case of unary versions that appear in [44]. (In the corresponding unary cases, the equational theories are also coNP-complete.)

Theorem 6.9.1. *Let* σ *be any signature whose symbols are a subset of* $\{\langle\ \rangle;, \cdot, 0, \pi_i, \mathrm{D}_i, \mathrm{A}_i, \mathrm{F}_i, \bowtie_i, \sqcup\}$. *Then the class of* σ*-algebras that are representable by* n*-ary partial functions has equational theory in* coNP. *If the signature contains* A_i *and either* $\langle\ \rangle;$ *or* $\cdot$ *then the equational theory is* coNP*-complete.*

Proof. For the first part we will show that if an equation $s = t$ is not valid then it can be refuted on an algebra of n-ary partial functions with a base of size linear in the length of the equation. Then a nondeterministic Turing machine can easily identify invalid equations in polynomial time by nondeterministically choosing an assignment of the variables to n-ary partial functions and then calculating the interpretations of the two terms.

Suppose $s = t$ is not valid. Then there is some algebra $\mathfrak{F}$ of n-ary partial functions, some assignment $\boldsymbol{f}$ of elements of $\mathfrak{F}$ to the variables in $s = t$ and some n-tuple $\boldsymbol{x}$ in the base of $\mathfrak{F}$ such that $s[\boldsymbol{f}](\boldsymbol{x}) \neq t[\boldsymbol{f}](\boldsymbol{x})$, meaning that either both sides are defined and they have different values, or one side is defined and the other not.[6] We will select a subset Y of the base of $\mathfrak{F}$, of size linear in the length of the equation, such that in any algebra of n-ary functions with base Y and containing the restrictions $\boldsymbol{f}|_Y$ of $\boldsymbol{f}$ to $Y \times Y$, we have $s[\boldsymbol{f}](\boldsymbol{x}) = s[\boldsymbol{f}|_Y](\boldsymbol{x})$ and $t[\boldsymbol{f}](\boldsymbol{x}) = t[\boldsymbol{f}|_Y](\boldsymbol{x})$ (or both sides are undefined). Then the equation is refuted in any such algebra, for example the algebra generated by the $\boldsymbol{f}|_Y$.

Define $Y(r, \boldsymbol{x})$ by structural induction on the term r as follows.

- For any variable a,
$$Y(a, \boldsymbol{x}) \coloneqq \begin{cases} \{x_1, \dots, x_n\} \cup \{a[\boldsymbol{f}](\boldsymbol{x})\} & \text{if } a[\boldsymbol{f}](\boldsymbol{x}) \text{ exists} \\ \{x_1, \dots, x_n\} & \text{otherwise} \end{cases}$$

- $$Y(\boldsymbol{u}\,;v, \boldsymbol{x}) \coloneqq \begin{cases} Y(u_1, \boldsymbol{x}) \cup \dots \cup Y(u_n, \boldsymbol{x}) \cup Y(v, (u_1[\boldsymbol{f}](\boldsymbol{x}), \dots, u_n[\boldsymbol{f}](\boldsymbol{x}))) \\ \qquad\qquad \text{if } u_1[\boldsymbol{f}](\boldsymbol{x}), \dots, u_n[\boldsymbol{f}](\boldsymbol{x}) \text{ exist} \\ Y(u_1, \boldsymbol{x}) \cup \dots \cup Y(u_n, \boldsymbol{x}) \quad \text{otherwise} \end{cases}$$

- $Y(0, \boldsymbol{x}) = Y(\pi_i, \boldsymbol{x}) \coloneqq \{x_1, \dots, x_n\}$

- $Y(\mathrm{D}_i u, \boldsymbol{x}) = Y(\mathrm{A}_i u, \boldsymbol{x}) = Y(\mathrm{F}_i u, \boldsymbol{x}) \coloneqq Y(u, \boldsymbol{x})$

- $Y(u \cdot v, \boldsymbol{x}) = Y(u \bowtie_i v, \boldsymbol{x}) = Y(u \sqcup v, \boldsymbol{x}) \coloneqq Y(u, \boldsymbol{x}) \cup Y(v, \boldsymbol{x})$

Then it follows by structural induction on terms that for any subset Y of the base of $\mathfrak{F}$ that contains $Y(r, \boldsymbol{x})$, we have $r[\boldsymbol{f}](\boldsymbol{x}) = r[\boldsymbol{f}|_Y](\boldsymbol{x})$. Hence we may take $Y \coloneqq Y(s, \boldsymbol{x}) \cup Y(t, \boldsymbol{x})$, which is clearly of size linear in the length of $s = t$.

For the second part, we describe a polynomial-time reduction from the coNP-complete problem of deciding the tautologies of propositional logic, to the problem of deciding equational validity in the representation class. To do this, we may assume the propositional formulas are formed using only the connectives $\neg$ and $\wedge$. Then replace every propositional letter, p say, in a given propositional formula, φ, with $\mathrm{D}_i p$ (for some fixed choice of i), every $\neg$ with A_i and every $\wedge$ with either the product $\bullet$ of Lemma 6.3.8 or with the operation $\cdot$ of the algebra, depending on availability in the signature. Denoting the resulting term φ^*, output the equation $\varphi^* = \pi_i$. This reduction is correct, since the A_i-elements form a Boolean algebra and there are assignments where $\mathrm{D}_i p$ is the bottom element and where it is the top. □

[6] The notation $s[\boldsymbol{f}]$ denotes the interpretation of the term s under the assignment $\boldsymbol{f}$.

Note that if we are interested in *injective* n-ary partial functions then the argument in the proof of Theorem 6.9.1 can be used to give the analogous result for this case so long as preferential union is not in the signature. Since the preferential union of two injective functions is not necessarily injective, restricted functions do not necessarily generate an algebra of injective functions when preferential union is present in the signature, invalidating the argument.

Problem 6.9.2. What are the computational complexities of the quasiequational theories of the representation classes in Theorem 6.9.1?

Chapter 7

Disjoint-union partial algebras

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ABSTRACT. Disjoint union is a partial binary operation returning the union of two sets if they are disjoint and undefined otherwise. A disjoint-union partial algebra of sets is a collection of sets closed under disjoint unions, whenever they are defined. We provide a recursive first-order axiomatisation of the class of partial algebras isomorphic to a disjoint-union partial algebra of sets but prove that no finite axiomatisation exists. We do the same for other signatures including one or both of disjoint union and subset complement, another partial binary operation we define.

Domain-disjoint union is a partial binary operation on partial functions, returning the union if the arguments have disjoint domains and undefined otherwise. For each signature including one or both of domain-disjoint union and subset complement and optionally including composition, we consider the class of partial algebras isomorphic to a collection of partial functions closed under the operations. Again the classes prove to be axiomatisable, but not finitely axiomatisable, in first-order logic.

We define the notion of pairwise combinability. For each of the previously considered signatures, we examine the class isomorphic to a partial algebra of sets/partial functions under an isomorphism mapping arbitrary suprema of pairwise-combinable sets to the corresponding disjoint unions. We prove that for each case the class is not closed under elementary equivalence.

However, when intersection is added to any of the signatures considered, the isomorphism class of the partial algebras of sets is finitely axiomatisable and in each case we give such an axiomatisation.

7.1 Introduction

Sets and functions are perhaps the two most fundamental and important types of object in all mathematics. Consequently, investigations into the first-order properties of collections of such objects have a long history. Boole, in 1847, was the first to focus attention directly on the algebraic properties of sets [12]. The outstanding result in this area is the Birkhoff–Stone representation theorem, completed in 1934, showing that Boolean algebra provides a first-order axiomatisation of the class of isomorphs of fields of sets [98].

For functions, the story starts around the same period, as we can view Cayley's theorem of 1854 as proof that the group axioms are in fact an axiomatisation of the isomorphism class of collections of bijective functions, closed under composition and inverse [17]. Schein's survey article of 1970 contains a summary of the many similar results about algebras of partial functions that were known by the time of its writing [91].

As we have explained in previous chapters, the past fifteen years have seen a revival of interest in algebras of partial functions, with results finding that such algebras are logically and computationally well behaved [52, 53, 55, 57, 44, 74]. In particular, algebras of partial functions with composition, intersection, domain, and range have the finite representation property [75]. Much more detail and many more references can be found in Section 3.2.1.

Separation logic is a formalism for reasoning about the state of dynamically-allocated computer memory [86]. In the standard 'stack-and-heap' semantics, dynamic memory states are modelled by (finite) partial functions. Thus statements in separation logic are statements about partial functions.

The logical connective common to all flavours of separation logic is the separating conjunction $*$. In the stack-and-heap semantics, the formulas are evaluated at a given heap (a partial function, h) and stack (a variable assignment, s). In this semantics $h, s \models \varphi * \psi$ if and only if there exist h_1, h_2 with disjoint domains, such that $h = h_1 \cup h_2$ and $h_1, s \models \varphi$ and $h_2, s \models \psi$. So lying behind the semantics of the separating conjunction is a partial operation on partial functions we call the domain-disjoint union, which returns the union when its arguments have disjoint domains and is undefined otherwise. Another logical connective that is often employed in separation logic is the separating implication and again a partial operation on partial functions lies behind its semantics.

Separation logic has enjoyed and continues to enjoy great practical successes [8, 16]. However, Brotherston and Kanovich have shown that, for propositional separation logic, the validity problem is undecidable for a variety of different semantics, including the stack-and-heap semantics [15]. The contrast between the aforementioned positive results concerning algebras of partial functions and the undecidability of a propositional logic whose semantics are based on partial algebras of partial functions, suggests a more detailed investigation into the computational and logical behaviour of collections of partial functions equipped with the partial operations arising from separation logic.

In this chapter we examine, from a first-order perspective, partial algebras of partial functions over separation logic signatures—signatures containing one or more of the partial operations underlying the semantics of separation logic. Specifically, we study, for each signature, the isomorphic closure of the class of partial algebras of partial functions. Because these partial operations have not previously been studied in a first-order context we also include an investigation into partial algebras of *sets* over these signatures.

In Section 7.2 we give the definitions needed to precisely define these classes of partial algebras. In Section 7.3 we show that each of our classes is first-order axiomatisable and in Section 7.4 we give a method to form recursive axiomatisations that are easily understandable as statements about certain two-player games.

In Section 7.5 we show that though our classes are axiomatisable, finite axiomatisations do not exist. In Section 7.6 we show that when ordinary intersection is added to the previously examined signatures, the classes of partial algebras become finitely axiomatisable. In Section 7.7 we examine decidability and complexity questions and then conclude with some open problems.

7.2 Disjoint-union partial algebras

In this section we give the fundamental definitions that are needed in order to state the results contained in this chapter. We first define the partial operations that we use.

Definition 7.2.1. Given two sets S and T the **disjoint union** $S \overset{\bullet}{\cup} T$ equals $S \cup T$ if $S \cap T = \emptyset$, else it is undefined. The **subset complement** $S \overset{\bullet}{\setminus} T$ equals $S \setminus T$ if $T \subseteq S$, else it is undefined.

Observe that $S \overset{\bullet}{\cup} T = U$ if and only if $U \overset{\bullet}{\setminus} S = T$.

The next definition involves partial functions. We take the set-theoretic view of a function as being a functional set of ordered pairs—we are back to unary functions—, rather than requiring a domain and codomain to be explicitly specified also. In this sense there is no notion of a function being 'partial'. But using the word partial serves to indicate that when we have a set of such functions they are not required to share a common domain (of definition)—they are 'partial functions' on (any superset of) the union of these domains.

Definition 7.2.2. Given two partial functions f and g the **domain-disjoint union** $f \overset{\bullet}{\smile} g$ equals $f \cup g$ if the domains of f and g are disjoint, else it is undefined. The symbol $|$ denotes the total operation on partial functions of (relational) composition.

Observe that if the domains of two partial functions are disjoint then their union is a partial function. So domain-disjoint union is a partial operation on partial functions. If f and g are partial functions with $g \subseteq f$ then $f \setminus g$ is also a partial function. Hence subset complement gives another partial operation on partial functions.

The reason for our interest in these partial operations is their appearance in the semantics of separation logic, which we now detail precisely.

The **separating conjunction** $*$ is a binary logical connective present in all forms of separation logic. As mentioned in the introduction, in the stack-and-heap semantics the formulas are evaluated at a given heap (a partial function, h) and stack (variable assignment, s). In this semantics $h, s \models \varphi * \psi$ if and only if there exist h_1, h_2 such that $h = h_1 \overset{\bullet}{\smile} h_2$ and both $h_1, s \models \varphi$ and $h_2, s \models \psi$.

The constant `emp` also appears in all varieties of separation logic. The semantics is $h, s \models \texttt{emp}$ if and only if $h = \emptyset$.

The **separating implication** $-\!*$ is another binary logical connective common in separation logic. The semantics is $h, s \models \varphi -\!* \psi$ if and only if for all h_1, h_2 such that $h = h_2 \overset{\bullet}{\setminus} h_1$ we have $h_1, s \models \varphi$ implies $h_2, s \models \psi$.

Because we are working with partial operations, the classes of structures we will examine are classes of partial algebras.

Definition 7.2.3. A **partial algebra** $\mathfrak{A} = (A, (\Omega_i)_{i<\beta})$ consists of a domain, A, together with a sequence $\Omega_0, \Omega_1, \ldots$ of partial operations on A, each of some finite arity $\alpha(i)$ that should be clear from the context.

Two partial algebras $\mathfrak{A} = (A, (\Omega_i)_{i<\beta})$ and $\mathfrak{B} = (B, (\Pi_i)_{i<\beta})$ are **similar** if for all $i < \beta$ the arities of Ω_i and Π_i are equal. (So in particular $\mathfrak{A}$ and $\mathfrak{B}$ must have the same ordinal indexing their partial operations.)

In this chapter we view signatures as sequences, rather than sets. And we use the word 'signature' flexibly. Depending on context it either means a sequence of symbols, each with a prescribed arity and each designated to be a function symbol, a partial-function symbol, or a relation symbol. Or, it means a sequence of actual operations/partial operations/relations.

Definition 7.2.4. Given two similar partial algebras $\mathfrak{A} = (A, (\Omega_i)_{i<\beta})$ and $\mathfrak{B} = (B, (\Pi_i)_{i<\beta})$, a map $\theta\colon A \to B$ is a **partial-algebra homomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ if for all $i < \beta$ and all $a_1, \dots, a_{\alpha(i)} \in A$ the value $\Omega_i(a_1, \dots, a_{\alpha(i)})$ is defined if and only if $\Pi_i(\theta(a_1), \dots, \theta(a_{\alpha(i)}))$ is defined, and in the case where they are defined we have $\theta(\Omega_i(a_1, \dots, a_{\alpha(i)})) = \Pi_i(\theta(a_1), \dots, \theta(a_{\alpha(i)}))$. If θ is surjective then we say $\mathfrak{B}$ is a **partial-algebra homomorphic image** of $\mathfrak{A}$. A **partial-algebra embedding** is an injective partial-algebra homomorphism. An **isomorphism** is a bijective partial-algebra homomorphism.

We are careful never to drop the words 'partial-algebra' when referring to the notions defined in Definition 7.2.4, since a bald 'homomorphism' is an ambiguous usage when speaking of partial algebras—at least three differing definitions have been given in the literature. What we call a partial-algebra homomorphism, Grätzer calls a strong homomorphism [33, Chapter 2].

Given a partial algebra $\mathfrak{A}$, when we write $a \in \mathfrak{A}$ or say that a is an element of $\mathfrak{A}$, we mean that a is an element of the domain of $\mathfrak{A}$. As is the case for total algebras, we require partial algebras to be nonempty.[1] When we want to refer to a signature consisting of a single symbol we will often abuse notation by using that symbol to denote the signature.

As in previous chapters, we write $\wp(X)$ for the power set of a set X.

Definition 7.2.5. Let σ be a signature whose symbols are members of $\{\dot{\cup}, \dot{\setminus}, \emptyset\}$. A **partial σ-algebra of sets**, $\mathfrak{A}$, with domain A, consists of a subset $A \subseteq \wp(X)$ (for some **base** set X), closed under the partial operations in σ, wherever they are defined, and containing the empty set if $\emptyset$ is in the signature. The particular case of $\sigma = (\dot{\cup})$ is called a **disjoint-union partial algebra of sets** and the case $\sigma = (\dot{\cup}, \emptyset)$ is a disjoint-union partial algebra of sets **with zero**.

Definition 7.2.6. Let σ be a signature whose symbols are members of $\{\dot{\smile}, \dot{\setminus}, |, \emptyset\}$. A **partial σ-algebra of partial functions**, $\mathfrak{A}$ consists of a set of partial functions closed under the partial and total operations in σ, wherever they are defined, and containing the empty set if $\emptyset$ is in the signature. The **base** of $\mathfrak{A}$ is the union of the domains and codomains of all the partial functions in $\mathfrak{A}$.

Definition 7.2.7. Let σ be a signature whose symbols are members of $\{\dot{\cup}, \dot{\setminus}, \emptyset\}$. A **$\sigma$-representation by sets** of a partial algebra is an isomorphism from that partial algebra to a partial σ-algebra of sets. The particular case of $\sigma = (\dot{\cup})$ is called a **disjoint-union representation** (by sets).

[1] This is a change from [45].

Definition 7.2.8. Let σ be a signature whose symbols are members of $\{\dot{\smile}, \dot{\setminus}, |, \emptyset\}$. A **$\sigma$-representation by partial functions** of a partial algebra is an isomorphism from that partial algebra to a partial σ-algebra of partial functions.

For a partial algebra $\mathfrak{A}$ and an element $a \in \mathfrak{A}$, we write a^θ for the image of a under a representation θ of $\mathfrak{A}$. We will be consistent about the symbols we use for abstract (partial) operations—those in the partial algebras being represented—employing them according to the correspondence indicated below.

$$\begin{array}{ccc} \dot{\sqcup} & \rightsquigarrow & \dot{\cup} \text{ or } \dot{\smile} \\ \dot{-} & \rightsquigarrow & \dot{\setminus} \\ ; & \rightsquigarrow & | \\ 0 & \rightsquigarrow & \emptyset \end{array}$$

For each notion of representability we are interested in the associated representation class—the class of all partial algebras having such a representation. It is usually clear whether we are talking about a representation by sets or a representation by partial functions. For example if the signature contains $\dot{\cup}$ we must be talking of sets and if it contains $\dot{\smile}$ we must be talking of partial functions. However, as part (1) of the next proposition shows, for the partial operations we are considering, representability by sets and representability by partial functions are the same thing.

Proposition 7.2.9.

1. *Let σ be a signature whose symbols are a subset of $\{\dot{\cup}, \dot{\setminus}, \emptyset\}$ and let σ' be the signature formed by replacing $\dot{\cup}$ (if present) by $\dot{\smile}$ in σ. A partial algebra is σ-representable by sets if and only if it is σ'-representable by partial functions.*

2. *Let $\mathfrak{A}$ be a partial $(\dot{\sqcup}, ;, 0)$-algebra. If the $(\dot{\sqcup}, 0)$-reduct of $\mathfrak{A}$ is $(\dot{\cup}, \emptyset)$-representable and $\mathfrak{A}$ validates $a ; b = 0$, then $\mathfrak{A}$ is $(\dot{\smile}, |, \emptyset)$-representable.*

Proof. For part (1), let σ be one of the signatures in question and let $\mathfrak{A}$ be a partial algebra. Suppose θ is a σ-representation of $\mathfrak{A}$ by sets over base X. Then the map ρ defined by $a^\rho = \{(x, x) \mid x \in a^\theta\}$ is easily seen to be a σ'-representation of $\mathfrak{A}$ by partial functions.

Conversely, suppose ρ is a σ'-representation of $\mathfrak{A}$ by partial functions over base X. Let Y be a set disjoint from X and of the same cardinality as X, and let $f : X \to Y$ be any bijection. Define θ by $a^\theta = a^\rho \cup \{(f(x), f(x)) \mid x \in \mathrm{dom}(a^\rho)\}$. Then it is easy to see that θ is *another* σ'-representation of $\mathfrak{A}$ by partial functions. By construction, θ has the property that any a^θ and b^θ have disjoint domains if and only if they are disjoint. Hence θ is also a σ-representation of $\mathfrak{A}$ by sets.

For part (2), let θ be a $(\dot{\cup}, \emptyset)$-representation of the $(\dot{\sqcup}, 0)$-reduct of $\mathfrak{A}$ over base set X. Let Y be a set disjoint from X and of the same cardinality as X, and let $f\colon X \to Y$ be any bijection. The map ρ defined by $a^\rho = f\restriction_{a^\theta}$ is easily seen to be a $(\dot{\smile}, \emptyset)$-representation of $\mathfrak{A}$. For all $a, b \in \mathfrak{A}$, the image of a^ρ (a subset of Y) is disjoint from the domain of b^ρ (a subset of X), and hence $a^\rho \mid b^\rho = \emptyset$. Since $\mathfrak{A}$ validates $a ; b = 0$, the map ρ also represents ; correctly as $|$. $\square$

Remark 7.2.10. In each of the following cases let the signature $\sigma_\emptyset$ be formed by the addition of $\emptyset$ to σ.

- Let σ be a signature containing $\dot{\cup}$. A partial algebra $\mathfrak{A}$ is $\sigma_\emptyset$-representable if and only its reduct to the signature without 0 is σ-representable and $\mathfrak{A}$ satisfies $0 \mathbin{\dot{\sqcup}} 0 = 0$.
- Let σ be a signature containing $\dot{\smile}$. A partial algebra $\mathfrak{A}$ is $\sigma_\emptyset$-representable if and only its reduct to the signature without 0 is σ-representable and $\mathfrak{A}$ satisfies $0 \mathbin{\dot{\sqcup}} 0 = 0$.
- Let σ be a signature containing $\dot{\setminus}$. A partial algebra $\mathfrak{A}$ is $\sigma_\emptyset$-representable if and only its reduct to the signature without 0 is σ-representable and $\mathfrak{A}$ satisfies $0 \mathbin{\dot{-}} 0 = 0$.

Hence axiomatisations of representation classes for signatures without $\emptyset$ would immediately yield axiomatisations for the case including $\emptyset$ also.

We now define a version of complete representability. For a partial $(\dot{\sqcup}, \dots)$-algebra $\mathfrak{A}$, define a relation $\lesssim$ over $\mathfrak{A}$ by letting $a \lesssim b$ if and only if either $a = b$ or there is $c \in \mathfrak{A}$ such that $a \mathbin{\dot{\sqcup}} c$ is defined and $a \mathbin{\dot{\sqcup}} c = b$.[2] By definition, $\lesssim$ is reflexive. If $\mathfrak{A}$ is $(\dot{\cup}, \dots)$-representable, then by elementary properties of sets, it is necessarily the case that if $(a \mathbin{\dot{\sqcup}} b) \mathbin{\dot{\sqcup}} c$ is defined then $a \mathbin{\dot{\sqcup}} (b \mathbin{\dot{\sqcup}} c)$ is also defined and equal to it, which is precisely what is required to see that $\lesssim$ is transitive. Antisymmetry of $\lesssim$ also follows by elementary properties of sets. Hence $\lesssim$ is a partial order.

Definition 7.2.11. A subset S of a partial $(\dot{\sqcup}, \dots)$-algebra $\mathfrak{A}$ is **pairwise combinable** if for all $s \neq t \in S$ the value $s \mathbin{\dot{\sqcup}} t$ is defined. A $(\dot{\cup}, \dots)$-representation θ of $\mathfrak{A}$ is **$\lesssim$-complete** if for any nonempty pairwise-combinable subset S of $\mathfrak{A}$ with a supremum a (with respect to the order $\lesssim$) we have $a^\theta = \bigcup_{s \in S} s^\theta$.

Proposition 7.2.12. *Let $\mathfrak{A}$ be a partial $(\dot{\sqcup}, \dots)$-algebra and θ be a $(\dot{\cup}, \dots)$-representation of $\mathfrak{A}$. Then for any finite nonempty pairwise-combinable subset S of $\mathfrak{A}$ with a supremum a with respect to $\lesssim$, we have $a^\theta = \bigcup_{s \in S} s^\theta$.*

Proof. Let $S \subseteq \mathfrak{A}$ be finite, nonempty, and pairwise combinable with supremum a. As S is pairwise combinable and θ is a $(\dot{\cup}, \dots)$-representation, we have that $s^\theta \mathbin{\dot{\cup}} t^\theta$ is defined for all $s \neq t \in S$. Then by the definition of $\dot{\cup}$, the set $\{s^\theta \mid s \in S\}$ is pairwise disjoint. Let $S = \{s_1, \dots, s_n\}$. By induction, for each k we have that $s_1 \mathbin{\dot{\sqcup}} \dots \mathbin{\dot{\sqcup}} s_k$ is defined and $(s_1 \mathbin{\dot{\sqcup}} \dots \mathbin{\dot{\sqcup}} s_k)^\theta = \bigcup_{i=1}^k s_i^\theta$. Hence $(s_1 \mathbin{\dot{\sqcup}} \dots \mathbin{\dot{\sqcup}} s_n)^\theta = \bigcup_{s \in S} s^\theta$. It is clear that for any $b_1, b_2 \in \mathfrak{A}$ the implication $b_1 \lesssim b_2 \implies b_1^\theta \subseteq b_2^\theta$ holds. Therefore a^θ must be a superset of each s_i^θ and must be a subset of b^θ for any upper bound b of S. But $s_1 \mathbin{\dot{\sqcup}} \dots \mathbin{\dot{\sqcup}} s_n$ is clearly an upper bound for S so we conclude that $a^\theta = (s_1 \mathbin{\dot{\sqcup}} \dots \mathbin{\dot{\sqcup}} s_n)^\theta = \bigcup_{s \in S} s^\theta$ as required. □

Corollary 7.2.13. *If $\mathfrak{A}$ is a finite partial $(\dot{\sqcup}, \dots)$-algebra then every $(\dot{\cup}, \dots)$-representation of $\mathfrak{A}$ is $\lesssim$-complete.*

Note that Proposition 7.2.12 would not have held if we naively defined $\lesssim$-completeness without the pairwise-combinable condition on the subset S, as the following example illustrates. Indeed, this is

[2] In the context of semigroup theory, where the operation is total, this is *Green's order*.

the reason the definition we gave is the appropriate one: it extends the behaviour on finite (nonempty) pairwise-combinable subsets to arbitrary (nonempty) pairwise-combinable subsets, as one would expect from a notion of completeness.

Example 7.2.14. Consider the disjoint-union partial algebra of sets whose domain consists of the five sets $\{1,2\},\{3,4\},\{1,3\},\{2,4\}$, and $\{1,2,3,4\}$. Then $\{1,2,3,4\}$ is the supremum of the set $\{\{1,2\},\{1,3\}\}$, but $\{1,2,3,4\} \neq \{1,2\} \cup \{1,3\}$.

Finally, a word about logic. In our meta-language, that is, English, we can talk in terms of partial operations and partial algebras, which is what we have been doing so far. However, the traditional presentation of first-order logic does not include partial-function symbols. Hence in order to examine the first-order logic of our partial algebras, we must view them formally as relational structures.

Let $\mathfrak{A} = (A, \overset{\bullet}{\sqcup})$ be a partial algebra. From the partial binary operation $\overset{\bullet}{\sqcup}$ over A we may define a ternary relation J over A by letting $J(a,b,c)$ if and only if $a \overset{\bullet}{\sqcup} b$ is defined and equal to c—that is, J is the **graph** of $\overset{\bullet}{\sqcup}$. Since a partial operation is (at most) single valued, we have

$$J(a,b,c) \wedge J(a,b,d) \ \rightarrow \ c = d. \tag{7.1}$$

Conversely, given any ternary relation J over A validating (7.1), we may define a partial operation $\overset{\bullet}{\sqcup}$ over A by letting $a \overset{\bullet}{\sqcup} b$ be defined if and only if there exists c such that $J(a,b,c)$ holds (unique, by (7.1)) and when this is the case we let $a \overset{\bullet}{\sqcup} b = c$. The definition of J from $\overset{\bullet}{\sqcup}$ and the definition of $\overset{\bullet}{\sqcup}$ from J are clearly inverses. Similarly, if $\overset{\bullet}{-}$ is in the signature we can define a corresponding ternary relation K in the same way.

To remain in the context of classical first-order logic we adopt languages that feature neither $\overset{\bullet}{\sqcup}$ nor $\overset{\bullet}{-}$ but have ternary relation symbols J and/or K as appropriate (as well as equality). In the relational language $\mathcal{L}(J)$, we may write $\exists a \overset{\bullet}{\sqcup} b$ as an abbreviation of the formula $\exists c J(a,b,c)$ and write $a \overset{\bullet}{\sqcup} b = c$ in place of $J(a,b,c)$. Similarly for $\overset{\bullet}{-}$ and K.

7.3 Axiomatisability

In this section we show there exists a first-order $\mathcal{L}(J)$-theory that axiomatises the class $\mathbf{J}$ of partial $\overset{\bullet}{\sqcup}$-algebras with $\overset{\bullet}{\cup}$-representations. Hence $\mathbf{J}$, viewed as a class of $\mathcal{L}(J)$-structures, is elementary. We do the same for the class $\mathbf{K}$ of partial $\overset{\bullet}{-}$-algebras with $\overset{\bullet}{\setminus}$-representations (as sets) and the class $\mathbf{L}$ of partial $(\overset{\bullet}{\sqcup}, \overset{\bullet}{-})$-algebras with $(\overset{\bullet}{\cup}, \overset{\bullet}{\setminus})$-representations.

Definition 7.3.1. If $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ are similar partial algebras and the inclusion map is a partial-algebra embedding then we say that $\mathfrak{A}_1$ is a **partial-subalgebra** of $\mathfrak{A}_2$. Let $\mathfrak{A}_i = (A_i, \Omega_0, \dots)$ be partial algebras, for $i \in I$, and let U be an ultrafilter over I. The **ultraproduct** $\Pi_{i \in I}\mathfrak{A}_i/U$ is defined in the normal way, noting that, for example, $[(a_i)_{i\in I}] \overset{\bullet}{\sqcup} [(b_i)_{i \in I}]$ (where $a_i, b_i \in \mathfrak{A}_i$ for all $i \in I$) is defined in the ultraproduct if and only if $\{i \in I \mid a_i \overset{\bullet}{\sqcup} b_i$ is defined in $\mathfrak{A}_i\} \in U$. Ultrapowers and ultraroots also have their normal definitions: an **ultrapower** is an ultraproduct of identical partial algebras and $\mathfrak{A}$ is an **ultraroot** of $\mathfrak{B}$ if $\mathfrak{B}$ is an ultrapower of $\mathfrak{A}$.

It is clear that a partial-subalgebra of $\mathfrak{A}$ is always a substructure of $\mathfrak{A}$, as relational structures, and also that any substructure of $\mathfrak{A}$ is a partial algebra, that is, validates (7.1). However, in order for a relational substructure of $\mathfrak{A}$ to be a partial-subalgebra it is necessary that it be closed under the partial operations, wherever they are defined in $\mathfrak{A}$.

It is almost trivial that the class of $\dot{\cup}$-representable partial algebras is closed under partial-subalgebras. This class is not however closed under substructures. Indeed it is easy to construct a partial $\dot{\sqcup}$-algebra $\mathfrak{A}$ with a disjoint-union representation but where an $\mathcal{L}(J)$-substructure of $\mathfrak{A}$ has no disjoint-union representation. We give an example now.

Example 7.3.2. The collection $\wp\{1,2,3\}$ of sets forms a disjoint-union partial algebra of sets and so is trivially a $\dot{\cup}$-representable partial $\dot{\sqcup}$-algebra, if we identify $\dot{\sqcup}$ with $\dot{\cup}$.

The substructure with domain $\wp\{1,2,3\} \setminus \{1,2,3\}$ is not $\dot{\cup}$-representable, because $\{1\} \mathbin{\dot{\sqcup}} \{2\}, \{2\} \mathbin{\dot{\sqcup}} \{3\}$, and $\{3\} \mathbin{\dot{\sqcup}} \{1\}$ all exist, so $\{1\}, \{2\}, \{3\}$ would have to be represented by pairwise-disjoint sets. But then $\{1,2\} \mathbin{\dot{\sqcup}} \{3\}$ would have to exist, which is not the case.

We obtain the following corollary.

Corollary 7.3.3. *The isomorphic closure of the class of disjoint-union partial algebras of sets is not axiomatisable by a universal first-order $\mathcal{L}(J)$-theory.*

We now return to our objective of proving that the classes $\mathbf{J}, \mathbf{K}$, and $\mathbf{L}$ are elementary.

Theorem 7.3.4. *Let σ be any one of the signatures $(\dot{\cup})$, $(\dot{\setminus})$, or $(\dot{\cup}, \dot{\setminus})$. The class of partial algebras σ-representable as sets, viewed as a class of relational structures, is elementary.*

Proof. The classes in question are $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{L}$. We are going to show that each of these classes is closed under isomorphisms, ultraproducts, and ultraroots. This is a well-known algebraic characterisation of elementarity (for example see [18, Theorem 6.1.16]).

We start with $\mathbf{J}$. By definition, $\mathbf{J}$ is closed under isomorphism. Next we show that $\mathbf{J}$ is pseudoelementary, hence also closed under ultraproducts (Theorem 2.3.22).

Consider a two-sorted language, with an algebra sort and a base sort. The signature consists of a ternary operation J on the algebra sort, and a binary predicate $\in$, written infix, of type $base \times algebra$. Consider the formulas

$$\begin{aligned} a \neq b &\rightarrow \exists x((x \in a \wedge x \notin b) \vee (x \notin a \wedge x \in b)) \\ \exists c Jabc &\leftrightarrow \neg\exists x(x \in a \wedge x \in b) \\ Jabc &\rightarrow ((x \in c) \leftrightarrow (x \in a \vee x \in b)) \end{aligned}$$

where a, b, c are algebra-sorted variables and x is a base-sorted variable.

These formulas merely state that the base-sorted elements form the base of a representation of the algebra-sorted elements. Hence $\mathbf{J}$ is the class of J-reducts of restrictions of models of the formulas to algebra-sorted elements, that is, $\mathbf{J}$ is pseudoelementary. Hence $\mathbf{J}$ is closed under ultraproducts.

To show that **J** is closed under ultraroots, we show that ultraroots are (isomorphic to) partial subalgebras. As we remarked earlier, **J** is closed under partial subalgebras.

Let $\mathfrak{A}$ be an ultraroot of $\mathfrak{U} \in \mathbf{J}$. Then $\mathfrak{A}$ is isomorphic to its image $\mathfrak{A}'$ under the diagonal embedding of $\mathfrak{A}$ into $\mathfrak{U}$ (by Corollary 2.3.20). To show $\mathfrak{A}'$ is a partial subalgebra of $\mathfrak{U}$, we need to show that for all $a_1, a_2 \in A'$, it holds that $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{A}'}} a_2$ is defined if and only $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{U}}} a_2$ is defined, and that when they are defined they are equal. The fact that whenever $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{A}'}} a_2$ is defined, $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{U}}} a_2$ is defined and equals $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{A}'}} a_2$, follows from the fact that, viewed as J-structures, $\mathfrak{A}'$ is a substructure of $\mathfrak{U}$ (since diagonal embeddings are embeddings). Now suppose $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{A}'}} a_2$ is undefined. Then $\mathfrak{A}', (a_1, a_2) \models \neg\exists y J x_1 x_2 y$. As diagonal embeddings are elementary embeddings, it follows that $\mathfrak{U}, (a_1, a_2) \models \neg\exists y J x_1 x_2 y$, and hence $a_1 \mathbin{\dot{\sqcup}_{\mathfrak{U}}} a_2$ is undefined. We conclude that **J** is closed under ultraroots.

We now know that **J** is closed under isomorphism, ultraproducts, and ultraroots. Then as **J** is elementary and closed under substructures it is universally axiomatisable, by the Łoś–Tarski preservation theorem.

For **K** and **L** the same line of reasoning applies. Each is by definition closed under isomorphism. For **K** we show closure under ultraproducts via pseudoelementarity, using the formulas

$$\begin{aligned} a \neq b &\;\rightarrow\; \exists x((x \in a \wedge x \notin b) \vee (x \notin a \wedge x \in b)) \\ \exists c K abc &\;\leftrightarrow\; (x \in b \rightarrow x \in a) \\ Kabc &\;\rightarrow\; ((x \in c) \leftrightarrow (x \in a \vee x \notin b)) \end{aligned}$$

and for **L** we do the same using the union of the formulas for **J** and the formulas for **K**. The proofs of closure under ultraroots are the same as for **J**. □

We can now easily establish elementarity in all cases without composition.

Corollary 7.3.5. *Let σ be any signature whose symbols are a subset of $\{\dot{\cup}, \dot{\setminus}, \emptyset\}$. The class of partial algebras that are σ-representable by sets is elementary.*

Proof. The previous theorem gives us the result for the three signatures $(\dot{\cup})$, $(\dot{\setminus})$, and $(\dot{\cup}, \dot{\setminus})$. Then as we noted in Remark 7.2.10, axiomatisations for these signatures yield axiomatisations for the signatures $(\dot{\cup}, \emptyset)$, $(\dot{\setminus}, \emptyset)$, and $(\dot{\cup}, \dot{\setminus}, \emptyset)$ with the addition of a single extra axiom, either $J(0,0,0)$ or $K(0,0,0)$. The remaining cases, the empty signature and the signature $(\emptyset)$, trivially are axiomatised by the empty theory. □

Corollary 7.3.6. *Let σ be any signature whose symbols are a subset of $\{\dot{\smile}, \dot{\setminus}, \emptyset\}$. The class of partial algebras that are σ-representable by partial functions is elementary.*

Proof. By Proposition 7.2.9(1) these representation classes are the same as those in Corollary 7.3.5. □

7.4 A recursive axiomatisation via games

In this section we describe a recursive axiomatisation of the class of $\dot{\cup}$-representable partial algebras. This axiomatisation can be understood quite simply, as a sequence of statements about a particular two-player game. The efficacy of this approach using games relies on our prior knowledge, obtained in the previous section, that the class in question is elementary. The reader should note that everything in this section can be adapted quite easily to $\dot\setminus$-representability by sets and $(\dot\cup, \dot\setminus)$-representability by sets.

Fix some partial $\dot\sqcup$-algebra $\mathfrak{A}$. The following definition and lemma are the motivation behind our two-player game.

Definition 7.4.1. We call a subset U of $\mathfrak{A}$

- **$\dot\sqcup$-prime** if $a \dot\sqcup b \in U$ implies either $a \in U$ or $b \in U$,
- **bi-closed** if the two conditions $a \in U$ or $b \in U$ and $a \dot\sqcup b$ defined, together imply $a \dot\sqcup b \in U$,
- **pairwise incombinable** if $a, b \in U$ implies $a \dot\sqcup b$ is undefined.

Lemma 7.4.2. *Let $\mathcal{F}(\mathfrak{A})$ be the set of all $\dot\sqcup$-prime, bi-closed, pairwise-incombinable subsets of $\mathfrak{A}$. Then $\mathfrak{A}$ has a disjoint-union representation if and only if there is a $B \subseteq \mathcal{F}(\mathfrak{A})$ such that*

(i) for all $a \neq b \in \mathfrak{A}$ there is $U \in B$ such that either $a \in U$ and $b \notin U$ or $b \in U$ and $a \notin U$,

(ii) for all $a, b \in \mathfrak{A}$ if $a \dot\sqcup b$ is undefined then there is $U \in B$ such that $a, b \in U$.

Proof. For the left-to-right implication, if θ is a disjoint-union representation of $\mathfrak{A}$ on a base set X then for each $x \in X$ let $U(x) = \{a \in \mathfrak{A} \mid x \in a^\theta\}$ and let $B = \{U(x) \mid x \in X\}$. It is easy to see that $U(x)$ is a $\dot\sqcup$-prime, bi-closed, pairwise-incombinable set for all $x \in X$, and that B includes all elements required by (i) and (ii) of this lemma.

Conversely, assuming that $B \subseteq \mathcal{F}(\mathfrak{A})$ has the required elements we can define a representation θ of $\mathfrak{A}$ by $a^\theta = \{U \in B \mid a \in U\}$. Condition (i) ensures that θ is faithful, that is, distinguishes distinct elements of $\mathfrak{A}$. Condition (ii) ensures a^θ and b^θ are disjoint only if $a \dot\sqcup b$ is defined. The pairwise-incombinability condition on each $U \in B$ ensures $a \dot\sqcup b$ is defined only if a^θ and b^θ are disjoint. The $\dot\sqcup$-prime and bi-closed conditions on elements of B ensure that when $a \dot\sqcup b$ is defined, $(a \dot\sqcup b)^\theta = a^\theta \cup b^\theta$. □

We define a two player game Γ_n over $\mathfrak{A}$ with $n \leq \omega$ rounds, played by players $\forall$ and $\exists$.[3] A position (Y, N) consists of two finite subsets Y and N of $\mathfrak{A}$. It might help to think of Y as a finite set of sets such that some given point belongs to each of them and N is a finite set of sets such that the same point belongs to none of them.

In the initial round (round 0) $\forall$ either

(i) picks $a \neq b \in \mathfrak{A}$, or

(ii) picks $a, b \in \mathfrak{A}$ such that $a \dot\sqcup b$ is undefined.

In the former case $\exists$ responds with an initial position, either $(\{a\}, \{b\})$ or $(\{b\}, \{a\})$, at her choice. In the latter case she must respond with the initial position $(\{a, b\}, \emptyset)$.

In all later rounds, if the position is (Y, N) then $\forall$ either

[3] Where ω denotes the first infinite ordinal.

(a) picks $a, b \in \mathfrak{A}$ such that $a \overset{\bullet}{\sqcup} b$ is defined and belongs to Y, or

(b) picks $a \in Y$ and $b \in \mathfrak{A}$ such that $a \overset{\bullet}{\sqcup} b$ is defined, or

(c) picks $a \in \mathfrak{A}$ and $b \in Y$ such that $a \overset{\bullet}{\sqcup} b$ is defined.

In case (a) player $\exists$ responds with either $(Y \cup \{a\}, N)$ or $(Y \cup \{b\}, N)$, in cases (b) and (c) she must respond with the position $(Y \cup \{a \overset{\bullet}{\sqcup} b\}, N)$. Observe that N never changes as the game proceeds, it is either a singleton or empty.

A position (Y, N) is a win for $\forall$ if either

1. $Y \cap N \neq \emptyset$, or
2. there are $a, b \in Y$ such that $a \overset{\bullet}{\sqcup} b$ is defined.

Player $\forall$ wins a play of Γ_n if he wins in some round $0 \leq i < n$, else $\exists$ wins the play of the game.

The game $\Gamma_n(Y, N)$ is similar (where Y, N are finite subsets of $\mathfrak{A}$), but the initial round is omitted and play begins from the position (Y, N).

Lemma 7.4.3. *If $\mathfrak{A}$ is representable then $\exists$ has a winning strategy for Γ_ω. If $\mathfrak{A}$ is countable and $\exists$ has a winning strategy for Γ_ω then $\mathfrak{A}$ has a representation on a base of size at most $2|\mathfrak{A}|^2$.*

Proof. First suppose $\mathfrak{A}$ has a representation, θ say. By Lemma 7.4.2 there is a set B of $\overset{\bullet}{\sqcup}$-prime, bi-closed, pairwise-incombinable subsets of $\mathfrak{A}$ such that (i) for all $a \neq b \in \mathfrak{A}$ there is $U \in B$ such that either $a \in U,\ b \notin U$ or $b \in U,\ a \notin U$ and (ii) whenever $a \overset{\bullet}{\sqcup} b$ is undefined there is $U \in B$ with $a, b \in U$. We describe a winning strategy for $\exists$. In response to any initial $\forall$-move she will select a suitable $U \in B$ and play an initial position (Y, N) such that

$$Y \subseteq U \text{ and } N \cap U = \emptyset. \tag{7.2}$$

and the remainder of her strategy will be to preserve this condition throughout the play.

In the initial round there are two possibilities.

(i) If $\forall$ plays $a \neq b \in \mathfrak{A}$ then there is a $U \in B$ with either $a \in U,\ b \notin U$ or $b \in U,\ a \notin U$. In the former case $\exists$ plays an initial position $(\{a\}, \{b\})$ and in the latter case she plays $(\{b\}, \{a\})$.

(ii) If $\forall$ plays (a, b) where $a \overset{\bullet}{\sqcup} b$ is undefined, there is $U \in B$ where $a, b \in U$ and $\exists$ selects such a U and plays $(\{a, b\}, \emptyset)$.

In each case, (7.2) holds.

In a subsequent round, if the current position (Y, N) satisfies (7.2) and $\forall$ plays a, b where $a \overset{\bullet}{\sqcup} b \in Y$ is defined then since U is $\overset{\bullet}{\sqcup}$-prime either $Y \cup \{a\} \subseteq U$ or $Y \cup \{b\} \subseteq U$, so $\exists$ may play either $(Y \cup \{a\}, N)$ or $(Y \cup \{b\}, N)$, as appropriate, preserving (7.2). Similarly, if $\forall$ plays a, b where $a \in Y$ and $a \overset{\bullet}{\sqcup} b$ is defined (or $b \in Y$ and $a \overset{\bullet}{\sqcup} b$ is defined), then since U is bi-closed we have $a \overset{\bullet}{\sqcup} b \in U$ so $\exists$ plays $(Y \cup \{a \overset{\bullet}{\sqcup} b\}, N)$, preserving condition (7.2). This condition suffices to prove that $\exists$ does not lose in any round of the play.

Conversely, suppose $\mathfrak{A}$ is countable and $\exists$ has a winning strategy for Γ_ω. Then for each $a \neq b \in \mathfrak{A}$ let $S_{a,b} = \bigcup_{i<\omega} Y_i$, where $(Y_0, N), (Y_1, N), \ldots$ is a play of Γ_ω in which $\forall$ plays the type (i) move (a, b) initially (so N is a singleton). For each $a, b \in \mathfrak{A}$ where $a \stackrel{\bullet}{\sqcup} b$ is undefined let $T_{a,b} = \bigcup_{i<\omega} Y_i$ be the limit of a play in which $\forall$ plays the type (ii) move (a, b) initially (so N is empty). In each case we suppose—here is where we use the hypothesis that $\mathfrak{A}$ is countable—that $\forall$ plays all possible moves subsequently. We also suppose that $\exists$ uses her winning strategy.

Each set $S_{a,b}$ (where $a \neq b$) or $T_{a,b}$ (where $a \stackrel{\bullet}{\sqcup} b$ is undefined) is $\stackrel{\bullet}{\sqcup}$-prime, bi-closed, and pairwise incombinable, since $\forall$ plays all possible moves in a play and $\exists$ never loses. Hence $B = \{S_{a,b} \mid a \neq b \in \mathfrak{A}\} \cup \{T_{a,b} \mid a \stackrel{\bullet}{\sqcup} b \text{ is undefined}\}$ satisfies the conditions of Lemma 7.4.2. Clearly the size of the base set B is at most $2|\mathfrak{A}|^2$. □

Lemma 7.4.4. *For each $n < \omega$ there is a first-order $\mathcal{L}(J)$-formula ρ_n such that $\mathfrak{A} \models \rho_n$ if and only if $\exists$ has a winning strategy in Γ_n.*

Proof. Let V and W be disjoint finite sets of variables. For each $n < \omega$ we define formulas $\mu_n(V, W)$ in such a way that for any partial $\stackrel{\bullet}{\sqcup}$-algebra $\mathfrak{A}$ and any variable assignment $\lambda\colon \mathit{vars} \to \mathfrak{A}$ we have

$$\mathfrak{A}, \lambda \models \mu_n(V, W) \iff \exists \text{ has a winning strategy in } \Gamma_n(\lambda[V], \lambda[W]). \tag{7.3}$$

Let

$$\mu_0(V, W) = \bigwedge_{v,v' \in V} \neg\exists c J(v, v', c) \ \wedge \bigwedge_{v \in V,\ w \in W} v \neq w$$

where c is a fresh variable. So (7.3) is clear when $n = 0$. For the recursive step let

$$\begin{aligned}\mu_{n+1}(V, W) = \forall a, b \Bigg(& \bigwedge_{v \in V} (J(a, b, v) \to \mu_n(V \cup \{a\}, W) \vee \mu_n(V \cup \{b\}, W)) \\ & \wedge \bigwedge_{v \in V} (J(a, v, b) \to \mu_n(V \cup \{b\}, W)) \\ & \wedge \bigwedge_{v \in V} (J(v, a, b) \to \mu_n(V \cup \{b\}, W))\Bigg)\end{aligned}$$

where a and b are fresh variables. By a simple induction on n we see that (7.3) holds for all n. Finally, (let $\rho_0 = \top$ and) let

$$\rho_{n+1} = \forall a, b\Big((a = b \vee \mu_n(\{a\}, \{b\}) \vee \mu_n(\{b\}, \{a\})) \wedge (\exists c J(a, b, c) \vee \mu_n(\{a, b\}, \emptyset))\Big)$$

where again a, b, and c are fresh variables. □

Observe that each formula $\mu_n(V, W)$ is equivalent to a universal formula and therefore ρ_n, but for the clause $\exists c J(a, b, c)$, is universal.

Theorem 7.4.5. *The isomorphic closure of the class of disjoint-union partial algebras of sets is axiomatised by $\{\rho_n \mid n < \omega\}$.*

Proof. We will use Lemma 7.4.3 and Lemma 7.4.4, but we must be slightly careful, because we chose to present the lemmas with the assumption that the $\mathcal{L}(J)$-structure in question is a partial algebra. Hence we must check that (7.1) holds before appealing to either lemma.

If an $\mathcal{L}(J)$-structure $\mathfrak{A}$ is isomorphic to a disjoint-union partial algebra of sets then certainly it validates (7.1). Then by Lemma 7.4.3, player $\exists$ has a winning strategy in the game of length n for each $n < \omega$. So $\mathfrak{A} \models \rho_n$ by Lemma 7.4.4.

Conversely, if $\mathfrak{A} \models \{\rho_n \mid n < \omega\}$ let $\mathfrak{B}$ be any countable elementary substructure of $\mathfrak{A}$. Then $\mathfrak{B} \models \{\rho_n \mid n < \omega\}$. The validity of ρ_3 tells us that (7.1) holds, as we now explain. For if $J(a, b, c)$ and $J(a, b, d)$, with $c \neq d$, then from ρ_3 we know that either $\mu_2(\{c\}, \{d\})$ holds or $\mu_2(\{d\}, \{c\})$ holds. Without loss of generality, we assume the former. From $\mu_2(\{c\}, \{d\})$, assigning c to v in the first conjunct, we deduce $\mu_1(\{c, a\}, \{d\})$ or $\mu_1(\{c, b\}, \{d\})$ and again we may assume the former. From the second conjunct in $\mu_1(\{c, a\}, \{d\})$ (assigning b to the variable v and d to the variable b) we deduce $\mu_0(\{c, a, d\}, \{d\})$, which is contradicted by the final inequality $v \neq w$, when v and w are both assigned d.

Hence we can use Lemma 7.4.4 and conclude that $\exists$ has a winning strategy in game Γ_n for each $n < \omega$. Then since $\exists$ has only finitely many choices open to her in each round (actually, at most two choices), by Kőnig's tree lemma [63], she also has a winning strategy in Γ_ω. So by Lemma 7.4.3 the partial algebra $\mathfrak{B}$ is isomorphic to a disjoint-union partial algebra of sets. Since $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$, we deduce $\mathfrak{A}$ is also isomorphic to a disjoint-union partial algebra of sets, by Theorem 7.3.4. □

7.5 Non-axiomatisability

In this section we show that for any of the signatures $(\dot{\cup})$, $(\dot{\setminus})$, $(\dot{\cup}, \dot{\setminus})$, $(\dot{\cup}, \emptyset)$, $(\dot{\setminus}, \emptyset)$, or $(\dot{\cup}, \dot{\setminus}, \emptyset)$ the class of partial algebras representable by sets is not finitely axiomatisable. Hence the same is true for representability by partial functions, when $\dot{\cup}$ is replaced by $\dot{\smile}$. For partial functions, we also show the same holds when we add composition to these signatures. Our strategy is to describe a set of non-representable partial algebras that has a representable ultraproduct, which, by Łoś's theorem, immediately rules out finite axiomatisability.

Let m and n be sets of cardinality greater than two. We will call a subset of $m \times n$ **axial** if it has the form $\{i\} \times J$ (for some $i \in m,\ J \subseteq n$) or the form $I \times \{j\}$ (for some $I \subseteq m,\ j \in n$). Observe that $\emptyset \times \{j\} = \{i\} \times \emptyset = \emptyset$ for any $i \in m,\ j \in n$.

Next we define a partial $(\dot{\sqcup}, 0)$-algebra $\mathfrak{X}(m, n)$. It has a domain consisting of all axial subsets of $m \times n$. The constant 0 is interpreted as the empty set and $S \mathbin{\dot{\sqcup}} T$ is defined and equal to $S \cup T$ if S is disjoint from T and $S \cup T$ is axial, else it is undefined.

Recall the notion of $\lesssim$-complete representability given in Definition 7.2.11. The following fact is not important for our results, but note that the algebra $\mathfrak{X}(m, n)$ is $\lesssim$-completely representable.[4] A $\lesssim$-complete representation has base $B = \{P \subseteq m \times n \mid |P| = 2 \text{ and } P \text{ is not axial}\}$, and maps each axial set S to $\{P \in B \mid P \cap S \neq \emptyset\}$. It is straightforward to check that this is indeed a $\lesssim$-complete representation.

Definition 7.5.1. Given a partial algebra $\mathfrak{A} = (A, (\Omega_i)_{i<\beta})$, a **partial-algebra congruence** on $\mathfrak{A}$ is an equivalence relation $\sim$ with the property that for each i and every $a_1, \ldots, a_{\alpha(i)}, b_1, \ldots, b_{\alpha(i)} \in \mathfrak{A}$, if

[4] Thanks to Ian Hodkinson for pointing out the $\lesssim$-complete aspect.

$a_1 \sim b_1, \ldots, a_{\alpha(i)} \sim b_{\alpha(i)}$ then $\Omega_i(a_1, \ldots, a_{\alpha(i)})$ is defined if and only if $\Omega_i(b_1, \ldots, b_{\alpha(i)})$ is defined and when these are defined $\Omega_i(a_1, \ldots, a_{\alpha(i)}) \sim \Omega_i(b_1, \ldots, b_{\alpha(i)})$.

Note our condition for being a partial-algebra congruence is strictly stronger than that obtained by viewing a partial algebra as a relational structure and then adopting the sometimes-used definition of 'congruence relation' that takes it to be synonymous with 'kernel of a homomorphism'—for signatures with no function symbols such a congruence relation is merely an equivalence relation. Our definition of a partial-algebra congruence takes the 'algebraic' rather than 'relational' view of the structure.

Definition 7.5.2. Given a partial algebra $\mathfrak{A} = (A, (\Omega_i)_{i<\beta})$ and a partial-algebra congruence $\sim$ on $\mathfrak{A}$, the **partial-algebra quotient** of $\mathfrak{A}$ by $\sim$, written $\mathfrak{A}/{\sim}$, is the partial algebra of the signature $(\Omega_i)_{i<\beta}$ with domain the set of $\sim$-equivalence classes and well-defined partial operations given by $\Omega_i([a_1], \ldots, [a_{\alpha(i)}])$= $[\Omega_i(a_1, \ldots, a_{\alpha(i)})]$ if $\Omega_i(a_1, \ldots, a_{\alpha(i)})$ is defined, else $\Omega_i([a_1], \ldots, [a_{\alpha(i)}])$ is undefined.

Note that partial-algebra quotients are indeed partial algebras. All the expected relationships between partial-algebra homomorphisms, partial-algebra congruences, and partial-algebra quotients hold.

Returning to our task, we define a binary relation $\sim$ over $\mathfrak{X}(m, n)$ as the smallest equivalence relation such that

$$\{i\} \times n \sim m \times \{j\}$$

$$\{i\} \times (n \setminus \{j\}) \sim (m \setminus \{i\}) \times \{j\}$$

for all $i \in m,\ j \in n$. The equivalence class of $\{i\} \times n$ (for any choice of $i \in m$) is denoted 1 and the equivalence class of $\{i\} \times (n \setminus \{j\})$ is denoted $\overline{(i, j)}$, for each $i \in m,\ j \in n$. All other equivalence classes are singletons, either $\{\{i\} \times J\}$ for some $i \in m,\ J \subsetneq n$ or $\{I \times \{j\}\}$ for some $I \subsetneq m,\ j \in n$. We show next that $\sim$ is a partial-algebra congruence. Clearly $\dot{\sqcup}$ is commutative in the sense that $S \dot{\sqcup} T$ is defined if and only if $T \dot{\sqcup} S$ is defined and then they are equal. Hence it suffices to show, for any $S \sim S'$, that $S \cap T = \emptyset$ and $S \cup T$ is axial if and only if $S' \cap T = \emptyset$ and $S' \cup T$ is axial, and if these statements are true then $S \cup T \sim S' \cup T$. Further, by symmetry, it suffices to prove only one direction of this biconditional.

Suppose then that $S \sim S'$, that $S \cap T = \emptyset$ and that $S \cup T$ is axial. We may assume $S \neq S'$, so without loss of generality there are two cases to consider: the case $S = \{i\} \times n$ and the case $S = \{i\} \times (n \setminus \{j\})$ and $S' = (m \setminus \{i\}) \times \{j\}$. In the first case, since $S \cup T$ is axial and $|n| > 1$ we know T must be a subset of S. But T is also disjoint from S, hence T is empty. Then it is clear that $S' \cap T = \emptyset$ and $S' \cup T$ is axial and that $S \cup T \sim S' \cup T$. In the second case, since $S \cup T$ is axial and $|n| > 2$ we know T must be a subset of $\{i\} \times n$. But T is also disjoint from S and so T is either $\{(i, j)\}$ or $\emptyset$. Either way, it is clear that $S' \cap T = \emptyset$ and $S' \cup T$ is axial and that $S \cup T \sim S' \cup T$.

Now define a partial $(\dot{\sqcup}, 0)$-algebra $\mathfrak{A}(m, n)$ as the partial-algebra quotient $\mathfrak{X}(m, n)/{\sim}$. Since the elements of $\mathfrak{A}(m, n)$ are $\sim$-equivalence classes and these are typically singletons, we will suppress the $[\cdot]$ notation and let the axial set S denote the equivalence class of S, taking care to identify $\sim$-equivalent axial sets.

Lemma 7.5.3. *For any sets m and n of cardinality greater than two, the partial algebra $\mathfrak{A}(m,n)$ is $\lesssim$-completely $(\dot{\cup}, 0)$-representable if and only if $|m| = |n|$.*

Proof. For the left-to-right implication let θ be a $\lesssim$-complete representation of $\mathfrak{A}(m,n)$ over the base X. The set 1^θ must be nonempty, because $1 \dot{\sqcup} 1$ is undefined. Fix some $x \in 1^\theta$ and define a subset R of $m \times n$ by letting $(i,j) \in R \iff x \in \{(i,j)\}^\theta$ for $i \in m,\ j \in n$. For each $i \in m$, since 1 is the supremum of $\{\{(i,j)\} \mid j \in n\}$ and θ is $\lesssim$-complete, there is $j \in n$ such that $x \in \{(i,j)\}^\theta$ and hence $(i,j) \in R$. Dually, for any $j \in n$, since 1 is the supremum of $\{\{(i,j)\} \mid i \in m\}$ there is $i \in m$ such that $(i,j) \in R$. We cannot have $(i,j),(i',j) \in R$, for distinct $i,i' \in m$, since θ is a representation and $\{(i,j)\} \dot{\sqcup} \{(i',j)\}$ is defined. Similarly, for distinct $j,j' \in n$ we cannot have $(i,j),(i,j') \in R$. Hence R is a bijection from m onto n. We deduce that $|m| = |n|$.

For the right-to-left implication suppose $|m| = |n|$. It suffices to describe a $\lesssim$-complete representation of $\mathfrak{A}(n,n)$.

The base of the representation is the set P_n of all permutations on n. If S is any axial set it has the form $\{i\} \times J$ for some $i \in n,\ J \subseteq n$ or the form $I \times \{j\}$ for some $I \subseteq n,\ j \in n$. Define a representation θ over P_n by letting $(\{i\} \times J)^\theta$ be the set of all permutations $\sigma \in P_n$ such that $\sigma(i) \in J$ and $(I \times \{j\})^\theta$ be the set of all permutations $\sigma \in P_n$ such that $\sigma^{-1}(j) \in I$. Observe this is well-defined, since firstly if an axial set is both of the form $\{i\} \times J$ and of the form $I \times \{j\}$ then the definitions agree, and secondly it is easily seen that $\sim$-equivalent axial sets are assigned the same set of permutations.

We now show that θ is a $(\dot{\cup}, 0)$-representation. To see that θ is faithful we show that $\sim$-inequivalent axial sets are represented as distinct sets of permutations. We may assume the axial sets are not in the equivalence class 1, since $1^\theta = P_n$ and all axial sets not in 1 are clearly assigned proper subsets of P_n. Similarly, we may assume the axial sets are not the empty set.

First suppose we have two inequivalent *vertical* sets $\{i\} \times J$ and $\{i'\} \times J'$. If $i = i'$ there must be a j in the symmetric difference of J and J'. Then any permutation with $i \mapsto j$ witnesses the distinction between $(\{i\} \times J)^\theta$ and $(\{i'\} \times J')^\theta$. Otherwise $i \neq i'$, and if we can choose $j \neq j'$ with $j \in J$ and $j' \notin J'$ then any permutation with $i \mapsto j$ and $i' \mapsto j'$ belongs to $(\{i\} \times J)^\theta \setminus (\{i'\} \times J')^\theta$. Since we assumed our axial sets are neither $\emptyset$ nor in 1 we can do this unless J and $n \setminus J'$ are the same singleton set, $\{j_0\}$ say. But then for any distinct $j, j' \in n \setminus \{j_0\}$ we have $j \notin J$ and $j' \in J'$ so any permutation with $i \mapsto j,\ i' \mapsto j'$ belongs to $(\{i'\} \times J')^\theta \setminus (\{i\} \times J)^\theta$. Hence θ always distinguishes inequivalent vertical sets. If we have two inequivalent *horizontal* sets $I \times \{j\}$ and $I' \times \{j\}'$ then the argument is similar.

Lastly, suppose we have inequivalent sets $\{i\} \times J$ and $I \times \{j\}$. If we can choose a $k \in J$ not equal to j and an $l \notin I$ not equal to i then there exist permutations with $i \mapsto k$ and $l \mapsto j$ and any such permutation belongs to $(\{i\} \times J)^\theta \setminus (I \times \{j\})^\theta$. We can do this unless either $J = \{j\}$, in which case we have two horizontal sets, which we have already considered, or $I = n \setminus \{i\}$. By a symmetrical argument, we can witness the distinction unless $J = n \setminus \{j\}$. Hence $(\{i\} \times J)^\theta \neq (I \times \{j\})^\theta$ unless $\{i\} \times J = \{i\} \times (n \setminus \{j\})$ and $I \times \{j\} = (n \setminus \{i\}) \times \{j\}$, contradicting the assumed inequivalence of $\{i\} \times J$ and $I \times \{j\}$. This completes the argument that θ is faithful.

It is clear that θ correctly represents 0 as $\emptyset$. Now to see that θ is a $(\dot{\cup}, \emptyset)$-representation it remains to show that θ represents $\dot{\sqcup}$ correctly as $\dot{\cup}$. If $S \dot{\sqcup} T$ is defined then we may assume $S = \{i\} \times J_1$ and $T = \{i\} \times J_2$ for some disjoint J_1 and J_2, since the case where $S \dot{\sqcup} T$ is a horizontal set is similar. Then it is clear from the definition of θ that S^θ and T^θ are disjoint and so $S^\theta \dot{\cup} T^\theta$ is defined and that $(S \dot{\sqcup} T)^\theta = S^\theta \dot{\cup} T^\theta$. If $S \dot{\sqcup} T$ is undefined then either there is some $(i, j) \in S \cap T$, in which case S^θ and T^θ clearly are non-disjoint, or $S \cup T$ is not axial, in which case there are $i \neq i'$ and $j \neq j'$ with $(i, j) \in S$ and $(i', j') \in T$. In the second case, any permutation with $i \mapsto j$ and $i' \mapsto j'$ witnesses that S^θ and T^θ are non-disjoint. Hence when $S \dot{\sqcup} T$ is undefined, $S^\theta \dot{\cup} T^\theta$ is undefined. This completes the proof that θ is a $(\dot{\cup}, \emptyset)$-representation.

Finally we show that θ is $\lesssim$-complete. Let γ be a pairwise-combinable subset of $\mathfrak{A}(n, n)$. If γ has supremum $\{i\} \times J$ for some J with $|n \setminus J| \geq 2$ then for all $S \in \gamma$, since the supremum is an upper bound and by the definition of $\lesssim$, either $S = \{i\} \times J$ or there is T such that $S \dot{\sqcup} T \sim \{i\} \times J$. It follows that each $S \in \gamma$ has the form $\{i\} \times J_S$ for some $J_S \subseteq J$, and since the $\{i\} \times J$ is the least upper bound we have $J = \bigcup_{S \in \gamma} J_S$. Then for any $\sigma \in P_n$ we have

$$\begin{aligned} \sigma \in (\{i\} \times J)^\theta &\iff \sigma(i) \in J \\ &\iff \sigma(i) \in J_S \text{ for some } S \in \gamma \\ &\iff \sigma \in (\{i\} \times J_S)^\theta \text{ for some } S \in \gamma \\ &\iff \sigma \in \bigcup_{S \in \gamma} (\{i\} \times J_S)^\theta = \bigcup_{S \in \gamma} S^\theta. \end{aligned}$$

Similarly if the supremum of γ is $I \times \{j\}$ for some I with $|m \setminus I| \geq 2$, then $(I \times \{j\})^\theta = \bigcup_{S \in \gamma} S^\theta$.

If the supremum of γ is $\overline{(i, j)}$ then either $\gamma = \{\overline{(i, j)}\}$, so the proof of the required equality is trivial, or, because γ is pairwise combinable, each $S \in \gamma$ has the form $\{i\} \times J_S$ or each $S \in \gamma$ has the form $I_S \times \{j\}$ in which cases the proof is similar to above. If the supremum of γ is 1, then either $\gamma = \{1\}$ or $\gamma = \{\{(i, j)\}, \overline{(i, j)}\}$ for some i, j, or each $S \in \gamma$ has the form $\{i\} \times J_S$, or each $S \in \gamma$ has the form $I_S \times \{j\}$. In every case the required equality is seen to hold. So θ is a $\lesssim$-complete representation. □

Remark 7.5.4. We have seen that $\mathfrak{X}(3, 4)$ has a $(\dot{\cup}, \emptyset)$-representation, but, by Lemma 7.5.3 and Corollary 7.2.13, the partial algebra $\mathfrak{A}(3, 4) = \mathfrak{X}(3, 4)/\sim$ does not. Since the latter is a partial-algebra homomorphic image of the former we see that the class of $(\dot{\cup}, \emptyset)$-representable partial algebras is not closed under partial-algebra homomorphic images, in contrast to the corresponding result for algebras representable as fields of sets, that is, Boolean algebras.

We now have a source of non-representable partial algebras with which to prove our first non-axiomatisability result.

Theorem 7.5.5. *The class of $(\dot{\cup}, \emptyset)$-representable partial algebras is not finitely axiomatisable.*

Proof. Write ν for $\omega \setminus \{0, 1, 2\}$ and let $m \in \nu$. By Lemma 7.5.3 the partial algebra $\mathfrak{A}(m, m + 1)$ has no $\lesssim$-complete $(\dot{\cup}, \emptyset)$-representation. Since this partial algebra is finite, it follows, by Corollary 7.2.13, that it has no $(\dot{\cup}, \emptyset)$-representation.

Let U be a non-principal ultrafilter over ν. We claim that the ultraproduct $\Pi_{m \in \nu} \mathfrak{A}(m, m + 1)/U$ is isomorphic to a partial-subalgebra of $\mathfrak{A}(\Pi_{m \in \nu} m/U,\ \Pi_{m \in \nu}(m + 1)/U)$. Note that every element of

$\Pi_{m\in\nu}\mathfrak{A}(m, m+1)/U$ is the equivalence class of a sequence of vertical sets $[(\{i_m\} \times J_m)_{m\in\nu}]$ where $i_m \in m$ and $J_m \subseteq m+1$ for each $m \in \nu$, or the equivalence class of a sequence of horizontal sets $[(I_m \times \{j_m\})_{m\in\nu}]$ where $I_m \subseteq m$ and $j \in m+1$ for each $m \in \nu$. The partial-algebra embedding θ maps $[(\{i_m\} \times J_m)_{m\in\nu}]$ to $\{[(i_m)_{m\in\nu}]\} \times \{[(j_m)_{m\in\nu}] \mid \{m \in \nu \mid j_m \in J_m\} \in U\}$, and it maps $[(I_m \times \{j_m\})_{m\in\nu}]$ to $\{[(i_m)_{m\in\nu}] \mid \{m \in \nu \mid i_m \in I_m\} \in U\} \times \{[(j_m)_{m\in\nu}]\}$.

It is easy to check that θ is a well-defined partial-algebra embedding. We limit ourselves to showing that if $a^\theta \overset{\bullet}{\sqcup} b^\theta$ is defined in $\mathfrak{A}(\Pi_{m\in\nu}m/U,\ \Pi_{m\in\nu}(m+1)/U)$ then $a \overset{\bullet}{\sqcup} b$ is defined in $\Pi_{m\in\nu}\mathfrak{A}(m, m+1)/U$, since it is this condition that distinguishes partial-algebra embeddings from embeddings of relational structures.

We prove the contrapositive. Suppose $a \overset{\bullet}{\sqcup} b$ is undefined and let $[(a_m)_{m\in\nu}] = a$ and $[(b_m)_{m\in\nu}] = b$. Then we can find $S \in U$ such that one of the following two possibilities holds. One, for each $m \in S$ there exists (i_m, j_m) belonging to both (a representative of) a_m and (a representative of) b_m. Or two, for each $m \in S$ there exists $i_m \neq i'_m$ and $j_m \neq j'_m$ such that (i_m, j_m) belongs to (a representative of) a_m and (i'_m, j'_m) belongs to (a representative of) b_m. Extend $(i_m)_{m\in S}, (j_m)_{m\in S}$ and, if appropriate, $(i'_m)_{m\in S}$ and $(j'_m)_{m\in S}$ to ν-sequences arbitrarily. If the first alternative holds then $([(i_m)_{m\in\nu}], [(j_m)_{m\in\nu}])$ belongs to (representatives of) both a^θ and b^θ. So $a^\theta \overset{\bullet}{\sqcup} b^\theta$ is undefined, since the representatives are non-disjoint. If the second alternative holds then $[(i_m)_{m\in\nu}] \neq [(i'_m)_{m\in\nu}]$, $[(j_m)_{m\in\nu}] \neq [(j'_m)_{m\in\nu}]$, and $([(i_m)_{m\in\nu}], [(j_m)_{m\in\nu}])$ belongs to (a representative of) a^θ and $([(i'_m)_{m\in\nu}], [(j'_m)_{m\in\nu}])$ belongs to (a representative of) b^θ. So $a^\theta \overset{\bullet}{\sqcup} b^\theta$ is undefined, since the union of the representatives is not axial.

We now argue that $\mathfrak{A}(\Pi_{m\in\nu}m/U,\ \Pi_{m\in\nu}(m+1)/U)$ is representable, by showing that the cardinalities of its two parameters are equal. The map $f\colon \Pi_{m\in\nu}m/U \to \Pi_{m\in\nu}(m+1)/U$ defined by $f([(i_m)_{m\in\nu}]) = [(i_m+1)_{m\in\nu}]$ is injective and its range is all of $\Pi_{m\in\nu}(m+1)/U$ except $[(0, 0, \dots)]$. Since these are infinite sets it follows that the cardinality of $\Pi_{m\in\nu}m/U$ equals the cardinality of $\Pi_{m\in\nu}(m+1)/U$. It follows by Lemma 7.5.3 that $\mathfrak{A}(\Pi_{m\in\nu}m/U,\ \Pi_{m\in\nu}(m+1)/U)$ is $(\overset{\bullet}{\cup}, \emptyset)$-representable.

Since the partial algebra $\Pi_{m\in\nu}\mathfrak{A}(m, m+1)/U$ has a partial algebra embedding into a representable partial algebra and the class of representable partial algebras is closed under partial subalgebras, we conclude that $\Pi_{m\in\nu}\mathfrak{A}(m, m+1)/U$ is itself representable. Hence we have an ultraproduct of unrepresentable partial algebras that is itself representable. It follows by Łoś's theorem that the class of $(\overset{\bullet}{\cup}, \emptyset)$-representable partial algebras cannot be defined by finitely many axioms. □

Corollary 7.5.6. *Let σ be any one of the signatures $(\overset{\bullet}{\cup})$, $(\overset{\bullet}{\smile})$, $(\overset{\bullet}{\cup}, \emptyset)$, $(\overset{\bullet}{\smile}, \emptyset)$, $(\overset{\bullet}{\smile}, |)$, or $(\overset{\bullet}{\smile}, |, \emptyset)$. The class of σ-representable partial algebras is not finitely axiomatisable in $\mathcal{L}(J)$, $\mathcal{L}(J, 0)$, $\mathcal{L}(J, ;)$, or $\mathcal{L}(J, ;, 0)$, as appropriate.*

Proof. The case $\sigma = (\overset{\bullet}{\cup}, \emptyset)$ is Theorem 7.5.5. The case $\sigma = (\overset{\bullet}{\smile}, \emptyset)$ follows by Proposition 7.2.9(1), which tells us that the representation classes for $(\overset{\bullet}{\cup}, \emptyset)$ and $(\overset{\bullet}{\smile}, \emptyset)$ coincide.

For the case $\sigma = (\overset{\bullet}{\smile}, |, \emptyset)$, for any sets m, n of cardinality greater than two, expand $\mathfrak{A}(m, n)$ to a partial $(\overset{\bullet}{\sqcup}, ;, 0)$-algebra $\mathfrak{B}(m, n)$ by defining $a \mathbin{;} b = 0$ for all a, b. As in the proof of Theorem 7.5.5, write ν for $\omega \setminus \{0, 1, 2\}$ and let U be a non-principal ultrafilter over ν. Then for every $m \in \nu$ the partial algebra $\mathfrak{B}(m, m{+}1)$ has no $(\overset{\bullet}{\smile}, |, \emptyset)$-representation, as its reduct to $(\overset{\bullet}{\sqcup}, 0)$ has no $(\overset{\bullet}{\smile}, \emptyset)$-representation. However, as we saw in the proof of Theorem 7.5.5, the reduct of $\Pi_{m\in\nu}\mathfrak{B}(m, m+1)/U$ to $(\overset{\bullet}{\sqcup}, 0)$ does have a $(\overset{\bullet}{\smile}, \emptyset)$-representation and moreover, by Łoś's theorem, it validates $a;b = 0$. By Proposition 7.2.9, these conditions ensure $\Pi_{m\in\nu}\mathfrak{B}(m, m + 1)/U$ has a $(\overset{\bullet}{\smile}, |, \emptyset)$-representation. Once again we have an ultraproduct of unrepresentable partial algebras that is itself representable. Hence the representation class is not finitely axiomatisable.

For each of the signatures not containing $\emptyset$ the result follows from the result for the corresponding signature with $\emptyset$, by Remark 7.2.10. Because if the representation class for the signature without $\emptyset$ were finitely axiomatisable we could finitely axiomatise the case with $\emptyset$ by the addition of the single extra axiom $J(0, 0, 0)$. □

We can prove a stronger negative result about $\lesssim$-complete representability.

Theorem 7.5.7. *The class of $\lesssim$-completely $(\overset{\bullet}{\cup}, \emptyset)$-representable partial algebras is not closed under elementary equivalence.*

Proof. Consider the two partial $(\overset{\bullet}{\sqcup}, \emptyset)$-algebras $\mathfrak{A}_1 = \mathfrak{A}(\omega_1, \omega)$ and $\mathfrak{A}_0 = \mathfrak{A}(\omega, \omega)$, where ω_1 denotes the first uncountable ordinal. By Lemma 7.5.3 the former is not $\lesssim$-completely $(\overset{\bullet}{\cup}, \emptyset)$-representable while the latter is. We prove these two partial algebras are elementarily equivalent by showing that the second player has a winning strategy in the Ehrenfeucht–Fräïssé game of length ω played over $\mathfrak{A}_1$ and $\mathfrak{A}_0$.[5]

Although elements of $\mathfrak{A}_1$ or $\mathfrak{A}_0$ are formally equivalence classes of axial sets, we may take $\{0\} \times \omega$ as the representative of 1 and $\{i\} \times (\omega \setminus \{j\})$ as the representative of $\overline{(i, j)}$, in either partial algebra. Since all elements are axial, each nonzero $a \in \mathfrak{A}_i$ uniquely determines (given this choice of representatives) sets $h_i(a)$ and $v_i(a)$ such that $a = h_i(a) \times v_i(a)$, for $i = 0, 1$. For example $h_1(\{i\} \times J) = \{i\},\ v_1(\{i\} \times J) = J,\ h_1(1) = \{0\}$, and $v_1(1) = \omega$. We will view 0 as $\emptyset \times \emptyset$, in that $h_i(0) = v_i(0) = \emptyset$.

For any sets X, Y we write $X \approx Y$ if

- either both X and Y contain 0 or neither contain 0

and

- either $|X| = |Y|$ or both sets are infinite.

Observe, for any X, Y, and $U \subseteq X$, that

$$X \approx Y \quad \Longleftrightarrow \quad \text{there is } V \subseteq Y \text{ with } U \approx V \text{ and } X \setminus U \approx Y \setminus V. \tag{7.4}$$

Initially there are no pebbles in play. After k rounds there will be k pebbles on $\bar{b} = (b_0, \ldots , b_{k-1}) \in$

[5] In fact, this proves that $\mathfrak{A}_1$ and $\mathfrak{A}_0$ are $L_{\infty\omega}$-equivalent, which is a stronger condition than elementary equivalence. See [62], for explanation of this notation, and the original proof.

$\mathfrak{A}_1^k$ and k matching pebbles on $\bar{a} = (a_0, \ldots, a_{k-1}) \in \mathfrak{A}_0^k$. For each $S \subseteq k$ let

$$h_1(\bar{b}, S) = \bigcap_{i \in S} h_1(b_i) \cap \bigcap_{i \in k \setminus S} (\omega_1 \setminus h_1(b_i)),$$
$$v_1(\bar{b}, S) = \bigcap_{i \in S} v_1(b_i) \cap \bigcap_{i \in k \setminus S} (\omega \setminus v_1(b_i)),$$

with similar definitions for $h_0(\bar{a}, S)$ and $v_0(\bar{a}, S)$. Observe that $\{h_1(\bar{b}, S) \mid S \subseteq k\} \setminus \{\emptyset\}$ is a finite partition of ω_1 and each of $\{v_1(\bar{b}, S) \mid S \subseteq k\} \setminus \{\emptyset\}$, $\{h_0(\bar{a}, S) \mid S \subseteq k\} \setminus \{\emptyset\}$, and $\{v_0(\bar{a}, S) \mid S \subseteq k\} \setminus \{\emptyset\}$ is a finite partition of ω.

As an induction hypothesis we assume, for each $S \subseteq k$, that $h_1(\bar{b}, S) \approx h_0(\bar{a}, S)$ and $v_1(\bar{b}, S) \approx v_0(\bar{a}, S)$. Initially, when $k = 0$, the only subset of k is $\emptyset$ and we have $h_1((\,), \emptyset) = \omega_1 \approx \omega = h_0((\,), \emptyset)$ and $v_1((\,), \emptyset) = \omega = v_0((\,), \emptyset)$.

In round k, suppose $\forall$ picks $b_k \in \mathfrak{A}_1$. The subsets of $k+1$ are $\{S \cup \{k\} \mid S \subseteq k\} \cup \{S \mid S \subseteq k\}$. For any $S \subseteq k$, since $h_0((a_0, \ldots, a_{k-1}), S) \approx h_1((b_0, \ldots, b_{k-1}), S)$, by (7.4) there is $X_S \subseteq h_0((a_0, \ldots, a_{k-1}), S)$ such that

$$\begin{aligned} X_S &\approx h_1((b_0, \ldots, b_k), S \cup \{k\}), \\ h_0((a_0, \ldots, a_{k-1}), S) \setminus X_S &\approx h_1((b_0, \ldots, b_k), S). \end{aligned} \tag{7.5}$$

Similarly there is $Y_S \subseteq v_0((a_0, \ldots, a_{k-1}), S)$ such that $Y_S \approx v_1((b_0, \ldots, b_k), S \cup \{k\})$ and $v_0((a_0, \ldots, a_{k-1}), S) \setminus Y_S \approx v_1((b_0, \ldots, b_k), S)$. Player $\exists$ lets a_k be the element of $\mathfrak{A}_0$ represented by $(\bigcup_{S \subseteq k} X_S) \times (\bigcup_{S \subseteq k} Y_S)$, which is an axial set since b_k is. In fact more is true: because b_k is the representative of its equivalence class, $(\bigcup_{S \subseteq k} X_S) \times (\bigcup_{S \subseteq k} Y_S)$ will be the representative of *its* equivalence class, so $h_0(a_k) = \bigcup_{S \subseteq k} X_S$ and $v_0(a_k) = \bigcup_{S \subseteq k} Y_S$. Then it follows that $h_0((a_0, \ldots, a_k), S \cup \{k\}) = X_S$ and $h_0((a_0, \ldots, a_k), S) = h_0((a_0, \ldots, a_{k-1}), S) \setminus X_S$ and similar identities hold for the vertical components. Hence, by (7.5), the induction hypothesis is maintained. Similarly if $\forall$ picks $a_k \in \mathfrak{A}_0$, we know $\exists$ can find $b_k \in \mathfrak{A}_1$ so as to maintain the induction hypothesis.

We claim the induction hypothesis ensures $\exists$ will not lose the play. To prove that $\exists$ does not lose, we must prove that $\{(a_i, b_i) \mid i < k\}$ is a partial isomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_0$ for every k. That is, we must prove for any $i, j, l < k$ that

1. $b_i = 0 \iff a_i = 0$,
2. $b_i = b_j \iff a_i = a_j$,
3. $J(b_i, b_j, b_l) \iff J(a_i, a_j, a_l)$.

Conditions (1) and (2) follow immediately from the induction hypothesis.

Given that (1) and (2) hold, it follows that (3) also holds whenever $0 \in \{b_i, b_j\}$. To prove (3) for the remaining cases, we assume $J(b_i, b_j, b_l)$ holds, where $0 \notin \{b_i, b_j\}$ and distinguish three cases: $b_l = 1$, $b_l = \overline{(i', j')}$ (for some $i' \in \omega_1$, $j' \in \omega$) and $b_l \notin \{1\} \cup \{\overline{(i', j')} \mid i' \in \omega_1,\ j' \in \omega\}$.

For $b_l = 1$ we have $h_1(b_l) = \{0\}$, $v_1(b_l) = \omega$, and either $h_1(b_i) = h_1(b_j)$ is a singleton and $v_1(b_i) \dot\cup v_1(b_j) = \omega$, or $v_1(b_i) = v_1(b_j)$ is a singleton and $h_1(b_i) \dot\cup h_1(b_j) = \omega_1$. The induction hypothesis shows that a similar condition holds for the vertical and horizontal components of a_i, a_j, a_l, hence $J(a_i, a_j, a_l)$ also holds.

For $b_l = \overline{(i', j')}$ we have $h_1(b_l) = \{i'\},\ v_1(b_l) = \omega \setminus \{j'\}$, and either $h_1(b_i) = h_1(b_j) = \{i'\}$ and $v_1(b_i) \overset{\bullet}{\cup} v_1(b_j) = \omega \setminus \{j'\}$, or $v_1(b_i) = v_1(b_j) = \{j'\}$ and $h_1(b_i) \overset{\bullet}{\cup} h_1(b_j) = \omega_1 \setminus \{i'\}$. Again, the induction hypothesis implies that a similar condition holds for the vertical and horizontal components of a_i, a_j, a_l, hence $J(a_i, a_j, a_l)$ holds.

When $b_l \notin \{1\} \cup \{\overline{(i', j')} \mid i' < \omega_1,\ j' < \omega\}$ (still with $0 \notin \{b_i, b_j\}$) then either $h_1(b_i) = h_1(b_j) = h_1(b_l)$ is a singleton and $v_1(b_i) \overset{\bullet}{\cup} v_1(b_j) = v_1(b_l)$, or a similar case, with h_1 and v_1 swapped. As before, an equivalent property holds on a_i, a_j, a_l and $J(a_i, a_j, a_l)$ follows. This completes the argument that the implication $J(b_i, b_j, b_l) \implies J(a_i, a_j, a_l)$ is valid. The implication $J(a_i, a_j, a_l) \implies J(b_i, b_j, b_l)$ is similar.

As $\exists$ can win all ω rounds of the play, the two structures $\mathfrak{A}_1$ and $\mathfrak{A}_0$ are elementarily equivalent. Hence the $\lesssim$-completely $(\overset{\bullet}{\cup}, \emptyset)$-representable partial algebras are not closed under elementary equivalence. □

Corollary 7.5.8. *Let σ be any one of the signatures $(\overset{\bullet}{\cup})$, $(\overset{\bullet}{\smile})$, $(\overset{\bullet}{\cup}, \emptyset)$, $(\overset{\bullet}{\smile}, \emptyset)$, $(\overset{\bullet}{\smile}, |)$, or $(\overset{\bullet}{\smile}, |, \emptyset)$. The class of $\lesssim$-completely σ-representable partial algebras is not closed under elementary equivalence.*

Proof. The case $\sigma = (\overset{\bullet}{\cup}, \emptyset)$ is Theorem 7.5.7. For the case $\sigma = (\overset{\bullet}{\smile}, \emptyset)$, note that the proof used in Proposition 7.2.9(1) of the equivalence of representability by sets and by partial functions extends to $\lesssim$-complete representability. Hence the $\lesssim$-complete representation classes for $(\overset{\bullet}{\cup}, \emptyset)$ and $(\overset{\bullet}{\smile}, \emptyset)$ coincide.

For the case $\sigma = (\overset{\bullet}{\smile}, |, \emptyset)$, let $\mathfrak{A}_1, \mathfrak{A}_0$ be as defined in Theorem 7.5.7. Expand $\mathfrak{A}_1$ and $\mathfrak{A}_0$ by adding a binary operation ; defined by $a \mathbin{;} b = 0$. It is clear that the two expansions are still elementarily equivalent, since we have given the same first-order definition of ; for both. The expansion of $\mathfrak{A}_1$ does not have a $\lesssim$-complete $(\overset{\bullet}{\smile}, |, \emptyset)$-representation as $\mathfrak{A}_1$ itself is not completely representable. The expansion of $\mathfrak{A}_0$ does have a $\lesssim$-complete $(\overset{\bullet}{\smile}, |, \emptyset)$-representation, which we can easily see via the same method employed in the proof of Proposition 7.2.9(2).

The results for signatures not including $\emptyset$ again follow straightforwardly from those for the corresponding signatures with $\emptyset$. For a signature with $\emptyset$, take any elementarily equivalent $\mathfrak{A}_1, \mathfrak{A}_2$ with $\mathfrak{A}_1$ $\lesssim$-completely representable and $\mathfrak{A}_2$ not. Let $\mathfrak{B}_1, \mathfrak{B}_2$ be the reducts of $\mathfrak{A}_1, \mathfrak{A}_2$ to the signature without 0. Then $\mathfrak{B}_1$ is $\lesssim$-completely representable since $\mathfrak{A}_1$ is. As $\mathfrak{A}_1$ is representable, it satisfies $J(0, 0, 0)$, so $\mathfrak{A}_2$ does too, by elementary equivalence. Now note that the content of Remark 7.2.10 applies to $\lesssim$-complete representability just as it does to representability. Hence if $\mathfrak{B}_2$ were $\lesssim$-completely representable then $\mathfrak{A}_2$ would have to be—a contradiction. Hence $\mathfrak{B}_2$ is not $\lesssim$-completely representable. So for the signature without $\emptyset$ we have elementarily equivalent $\mathfrak{B}_1, \mathfrak{B}_2$ with the first $\lesssim$-completely representable and the second not. □

Finally we prove that all the negative results concerning representability for signatures containing $\overset{\bullet}{\cup}$ carry over to signatures containing $\overset{\bullet}{\setminus}$. First note that if a partial algebra $\mathfrak{A} = (A, \overset{\bullet}{\sqcup}, \overset{\bullet}{-})$ has a $(\overset{\bullet}{\cup}, \overset{\bullet}{\setminus})$-representation then it validates

$$a \overset{\bullet}{-} b = c \iff b \overset{\bullet}{\sqcup} c = a. \tag{7.6}$$

However, as we see in the following example there exist partial $(\dot\sqcup, \dot{-})$-algebras validating (7.6), whose $\dot\sqcup$-reduct is $\dot\cup$-representable but whose $\dot{-}$-reduct has no $\dot\setminus$-representation. Similarly there exist partial $(\dot\sqcup, \dot{-})$-algebras validating (7.6), whose $\dot{-}$-reduct is $\dot\setminus$-representable but whose $\dot\sqcup$-reduct has no $\dot\cup$-representation.

Example 7.5.9. Our first partial algebra can be quite simple: a partial algebra consisting of a single element a, with $a \dot\sqcup a$ and $a \dot{-} a$ both undefined. It validates (7.6) and is $\dot\cup$-representable but not $\dot\setminus$-representable. Moreover, we give an example of a partial algebra containing a zero element. The domain is $\wp\{1,2,3\} \setminus \{3\}$ and we define $\dot\sqcup$ as $\dot\cup$ and then define $\dot{-}$ using (7.6). The identity map is a $\dot\cup$-representation of the $\dot\sqcup$-reduct (in fact, a $(\dot\cup, \emptyset)$-representation of the $(\dot\sqcup, 0)$-reduct). Suppose θ is a $\dot\setminus$-representation of the $\dot{-}$-reduct. We show that $\{1\}^\theta \subseteq \{1,3\}^\theta$, which is a contradiction as $\{1,3\} \dot{-} \{1\}$ is undefined. Let $x \in \{1\}^\theta$. Then $x \in \{1,2,3\}^\theta$, since $\{1,2,3\} \dot{-} \{1\}$ is defined. As $\{1,2\} \dot{-} \{1\} = \{2\}$ and $x \in \{1\}^\theta$ we cannot have $x \in \{2\}$. From $\{1,2,3\} \dot{-} \{2\} = \{1,3\}$ we deduce that $x \in \{1,3\}^\theta$.

Similarly, if we take a partial algebra with domain $\wp\{1,2,3\} \setminus \{1,2,3\}$, define $\dot{-}$ as $\dot\setminus$ and define $\dot\sqcup$ using (7.6), the identity map represents the $\dot{-}$-reduct, but the $\dot\sqcup$-reduct of the partial algebra has no $\dot\cup$-representation. To see this, note that, since $\{1,3\} = \{1\} \dot\sqcup \{3\}$, in any $\dot\cup$-representation $\{1\}$ and $\{3\}$ would have to be represented by disjoint sets. By similar arguments, $\{1\}, \{2\}$, and $\{3\}$ would have to be represented by pairwise-disjoint sets, contradicting the fact that $\{1\} \dot\sqcup \{2\} \dot\sqcup \{3\}$ is undefined.

Notwithstanding Example 7.5.9 there is a simple condition that ensures a $\dot\cup$-representation is always a $\dot\setminus$-representation, and vice versa.

Definition 7.5.10. A partial algebra $\mathfrak{A} = (A, \dot\sqcup, \dot{-}, \dots)$ is **complemented** if it validates (7.6) and there is a unique $1 \in \mathfrak{A}$ such that $1 \dot{-} a$ is defined for all $a \in \mathfrak{A}$. We write $\overline{a}$ for $1 \dot{-} a$.

Observe by (7.6) that

$$a \dot\sqcup \overline{a} = 1. \tag{7.7}$$

Hence in a complemented partial algebra $\mathfrak{A}$, if σ is a signature containing either $\dot\cup$ or $\dot\setminus$ and θ is a σ-representation of $\mathfrak{A}$ then

$$\overline{a}^\theta = \overline{a^\theta}, \tag{7.8}$$

where $\overline{Y} = 1^\theta \dot\setminus Y$ for any $Y \subseteq 1^\theta$.

Before we articulate the consequences of a partial algebra being complemented, we describe a $\dot\setminus$-analogue of $\lesssim$-completeness. In any partial $(\dots, \dot{-}, \dots)$-algebra $\mathfrak{A}$, define a relation $\lesssim'$ by letting $a \lesssim' b$ if and only if $a = b$ or $b \dot{-} a$ is defined. If $\mathfrak{A}$ is $(\dots, \dot\setminus, \dots)$-representable then it is clear that $\lesssim'$ is a partial order. For $(\dot\sqcup, \dot{-})$ structures validating (7.6), observe that $\lesssim' = \lesssim$.

Definition 7.5.11. A subset S of a partial $(\dots, \dot{-}, \dots)$-algebra $\mathfrak{A}$ is **$\dot{-}$-pairwise combinable** if for all distinct $s, t \in S$ there exists $u \in \mathfrak{A}$ such that $u \dot{-} s = t$. As in Definition 7.2.11 we may define a $(\dots, \dot\setminus, \dots)$-representation to be **$\lesssim'$-complete** if it maps $\lesssim'$-suprema of $\dot{-}$-pairwise-combinable sets to (necessarily disjoint) unions.

Lemma 7.5.12. *Let $\mathfrak{A} = (A, \dot{\sqcup}, \dot{-}, \dots)$ be complemented and let θ be a map from $\mathfrak{A}$ to a subset of $\wp(X)$ (for some X). Then θ is a $\dot{\cup}$-representation (of the $\dot{\sqcup}$-reduct) if and only if it is a $\dot{\setminus}$-representation (of the $\dot{-}$-reduct). Moreover, if θ is a representation it is $\lesssim$-complete if and only if it is $\lesssim'$-complete.*

Proof. Suppose $\mathfrak{A}$ is complemented and let θ be a $\dot{\setminus}$-representation. For any $a, b \in \mathfrak{A}$, if $a \dot{\sqcup} b$ is defined then by (7.6) we know that $(a \dot{\sqcup} b) \dot{-} a = b$, which, by our hypothesis about θ, implies that a^θ is disjoint from b^θ, so $a^\theta \dot{\cup} b^\theta$ is defined. We now show that, conversely, if $a^\theta \dot{\cup} b^\theta$ is defined then $a \dot{\sqcup} b$ is defined and $(a \dot{\sqcup} b)^\theta = a^\theta \dot{\cup} b^\theta$. Using equations to mean both sides are defined and equal, assuming $a^\theta \dot{\cup} b^\theta$ is defined, we have

$$
\begin{aligned}
a^\theta \cap b^\theta &= \emptyset && \text{by the definition of } \dot{\cup} \\
\overline{a}^\theta &\supseteq b^\theta && \text{as } \overline{a}^\theta = \overline{a^\theta} \text{ and } b^\theta \subseteq 1^\theta \\
\overline{a} \dot{-} b &\text{ is defined} && \text{as } \theta \text{ is a } \dot{\setminus}\text{-representation} \\
\overline{\overline{a} \dot{-} b} &\text{ is defined} && \text{as } \mathfrak{A} \text{ is complemented} \\
\overline{(\overline{a} \dot{-} b)}^{\,\theta} &= \overline{\overline{a^\theta} \dot{\setminus} b^\theta} && \text{as } \theta \text{ is a } \dot{\setminus}\text{-representation and by (7.8)} \\
&= a^\theta \dot{\cup} b^\theta && \text{by elementary set theory} \\
\overline{\overline{a} \dot{-} b} \dot{-} a &= b && \text{as } \theta \text{ is a } \dot{\setminus}\text{-representation} \\
a \dot{\sqcup} b &= \overline{\overline{a} \dot{-} b} && \text{by (7.6)} \\
(a \dot{\sqcup} b)^\theta &= a^\theta \dot{\cup} b^\theta && \text{by the calculation of } \overline{(\overline{a} \dot{-} b)}^{\,\theta} \text{ above}
\end{aligned}
$$

and hence θ represents $\dot{\sqcup}$ correctly as $\dot{\cup}$.

Dually, if θ is a $\dot{\cup}$-representation and $a \dot{-} b$ is defined then we know by (7.6) that $b \dot{\sqcup} (a \dot{-} b) = a$, implying $a^\theta \dot{\setminus} b^\theta$ is defined. For the converse and for showing that when both are defined they are equal, assume $a^\theta \dot{\setminus} b^\theta$ is defined, so

$$
\begin{aligned}
a^\theta &\supseteq b^\theta && \text{by the definition of } \dot{\setminus} \\
\overline{a}^\theta \cap b^\theta &= \emptyset && \text{as } \overline{a}^\theta = \overline{a^\theta} \\
\overline{a} \dot{\sqcup} b &\text{ is defined} && \text{as } \theta \text{ is a } \dot{\cup}\text{-representation} \\
(\overline{a} \dot{\sqcup} b) \dot{\sqcup} \overline{\overline{a} \dot{\sqcup} b} &= 1 = \overline{a} \dot{\sqcup} a && \text{by (7.7)} \\
b \dot{\sqcup} \overline{\overline{a} \dot{\sqcup} b} &= a && \text{cancelling the } \overline{a}\text{'s, as } \theta \text{ is a } \dot{\cup}\text{-representation} \\
(a \dot{-} b)^\theta &= \overline{\overline{a} \dot{\sqcup} b}^{\,\theta} && \text{by (7.6)} \\
&= \overline{\overline{a^\theta} \dot{\sqcup} b^\theta} && \text{as } \theta \text{ is a } \dot{\setminus}\text{-representation and by (7.8)} \\
&= a^\theta \dot{\setminus} b^\theta && \text{by elementary set theory}
\end{aligned}
$$

and so $\dot{-}$ is correctly represented as $\dot{\setminus}$.

For the final sentence of this lemma we do not need $\mathfrak{A}$ to be complemented, only that it validates (7.6). Then the concepts 'pairwise combinable' and '$\dot{-}$-pairwise combinable' coincide and the relations $\lesssim$ and $\lesssim'$ are equal. Hence the concepts '$\lesssim$-complete' and '$\lesssim'$-complete' coincide. $\square$

Theorem 7.5.13. *Suppose $\overset{\bullet}{\setminus}$ is included in σ and all symbols in σ are from $\{\overset{\bullet}{\cup}, \overset{\bullet}{\setminus}, \emptyset\}$. The class of partial algebras σ-representable by sets is not finitely axiomatisable. The class of partial algebras $\lesssim'$-completely σ-representable by sets is not closed under elementary equivalence.*

Proof. For m, n of cardinality greater than two, let $\mathfrak{A}'(m,n)$ be the expansion of $\mathfrak{A}(m,n)$ to $(\overset{\bullet}{\sqcup}, \overset{\bullet}{-}, 0)$ where $\overset{\bullet}{-}$ is defined by (7.6). Observe that $\mathfrak{A}'(m,n)$ is complemented. Let $\mathfrak{A}_\sigma(m,n)$ be the reduct of $\mathfrak{A}'(m,n)$ to the abstract analogue of σ. By Lemma 7.5.12 (and the fact that $\mathfrak{A}'(m,n)$ satisfies $0 \overset{\bullet}{\sqcup} 0 = 0$) we see that $\mathfrak{A}_\sigma(m,n)$ is $\lesssim'$-completely σ-representable if and only if $\mathfrak{A}'(m,n)$ is $\lesssim'$-completely $(\overset{\bullet}{\sqcup}, \overset{\bullet}{\setminus}, \emptyset)$-representable, which is true if and only if $\mathfrak{A}(m,n)$ is $\lesssim$-completely $(\overset{\bullet}{\sqcup}, \emptyset)$-representable. By Lemma 7.5.3 this is the case precisely when $|m| = |n|$. So $\mathfrak{A}_\sigma(m, m+1)$ is not σ-representable for $2 < m < \omega$.

As before, write ν for $\omega \setminus \{0,1,2\}$ and let U be any non-principal ultrafilter over ν. We will argue that $\Pi_{m\in\nu}\mathfrak{A}_\sigma(m, m+1)/U$ is σ-representable. From $\Pi_{m\in\nu}\mathfrak{A}_\sigma(m, m+1)/U$, form the partial algebra $\mathfrak{B}'$ by expanding to $(\overset{\bullet}{\sqcup}, \overset{\bullet}{-}, 0)$ using (7.6) and defining 0 in the obvious way, if necessary. Then let $\mathfrak{B}$ be the $(\overset{\bullet}{\sqcup}, 0)$-reduct of $\mathfrak{B}'$. We can easily see that, $\mathfrak{B}'$ is complemented and in particular it validates (7.6). Hence $\Pi_{m\in\nu}\mathfrak{A}_\sigma(m, m+1)/U$ is σ-representable if and only if $\mathfrak{B}$ is $(\overset{\bullet}{\sqcup}, 0)$-representable. It is easy to check that $\mathfrak{B} = \Pi_{m\in\nu}\mathfrak{A}(m, m+1)/U$, which we know, by the proof of Theorem 7.5.5, is $(\overset{\bullet}{\sqcup}, 0)$-representable. Hence the ultraproduct $\Pi_{m\in\nu}\mathfrak{A}_\sigma(m, m+1)/U$ of non-σ-representable partial algebras is itself σ-representable and so the class of σ-representable partial algebras is not finitely axiomatisable.

For the second half of the theorem, we know, from the proof of Theorem 7.5.7, that $\mathfrak{A}(\omega_1, \omega) \equiv \mathfrak{A}(\omega, \omega)$, where $\equiv$ denotes elementary equivalence. Hence $\mathfrak{A}'(\omega_1, \omega) \equiv \mathfrak{A}'(\omega, \omega)$, since both expansions use the same first-order definition of $\overset{\bullet}{-}$. The elementary equivalence of the reducts $\mathfrak{A}_\sigma(\omega_1, \omega)$ and $\mathfrak{A}_\sigma(\omega, \omega)$ follows. We established earlier in this proof that $\mathfrak{A}_\sigma(\omega_1, \omega)$ is not $\lesssim'$-completely σ-representable, while $A_\sigma(\omega, \omega)$ is. Hence the class of $\lesssim'$-completely σ-representable partial algebras is not closed under elementary equivalence. □

Corollary 7.5.14. *Suppose $\overset{\bullet}{\setminus}$ is included in σ and all symbols in σ are from $\{\overset{\bullet}{\smile}, \overset{\bullet}{\setminus}, |, \emptyset\}$. The class of partial algebras σ-representable by partial functions is not finitely axiomatisable. The class of partial algebras $\lesssim'$-completely σ-representable by partial functions is not closed under elementary equivalence.*

Proof. Proposition 7.2.9(1) tells us that when all symbols are from $\{\overset{\bullet}{\smile}, \overset{\bullet}{\setminus}, \emptyset\}$, representability by partial functions is the same as representability by sets (with $\overset{\bullet}{\smile}$ in place of $\overset{\bullet}{\cup}$). The proof of Proposition 7.2.9(1) extends to equality of $\lesssim'$-complete representability by partial functions and by sets. Hence for these signatures the results are immediate corollaries of Theorem 7.5.13.

For signatures σ including both $|$ and $\emptyset$ we use the same methods as in the proofs of Corollary 7.5.6 and Corollary 7.5.8. Let σ^- be the signature formed by removing $|$ from σ. First we expand the partial algebras $\mathfrak{A}_{\sigma^-}(m, m+1)$ described in the proof of Theorem 7.5.13 to a signature including ; by defining $a \,;\, b = 0$ for all a, b. The expanded partial algebras are not representable since the $\mathfrak{A}_{\sigma^-}(m, m+1)$'s are not. The ultraproduct of the expanded partial algebras validates $a \,;\, b = 0$, by Łoś's theorem and so is representable, by the same method as in the proof of Proposition 7.2.9(2). This refutes finite axiomatisability. For $\lesssim'$-complete representability, again define ; by $a \mathbin{;} b = 0$, to expand both of the two elementarily equivalent partial algebras $\mathfrak{A}_{\sigma^-}(\omega_1, \omega)$ and $\mathfrak{A}_{\sigma^-}(\omega, \omega)$. The expansions $\mathfrak{B}_1$ and $\mathfrak{B}_0$ remain elementarily equivalent and the first is not $\lesssim'$-completely representable whilst the second is, by the same method as in the proof of Proposition 7.2.9(2).

The remaining cases are signatures including $|$ but not $\emptyset$, that is, $(\overset{\bullet}{\setminus}, |)$ and $(\overset{\bullet}{\smile}, \overset{\bullet}{\setminus}, |)$. For these the results follow from the corresponding signatures that include $\emptyset$, by the now-familiar arguments involving Remark 7.2.10 and its generalisation to $\lesssim'$-complete representability. □

7.6 Signatures including intersection

In this section we consider signatures including a total operation $\cdot$ to be represented as intersection. In contrast to the results of the previous section, the classes of partial algebras representable by sets are finitely axiomatisable. This is true for all signatures containing intersection and with other operations members of $\{\overset{\bullet}{\cup}, \overset{\bullet}{\setminus}, \emptyset\}$. In order to control the size of this chapter we do not consider representability by partial functions, only noting that the proofs in this section are not immediately adaptable to that setting.

We start with the signatures $(\overset{\bullet}{\cup}, \cap, \emptyset)$ and $(\overset{\bullet}{\cup}, \cap)$. Consider the following finite set $\mathbf{Ax}(J, \cdot, 0)$ of $\mathcal{L}(J, \cdot, 0)$ axioms.

$\overset{\bullet}{\sqcup}$ is single valued $J(a, b, c) \wedge J(a, b, c') \rightarrow c = c'$

$\overset{\bullet}{\sqcup}$ is commutative $J(a, b, c) \rightarrow J(b, a, c)$

$\cdot$-semilattice $\cdot$ is commutative, associative, and idempotent

$\cdot$ distributes over $\overset{\bullet}{\sqcup}$ $J(b, c, d) \rightarrow J(a \cdot b, a \cdot c, a \cdot d)$

0 is identity for $\overset{\bullet}{\sqcup}$ $J(a, 0, a)$

domain of $\overset{\bullet}{\sqcup}$ $\exists c J(a, b, c) \leftrightarrow a \cdot b = 0$

Let $\mathbf{Ax}(J, \cdot)$ be obtained from $\mathbf{Ax}(J, \cdot, 0)$ by replacing the axioms concerning 0 (the '0 is identity for $\overset{\bullet}{\sqcup}$' and 'domain of $\overset{\bullet}{\sqcup}$' axioms) by the following axiom stating that either there exists an element z that acts like 0, or else the partial operation $\overset{\bullet}{\sqcup}$ is nowhere defined.

$$\begin{array}{l} \exists z(\forall a J(a, z, a) \wedge \forall a, b(a \cdot b = z \leftrightarrow \exists c J(a, b, c)) \\ \vee \\ \forall a, b, c\, \neg J(a, b, c) \end{array} \tag{7.9}$$

Theorem 7.6.1. *The class of $(J, \cdot, 0)$-structures that are $(\overset{\bullet}{\cup}, \cap, \emptyset)$-representable by sets is axiomatised by $\mathbf{Ax}(J, \cdot, 0)$. The class of $(J, \cdot)$-structures that are $(\overset{\bullet}{\cup}, \cap)$-representable by sets is axiomatised by $\mathbf{Ax}(J, \cdot)$.*

Proof. We first give a quick justification for the axioms being sound in both cases. It suffices to argue that the axioms are sound for disjoint-union partial algebras of sets, with or without zero respectively.

Let $\mathfrak{A}$ be a disjoint-union partial algebra of sets with zero. We attend to each axiom of $\mathbf{Ax}(J, \cdot, 0)$ in turn.

$\dot{\sqcup}$ is single valued if $J(a,b,c)$ and $J(a,b,c')$ hold then $a \mathbin{\dot{\cup}} b$ is defined and is equal to both c and c'. Hence $c = c'$.

$\dot{\sqcup}$ is commutative $J(a,b,c)$ holds if and only if $a \cap b = \emptyset$ and $a \cup b = c$. By commutativity of intersection and union this is equivalent to the conjunction $b \cap a = \emptyset$ and $b \cup a = c$, which holds if and only if $J(b,a,c)$ holds.

$\cdot$-semilattice the easily verifiable facts that intersection is commutative, associative, and idempotent are well known.

$\cdot$ distributes over $\dot{\sqcup}$ if $J(b,c,d)$ then $b \cap c = \emptyset$, so certainly $(a \cap b) \cap (a \cap c) = \emptyset$. The other condition necessary for $J(a \cdot b, a \cdot c, a \cdot d)$ to hold is that $(a \cap b) \cup (a \cap c) = a \cap d$. The left-hand side equals $a \cap (b \cup c)$ and by our hypothesis $b \cup c = d$, so we are done.

0 is identity for $\dot{\sqcup}$ for any set a we have $a \cap \emptyset = \emptyset$ and $a \cup \emptyset = a$, which are the two conditions needed to establish $J(a, 0, a)$.

domain of $\dot{\sqcup}$ for any sets a and b there exists a set c such that $J(a,b,c)$ if and only if $a \mathbin{\dot{\cup}} b$ is defined, which is true if and only if $a \cap b = \emptyset$.

Now let $\mathfrak{A}$ be a disjoint-union partial algebra of sets *without* zero. It is clear that for all the axioms not concerning 0 the above soundness arguments still hold. To see that axiom (7.9) holds, note that if $\emptyset \in \mathfrak{A}$ then $\emptyset$ is an element z that acts like 0, as the first clause of (7.9) asks for. Alternatively, if $\emptyset \notin \mathfrak{A}$, then for any sets a and b the intersection $a \cap b$, which is an element of $\mathfrak{A}$, must be nonempty. Hence $a \mathbin{\dot{\cup}} b$ is undefined and so for any c we have $\neg J(a,b,c)$, meaning the second clause of (7.9) holds.

The sufficiency of the axioms is proved for $(J, \cdot, 0)$-structures by a modification of the proof of Birkhoff's representation theorem for distributive lattices. Assume that $\mathbf{Ax}(J, \cdot, 0)$ is valid on a $(J, \cdot, 0)$-structure $\mathfrak{A}$. By the '$\dot{\sqcup}$ is single valued' axiom we can view $\mathfrak{A}$ as a partial $(\dot{\sqcup}, \cdot, 0)$-algebra. A **filter** F is a nonempty subset of $\mathfrak{A}$ such that $a \cdot b \in F \iff (a \in F \text{ and } b \in F)$. For any nonempty subset S of $\mathfrak{A}$ let $\langle S \rangle$ be the filter generated by S, that is, $\{a \in \mathfrak{A} \mid \exists s_1, s_2, \ldots, s_n \in S \text{ (some finite } n),\ a \geq s_1 \cdot s_2 \cdot \ldots \cdot s_n\}$, where $\leq$ is the partial ordering given by the $\cdot$-semilattice.[6] A filter is **proper** if it is a proper subset of $\mathfrak{A}$. Recall that a set F is $\dot{\sqcup}$-prime if $a \mathbin{\dot{\sqcup}} b \in F$ implies either $a \in F$ or $b \in F$.

Let Φ be the set of all proper $\dot{\sqcup}$-prime filters of $\mathfrak{A}$. Define a map θ from $\mathfrak{A}$ to $\wp(\Phi)$ by letting $a^\theta = \{F \in \Phi \mid a \in F\}$. We will show that θ is a representation of $\mathfrak{A}$.

The requirement that $(a \cdot b)^\theta = a^\theta \cap b^\theta$ follows directly from the filter condition $a \cdot b \in F \iff (a \in F \text{ and } b \in F)$. It follows easily from the axioms concerning 0 that 0 is the minimal element with respect to $\leq$. Hence a filter is proper if and only if it does not contain 0. Then the requirement that $0^\theta = \emptyset$ follows directly from the condition that the filters in Φ be proper.

We next show that θ is faithful. For this we show that if $a \not\leq b$ then there is a proper $\dot{\sqcup}$-prime filter F such that $a \in F$, but $b \notin F$. The filters containing a but not b, ordered by inclusion, form an inductive

[6] There might be no 'filter generated by the empty set', that is, no smallest filter, as the intersection of two or more filters can be empty.

poset, that is, a poset in which every chain has an upper bound. (The empty chain has an upper bound, since the up-set of a is an example of a filter containing a but not b.) Hence, by Zorn's lemma, there exists a maximal such filter, F say. We claim that F is proper and $\dot{\sqcup}$-prime.

Suppose, for contradiction, that $c \mathbin{\dot{\sqcup}} d$ is defined and belongs to F, but neither $c \in F$ nor $d \in F$. By maximality of F we have $b \in \langle F \cup \{c\}\rangle$ and $b \in \langle F \cup \{d\}\rangle$. Then there is an $f \in F$ such that $f \cdot c \leq b$ and $f \cdot d \leq b$. Then by the definition of $\leq$ and the distributive axiom, $b \cdot f \cdot (c \mathbin{\dot{\sqcup}} d) = (b \cdot f \cdot c) \mathbin{\dot{\sqcup}} (b \cdot f \cdot d) = (f \cdot c) \mathbin{\dot{\sqcup}} (f \cdot d) = f \cdot (c \mathbin{\dot{\sqcup}} d)$. Hence $b \geq f \cdot (c \mathbin{\dot{\sqcup}} d)$, and since both f and $c \mathbin{\dot{\sqcup}} d$ are in F we get that b should be too—a contradiction. Thus either $c \in F$ or $d \in F$. We conclude that F satisfies the $\dot{\sqcup}$-prime condition. Clearly F is proper, as $b \notin F$. Hence F is a proper and $\dot{\sqcup}$-prime filter and so θ is faithful.

To complete the proof that θ is a representation we show that $\dot{\sqcup}$ is correctly represented as $\dot{\cup}$. That is, $a \mathbin{\dot{\sqcup}} b$ is defined if and only if $a^\theta \mathbin{\dot{\cup}} b^\theta$ is defined, and when they are defined $(a \mathbin{\dot{\sqcup}} b)^\theta = a^\theta \mathbin{\dot{\cup}} b^\theta$. We have that

$$\begin{aligned}
a \mathbin{\dot{\sqcup}} b \text{ is defined} &\iff a \cdot b = 0 && \text{by the domain of } \dot{\sqcup} \text{ axiom}\\
&\iff (a \cdot b)^\theta = 0^\theta && \text{as } \theta \text{ is faithful}\\
&\iff a^\theta \cap b^\theta = \emptyset && \text{as } 0 \text{ and } \cdot \text{ are represented correctly}\\
&\iff a^\theta \mathbin{\dot{\cup}} b^\theta \text{ is defined} && \text{by the definition of } \dot{\cup}.
\end{aligned}$$

Further, when both $a \mathbin{\dot{\sqcup}} b$ and $a^\theta \mathbin{\dot{\cup}} b^\theta$ are defined it follows easily from $\mathbf{Ax}(J, \cdot, \emptyset)$ that $a = a \cdot (a \mathbin{\dot{\sqcup}} b)$. So if a is in a filter then by the filter condition $a \mathbin{\dot{\sqcup}} b$ is too. Hence $a^\theta \subseteq (a \mathbin{\dot{\sqcup}} b)^\theta$, and similarly $b^\theta \subseteq (a \mathbin{\dot{\sqcup}} b)^\theta$, giving us $a^\theta \mathbin{\dot{\cup}} b^\theta \subseteq (a \mathbin{\dot{\sqcup}} b)^\theta$. By the $\dot{\sqcup}$-prime condition on filters we get the reverse inclusion $(a \mathbin{\dot{\sqcup}} b)^\theta \subseteq a^\theta \mathbin{\dot{\cup}} b^\theta$. Hence $(a \mathbin{\dot{\sqcup}} b)^\theta = a^\theta \mathbin{\dot{\cup}} b^\theta$.

For a $(J, \cdot)$-structure $\mathfrak{A}$, if $\mathbf{Ax}(J, \cdot)$ is valid in $\mathfrak{A}$ then (7.9) holds. If the first alternative of (7.9) holds then we may form an expansion of $\mathfrak{A}$ to a $(J, \cdot, 0)$-structure, interpreting 0 as the z given by this clause. Then by the above proof for $(J, \cdot, 0)$-structures we can find a $(\dot{\cup}, \cap, \emptyset)$-representation of the expansion. By ignoring the constant 0 we obtain a $(\dot{\cup}, \cap)$-representation of $\mathfrak{A}$.

Otherwise, the second alternative in (7.9) is true and $J(a, b, c)$ never holds, so we may define a representation θ of $\mathfrak{A}$ by letting $a^\theta = \{b \in \mathfrak{A} \mid b \leq a\}$. Clearly θ represents $\cdot$ as $\cap$ correctly, by the $\cdot$-semilattice axioms. Since $a \cdot b \in a^\theta \cap b^\theta$ for any $a, b \in \mathfrak{A}$ and $a \mathbin{\dot{\sqcup}} b$ is never defined, θ also represents $\dot{\sqcup}$ as $\dot{\cup}$ correctly. □

From the previous theorem we can easily obtain finite axiomatisations for the signatures $(\dot{\cup}, \dot{\setminus}, \cap, \emptyset)$ and $(\dot{\cup}, \dot{\setminus}, \cap)$. Recall that we use the ternary relation K to make first-order statements about the partial binary operation $\dot{-}$.

Corollary 7.6.2. *The class of $(J, K, \cdot, 0)$-structures that are $(\dot{\cup}, \dot{\setminus}, \cap, \emptyset)$-representable by sets is finitely axiomatisable. The class of $(J, K, \cdot)$-structures that are $(\dot{\cup}, \dot{\setminus}, \cap)$-representable by sets is finitely axiomatisable.*

Proof. To $\mathbf{Ax}(J, \cdot, 0)$ and $\mathbf{Ax}(J, \cdot)$ add the formulas $a \cdot b = b \rightarrow \exists c K(a, b, c)$ and the relational form of (7.6) (that is, $K(a, b, c) \leftrightarrow J(b, c, a)$), which are valid on the representable partial algebras. Then when these axiomatisations hold, the representations in the proof of Theorem 7.6.1 will correctly represent $\dot{-}$ as $\dot{\setminus}$. □

We claimed finite representability for all signatures containing intersection and with other operations coming from $\{\dot{\cup}, \dot{\setminus}, \emptyset\}$. For the signatures $(\cap)$ and $(\cap, \emptyset)$ finite axiomatisability is easy and well known. So the signatures remaining to be examined are $(\dot{\setminus}, \cap, \emptyset)$ and $(\dot{\setminus}, \cap)$.[7] Our treatment is very similar to the cases $(\dot{\cup}, \cap, \emptyset)$ and $(\dot{\cup}, \cap)$—no new ideas are needed—but we provide the details anyway. Consider the following finite set $\mathbf{Ax}(K, \cdot, 0)$ of $\mathcal{L}(K, \cdot, 0)$ axioms.

$\dot{-}$ **is single valued** $K(a, b, c) \wedge K(a, b, c') \to c = c'$

$\dot{-}$ **is left injective** $K(a, b, c) \wedge K(a', b, c) \to a = a'$

$\dot{-}$ **is subtractive** $K(a, b, c) \leftrightarrow K(a, c, b)$

$\cdot$**-semilattice** $\cdot$ is commutative, associative, and idempotent

$\cdot$ **distributes over** $\dot{-}$ $K(b, c, d) \to K(a \cdot b, a \cdot c, a \cdot d)$

0 **is identity for** $\dot{-}$ $K(a, 0, a)$

domain of $\dot{-}$ $\exists c K(a, b, c) \leftrightarrow a \cdot b = b$

Let $\mathbf{Ax}(K, \cdot)$ be obtained from $\mathbf{Ax}(K, \cdot, 0)$ by replacing the '0 is identity for $\dot{-}$' axiom by the axiom

$$\exists z \forall a K(a, z, a) \tag{7.10}$$

stating that there exists an element z that acts like 0.

Theorem 7.6.3. *The class of* $(K, \cdot, 0)$*-structures that are* $(\dot{\setminus}, \cap, \emptyset)$*-representable by sets is axiomatised by* $\mathbf{Ax}(K, \cdot, 0)$*. The class of* $(K, \cdot)$*-structures that are* $(\dot{\setminus}, \cap)$*-representable by sets is axiomatised by* $\mathbf{Ax}(K, \cdot)$*.*

Proof. Again we give a quick justification for the soundness of the axioms. It suffices to argue that the axioms are sound for partial $(\dot{\setminus}, \cap, \emptyset)$-algebras of sets and for partial $(\dot{\setminus}, \cap)$-algebras of sets respectively.

Let $\mathfrak{A}$ be a partial $(\dot{\setminus}, \cap, \emptyset)$-algebra of sets. We attend to each axiom of $\mathbf{Ax}(K, \cdot, 0)$ in turn.

$\dot{-}$ **is single valued** if $K(a, b, c)$ and $K(a, b, c')$ hold then $a \dot{\setminus} b$ is defined and is equal to both c and c'. Hence $c = c'$.

$\dot{-}$ **is left injective** re-write the axiom with the predicate J, using (7.6), then it becomes '$\dot{\sqcup}$ is single valued', which we verified in Theorem 7.6.1.

$\dot{-}$ **is subtractive** re-write with J, then it becomes '$\dot{\sqcup}$ is commutative'.

$\cdot$**-semilattice** as in proof of Theorem 7.6.1.

[7] As an aside, note these are signatures for which representability by sets and by partial functions are easily seen to be the same thing.

$\cdot$ **distributes over** $\stackrel{\bullet}{-}$ re-write with J, then it becomes '$\cdot$ distributes over $\stackrel{\bullet}{\sqcup}$'.

0 **is identity for** $\stackrel{\bullet}{-}$ clear.

domain of $\stackrel{\bullet}{-}$ for any sets a and b there exists a set c such that $K(a, b, c)$ if and only if $a \stackrel{\bullet}{\setminus} b$ is defined, which is true if and only if $b \subseteq a$, true if and only if $a \cap b = b$.

Now let $\mathfrak{A}$ be a partial $(\stackrel{\bullet}{\setminus}, \cap)$-algebra. It is clear that for all the axioms not concerning 0 the above soundness arguments still hold. To see that axiom (7.10) holds we can take any $a \in \mathfrak{A}$ and find that $a \stackrel{\bullet}{\setminus} a$ is defined and hence its value, $\emptyset$, is a member of $\mathfrak{A}$ and witnesses the existence of a z such that $\forall a K(a, z, a)$.

To prove the sufficiency of the axioms for $(K, \cdot, 0)$-structures we use the same method employed in the proof of Theorem 7.6.1. The definitions of the ordering $\leq$, of filters, and of proper filters remain the same. This time however, we define a filter to be $\stackrel{\bullet}{-}$**-prime** if $a \in F$ and $\exists a \stackrel{\bullet}{-} b$ together imply either $b \in F$ or $a \stackrel{\bullet}{-} b \in F$.

Similarly to before, Φ is the set of all proper $\stackrel{\bullet}{-}$-prime filters of $\mathfrak{A}$ and our representation will be the map θ from $\mathfrak{A}$ to $\wp(\Phi)$ defined by $a^\theta = \{F \in \Phi \mid a \in F\}$. That $\cdot$ is correctly represented as intersection is again immediate from the (unchanged) definition of a filter. It follows from the '0 is identity for $\stackrel{\bullet}{-}$' and 'domain of $\stackrel{\bullet}{-}$' axioms that once again a filter is proper if and only if it does not contain 0. Hence 0 is represented correctly as the empty set.

To show that θ is faithful, given $a \not\leq b$, as before, we can find a maximal filter F containing a but not b and we show F is proper and $\stackrel{\bullet}{-}$-prime.

Suppose, for contradiction, that $c \in F$ and $c \stackrel{\bullet}{-} d$ is defined, but neither $d \in F$ nor $c \stackrel{\bullet}{-} d \in F$. By maximality of F we have $b \in \langle F \cup \{d\} \rangle$ and $b \in \langle F \cup \{c \stackrel{\bullet}{-} d\} \rangle$. So there is an $f \in F$ such that $f \cdot d \leq b$ and $f \cdot (c \stackrel{\bullet}{-} d) \leq b$. Then by the definition of $\leq$ and the distributive axiom, $b \cdot f \cdot c \stackrel{\bullet}{-} f \cdot d = b \cdot f \cdot c \stackrel{\bullet}{-} b \cdot f \cdot d = b \cdot f \cdot (c \stackrel{\bullet}{-} d) = f \cdot (c \stackrel{\bullet}{-} d) = f \cdot c \stackrel{\bullet}{-} f \cdot d$. Then by left-injectivity of $\stackrel{\bullet}{-}$ we obtain $b \cdot f \cdot c = f \cdot c$, that is, $b \geq f \cdot c$. Since both f and c are in F we see that b should be too—a contradiction. Thus either $d \in F$ or $c \stackrel{\bullet}{-} d \in F$. We conclude that F satisfies the $\stackrel{\bullet}{-}$-prime condition. Clearly F is proper, as $b \notin F$. Hence F is a proper $\stackrel{\bullet}{-}$-prime filter and so θ is faithful.

Finally, we show that $\stackrel{\bullet}{-}$ is correctly represented as $\stackrel{\bullet}{\setminus}$. We have that

$$\begin{aligned}
a \stackrel{\bullet}{-} b \text{ is defined} &\iff a \cdot b = b && \text{by the domain of } \stackrel{\bullet}{-} \text{ axiom}\\
&\iff (a \cdot b)^\theta = b^\theta && \text{as } \theta \text{ is faithful}\\
&\iff b^\theta \subseteq a^\theta && \text{as } \cdot \text{ is represented correctly}\\
&\iff a^\theta \stackrel{\bullet}{\setminus} b^\theta \text{ is defined} && \text{by the definition of } \stackrel{\bullet}{\setminus}.
\end{aligned}$$

Further, when both $a \stackrel{\bullet}{-} b$ and $a^\theta \stackrel{\bullet}{\setminus} b^\theta$ are defined it follows easily from $\mathbf{Ax}(K, \cdot, \emptyset)$ that $a \stackrel{\bullet}{-} b = a \cdot (a \stackrel{\bullet}{-} b)$. So if $a \stackrel{\bullet}{-} b$ is in a filter then by the filter condition a is too. Hence $(a \stackrel{\bullet}{-} b)^\theta \subseteq a^\theta$. Similarly, it is easy to show that $(a \stackrel{\bullet}{-} b) \cdot b = 0$, so if $a \stackrel{\bullet}{-} b$ is in a proper filter then b is not. Hence $(a \stackrel{\bullet}{-} b)^\theta \cap b^\theta = \emptyset$, giving us $(a \stackrel{\bullet}{-} b)^\theta \subseteq a^\theta \stackrel{\bullet}{\setminus} b^\theta$. By the $\stackrel{\bullet}{-}$-prime condition on filters we get the reverse inclusion $(a \stackrel{\bullet}{-} b)^\theta \supseteq a^\theta \stackrel{\bullet}{\setminus} b^\theta$. Hence $(a \stackrel{\bullet}{-} b)^\theta = a^\theta \stackrel{\bullet}{\setminus} b^\theta$.

For a $(K, \cdot)$-structure $\mathfrak{A}$, if $\mathbf{Ax}(K, \cdot)$ is valid in $\mathfrak{A}$ then (7.10) holds. Then we may form an expansion of $\mathfrak{A}$ to a $(K, \cdot, 0)$-structure, interpreting 0 as the z given by this formula. Then by the above proof for $(K, \cdot, 0)$-structures we can find a $(\dot{\setminus}, \cap, \emptyset)$-representation of the expansion. By ignoring the constant 0 we obtain a $(\dot{\setminus}, \cap)$ representation of $\mathfrak{A}$. □

7.7 Decidability and complexity

We finish with a discussion of the decidability and complexity of problems of representability and validity. We also highlight some still-open questions.

Theorem 7.7.1. *The problem of determining whether a finite partial $\dot{\sqcup}$-algebra has a disjoint-union representation is in* NP.

Proof. Given a finite partial $\dot{\sqcup}$-algebra $\mathfrak{A} = (A, \dot{\sqcup})$, a non-deterministic polynomial-time algorithm based on the proof of Lemma 7.4.3 runs as follows. For each distinct pair $a \neq b$ it creates a set $S_{a,b}$ and for each pair a, b where $a \dot{\sqcup} b$ is undefined it creates a set $T_{a,b}$ (all these sets are initially empty). Then for each $c \in \mathfrak{A}$, each set $S_{a,b}$ and each set $T_{a,b}$ it guesses whether $c \in S_{a,b}$ and whether $c \in T_{a,b}$. Once this is done, the algorithm then verifies that exactly one of a and b belongs to $S_{a,b}$, that both a and b belong to $T_{a,b}$ and that each of these sets is a $\dot{\sqcup}$-prime, bi-closed, pairwise-incombinable set (to verify this for any single set takes quadratic time, in terms of the size of the input $(A, \dot{\sqcup})$). This takes quartic time. By Lemma 7.4.3 this non-deterministic algorithm solves the problem. □

Problem 7.7.2. Is the problem of determining whether a finite partial $\dot{\sqcup}$-algebra has a disjoint-union representation NP-complete?

For signatures including intersection, a polynomial time bound follows immediately (by Theorem 2.4.9), from our finite axiomatisability results of the previous section.

Corollary 7.7.3 (of Theorem 7.6.1, Corollary 7.6.2, and Theorem 7.6.3)**.** *Let σ be any one of the signatures $(\dot{\cup}, \cap, \emptyset)$, $(\dot{\cup}, \cap)$, $(\dot{\setminus}, \cap, \emptyset)$, $(\dot{\setminus}, \cap)$, $(\dot{\cup}, \dot{\setminus}, \cap)$, or $(\dot{\cup}, \dot{\setminus}, \cap, \emptyset)$. The problem of determining whether a finite partial algebra has a σ-representation by sets can be solved in polynomial time.*

Now turning our attention to validity, let $s(\bar{a}), t(\bar{a})$ be terms built from variables in $\bar{a}$ and the constant 0, using $\dot{\sqcup}$. We take the view that the equation $s(\bar{a}) = t(\bar{a})$ is valid if for every disjoint-union partial algebra of sets with zero, $\mathfrak{A}$, and every assignment of the variables in $\bar{a}$ to sets in $\mathfrak{A}$, either both $s(\bar{a})$ and $t(\bar{a})$ are undefined or they are both defined and are equal. The following result is rather trivial but worth noting. It contrasts with Theorem 7.5.5 by showing that the equational fragment of the first-order theory of partial $(\dot{\cup}, \emptyset)$-algebras is a rather simple object.

Theorem 7.7.4. *The validity problem for $(\dot{\sqcup}, 0)$-equations can be solved in polynomial time.*

Proof. A $(\dot{\sqcup}, 0)$-term is formed from variables and 0, using $\dot{\sqcup}$. Now $\dot{\cup}$ is associative in the sense that either both sides of $(a \dot{\cup} b) \dot{\cup} c = a \dot{\cup} (b \dot{\cup} c)$ are defined and equal, or neither is defined. In the same sense, $\dot{\cup}$ is also commutative. Hence in representable partial $(\dot{\sqcup}, 0)$-algebras, the bracketing and order of variables in a term does not affect whether a term is defined, under a given variable assignment, or the value it denotes when it is defined. Similarly, any zeros occurring in a term may be deleted from the term without altering its denotation. If a variable a occurs more than once in a term then the term can only be defined if a is assigned the value 0. Hence an equation $s(\bar{a}) = t(\bar{a})$ is valid if and only if

(a) the set of variables occurring in $s(\bar{a})$ is the same as the set of variables occurring in $t(\bar{a})$,

(b) the set of variables occurring more than once in $s(\bar{a})$ is the same as the set of variables occurring more than once in $t(\bar{a})$.

This can be tested in polynomial time. □

Problem 7.7.5. Consider the set Σ of all first-order $\mathcal{L}(J)$-formulas satisfiable over some disjoint-union partial algebras of sets. Is this language decidable and if so, what is its complexity?

We have seen that the class of partial algebras with σ-representations by sets is not finitely axiomatisable, provided either $\overset{\bullet}{\cup}$ or $\overset{\bullet}{\setminus}$ is in σ and all symbols in σ are from $\{\overset{\bullet}{\cup}, \overset{\bullet}{\setminus}, \emptyset\}$, and the same negative result holds for representations by partial functions (with $\overset{\bullet}{\smile}$ in place of $\overset{\bullet}{\cup}$). However, when intersection is added to these signatures the representation classes *are* finitely axiomatisable by sets. This leaves some cases in question, with regard to finite axiomatisability.

Problem 7.7.6. Determine whether the class of partial algebras σ-representable by partial functions is finitely axiomatisable for signatures σ containing $\cap$ and either $\overset{\bullet}{\smile}$ or $\overset{\bullet}{\setminus}$, where symbols in σ are from $\{\overset{\bullet}{\smile}, \overset{\bullet}{\setminus}, \cap, |, \emptyset\}$.

Chapter 8

Conclusion

In this short final chapter, we give an overall assessment of the logic of partial functions, as currently understood, and make some suggestions for future research.

First, we reiterate what was said in the introduction to this thesis: that partial functions have, in general, more favourable logical and computational properties than binary relations. The results in this thesis only reinforce this viewpoint. Consider those operations with a first-order definition—by which we mean definable in the manner required by the fundamental theorem, Theorem 3.1.6. It had already been established that when considering these types of operations, generally the representation classes are finitely axiomatisable and have equational theories of low complexity, the finite representation property is satisfied, and representability of finite algebras is simple to decide. And it had been found that these remarks extend to multiplace functions as well. This is all in contrast to how relations behave.

We have added to the signatures for which the finite representation property is known, in Chapter 5, including a signature expressing almost every operation that has been considered. For multiplace functions, we have added finite axiomatisability results and results on the complexity of equational theories, in Chapter 6. And we have begun the investigation of complete representability for partial functions, obtaining a finite axiomatisability result, in Chapter 4. This again contrasts with relations. These are all positive results, and in Chapter 7 we obtained some more finite axiomatisations, but also showed some representation classes are not finitely axiomatisable. It would be interesting to investigate exactly what causes this divergence from our other results and all those that have come before.

For reasoning with partial functions, the application we gave the most prominence to was using functions to model the dynamic action of computer programs. So it is worth discussing what might be necessary to reason in a way that has practical value, and whether this is feasible.

For applications, it is deciding the validity of formulas, more than having axiomatisations or deciding representability, that is useful. Of course, the fewer the syntactic restrictions on the formulas under consideration, the more complex deciding validity is likely to become. Quasiequations seem to strike a good balance between expressiveness and tractability. In Section 7 of [50], Höfner and Struth give a simple example of an automated verification task: verifying code for integer division. Reducing this to proving an equation, the task cannot be fully automated, for the equation is only valid on condition of the validity of two simpler equations, which then have to be verified by hand after instantiating variables to atomic programming statements. Viewed another way, if the right relationships between atomic statements are known and supplied to the automated prover, then being able to deduce a quasiequational validity is precisely what is needed for the prover to perform the verification task.

If we are to reason about programs specified by code written in any general-purpose (that is to say, Turing-complete) language, then we are certainly going to need to be able to express some kind of unbounded iteration operation. There are negative results in this area, which we reported in Section 3.2.1, but a worthwhile state of affairs has not yet been completely ruled out. Regarding validities, the important paper is the Goldblatt and Jackson paper [30]. They rule out decidability for a whole family of propositional dynamic logics, and this immediately translates into undecidability of equational theories for partial functions over a range of signatures. However, obtaining results by translating in this way necessarily requires antidomain in the signature. If we have in mind to model partial recursive functions without any restrictions, then it is difficult to justify including antidomain, as identifying the points where partial recursive functions are undefined is not in general an effectively computable operation and so not expressible in any programming language. We therefore pose the following informally specified problem. This has similarities to [58, Problem 4.6] by Jackson and Stokes.

Problem 8.1.1. Find a collection of effectively computable operations on partial functions including composition, some kind of conditional, and some kind of unbounded iteration, for which the quasiequational theory of the representation class is decidable, or prove that no such collection exists.

We would probably want to include a test sort that forms an embedded Boolean subalgebra. Including the domain operation would be reasonable, but it must not be test-valued, otherwise antidomain is expressible. If Problem 8.1.1 is resolved negatively, it may be fruitful to instead consider a restricted form of iteration that nevertheless still suffices in most instances, along the same lines as primitive recursion.

Beyond this core problem, it may prove valuable to investigate various extensions, such as we have in this thesis. Generalising results for unary functions with iteration to multiplace functions is yet to be done. For complete representability, there is still much to be done even in the absence of iteration, for the results in Chapter 4 covered only a single signature.

Problem 8.1.2. Axiomatise the complete representation classes for algebras of partial functions for signatures containing operations from $\{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \bowtie, \sqcup, \texttt{if-then-else}, {}^{\uparrow}, \texttt{while}, {}^{-1}\}$, or prove the classes to be nonelementary. Determine the decidability/complexity of their equational, quasiequational and first-order theories.

For the finite representation property, the program of checking all combinations of commonly considered operations is now more-or-less complete. To advance understanding further, it would be helpful to have a meta-theorem identifying conditions on signatures that ensure the finite representation property. In order to be interesting, it should encompass not just 'forward-looking' operations, but operations such as range as well.

Problem 8.1.3. Identify conditions on a collection of set-theoretic operations on partial functions that ensure the finite representation property holds for the corresponding representation class. The conditions should hold true for all combinations of operations from $\{;, \cdot, \mathrm{D}, \mathrm{R}, 0, 1', \mathrm{A}, \mathrm{F}, \bowtie, \sqcup\}$.

In summary, a consistent picture of partial functions as logically well behaved has emerged, yet there remain a number of interesting research directions still to be pursued.

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