Multilevel Structural Equation Modeling with lavaan

Yves Rosseel
Department of Data Analysis
Ghent University

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1 Multilevel regression

1.1 Brief overview
different types of data with non-independent observations

• clustered data (family members, teeth in a mouth)
• dyadic data (romantic couples)
• hierarchical data (students within schools within regions)
• matched data (case-control studies)
• survey data (nested sampling)
• longitudinal data (blood pressure of patients measured every week)
• repeated measures (within-subjects design)
• …
balanced versus unbalanced data

• when the data is balanced, we have the same number of units within each cluster

• typical examples of balanced data:
  – dyadic data: always two units per cluster
  – repeated measures data: everyone has scores for the same set of conditions
  – longitudinal data where the number of observations (over time) is the same for all individuals (often called panel data)
  – hierarchical data where a fixed number of units was sampled for each cluster

• when the data is unbalanced, we have different cluster sizes
  – this may be due to missing values
  – in hierarchical data, the number of units for each cluster may vary considerably from cluster to cluster
wide versus long data

- when data is arranged in ‘wide’ format, each row corresponds to a single cluster
  - we may end up with many columns (one for each measure/variable, for each unit)
  - rows are independent
  - unbalanced data can be handled by filling in missing values for the smaller clusters
- when data is arranged in ‘long’ format, each row corresponds to a single unit
  - the columns contain the variables for that unit (only)
  - multiple rows belong to the same clusters
  - rows are not independent
  - higher-level variables (for example school characteristics) are duplicated for each unit
example wide format

```
cluster.id y1 m1 x1 y2 m2 x2 y3 m3 x3 schoolsize
1   1 16 4 60 28 36 6 4 22 12  large
2   2 24 14 10 18 6 20 38 28 22  medium
3   3 26 2 2 32 4 8 4 4 10  medium
4   4 4 36 14 2 2 0 8 8 10  small
5   5 14 10 16 28 2 4 8 22 6  small
6   6 24 20 16 42 18 2 2 28 18  large
7   7 22 0 14 32 6 2 18 18 10  medium
8   8 0 8 34 16 16 14 8 28 18  large
```
example long format

<table>
<thead>
<tr>
<th>cluster.id</th>
<th>y</th>
<th>m</th>
<th>x</th>
<th>schoolsize</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>16</td>
<td>4</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>28</td>
<td>36</td>
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<td>4</td>
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<td>12</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>24</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>18</td>
<td>6</td>
<td>20</td>
</tr>
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<td>22</td>
</tr>
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<td>7</td>
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<td>26</td>
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<td>2</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>32</td>
<td>4</td>
<td>8</td>
</tr>
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<td>2</td>
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<td>0</td>
</tr>
<tr>
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<td>8</td>
<td>8</td>
<td>10</td>
</tr>
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<td>14</td>
<td>5</td>
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<td>4</td>
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<tr>
<td>15</td>
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<td>22</td>
<td>6</td>
</tr>
<tr>
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<td>6</td>
<td>24</td>
<td>20</td>
<td>16</td>
</tr>
<tr>
<td>17</td>
<td>6</td>
<td>42</td>
<td>18</td>
<td>2</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>2</td>
<td>28</td>
<td>18</td>
</tr>
<tr>
<td>19</td>
<td>7</td>
<td>22</td>
<td>0</td>
<td>14</td>
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<td>18</td>
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<td>22</td>
<td>8</td>
<td>0</td>
<td>8</td>
<td>34</td>
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<td>16</td>
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<tr>
<td>24</td>
<td>8</td>
<td>8</td>
<td>28</td>
<td>18</td>
</tr>
</tbody>
</table>
ignoring the dependency structure

- we could treat the sample as a simple random sample with $N$ independent observations

- in the multilevel context, this is often called a ‘disaggregated analysis’, as higher-level variables (e.g., school characteristics) are assigned to the individual level

- although (still) widely used, ignoring the clustering in the data may have severe consequences:
  - wrong standard errors
  - inflated type I error rates

- what about reviewers?
  - the ‘tolerance’ for ignoring the clustering in data is now almost non-existing in most fields
solutions

- because clustered data is everywhere, a wide spectrum of ‘solutions’ have been proposed:
  - avoiding the clustering (only pick one individual per cluster)
  - aggregating the data (may lead to ecological fallacies)
  - cluster-robust standard errors (clustering is just a nuisance)
  - fixed-effects approach (school as a fixed factor)
  - mixed-effects approach (school as a random factor)
  - ...

- some solutions are naive, and some may lead to wrong conclusions
the many faces of mixed-effects models

• mixed-effects models have been developed in a variety of disciplines, with varying names and terminology:
  – random-effects (ANOVA) models (statistics, econometrics)
  – linear mixed models (statistics)
  – variance components models (statistics)
  – hierarchical linear models (education, Bayesian)
  – multilevel models (sociology, education)
  – contextual-effects models (sociology)
  – random-coefficient models (econometrics)
  – repeated-measures models, repeated measures ANOVA (statistics, psychology)
  – …

• the different terminology is still a source of much confusion
multilevel regression

- multilevel regression is the application of mixed-effects statistical models to analyze hierarchical (or multilevel) data

- this branch of statistics was mainly developed in the educational sciences, and in quantitative sociology

- Blalock (1984) introduced ‘contextual effect models’ in sociology

- school effectiveness researchers realized early on (’70s, ’80s) that taking the cluster structure into account was important
  
  - a regression analysis per school was one solution, but this ignored the fact that many regression coefficients (across schools) should be similar; this similarity should be used (‘borrowing strength’)
  
  - on the other hand, requiring regression coefficients in all schools to be the same, was regarded as too restrictive
  
  - clearly, some intermediate form of analysis was needed
• this led to the idea of random coefficient models, but it left open the problem of combining predictors of different levels

• Burstein (and others) suggested in the early ’80s to proceed in two stages:
  – in a first stage, a regression analysis was done for each school
  – in a second stage, the resulting regression coefficients were entered as outcome variables in a regression, where the predictors were cluster variables
  – this became known as the ‘slopes-as-outcomes’ approach

• in the mid ’80s, it became clear that the models that educational researchers were looking for had been around for quite some time in other branches of statistics (e.g., linear mixed models)

• a number of authors published a series of papers that would eventually lead to what we now call today ‘multilevel regression’ (Mason et al., 1983; Aitkin and Longford, 1986; de Leeuw and Kreft, 1986; Goldstein, 1986; Raudenbush and Bryk, 1986)
• some important textbooks paved the way for a wide adoption of multilevel regression in the social and behavioural sciences:


1.2 Example: the “High School and Beyond” data

- the following example is borrowed from Raudenbusch and Bryk (2001)

- the data are from the 1982 “High School and Beyond” survey, and pertain to 7185 U.S. high-school students from 160 schools (70 catholic, 90 public)

- these are the variables that we will use:
  - school an ordered factor designating the school that the student attends.
  - ses a numeric vector of socio-economic scores
  - mAch a numeric vector of Mathematics achievement scores
  - meanses a numeric vector of mean ses for the school
  - sector a factor with levels Public and Catholic
  - cses a numeric vector of centered ses values where the centering is with respect to the meanses for the school

- the aim of the analysis is to determine how students’ math achievement scores are related to their family socioeconomic status
• but this relationship may very well vary among schools

• if there is indeed variation among schools, can we find any school characteristics that ‘explain’ this variation? the two school characteristics that we will use are:

  – **sector**: public school or Catholic school
  – **meanses**: the average SES of students in the school

• dataset is included in the R package ‘mlmRev’
exploring the data

```r
> library(mlmRev)
> summary(Hsb82)
```

<table>
<thead>
<tr>
<th>school</th>
<th>minrty</th>
<th>sx</th>
<th>ses</th>
<th>mAch</th>
</tr>
</thead>
<tbody>
<tr>
<td>2305</td>
<td>67</td>
<td>No :5211 Male :3390</td>
<td>Min. :-3.758000</td>
<td>Min. :-2.832</td>
</tr>
<tr>
<td>5619</td>
<td>66</td>
<td>Yes:1974 Female:3795</td>
<td>1st Qu.: -0.538000</td>
<td>1st Qu.: 7.275</td>
</tr>
<tr>
<td>4292</td>
<td>65</td>
<td></td>
<td>Median : 0.002000</td>
<td>Median : 13.131</td>
</tr>
<tr>
<td>8857</td>
<td>64</td>
<td></td>
<td>Mean : 0.000143</td>
<td>Mean : 12.748</td>
</tr>
<tr>
<td>4042</td>
<td>64</td>
<td></td>
<td>3rd Qu.: 0.602000</td>
<td>3rd Qu.: 18.317</td>
</tr>
<tr>
<td>3610</td>
<td>64</td>
<td></td>
<td>Max. : 2.692000</td>
<td>Max. : 24.993</td>
</tr>
<tr>
<td>(Other):6795</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>meanses</th>
<th>sector</th>
<th>cses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min. :-1.1939459</td>
<td>Public :3642</td>
<td>Min. :-3.6507</td>
</tr>
<tr>
<td>1st Qu.: -0.3230000</td>
<td>Catholic :3543</td>
<td>1st Qu.: -0.4479</td>
</tr>
<tr>
<td>Median : 0.0320000</td>
<td>Median : 0.0160</td>
<td></td>
</tr>
<tr>
<td>Mean : 0.0001434</td>
<td>Mean : 0.0000</td>
<td></td>
</tr>
<tr>
<td>3rd Qu.: 0.3269123</td>
<td>3rd Qu.: 0.4694</td>
<td></td>
</tr>
<tr>
<td>Max. : 0.8249825</td>
<td>Max. : 2.8561</td>
<td></td>
</tr>
</tbody>
</table>
exploring the data (2)

```r
> head(Hsb82, n = 8)

<table>
<thead>
<tr>
<th>school</th>
<th>minrty</th>
<th>sx</th>
<th>ses</th>
<th>mAch</th>
<th>meanses</th>
<th>sector</th>
<th>cses</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1224</td>
<td>No</td>
<td>Female</td>
<td>-1.528</td>
<td>5.876</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>2</td>
<td>1224</td>
<td>No</td>
<td>Female</td>
<td>-0.588</td>
<td>19.708</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>3</td>
<td>1224</td>
<td>No</td>
<td>Male</td>
<td>-0.528</td>
<td>20.349</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>4</td>
<td>1224</td>
<td>No</td>
<td>Male</td>
<td>-0.668</td>
<td>8.781</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>5</td>
<td>1224</td>
<td>No</td>
<td>Male</td>
<td>-0.158</td>
<td>17.898</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>6</td>
<td>1224</td>
<td>No</td>
<td>Male</td>
<td>0.022</td>
<td>4.583</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>7</td>
<td>1224</td>
<td>No</td>
<td>Female</td>
<td>-0.618</td>
<td>-2.832</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
<tr>
<td>8</td>
<td>1224</td>
<td>No</td>
<td>Male</td>
<td>-0.998</td>
<td>0.523</td>
<td>-0.434383</td>
<td>Public</td>
</tr>
</tbody>
</table>

> tail(Hsb82, n = 8)

<table>
<thead>
<tr>
<th>school</th>
<th>minrty</th>
<th>sx</th>
<th>ses</th>
<th>mAch</th>
<th>meanses</th>
<th>sector</th>
<th>cses</th>
</tr>
</thead>
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<td>7178</td>
<td>9586</td>
<td>No</td>
<td>Female</td>
<td>1.212</td>
<td>15.260</td>
<td>0.6211525</td>
<td>Catholic</td>
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<td>7179</td>
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<td>Female</td>
<td>1.022</td>
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<td>Catholic</td>
</tr>
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<td>Yes</td>
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<td>1.612</td>
<td>20.967</td>
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<td>Catholic</td>
</tr>
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<td>No</td>
<td>Female</td>
<td>1.512</td>
<td>20.402</td>
<td>0.6211525</td>
<td>Catholic</td>
</tr>
<tr>
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<td>No</td>
<td>Female</td>
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<td>14.794</td>
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<td>Catholic</td>
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<tr>
<td>7183</td>
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<td>No</td>
<td>Female</td>
<td>1.332</td>
<td>19.641</td>
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<td>Catholic</td>
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<tr>
<td>7184</td>
<td>9586</td>
<td>No</td>
<td>Female</td>
<td>-0.008</td>
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<td>Catholic</td>
</tr>
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<td>7185</td>
<td>9586</td>
<td>No</td>
<td>Female</td>
<td>0.792</td>
<td>22.733</td>
<td>0.6211525</td>
<td>Catholic</td>
</tr>
</tbody>
</table>
```
25 Catholic schools

Math Achievement

student SES
25 Public schools

![Graph showing scatter plots for 25 public schools with student SES on the x-axis and Math Achievement on the y-axis.](image)
model 1: a random-effects one-way ANOVA

• this is often called the ‘empty’ model, since it contains no predictors, but simply reflects the nested structure

• no level-1 predictors, no level-2 predictors

• in the ‘multilevel’ notation we specify a model for each level

• model for the first (student) level:

\[ y_{ij} = \alpha_{0i} + \epsilon_{ij} \]

• model for the second (school) level:

\[ \alpha_{0i} = \gamma_{00} + u_{0i} \]

• the combined model and the Laird-Ware form:

\[ y_{ij} = \gamma_{00} + u_{0i} + \epsilon_{ij} \]
\[ = \beta_0 + b_{0i} + \epsilon_{ij} \]
• this is an example of a random-effects one-way ANOVA model with one fixed effect (the intercept, $\beta_0$) representing the general population mean of math achievement, and two random effects:

  – $b_{0i}$ representing the deviation of math achievement in school $i$ from the general mean

  – $\epsilon_{ij}$ representing the deviation of individual $j$’s math achievement in school $i$ from the school mean

• there are two variance components for this model:

  – $\text{Var}(b_{0i}) = d^2$: the variance among school means

  – $\text{Var}(\epsilon_{ij}) = \sigma^2$: the variance among individuals in the same school

• since $b_{0i}$ and $\epsilon_{ij}$ are assumed to be independent, the variation in math scores among individuals can be decomposed into these two variance components:

$$\text{Var}(y_{ij}) = d^2 + \sigma^2$$
**R code**

```r
> fit.model1 <- lmer(mAch ~ 1 + (1 | school), data = Hsb82, REML = FALSE)
> summary(fit.model1)

Linear mixed model fit by maximum likelihood ['lmerMod']
Formula: mAch ~ 1 + (1 | school)
   Data: Hsb82

     AIC   BIC  logLik deviance df.resid
47121.8 47142.4 -23557.9 47115.8    7182

Scaled residuals:
  Min       1Q   Median       3Q      Max
-3.06262 -0.75365  0.02676  0.76070  2.74184

Random effects:
     Groups   Name   Variance Std.Dev.
       school (Intercept)    8.553    2.925
       Residual            39.148    6.257
Number of obs: 7185, groups: school, 160

Fixed effects:
                  Estimate Std. Error t value
(Intercept)       12.6371     0.2436   51.87
```

Yves Rosseel

Multilevel Structural Equation Modeling with lavaan
intra-class correlation

• the *intra-class correlation coefficient* is the proportion of variation in individuals’ scores due to differences among schools:

\[
\frac{d^2}{\text{Var}(y_{ij})} = \frac{d^2}{d^2 + \sigma^2} = \rho
\]

• \(\rho\) may also be interpreted as the correlation between the math scores of two individuals from the same school:

\[
\text{Cor}(y_{ij}, y_{ij'}) = \frac{\text{Cov}(y_{ij}, y_{ij'})}{\sqrt{\text{Var}(y_{ij}) \times \text{Var}(y_{ij'})}} = \frac{d^2}{d^2 + \sigma^2} = \rho
\]

```r
> d2 <- as.numeric(VarCorr(fit.model1)$school)
> s <- as.numeric(attr(VarCorr(fit.model1), "sc"))
> rho <- d2/(d2 + s^2); rho

[1] 0.1793109
```

• about 18 percent of the variation in students’ match-achievement scores is “attributable” to differences among schools
model 2: a random-effects one-way ANCOVA

• 1 level-1 predictor (SES, centered within school), no level-2 predictors
• random intercept, no random slopes
• model for the first (student) level:

\[ y_{ij} = \alpha_0 + \alpha_1 \text{cses}_{ij} + \epsilon_{ij} \]

• model for the second (school) level:

\[ \alpha_0 = \gamma_0 + u_0 \]  
\[ \alpha_1 = \gamma_1 \]  

(the random intercept)  

(the constant slope)

• the combined model and the Laird-Ware form:

\[ y_{ij} = (\gamma_0 + u_0) + \gamma_1 \text{cses}_{ij} + \epsilon_{ij} \]
\[ = \gamma_0 + \gamma_1 \text{cses}_{ij} + u_0 + \epsilon_{ij} \]
\[ = \beta_0 + \beta_1 x_{1ij} + b_0 + \epsilon_{ij} \]
• the fixed-effect coefficients $\beta_0$ and $\beta_1$ represent the average within-schools population intercept and slope respectively
  
  – note: because SES is centered within schools, the intercept $\beta_0$ represents the ‘average’ level of math achievement in the population

• the model has two variance-covariance components:
  
  – $\text{Var}(b_{0i}) = d^2$: the variance among school intercepts
  – $\text{Var}(\epsilon_{ij}) = \sigma^2$: the error variance around the within-school regressions
R code

```r
> fit.model2 <- lmer(mAch ~ 1 + cses + (1 | school), data = Hsb82, REML = FALSE)
> summary(fit.model2, correlation = FALSE)
```

Linear mixed model fit by maximum likelihood ['lmerMod']
Formula: mAch ~ 1 + cses + (1 | school)
Data: Hsb82

```
AIC       BIC     logLik deviance df.resid
46728.4    46755.9  -23360.2  46720.4     7181
```

Scaled residuals:

```
        Min       1Q   Median       3Q      Max
-3.09692 -0.73195  0.01945  0.75738  2.91422
```

Random effects:

```
  Groups   Name      Variance  Std.Dev.  
school   (Intercept) 8.612      2.935
      Residual       37.005     6.083
Number of obs: 7185, groups: school, 160
```

Fixed effects:

```
                 Estimate  Std. Error    t value
(Intercept)   12.636231   0.24374901    51.8485
  cses        2.191175   0.10864043    20.1711
```

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Multilevel Structural Equation Modeling with lavaan
model 3: a random-coefficients regression model

• 1 level-1 predictor (SES, centered within school), no level-2 predictors

• random intercept and random slopes

• model for the first (student) level:

\[ y_{ij} = \alpha_0 + \alpha_1 \text{cses}_{ij} + \epsilon_{ij} \]

• model for the second (school) level:

\[ \alpha_0 = \gamma_0 + u_0 \]  
\[ \alpha_1 = \gamma_1 + u_1 \]

(the random intercept)

(the random slope)

• the combined model and the Laird-Ware form:

\[ y_{ij} = (\gamma_0 + u_0) + (\gamma_1 + u_1) \text{cses}_{ij} + \epsilon_{ij} \]
\[ = \gamma_0 + \gamma_1 \text{cses}_{ij} + u_0 + u_1 \text{cses}_{ij} + \epsilon_{ij} \]
\[ = \beta_0 + \beta_1 x_{1ij} + b_0 + b_0 z_{1ij} + \epsilon_{ij} \]
• the fixed-effect coefficients $\beta_0$ and $\beta_1$ again represent the average within-schools population intercept and slope respectively

• the model has four variance-covariance components:
  
  – $\text{Var}(b_{0i}) = d_0^2$: the variance among school intercepts
  – $\text{Var}(b_{1i}) = d_1^2$: the variance among school slopes
  – $\text{Cov}(b_{0i}, b_{1i}) = d_{01}$: the covariance between within-school intercepts and slopes
  – $\text{Var}(\epsilon_{ij}) = \sigma^2$: the error variance around the within-school regressions
R code

```r
> fit.model3 <- lmer(mAch ~ 1 + cses + (1 + cses | school), data = Hsb82,
+ REML = FALSE)
> summary(fit.model3, correlation = FALSE)

Linear mixed model fit by maximum likelihood ['lmerMod']
Formula: mAch ~ 1 + cses + (1 + cses | school)
   Data: Hsb82

             AIC     BIC   logLik deviance df.resid
46723.0 46764.3  -23355.5 46711.0     7179

Scaled residuals:
    Min      1Q  Median      3Q     Max
-3.09688 -0.73199  0.01794  0.75445  2.89901

Random effects:
   Groups   Name       Variance  Std.Dev.  Corr
school   (Intercept)  8.6204     2.9361
          cses        0.6782     0.8236  0.02
Residual             36.7000    6.0581
Number of obs: 7185, groups: school, 160

Fixed effects:
             Estimate Std. Error   t value
(Intercept)   12.6363     0.2437   51.85
cses           2.1932     0.1278   17.15
```

Yves Rosseel
model 4: random intercept + level-2 predictor

- we drop the level-1 predictor (cses), but add a level-2 predictor (meanses)
- random intercept, no random slopes
- model for the first (student) level:

\[ y_{ij} = \alpha_0 + \epsilon_{ij} \]

- model for the second (school) level:

\[ \alpha_{0i} = \gamma_{00} + \gamma_{01} \text{meanses}_i + u_{0i} \]

- the combined model and the Laird-Ware form:

\[ y_{ij} = \gamma_{00} + \gamma_{01} \text{meanses}_i + u_{0i} + \epsilon_{ij} \]

\[ = \beta_0 + \beta_1 x_{1ij} + b_0i + \epsilon_{ij} \]

- note that in the Laird-Ware notation, we use a double index for the fixed-effects \( x_{ij} \), even if the variable (meanses) does not change over students
• this model has two fixed effects ($\beta_0$ and $\beta_1$) and two random effects ($b_{0i}$ and $\epsilon_{ij}$)

• the interpretation of $b_{0i}$ has changed: whereas in the previous model it had been the deviation of school $i$’s mean from the grand mean, it now represents the residual ($\alpha_{0i} - \gamma_{00} - \gamma_{10}$ meanses$_j$); correspondingly, the variance component $d^2$ is now a conditional variance (conditional on the school mean SES)

• in Raudenbush and Bryk, this is called a ‘Regression with means-as-outcomes’ model, because the school’s mean ($\alpha_{0i}$) is predicted by the means SES of the school

• note in the following output that the residual variance between schools is substantially smaller than the original
R code

> fit.model4 <- lmer(mAch ~ 1 + meanses + (1 | school), data = Hsb82, +    REML = FALSE)
> summary(fit.model4, correlation = FALSE)

Linear mixed model fit by maximum likelihood ['lmerMod']
Formula: mAch ~ 1 + meanses + (1 | school)
Data: Hsb82

      AIC BIC logLik deviance df.resid
46967.1 46994.6  -23479.6 46959.1     7181

Scaled residuals:
       Min     1Q   Median     3Q    Max
-3.13480 -0.75252  0.02331  0.76833  2.78413

Random effects:
   Groups   Name        Variance  Std.Dev.
school   (Intercept)  2.593     1.610
   Residual                  39.157    6.258
Number of obs: 7185, groups: school, 160

Fixed effects:
        Estimate Std. Error t value
(Intercept)  12.6849    0.1483   85.52
meanses      5.8629    0.3591   16.32
model 5: intercepts-and-slopes-as-outcomes model

- we expand the model by including two level-2 predictors: meanses and sector; the slopes are again allowed to vary randomly

- model for the first (student) level:

\[ y_{ij} = \alpha_{0i} + \alpha_{1i} \text{cse}_i + \epsilon_{ij} \]

- model for the second (school) level:

\[ \alpha_{0i} = \gamma_{00} + \gamma_{01} \text{meanses}_i + \gamma_{02} \text{sector}_i + u_{0i} \]
\[ \alpha_{1i} = \gamma_{10} + \gamma_{11} \text{meanses}_i + \gamma_{12} \text{sector}_i + u_{1i} \]
the combined model and the Laird-Ware form:

\[ y_{ij} = (\gamma_0 + \gamma_0 \text{meanses}_i + \gamma_0 \text{sector}_i + u_{0i}) + \\
(\gamma_{10} + \gamma_{11} \text{meanses}_i + \gamma_{12} \text{sector}_i + u_{1i}) \text{cses}_{ij} + \epsilon_{ij} \\
= \gamma_0 + \gamma_0 \text{meanses}_i + \gamma_0 \text{sector}_i + \gamma_{10} \text{cses}_{ij} + \\
\gamma_{11} \text{meanses}_i \text{cses}_{ij} + \gamma_{12} \text{sector}_i \text{cses}_{ij} + u_{0i} + u_{1i} \text{cses}_{ij} + \epsilon_{ij} \\
= \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + \beta_4 (x_{1ij} x_{3ij}) + \beta_5 (x_{2ij} x_{3ij}) + \\
b_{0i} + b_{1i} z_{1ij} + \epsilon_{ij}

R code

```r
> fit.model5 <- lmer(mAch ~ 1 + meanses*cses + sector*cses + (1 + cses | school),
+                     data = Hsb82, REML = FALSE)
> summary(fit.model5, correlation = FALSE)
```

Linear mixed model fit by maximum likelihood  ['lmerMod']
Formula: mAch ~ 1 + meanses * cses + sector * cses + (1 + cses | school)
Data: Hsb82

<p>| | | | | |</p>
<table>
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<tr>
<th></th>
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<th></th>
<th></th>
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<td>AIC</td>
<td>BIC</td>
<td>logLik</td>
<td>deviance</td>
<td>df.resid</td>
</tr>
<tr>
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<td>46585.2</td>
<td>-23248.2</td>
<td>46496.4</td>
<td>7175</td>
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Scaled residuals:

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<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.1610</td>
<td>-0.7244</td>
<td>0.0168</td>
<td>0.7549</td>
<td>2.9581</td>
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</tbody>
</table>

Random effects:

<table>
<thead>
<tr>
<th>Groups</th>
<th>Name</th>
<th>Variance</th>
<th>Std.Dev.</th>
<th>Corr</th>
</tr>
</thead>
<tbody>
<tr>
<td>school</td>
<td>(Intercept)</td>
<td>2.31667</td>
<td>1.5221</td>
<td></td>
</tr>
<tr>
<td></td>
<td>cses</td>
<td>0.06507</td>
<td>0.2551</td>
<td>0.48</td>
</tr>
<tr>
<td>Residual</td>
<td></td>
<td>36.72118</td>
<td>6.0598</td>
<td></td>
</tr>
</tbody>
</table>

Number of obs: 7185, groups: school, 160

Fixed effects:

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<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Std. Error</th>
<th>t value</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>12.1279</td>
<td>0.1974</td>
<td>61.44</td>
<td></td>
</tr>
<tr>
<td>meanses</td>
<td>5.3317</td>
<td>0.3655</td>
<td>14.59</td>
<td></td>
</tr>
<tr>
<td>cses</td>
<td>2.9457</td>
<td>0.1540</td>
<td>19.13</td>
<td></td>
</tr>
<tr>
<td>sectorCatholic</td>
<td>1.2269</td>
<td>0.3033</td>
<td>4.05</td>
<td></td>
</tr>
<tr>
<td>meanses:cses</td>
<td>1.0427</td>
<td>0.2960</td>
<td>3.52</td>
<td></td>
</tr>
<tr>
<td>cses:sectorCatholic</td>
<td>-1.6440</td>
<td>0.2373</td>
<td>-6.93</td>
<td></td>
</tr>
</tbody>
</table>
do we need the random slopes?

> fit.model5bis <- lmer(mAch ~ 1 + meanses*cse + sector*cse + (1 | school), +                          data = Hsb82, REML = FALSE)
> anova(fit.model5bis, fit.model5)

Data: Hsb82
Models:
fit.model5bis: mAch ~ 1 + meanses * cse + sector * cse + (1 | school)
fit.model5: mAch ~ 1 + meanses * cse + sector * cse + (1 + cse | school)

             Df  AIC     BIC deviance Chi Df Pr(>Chisq)
fit.model5bis 8 46513 46568    -23249   46497
fit.model5  10 46516 46585    -23248   46496  2  0.6061

• no: apparently, the level-2 predictors do a sufficiently good job of accounting for differences in slopes

• statistical note: using a LRT for comparing two models with a different random structure is conservative; better approaches exist (e.g. in the R package RLRsim)
summary of the models

> # model 1:
> lmer(mAch ~ 1 + (1 | school), data = Hsb82, REML = FALSE)

> # model 2:
> lmer(mAch ~ 1 + cses + (1 | school), data = Hsb82, REML = FALSE)

> # model 3:
> lmer(mAch ~ 1 + cses + (1 + cses | school), data = Hsb82, REML = FALSE)

> # model 4:
> lmer(mAch ~ 1 + meanses + (1 | school), data = Hsb82, REML = FALSE)

> # model 5:
> lmer(mAch ~ 1 + meanses*cses + sector*cses + (1 + cses | school),
>     data = Hsb82, REML = FALSE)

> # model 5bis:
> lmer(mAch ~ 1 + meanses*cses + sector*cses + (1 | school),
>     data = Hsb82, REML = FALSE)
2 Multilevel SEM

2.1 Introduction

- limitations of the multilevel regression model:
  - (mostly) univariate perspective (multivariate is possible but awkward)
  - no measurement models (latent variables)
  - no mediators (only strictly dependent or independent variables)
  - no reciprocal effects, no goodness-of-fit measures, …

- two evolutions since the late 1980s:
  - the multilevel regression framework was extended to include measurement errors and latent variables (cfr. HLM and MLwiN software)
  - the traditional SEM framework started to incorporate random intercepts and random slopes

- the boundaries between SEM and multilevel regression have gradually disappeared
2.2 History


  - full modeling of within and between covariance matrices
  - provided a computer program for ML estimation
  - balanced data only, no level-2 variables, no meanstructure
  - structured case is described in Schmidt & Wisenbaker (1986)


LISREL also offers great possibilities for conducting such multilevel analyses. It has been shown by Schmidt (1969) that maximum likelihood estimates can be derived of the within-class and between-class covariance matrices, and these can
be parameterized in LISREL models, to allow separate estimates of parameters at the two levels [...] A great problem, of course, is that there in most studies tend to be few classes (or other higher level units) only, which precludes the possibility of obtaining any stable estimates at the class level. We would like to suggest, however, that in the least within-class analyses should be performed to guard against the possibility that results obtained in non-hierarchical analyses can in fact he accounted for by effects at the class level, which may be more or less artifactual.

- his work was also picked up by Leigh Burstein
  - Burstein worked at the Graduate School of Education (UCLA)

  - he reformulated Schmidt’s fitting function so that it could be estimated using existing software for multiple-group SEM (e.g., LISCOMP)
Goldstein & McDonald

  - very general formulation, including multilevel SEM
  - univariate perspective (multivariate vector = 1st level)
  - can handle missing data, hierarchical data, cross-classified data
  - expression of the likelihood, IGLS algorithm is suggested

  - multivariate perspective, within-and-between formulation
  - likelihood expression + a computationally tractable (re)expression
  - both for balanced and unbalanced clusters

Muthén

  
  – re-expresses the within-part of the likelihood as a sum over different cluster sizes
  
  – in the balanced case, this leads to a multiple-group SEM fitting function with two groups

  
  – derivations of Muthén (1989)
  
  – suggestion: we can use the balanced solution even in the unbalanced case (using an estimate of the average cluster size): estimator = MUML
  

• standard SEM software could be used (at least for the balanced case)
Lee

  - statistically more rigorous development of multilevel SEM theory: ML and GLS estimation, inference, goodness-of-fit statistics, constraints
  - suggested using Fisher scoring and Gauss-Newton for optimization
  - no level-2 variables

- Poon & Lee (1992): within-part as sum over different cluster sizes


- Lee & Poon (1998): using the EM algorithm (by treating the the latent random vectors at the cluster level as missing data)

  - Chapter 9: Bayesian methods for analyzing various two-level SEMs
Bentler

  
  
  – extend the EM algorithm of Lee & Poon (1998) to handle level-2 predictors
  
  – clever way to avoid a large number of matrix inversions
  
  – often considered to be the state-of-the-art algorithm for estimating 2-level SEMs with continuous responses
  
  – no missing data, no random slopes

• perhaps the last technical paper on (continuous) two-level SEM (in the frequentist framework)
2.3 Frameworks (and software) for multilevel SEM

overview

• two-level SEM with random intercepts
  – Mplus (type = twolevel), LISREL, EQS, lavaan

• the gllamm framework: gllamm, (related approach: Latent Gold)

• the Mplus framework: Mplus

• the case-wise likelihood based approach (e.g., Mehta & Neale, 2005)
  – Mplus (type = random), Mx, OpenMx (definition variables)
  – in principle: both continuous and categorical outcomes; random slopes
  – xxM?

• the Bayesian framework
  – Mplus
  – (Open)BUGS, JAGS, Stan
two-level SEM with random intercepts

- an extension of single-level SEM to incorporate random intercepts
- extensive technical literature, starting from the late 1980s (until about 2004)
- available in Mplus, EQS, LISREL, lavaan, …
- this is by far the most widely used framework in the applied literature

- advantages:
  - fast, simple, well-understood, plenty of examples
  - well-documented

- disadvantages:
  - continuous outcomes only
  - no random slopes
the Mplus framework

- the Mplus framework has added many extensions to the two-level within/between approach in the last 17 years
  - EM algorithm can handle random slopes and missing data
  - categorical outcomes (with numerical quadrature)
  - multilevel (robust) (D)WLS
  - combination multilevel with complex survey data, mixture modeling, ...

- advantages:
  - superb implementation
  - user-friendly, familiar (‘multivariate’) approach

- disadvantages:
  - NO technical documentation (about the extensions)
  - black box software
the gllamm framework

• Sophia Rabe-Hesketh, Anders Skrondal and Andrew Pickles

• see http://www.gllamm.org/

• an extension of generalized linear mixed models to include (continuous and discrete) latent variables (including a structural part)

• advantages:
  – very well documented, open-source code (written in Stata)
  – handles a wide range of outcome types (normal, categorical, …)
  – very general, very flexible

• disadvantages:
  – not easy to specify (complex) models, univariate perspective
  – needs Stata
  – very, very slow (even in the continuous case)
lavaan

- multilevel SEM development just started (jan 2017)
- implemented in the development version (0.6-1):
  - standard two-level ‘within-and-between’ approach
  - continuous responses only, no missing data (for now)
  - no random slopes (for now)
  - using quasi-newton optimization (for now)

- future plans: many
  - gllamm framework (but more user-friendly)
  - case-wise likelihood approach
  - hybrids
lavaan syntax setup for two-level SEM

$$\Sigma_B$$

Between

$$\Sigma_W$$

Within

```r
model <- ' 
  level: 1
    # here comes the within level
  level: 2
    # here comes the between level
',

fit <- sem(myModel, myData, 
  cluster = "school")
```
useful literature

• the relationship between SEM and multilevel regression:


• books:


2.4 The two-level SEM model with random intercepts

- we assume two-level data with individuals (students) nested within clusters (schools)

- in this framework, we decompose the total score of each variable into two parts: a within part, and a between part (Cronbach & Webb, 1979):

\[
y_{ji} = (y_{ji} - \bar{y}_g) + \bar{y}_g
\]

\[
y_T = y_W + y_B
\]

where \( j = 1, \ldots, J \) is an index for the clusters, and \( i = 1, \ldots, n_j \) is an index for the units within a cluster; \( \bar{y}_j \) is the cluster mean of cluster \( j \)

- both components are treated as unknown (latent) variables
- the two parts are orthogonal and additive; one of the parts can be zero

- the total covariance (at the population level) can be decomposed as

\[
\text{Cov}(y) = \Sigma_T = \Sigma_W + \Sigma_B
\]
two-level SEM: specifying a model for each level

- for a two-level CFA model, we can use

\[ \Sigma_W = \Lambda_W \Psi_W \Lambda'_W + \Theta_W \]

and

\[ \Sigma_B = \Lambda_B \Psi_B \Lambda'_B + \Theta_B \]

- if we add a structural (regression) part, we need to add the \((I - B)^{-1}\) term to the matrix formulation (as in regular SEM)

- no meanstructure is needed for the within part (as the level-1 variables are cluster-centered)

- a meanstructure \(\mu_B\) can be added for the between part of the model

- in addition, we can add level-2 covariates \((z_j)\) to the model
2.5 Loglikelihood of a two-level SEM

notation

- number of clusters: \( J \), number of units per cluster: \( n_j \)
- data for cluster \( j \):
  \[
  \mathbf{v}_j = [\mathbf{z}_j, \mathbf{y}_{j1}, \mathbf{y}_{j2}, \ldots, \mathbf{y}_{jn_j}]^T
  \]
- model implied matrices/vectors: \( \Sigma_{zz}, \Sigma_{zy}, \Sigma_w, \Sigma_b \) and \( \mu_b = [\mu_z, \mu_y]^T \)
- expectation of \( \mathbf{v}_j \):
  \[
  E[\mathbf{v}_j] = \hat{\mathbf{v}}_j = [\mu_z, \mu_y, \mu_y, \ldots, \mu_y]^T
  \]
- covariance matrix for \( \mathbf{v}_j \):
  \[
  \text{Cov}[\mathbf{v}_j] = \mathbf{V}_j = \begin{bmatrix}
  \Sigma_{zz} & 1_{n_j}^T \otimes \Sigma_{zy} \\
  1_{n_j} \otimes \Sigma_{yz} & \Sigma_{yy}
  \end{bmatrix}
  \]
  where
  \[
  \Sigma_{yy} = \mathbf{I}_{n_j} \otimes \Sigma_w + 1_{n_j} 1_{n_j}^T \otimes \Sigma_b
  \]
loglikelihood

- assuming multivariate normality, we can write the loglikelihood for cluster $j$ as follows:

$$
\text{loglik}_j = -\frac{O_j}{2} \ln(2\pi) - \frac{1}{2} \ln |V_j| - \frac{1}{2} (v_j - \hat{v}_j)^T V_j^{-1} (v_j - \hat{v}_j)
$$

where $O_j$ is the length of $v_j$, usually $p_z + (n_j \times p_y)$

- the total likelihood over all $J$ clusters:

$$
\text{loglik} = \sum_{j=1}^{J} \text{loglik}_j
$$

- we can find ML estimates by minimizing the objective function $F_{ML}$ which is minus two times the loglikelihood function, ignoring the constant:

$$
F_{ML} = \sum_{j=1}^{J} \ln |V_j| + (v_j - \hat{v}_j)^T V_j^{-1} (v_j - \hat{v}_j)
$$
objective function (optional)

• the original objective function:

\[
F_{ML} = \sum_{j=1}^{J} \ln |V_j| + (v_j - \hat{v}_j)^T V_j^{-1} (v_j - \hat{v}_j)
\]

• for large clusters, the size of \( V_j \) can be formidable

• we should exploit the block-diagonal structure of \( V \)

• we define:

\[
\Sigma_{b.z} = (\Sigma_b - \Sigma_{yz} \Sigma_{zz}^{-1} \Sigma_{zy})
\]
• version 1: McDonald & Goldstein (1989), per cluster, using $\Sigma_{b,z}$:

$$F_{ML} = \sum_{j=1}^{J} \left[ \ln |\Sigma_{zz}| + (n_j - 1) \ln |\Sigma_w| + \ln |\Sigma_w + n_j \cdot \Sigma_{b,z}| ight]$$

$$+ \text{tr} \left[ \left( \Sigma_{zz}^{-1} + n_j \Sigma_{zz}^{-1} \Sigma_{zy} (n_j \Sigma_{b,z} + \Sigma_w)^{-1} \Sigma_{yz} \Sigma_{zz}^{-1} \right) (z_j - \mu_z)(z_j - \mu_z)^T \right]$$

$$+ 2n_j \text{tr} \left[ -\Sigma_{zz}^{-1} \Sigma_{zy} (n_j \Sigma_{b,z} + \Sigma_w)^{-1} (\bar{y}_j - \mu_y)(z_j - \mu_z)^T \right]$$

$$+ \text{tr} \left[ \Sigma_w^{-1} Y_j^{(c)} Y_j^{(c)^T} \right]$$

$$- n_j \text{tr} \left[ \Sigma_w^{-1} (\bar{y}_j - \mu_y)(\bar{y}_j - \mu_y)^T \right]$$

$$+ n_j \text{tr} \left[ (n_j \Sigma_{b,z} + \Sigma_w)^{-1} (\bar{y}_j - \mu_y)(\bar{y}_j - \mu_y)^T \right]$$
• version 2: lavaan = McDonald & Goldstein (1989), per cluster size,

\[ F_{ML} = (N - J) \left( \ln |\Sigma_w| + \text{tr} \left[ \Sigma_w^{-1} S_{pw} \right] \right) + \]

\[ \sum_{s=1}^{S} n_s \cdot \left[ (\ln |\Sigma_{zz}| + \ln |\Sigma_w + n_j \cdot \Sigma_{b,z}|) + \right. \]

\[ \text{tr} \left[ (\Sigma_{zz}^{-1} + n_j \Sigma_{zz}^{-1} \Sigma_{zy}(n_j \Sigma_{b,z} + \Sigma_w)^{-1} \Sigma_{yz} \Sigma_{zz}^{-1}) (z_j - \mu_z)(z_j - \mu_z)^T \right] \]

\[ + 2n_j \text{tr} \left( -\Sigma_{zz}^{-1} \Sigma_{zy}(n_j \Sigma_{b,z} + \Sigma_w)^{-1} (\bar{y}_j - \mu_y)(z_j - \mu_z)^T \right) \]

\[ + n_j \text{tr} \left( (n_j \Sigma_{b,z} + \Sigma_w)^{-1} (\bar{y}_j - \mu_y)(\bar{y}_j - \mu_y)^T \right) \]

where \( S_{pw} \) is the pooled within-clusters covariance matrix:

\[ S_{pw} = \frac{\sum_{j=1}^{J} \sum_{i=1}^{n_j} (y_{ji} - \bar{y}_j)(y_{ji} - \bar{y}_j)^T}{N - J} \]
optimization techniques for two-level SEM (optional)

- quasi-newton methods (lavaan, 0.6-1)

- Fisher scoring, Gauss-Newton (LISREL)

- Expectation-Maximization (EM) (Mplus, EQS)
  - many variants exist

- hybrid optimization schemes (EM + quasi-newton)
Example: Mplus ex9.6 (simulated data)

- data: 110 clusters, 1000 observations, cluster sizes: 5, 10, 15
- 4 measures at the within level $y_1, y_2, y_3, y_4$
- 2 covariates at the within level $x_1, x_2$
- 1 covariate at the between level $w$
- reading in the data:

```r
> names(Data) <- c("y1", "y2", "y3", "y4", "x1", "x2", "w", "clus")
> head(Data)
```

```
  y1  y2  y3  y4  x1  x2  w  clus
1 2.20 1.86 1.74 2.24 1.14 -0.80 -0.15 1
2 1.93 2.13 0.08 2.51 1.95 -0.12 -0.15 1
3 0.32 0.98 -0.84 0.56 -0.72 -0.77 -0.15 1
4 0.07 -1.74 -2.31 -1.51 -2.64 0.64 -0.15 1
5 -1.21 0.45 0.37 -1.79 -0.26 0.30 -0.15 1
6 0.29 -1.82 0.56 -2.09 -0.94 1.36 0.32 2
```
library(lavaan)

model <- '  
level: 1
  fw =˜ y1 + y2 + y3 + y4
  fw ~ x1 + x2

level: 2
  fb =˜ y1 + y2 + y3 + y4

  # optional
  y1 ~ 0*y1
  y2 ~ 0*y2
  y3 ~ 0*y3
  y4 ~ 0*y4

  fb ~ w
',

fit <- sem(model, data = Data,
  cluster = "clus",
  fixed.x = FALSE)
> summary(fit)

lavaan (0.6-1.1173) converged normally after 27 iterations

Number of observations 1000
Number of clusters [clus] 110

Estimator ML
Model Fit Test Statistic 3.863
Degrees of freedom 17
P-value (Chi-square) 1.000

Parameter Estimates:

Information Observed
Observed information based on Hessian
Standard Errors Standard

Level 1 [within]:

Latent Variables:

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| fw =~ |          |         |         |         |
| y1  | 1.000    |         |         |         |
| y2  | 0.999    | 0.033   | 30.735  | 0.000   |
| y3  | 0.995    | 0.033   | 29.804  | 0.000   |
| y4  | 1.017    | 0.033   | 30.364  | 0.000   |
**Regressions:**

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| fw ~ |          |         |         |         |
| x1  | 0.973    | 0.042   | 23.287  | 0.000   |
| x2  | 0.510    | 0.038   | 13.422  | 0.000   |

**Covariances:**

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| x1 ~ x2 | 0.032    | 0.032   | 1.014   | 0.311   |

**Intercepts:**

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| .y1 | 0.000    |         |         |         |
| .y2 | 0.000    |         |         |         |
| .y3 | 0.000    |         |         |         |
| .y4 | 0.000    |         |         |         |
| x1  | 0.007    | 0.032   | 0.215   | 0.830   |
| x2  | 0.014    | 0.032   | 0.436   | 0.663   |
| .fw | 0.000    |         |         |         |

**Variances:**

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| .y1 | 0.981    | 0.057   | 17.151  | 0.000   |
| .y2 | 0.948    | 0.056   | 17.015  | 0.000   |
| .y3 | 1.070    | 0.060   | 17.700  | 0.000   |
| .y4 | 1.014    | 0.059   | 17.182  | 0.000   |
Level 2 [clus]:

Latent Variables:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| fb =~    |         |         |         |
| y1       | 1.000   |         |         |
| y2       | 0.960   | 0.073   | 13.078  | 0.000  |
| y3       | 0.924   | 0.074   | 12.452  | 0.000  |
| y4       | 0.949   | 0.075   | 12.631  | 0.000  |

Regressions:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| fb ~     |         |         |         |
| w        | 0.344   | 0.078   | 4.429   | 0.000  |

Intercepts:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| y1       | -0.083  | 0.076   | -1.095  | 0.274  |
| y2       | -0.077  | 0.074   | -1.047  | 0.295  |
| y3       | -0.045  | 0.073   | -0.617  | 0.537  |
| y4       | -0.030  | 0.074   | -0.405  | 0.686  |
| w        | 0.006   | 0.086   | 0.070   | 0.944  |
| fb       | 0.000   |         |         |        |
### Variances:

|      | Estimate | Std.Err | z-value | P(>|z|) |
|------|----------|---------|---------|---------|
| .y1  | 0.000    |         |         |         |
| .y2  | 0.000    |         |         |         |
| .y3  | 0.000    |         |         |         |
| .y4  | 0.000    |         |         |         |
| .fb  | 0.361    | 0.078   | 4.643   | 0.000   |
| w    | 0.815    | 0.110   | 7.416   | 0.000   |

```r
> fitMeasures(fit)

<table>
<thead>
<tr>
<th>npar</th>
<th>fmin</th>
<th>chisq</th>
<th>df</th>
</tr>
</thead>
<tbody>
<tr>
<td>26.000</td>
<td>3.913</td>
<td>3.863</td>
<td>17.000</td>
</tr>
<tr>
<td>pvalue</td>
<td>baseline.chisq</td>
<td>baseline.df</td>
<td>baseline.pvalue</td>
</tr>
<tr>
<td>1.000</td>
<td>3280.729</td>
<td>25.000</td>
<td>0.000</td>
</tr>
<tr>
<td>cfi</td>
<td>tli</td>
<td>nnfi</td>
<td>rfi</td>
</tr>
<tr>
<td>1.000</td>
<td>1.006</td>
<td>1.006</td>
<td>0.998</td>
</tr>
<tr>
<td>nfi</td>
<td>pnfi</td>
<td>ifi</td>
<td>rni</td>
</tr>
<tr>
<td>0.999</td>
<td>0.679</td>
<td>1.004</td>
<td>1.004</td>
</tr>
<tr>
<td>logl</td>
<td>unrestricted.logl</td>
<td>aic</td>
<td>bic</td>
</tr>
<tr>
<td>-9527.429</td>
<td>-9525.497</td>
<td>19106.857</td>
<td>19234.459</td>
</tr>
<tr>
<td>ntotal</td>
<td>bic2</td>
<td>rmsea</td>
<td>rmsea.ci.lower</td>
</tr>
<tr>
<td>1000.000</td>
<td>19151.882</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>rmsea.ci.upper</td>
<td>rmsea.pvalue</td>
<td>srmr</td>
<td>srmr_within</td>
</tr>
<tr>
<td>0.000</td>
<td>1.000</td>
<td>0.022</td>
<td>0.004</td>
</tr>
<tr>
<td>srmr_between</td>
<td>0.017</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
> `lavInspect(fit, "h1")`

```r
within
within$cov
   y1   y2   y3   y4   x1   x2
y1 3.191
y2 2.216 3.144
y3 2.203 2.197 3.257
y4 2.233 2.242 2.236 3.284
x1 0.966 0.962 0.969 1.011 0.985
x2 0.566 0.553 0.535 0.552 0.032 1.017

within$mean
   y1   y2   y3   y4   x1   x2
0.000 0.000 0.000 0.000 0.007 0.014

clus
clus$cov
   y1   y2   y3   y4   w
y1 0.456
y2 0.440 0.432
y3 0.419 0.406 0.387
y4 0.433 0.418 0.398 0.425
w  0.296 0.247 0.273 0.264 0.815

clus$mean
   y1   y2   y3   y4   w
```
\begin{verbatim}
-0.067 -0.062 -0.030 -0.013 0.006

> lavInspect(fit, "implied")

$within
$within$cov
  y1  y2  y3  y4  x1  x2
y1 3.190
y2 2.207 3.152
y3 2.198 2.195 3.256
y4 2.247 2.245 2.235 3.300
x1 0.975 0.974 0.970 0.992 0.985
x2 0.550 0.550 0.548 0.560 0.032 1.017

$within$mean
  y1  y2  y3  y4  x1  x2
0.014 0.014 0.014 0.014 0.007 0.014

$clus
$clus$cov
  y1  y2  y3  y4  w
y1 0.458
y2 0.439 0.421
y3 0.423 0.406 0.391
y4 0.434 0.417 0.401 0.412
w 0.281 0.269 0.259 0.266 0.815
\end{verbatim}
\$clus\$mean

\begin{verbatim}
  y1  y2  y3  y4  w
-0.081 -0.075 -0.043 -0.028  0.006
\end{verbatim}

\>	exttt{lavInspect(fit, "icc")}

\begin{verbatim}
  y1  y2  y3  y4  x1  x2
0.125 0.121 0.106 0.115  0.000  0.000
\end{verbatim}
2.6 The status of a latent variable in a two-level SEM

- when a latent variable, representing a hypothetical construct, is introduced in a two-level model, we need to carefully reflect on the ‘status’ of this latent variable
  
  – are the indicators measured at the within or the between level?
  
  – is the construct of (theoretical) interest at the within level, the between level, or both?
  
  – how can we interpret the ‘meaning’ of the construct at the within/between level?

- based on the answers on these questions, we need to create the latent variable in a different way at the within and/or the between level

- this is (still today) a big source of confusion (and bad practices) in the educational sciences
different types of latent variables

- we will discuss five different construct types:
  
  1. within-only construct
     - in this case, if we have no other level-2 variables, we may as well use a single-level SEM based on a pooled within-cluster covariance matrix
  2. between-only construct
  3. shared between-level construct
  4. configural (or contextual) construct
  5. shared and configural construct

- reference:

**within-only construct**

- indicators of the latent variable are measured at the within level
- level at which construct is of interest: within level only
- interpretation at the within level: construct explains the covariances between its indicators measured at the within level
- interpretation at the between level: not relevant
- although the construct only ‘exists’ at the within level, we may still observe ‘spurious’ between-level variation in the sample
- example: construct represents ‘lactose intolerance’
  - items inquire about the degree of severity of physical reactions after consuming products containing lactose
  - construct can not be a school-level characteristic, although we may observe differences (on average) across schools
model <- '  
level: 1  
  fw =~ y1 + y2 + y3 + y4  
level: 2  
  y1 ~~ y1 + y2 + y3 + y4  
  y2 ~~ y2 + y2 + y3  
  y3 ~~ y3 + y4  
  y4 ~~ y4  
'
**between-only construct**

- indicators of the latent variable are measured at the between level
- level at which construct is of interest: between level only
- interpretation at the within level: not relevant (does not ‘exist’ at the within level)
- interpretation at the between level: construct explains the covariances between its indicators measured at the between level
- example: construct reflects self-reported ‘leadership style’ measured by a questionnaire filled in by the school principles
diagram and lavaan syntax

\[ \text{model} \leftarrow ' \]

\[ \text{level: 1} \]

\[ \text{# perhaps other level-1 variables} \]

\[ \text{level: 2} \]

\[ \text{fb} = \sim y_1 + y_2 + y_3 + y_4 \]
shared (or reflective) between-level construct

- indicators of the latent variable are measured at the within level
- level at which construct is of interest: between level only
- interpretation at the within level: none
- interpretation at the between level: construct represents a characteristic of the cluster
- example: construct reflects ‘instructional quality’ (a classroom characteristic) as perceived by students
  - each student in each classroom was asked to judge the ‘instructional quality’ of the teacher of that classroom
  - we are interested in the ‘average’ responses of the individual students within each classroom
  - responses within each classroom should be highly correlated (high agreement) if indeed ‘instructional quality’ is a shared construct
diagram and lavaan syntax

```
model <- '

  level: 1

    y1  ~ y1 + y2 + y3 + y4
    y2  ~ y2 + y2 + y3
    y3  ~ y3 + y4
    y4  ~ y4

  level: 2

    fs = ~ y1 + y2 + y3 + y4

',
```
**configural (or formative) construct**

- indicators of the latent variable are measured at the within level
- level at which construct is of interest: both within and between level
- interpretation at the within/between level: construct explains the covariances of the within/between part of its indicators
- the configural construct (at the between level) represents the *aggregate* of the measurements of individuals within a cluster
- example: reading motivation:
  - at the individual level (within cluster)
  - at the school level (average student motivation within a school)
- the cluster itself is not seen as the source/reason for variability of an individual construct
- therefore, between-cluster loadings are fixed to be the same as within-cluster loadings (cross-level measurement invariance)
model <- ' 
   level: 1
   fw =~ a*y1 + b*y2 + c*y3 + d*y4
   level: 2
   fb =~ a*y1 + b*y2 + c*y3 + d*y4',
**shared + configural construct**

- indicators of the latent variable are measured at the within level

- level at which construct is of interest: within and between level

- interpretation at the within level: construct explains the covariances of the within part of its indicators

- interpretation at the between level: both the configural construct and the shared construct explain the covariances of the within/between part of its indicators

- example: reading motivation for each child in a classroom is rated by the classroom teacher (using multiple items)
  - some teachers tend to rate more positively as compared to others
  - the ‘shared’ construct reflects the rater effect
  - the ‘configural’ construct reflects the average reading motivation in a classroom
diagram and lavaan syntax

\[
\text{model} \leftarrow '\
\text{level: 1}
\text{fw} = \sim a*y_1 + b*y_2 + c*y_3 + d*y_4
\text{level: 2}
\text{fb} = \sim a*y_1 + b*y_2 + c*y_3 + d*y_4
\text{fs} = \sim y_1 + y_2 + y_3 + y_4
\text{fs} \sim 0*\text{fb}
',
\]
2.7 The status of observed covariates in a two-level SEM

• when observed covariates are added in a two-level model, we again need to carefully reflect on the ‘status’ of these covariates
  – are the covariates measured at the within or the between level?
  – if they are measured at the within level, does it make sense to split this covariate into a within and a between part?

• based on the answers on these questions, we can make a distinction between three types of covariates:
  1. within-only covariates
  2. between-only covariates
  3. level-1 covariates with a within and a between part
adding a within-only covariate

\[
\begin{align*}
\text{level: 1} \\
fw = & a \cdot y_1 + b \cdot y_2 + c \cdot y_3 + d \cdot y_4 \\
fw & \sim x_1 \\
\text{level: 2} \\
fb = & a \cdot y_1 + b \cdot y_2 + c \cdot y_3 + d \cdot y_4
\end{align*}
\]
adding a between-only covariate

\[
\begin{align*}
& z_1 
\quad \downarrow 
\quad \uparrow \quad f_b \\
& y_1 \quad y_2 \quad y_3 \quad y_4 \\
\text{Between} \\
& f_w = a*y_1 + b*y_2 + c*y_3 + d*y_4 \\
\text{Within} \\
& f_w \leftarrow z_1 
\end{align*}
\]

```
model <- ' 
level: 1
  fw =~ a*y1 + b*y2 + c*y3 + d*y4

level: 2
  fb =~ a*y1 + b*y2 + c*y3 + d*y4
  fb ~ z1 
',
```
adding a level-1 covariate with a within and a between part

```
model <- '  
level: 1
  fw = a*y1 + b*y2 + c*y3 + d*y4
  fw ~ x1
level: 2
  fb = a*y1 + b*y2 + c*y3 + d*y4
  fb ~ x1',
```

Yves Rosseel
Multilevel Structural Equation Modeling
with lavaan
adding a level-1 covariate with a within and a between part (2)

- decomposition of a level-1 covariate (say, $x_1$) into its within part and between part:
  - the level-1 covariate is centered using cluster/group-mean centering
  - the cluster/group means are treated as (unknown, latent) population parameters that need to be estimated
  - this implies that we assume a random intercept for the level-1 covariate

- note that this is not the same as creating aggregated versions of the level-1 covariates manually, and adding them to the datafile so they can be used in the between part of the model, see:

2.8 Evaluating model fit

- if no random slopes are involved, we can fit an unrestricted (saturated) model: we estimate all the elements of $\Sigma_W$, $\Sigma_B$ and $\mu_B$

- then, we can compute the standard $\chi^2$ goodness-of-fit test statistic as:

$$T = -2(L_0 - L_1)$$

where $L_0$ and $L_1$ are the loglikelihood of the restricted (user-specified) model (h0) and the unrestricted model (h1) respectively

- under various optimal conditions, this statistic follows a chi-square distribution

- the degrees of freedom are computed as in a two-group SEM model: the difference between the number of (non-redundant) sample statistics for each level, and the number of free model parameters

- in principle, fit measures like CFI/TLI, RMSEA, SRMR, … can be computed in a similar way as in a single-level SEM
evaluating fit (2)

• unfortunately, a recent simulation study showed that CFI, TLI, and RMSEA were not sensitive to Level-2 model misspecification:


• there seems to be a growing sentiment that ‘global’ fit indices may not be very useful in a multilevel setting

• an alternative approach is to assess the fit per level:
  
  – we could compute the SRMR for each level
  
  – we could fit a single-level model separately for each level, and look at the traditional fit measures to judge the model fit for that level
2.9 Example: two-level CFA

- we use an example from this book (Chapter 14):


- the (simulated) data are the scores on six intelligence measures of 399 children from 60 (large) families, patterned after a real dataset collected by Van Peet, A.A.J. (1992)

- the six intelligence measures are: word list, cards, matrices, figures, animals, and occupations

- the data have a two-level structure, with children nested within families

- if intelligence is strongly influenced by shared genetic and environmental influences in the families, we may expect strong between-family effects

- the ICCs of the 6 measures range from 0.36 to 0.49
exploring the data

```r
> FamIQData <- read.table("FamIQData.dat")
> names(FamIQData) <- c("family", "child", "wordlist", "cards", "matrices", +                          "figures", "animals", "occupats")
> summary(FamIQData)

  family   child  wordlist   cards
 Min.   : 1.00  Min. : 1.00  Min. :12.00  Min. :11.00
  1st Qu.:16.00  1st Qu.: 2.00  1st Qu.:27.00  1st Qu.:26.50
  Median :33.00  Median : 4.00  Median :30.00  Median :30.00
  Mean   :31.78  Mean   : 4.04  Mean   :29.95  Mean   :29.84
  3rd Qu.:48.00  3rd Qu.: 6.00  3rd Qu.:33.00  3rd Qu.:33.00
  Max.   :60.00  Max.   :12.00  Max.   :45.00  Max.   :44.00

  matrices   figures  animals  occupats
 Min.   :15.00  Min.   :17.00  Min.   :15.00  Min.   :15.00
  1st Qu.:26.00  1st Qu.:27.00  1st Qu.:27.00  1st Qu.:27.00
  Median :30.00  Median :30.00  Median :30.00  Median :30.00
  Mean   :29.73  Mean   :30.00  Mean   :30.11  Mean   :30.01
  3rd Qu.:33.00  3rd Qu.:33.00  3rd Qu.:34.00  3rd Qu.:33.00
  Max.   :46.00  Max.   :44.00  Max.   :46.00  Max.   :43.00

> # various cluster sizes
> table(table(FamIQData$family))

   4  5  6  7  8  9 10 11 12
 3 16 11 12 11 4 1 1 1
```
analytic procedure

- fitting a two-level model is often a stepwise procedure; below are the steps used by Joop Hox

- model 0: as a preliminary step, an EFA was carried out on the pooled within-clusters covariance matrix $S_{PW}$
  - it was concluded that a 2-factor model fitted well at the within level
  - not shown here

- model 1: a two-factor model at the within level + a null model at the between level
  - a null model implies: zero variances and covariances for all (6) variables
  - if this model fits well, we would conclude that there is no between family structure at all: we may as well continue with a single-level analysis
• model 2: a two-factor model at the within level + an independence model at the between level
  
  – independence model implies: estimated variances but zero covariances
  
  – if this model holds, there is family-level variance, but no substantively interesting structural model

• model 3: a two-factor model at the within level + a saturated model at the between level
  
  – the factors at the within-level in this model correspond to what we have called ‘within-only’ constructs

• models 4a and 4b: in his book, Joop Hox goes on and fits a model with a one-factor model for the between part (4a), and a model with a two-factor model for the between part (4b)
  
  – the two-factor model seems no improvement over the one-factor model
  
  – model 4a (with a general factor at the between level) is kept as the final model
model 1: a 2-factor within model + null between model
lavaan syntax

> modell <- ' 
+  level: 1 
+    numeric =~ wordlist + cards + matrices 
+    perception =~ figures + animals + occupats 
+  level: 2 
+    wordlist ~ 0*wordlist 
+    cards ~ 0*cards 
+    matrices ~ 0*matrices 
+    figures ~ 0*figures 
+    animals ~ 0*animals 
+    occupats ~ 0*occupats 
+ ', 
> fit1 <- sem(modell, data = FamIQData, cluster = "family", 
+      std.lv = TRUE, verbose = FALSE) 
> # summary(fit1) 
> fit1

lavaan (0.6–1.1173) converged normally after 53 iterations

Number of observations 399
Number of clusters [family] 60

Estimator ML
Model Fit Test Statistic 323.649
Degrees of freedom 29
P-value (Chi-square) 0.000
model 2: a 2-factor within model + independence between model

Between

wordlist  cards  matrices  figures  animals  occupats

Within

wordlist  cards  matrices  figures  animals  occupats

numeric  perception

Yves Rosseel
Multilevel Structural Equation Modeling with lavaan
lavaan syntax model 2

```r
> model2 <-'
+   level: 1
+     numeric =~ wordlist + cards + matrices
+     perception =~ figures + animals + occupats
+   level: 2
+     wordlist ~ wordlist
+     cards ~ cards
+     matrices ~ matrices
+     figures ~ figures
+     animals ~ animals
+     occupats ~ occupats
+ ',

> fit2 <- sem(model2, data = FamIQData, cluster = "family",
+             std.lv = TRUE, verbose = FALSE)

> # summary(fit2)
> fit2

lavaan (0.6-1.1173) converged normally after 50 iterations

  Number of observations                     399
  Number of clusters [family]                60

  Estimator                         ML
  Model Fit Test Statistic           177.206
  Degrees of freedom                23
  P-value (Chi-square)              0.000
```
model 3: a 2-factor within model, with saturated between part
lavaan syntax model 3

> model3 <- '  + level: 1
+   numeric =~ wordlist + cards + matrices
+   perception =~ figures + animals + occupats
+ level: 2
+   # saturated
+   wordlist ~~ cards + matrices + figures + animals + occupats
+   cards ~~ matrices + figures + animals + occupats
+   matrices ~~ figures + animals + occupats
+   figures ~~ animals + occupats
+   animals ~~ occupats
+
> fit3 <- sem(model3, data = FamIQData, cluster = "family",
+   std.lv = TRUE, verbose = FALSE)
> summary(fit3)

lavaan (0.6-1.1173) converged normally after 174 iterations

Number of observations 399
Number of clusters [family] 60

Estimator ML
Model Fit Test Statistic 6.716
Degrees of freedom 8
P-value (Chi-square) 0.568
Parameter Estimates:

<table>
<thead>
<tr>
<th>Information</th>
<th>Observed information based on Hessian</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Errors</td>
<td>Standard</td>
</tr>
</tbody>
</table>

Level 1 [within]:

Latent Variables:

| numeric =˜ | Estimate | Std.Err | z-value | P(>|z|) |
|------------|----------|---------|---------|--------|
| wordlist   | 3.155    | 0.203   | 15.558  | 0.000  |
| cards      | 3.156    | 0.196   | 16.113  | 0.000  |
| matrices   | 3.032    | 0.199   | 15.207  | 0.000  |

| perception =˜ | Estimate | Std.Err | z-value | P(>|z|) |
|---------------|----------|---------|---------|--------|
| figures       | 3.091    | 0.205   | 15.069  | 0.000  |
| animals       | 3.192    | 0.195   | 16.397  | 0.000  |
| occupats      | 2.774    | 0.183   | 15.139  | 0.000  |

Covariances:

| numeric ~ | perception | Estimate | Std.Err | z-value | P(>|z|) |
|-----------|------------|----------|---------|---------|--------|
| perception| 0.386      | 0.058    | 6.691   | 0.000   |

Intercepts:

| .wordlist | Estimate | Std.Err | z-value | P(>|z|) |
|-----------|----------|---------|---------|--------|
| .wordlist | 0.000    |         |         |        |
### Variances:

|              | Estimate | Std.Err | z-value | P(>|z|) |
|--------------|----------|---------|---------|---------|
| wordlist     | 6.234    | 0.739   | 8.433   | 0.000   |
| cards        | 5.344    | 0.693   | 7.705   | 0.000   |
| matrices     | 6.443    | 0.714   | 9.025   | 0.000   |
| figures      | 6.856    | 0.757   | 9.053   | 0.000   |
| animals      | 4.851    | 0.696   | 6.968   | 0.000   |
| occupats     | 5.338    | 0.604   | 8.835   | 0.000   |
| numeric      | 1.000    |         |         |         |
| perception   | 1.000    |         |         |         |

### Level 2 [family]:

### Covariances:

|              | Estimate | Std.Err | z-value | P(>|z|) |
|--------------|----------|---------|---------|---------|
| wordlist     |          |         |         |         |
| cards        | 9.272    | 2.225   | 4.168   | 0.000   |
| matrices     | 8.515    | 2.077   | 4.100   | 0.000   |
| figures      | 8.410    | 2.053   | 4.097   | 0.000   |
| .animals     | 9.700 | 2.195 | 4.419 | 0.000 |
| .occupats    | 10.428 | 2.357 | 4.425 | 0.000 |
| .cards ~    |
| .matrices   | 7.997 | 2.018 | 3.964 | 0.000 |
| .figures    | 8.424 | 2.035 | 4.140 | 0.000 |
| .animals    | 10.000 | 2.203 | 4.540 | 0.000 |
| .occupats   | 10.418 | 2.337 | 4.457 | 0.000 |
| .matrices ~  |
| .figures    | 7.733 | 1.902 | 4.067 | 0.000 |
| .animals    | 8.022 | 1.966 | 4.081 | 0.000 |
| .occupats   | 9.000 | 2.142 | 4.203 | 0.000 |
| .figures ~  |
| .animals    | 8.980 | 2.177 | 4.125 | 0.000 |
| .occupats   | 9.750 | 2.333 | 4.179 | 0.000 |
| .animals ~  |
| .occupats   | 11.080 | 2.489 | 4.451 | 0.000 |

**Intercepts:**

|          | Estimate  | Std.Err | z-value | P(>|z|) |
|----------|-----------|---------|---------|--------|
| .wordlist| 29.890    | 0.470   | 63.547  | 0.000  |
| .cards   | 29.892    | 0.465   | 64.308  | 0.000  |
| .matrices| 29.732    | 0.439   | 67.746  | 0.000  |
| .figures | 30.047    | 0.459   | 65.476  | 0.000  |
| .animals | 30.135    | 0.471   | 63.956  | 0.000  |
| .occupats| 29.967    | 0.509   | 58.891  | 0.000  |

**Variances:**

<p>|          | Estimate  | Std.Err | z-value | P(&gt;|z|) |
|----------|-----------|---------|---------|--------|
| .animals | 9.700     | 2.195   | 4.419   | 0.000  |
| .occupats| 10.428    | 2.357   | 4.425   | 0.000  |</p>
<table>
<thead>
<tr>
<th>wordlist</th>
<th>10.727</th>
<th>2.456</th>
<th>4.368</th>
<th>0.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>cards</td>
<td>10.558</td>
<td>2.397</td>
<td>4.404</td>
<td>0.000</td>
</tr>
<tr>
<td>matrices</td>
<td>9.097</td>
<td>2.123</td>
<td>4.285</td>
<td>0.000</td>
</tr>
<tr>
<td>figures</td>
<td>10.051</td>
<td>2.321</td>
<td>4.330</td>
<td>0.000</td>
</tr>
<tr>
<td>animals</td>
<td>10.956</td>
<td>2.466</td>
<td>4.442</td>
<td>0.000</td>
</tr>
<tr>
<td>occupats</td>
<td>13.473</td>
<td>2.874</td>
<td>4.688</td>
<td>0.000</td>
</tr>
</tbody>
</table>
model 4a: a 2-factor within model + general factor between
lavaan syntax model 4a

```r
> model4a <- '
+   level: 1
+     numeric =~ wordlist + cards + matrices
+     perception =~ figures + animals + occupats
+   level: 2
+     general =~ wordlist + cards + matrices +
+           figures + animals + occupats
+ '
> fit4a <- sem(model4a, data = FamIQData, cluster = "family",
+               std.lv = TRUE, verbose = FALSE)
> summary(fit4a)

lavaan (0.6-1.1173) converged normally after  64  iterations

Number of observations                       399
Number of clusters [family]                 60

Estimator             ML
Model Fit Test Statistic   11.927
Degrees of freedom        17
P-value (Chi-square)      0.805

Parameter Estimates:

Information    Observed
Observed information based on Hessian
```
Level 1 [within]:

Latent Variables:

| Latent Variable | Estimate | Std.Err | z-value | P(>|z|) |
|-----------------|----------|---------|---------|---------|
| numeric =~      |          |         |         |         |
| wordlist        | 3.175    | 0.202   | 15.710  | 0.000   |
| cards           | 3.144    | 0.194   | 16.167  | 0.000   |
| matrices        | 3.054    | 0.199   | 15.349  | 0.000   |
| perception =~   |          |         |         |         |
| figures         | 3.095    | 0.204   | 15.147  | 0.000   |
| animals         | 3.188    | 0.194   | 16.438  | 0.000   |
| occupats        | 2.782    | 0.183   | 15.215  | 0.000   |

Covariances:

| Latent Variable | Estimate | Std.Err | z-value | P(>|z|) |
|-----------------|----------|---------|---------|---------|
| numeric ~ ~     |          |         |         |         |
| perception      | 0.382    | 0.057   | 6.740   | 0.000   |

Intercepts:

| Intercept       | Estimate | Std.Err | z-value | P(>|z|) |
|-----------------|----------|---------|---------|---------|
| .wordlist       | 0.000    |         |         |         |
| .cards          | 0.000    |         |         |         |
| .matrices       | 0.000    |         |         |         |
| .figures        | 0.000    |         |         |         |
| .animals        | 0.000    |         |         |         |
.occupats 0.000
numeric 0.000
perception 0.000

Variances:

|            | Estimate | Std.Err | z-value | P(>|z|) |
|------------|----------|---------|---------|---------|
|.wordlist   | 6.194    | 0.737   | 8.406   | 0.000   |
|.cards      | 5.403    | 0.692   | 7.804   | 0.000   |
|.matrices   | 6.417    | 0.714   | 8.992   | 0.000   |
|.figures    | 6.847    | 0.757   | 9.049   | 0.000   |
|.animals    | 4.881    | 0.696   | 7.009   | 0.000   |
|.occupats   | 5.324    | 0.603   | 8.823   | 0.000   |
numeric   | 1.000    |
perception| 1.000    |

Level 2 [family]:

Latent Variables:

|            | Estimate | Std.Err | z-value | P(>|z|) |
|------------|----------|---------|---------|---------|
|general =~  |          |         |         |         |
|wordlist    | 3.057    | 0.393   | 7.785   | 0.000   |
cards      | 3.054    | 0.389   | 7.843   | 0.000   |
matrices   | 2.632    | 0.381   | 6.904   | 0.000   |
figures    | 2.806    | 0.398   | 7.048   | 0.000   |
animals    | 3.204    | 0.383   | 8.371   | 0.000   |
occupats   | 3.439    | 0.415   | 8.292   | 0.000   |
### Intercepts:

|          | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| .wordlist| 29.891   | 0.468   | 63.847  | 0.000   |
| .cards   | 29.890   | 0.466   | 64.097  | 0.000   |
| .matrices| 29.749   | 0.435   | 68.462  | 0.000   |
| .figures | 30.044   | 0.458   | 65.536  | 0.000   |
| .animals | 30.134   | 0.471   | 64.041  | 0.000   |
| .occupats| 29.967   | 0.508   | 59.012  | 0.000   |
| general  | 0.000    |         |         |         |

### Variances:

|          | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| .wordlist| 1.253    | 0.569   | 2.201   | 0.028   |
| .cards   | 1.323    | 0.586   | 2.258   | 0.024   |
| .matrices| 1.935    | 0.669   | 2.891   | 0.004   |
| .figures | 2.158    | 0.714   | 3.022   | 0.003   |
| .animals | 0.656    | 0.487   | 1.347   | 0.178   |
| .occupats| 1.581    | 0.624   | 2.536   | 0.011   |
| general  | 1.000    |         |         |         |
2.10 Example: two-level SEM

- we use an example from this book (Chapter 15):


- based on a study by Schijf and Dronker (1991): they collected data from 1559 pupils (1382 after listwise deletion) in 58 schools

- pupil variables: father’s occupational status (focc), father’s education (feduc), mother’s education (meduc), the result of the GALO school achievement test (galo), and the teacher’s advice about secondary education (advice)

- at the school level, we have one variable: the school’s denomination (denom) coded as 1=Protestant, 2=Nondenominational, 3=Catholic

- the main research question is whether the school’s denomination affects the GALO score and (indirectly) the teacher’s advice, after the other variables have been accounted for
modeling strategy

- a latent variable is constructed to reflect the socio-economic status (ses) using the variables focc, meduc and feduc as indicators
  - we will construct a configural latent variable for ses at the between level (using equality constraints for the loadings)

- preliminary analysis (using the pooled within-clusters covariance matrix only) revealed that a residual correlation is needed between the indicators focc and feduc at the within level

- in addition, it was decided to fix the residual variance of feduc to zero at the between level

- a secondary question is whether the effect of ses on advice is direct or indirect
  - we label the various regression paths, and compute product terms to compute the indirect effect
  - both at the within and the between level
the model

Between

Within

Yves Rosseel
Multilevel Structural Equation Modeling with lavaan
exploring the data

\[
\begin{align*}
> & \text{Galo} \leftarrow \text{read.table}("Galo.dat") \\
> & \text{names(Galo)} \leftarrow c(\text{"school"}, \text{"sex"}, \text{"galo"}, \text{"advice"}, \text{"feduc"}, \text{"meduc"}, \text{"focc"}, \text{"denom"}) \\
> & \text{Galo}[\text{Galo == 999}] \leftarrow \text{NA} \\
> & \text{Galo$denom1} \leftarrow \text{ifelse(Galo$denom == 1, 1, 0)} \\
> & \text{Galo$denom2} \leftarrow \text{ifelse(Galo$denom == 2, 1, 0)} \\
> & \text{summary(Galo)}
\end{align*}
\]

<table>
<thead>
<tr>
<th>school</th>
<th>sex</th>
<th>galo</th>
<th>advice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min. : 1.00</td>
<td>Min. :1.000</td>
<td>Min. : 53.0</td>
<td>Min. : 0.000</td>
</tr>
<tr>
<td>1st Qu.:16.00</td>
<td>1st Qu.:1.000</td>
<td>1st Qu.: 94.0</td>
<td>1st Qu.: 2.000</td>
</tr>
<tr>
<td>Median :30.00</td>
<td>Median :2.000</td>
<td>Median :103.0</td>
<td>Median :2.000</td>
</tr>
<tr>
<td>Mean :29.87</td>
<td>Mean :1.509</td>
<td>Mean : 102.3</td>
<td>Mean : 3.121</td>
</tr>
<tr>
<td>3rd Qu.:43.00</td>
<td>3rd Qu.:2.000</td>
<td>3rd Qu.:111.0</td>
<td>3rd Qu.:4.000</td>
</tr>
<tr>
<td>Max. :58.00</td>
<td>Max. : 2.000</td>
<td>Max. : 143.0</td>
<td>Max. : 6.000</td>
</tr>
</tbody>
</table>

| NA's :7 |

<table>
<thead>
<tr>
<th>feduc</th>
<th>meduc</th>
<th>focc</th>
<th>denom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min. :1.000</td>
<td>Min. :1.000</td>
<td>Min. :1.000</td>
<td>Min. :1.000</td>
</tr>
<tr>
<td>1st Qu.:1.000</td>
<td>1st Qu.:1.000</td>
<td>1st Qu.:2.000</td>
<td>1st Qu.:2.000</td>
</tr>
<tr>
<td>Median :4.000</td>
<td>Median :2.000</td>
<td>Median :3.000</td>
<td>Median :2.000</td>
</tr>
<tr>
<td>Mean :4.002</td>
<td>Mean :2.966</td>
<td>Mean : 3.336</td>
<td>Mean : 2.007</td>
</tr>
<tr>
<td>3rd Qu.:6.000</td>
<td>3rd Qu.:5.000</td>
<td>3rd Qu.:5.000</td>
<td>3rd Qu.:2.000</td>
</tr>
<tr>
<td>Max. :9.000</td>
<td>Max. : 9.000</td>
<td>Max. : 6.000</td>
<td>Max. : 3.000</td>
</tr>
<tr>
<td>NA's :89</td>
<td>NA's : 61</td>
<td>NA's :117</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>denom1</th>
<th>denom2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min. :0.0000</td>
<td>Min. :0.0000</td>
</tr>
</tbody>
</table>
1st Qu.:0.0000 1st Qu.:0.0000
Median :0.0000 Median :1.0000
Mean :0.1501 Mean :0.6928
3rd Qu.:0.0000 3rd Qu.:1.0000
Max. :1.0000 Max. :1.0000

> table(table(Galo$school))

10 12 13 14 19 20 21 22 23 24 25 26 27 28 29 30 32 33 34 35 36 37 42 46
1 2 1 3 1 2 3 3 1 6 3 3 4 2 1 4 1 4 5 1 2 2 1 2
lavaan syntax

> model <- '
+  level: within
+     wses =~ a*focc + b*meduc + c*feduc
+     # residual correlation
+     focc ~~ feduc
+
+     advice ~ wc*wses + wb*galo
+     galo ~ wa*wses
+
+  level: between
+     bses =~ a*focc + b*meduc + c*feduc
+     feduc ~~ 0*feduc
+
+     advice ~ bc*bses + bb*galo
+     galo ~ ba*bses + denom1 + denom2
+
+     # defined parameters
+     wi := wa * wb
+     bi := ba * bb
+ '
> fit <- sem(model, data = Galo, cluster = "school", std.lv = TRUE)
> summary(fit)

lavaan (0.6-1.1173) converged normally after 105 iterations
Number of observations           1382 1559
Number of clusters [school]       58

Estimator                       ML
Model Fit Test Statistic        26.221
Degrees of freedom              19
P-value (Chi-square)             0.124

Parameter Estimates:

Information                     Observed
   Observed information based on  Hessian
   Standard Errors               Standard

Level 1 [within]:

Latent Variables:

   wses =~
      focc  (a)  0.748  0.038  19.558  0.000
      meduc (b) 1.282  0.047  27.570  0.000
      feduc (c) 1.674  0.057  29.205  0.000

Regressions:

   advice ~
      wses  (wc)  0.119  0.027  4.489  0.000
### Covariances:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| .focc ~ .feduc | 0.257 | 0.086 | 2.986 | 0.003 |

### Intercepts:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| .focc    | 0.000   |         |         |
| .meduc   | 0.000   |         |         |
| .feduc   | 0.000   |         |         |
| .advice  | 0.000   |         |         |
| .galo    | 0.000   |         |         |
| wses     | 0.000   |         |         |

### Variances:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| .focc    | 1.186   | 0.065   | 18.132  | 0.000  |
| .meduc   | 2.021   | 0.120   | 16.900  | 0.000  |
| .feduc   | 1.582   | 0.167   | 9.462   | 0.000  |
| .advice  | 0.574   | 0.022   | 25.512  | 0.000  |
| .galo    | 125.024 | 5.123   | 24.403  | 0.000  |
| wses     | 1.000   |         |         |
Level 2 [school]:

Latent Variables:

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| bses  |          |         |         |         |
| focc  | 0.748    | 0.038   | 19.558  | 0.000   |
| meduc | 1.282    | 0.047   | 27.570  | 0.000   |
| feduc | 1.674    | 0.057   | 29.205  | 0.000   |

Regressions:

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| advice|          |         |         |         |
| bses  | 0.274    | 0.069   | 3.958   | 0.000   |
| galo  | 0.062    | 0.011   | 5.443   | 0.000   |
| galo  |          |         |         |         |
| bses  | 5.121    | 0.591   | 8.672   | 0.000   |
| denom1| -5.152   | 1.602   | -3.216  | 0.001   |
| denom2| -0.511   | 1.267   | -0.403  | 0.687   |

Intercepts:

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| .focc | 3.251    | 0.108   | 30.201  | 0.000   |
| .meduc| 2.839    | 0.178   | 15.965  | 0.000   |
| .feduc| 3.862    | 0.228   | 16.948  | 0.000   |
| .advice| -3.238  | 1.164   | -2.782  | 0.005   |
| .galo | 103.333  | 1.335   | 77.379  | 0.000   |
| bses  | 0.000    |         |         |         |
Variances:

|     | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| .feduc | 0.000    |         |         |         |
| .focc | 0.032    | 0.016   | 2.010   | 0.044   |
| .meduc | 0.021    | 0.025   | 0.844   | 0.398   |
| .advice | 0.015    | 0.008   | 1.905   | 0.057   |
| .galo  | 5.745    | 2.057   | 2.793   | 0.005   |
| bses   | 1.000    |         |         |         |

Defined Parameters:

|     | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| wi  | 0.360    | 0.032   | 11.146  | 0.000   |
| bi  | 0.317    | 0.068   | 4.698   | 0.000   |
3 Alternative ways to analyze multilevel data with SEM

• some alternative ways to analyze multilevel data with SEM:

1. the ‘wide data’ approach: we arrange data in the wide format, and then use single-level SEM to analyze our model

2. the ‘survey’ approach: we analyze the data (in long format) as if there where no clusters, but we use cluster-robust standard errors

3. the two-stage approach: multilevel software (e.g., MLwiN) is used to estimate the (saturated) within and between covariance matrix; analysis by multigroup SEM (Goldstein, 1987)

4. the pseudo-balanced approach: we pretend the data is balanced, and use a special estimator to fit a multigroup SEM (MUML)

5. …
why should you know about these alternatives?

• they may enhance your understanding of:
  – SEM
  – multilevel regression
  – multilevel SEM
  – the relationships between the different modeling frameworks

• depending on your data, model and research questions, they may be easier to set up, have less convergence problems, and the results may be easier to interpret and report

• in some cases, they may save the day
3.1 The ‘wide data’ approach

- wonderful paper about this:


- first approach: using classic SEM to mimic multilevel regression models
  - the random intercepts and random slopes are represented by latent variables
  - the factor loadings of the random intercept are fixed to 1.0
  - the factor loadings of the random slope are fixed to the values of the predictor
  - only feasible if the predictor has a limited number of possible values (e.g. binary, or timepoint 1, 2, 3, or 4)
  - most importantly: only if the values for the predictor are the same for all units (‘balanced design’)

– typical example: growth curve model
– advantage: single-level analysis, model fit (although care is needed to specify the saturated model), flexible error structure, …

• second approach: calculate a model-implied covariance matrix (and mean vector) for each individual
  – needs special software (like OpenMx or Mplus)
  – predictor can be continuous, design does not need to be balanced

• because we are in the SEM context, we can extend these approaches to include latent variables, mediators, …

• can be useful if:
  – the cluster sizes are (very) small
  – the number of variables (per unit) is relatively small
  – the data is (almost) balanced
  – the wide data still has many more rows ($N$) then columns ($P$)
example: a growth curve model with 4 time-points

- random intercept and random slope

\[ y_t = (\text{initial time at time 1}) + (\text{growth per unit time}) \times \text{time} + \text{error} \]

\[ y_t = \text{intercept} + \text{slope} \times \text{time} + \text{error} \]
R code: using SEM in wide format

```r
> library(lavaan)
> head(Demo.growth[,c("t1","t2","t3","t4")], n = 4)

   t1      t2      t3      t4
1 1.726  2.142  2.773  2.516
2-1.984-4.401-6.017-7.030
3 0.319-1.269  1.560  2.868
4 0.777  3.531  3.138  5.364

> model.slope <- '
+    int    =~ 1*t1 + 1*t2 + 1*t3 + 1*t4
+    slope =~ 0*t1 + 1*t2 + 2*t3 + 3*t4
+
+    # intercepts (fixed effects)
+    int   ~ 1
+    slope ~ 1
+
+    # random intercept, random slope
+    int   ~~ int
+    slope ~~ slope
+    int   ~~ slope
+
+    # force same variance for all (compound symmetry)
+    t1    ~~ v1*t1
+    t2    ~~ v1*t2
```
```r
+ t3 ~ v1*t3
+ t4 ~ v1*t4
+
> fit.slope <- lavaan(model.slope, data = Demo.growth)
> summary(fit.slope)

lavaan (0.6-1.1173) converged normally after 24 iterations

<table>
<thead>
<tr>
<th>Parameter Estimates:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Information</td>
</tr>
<tr>
<td>Information saturated (h1) model</td>
</tr>
<tr>
<td>Standard Errors</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Latent Variables:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate       Std.Err   z-value   P(&gt;</td>
</tr>
<tr>
<td>----------------------------------------------------------</td>
</tr>
<tr>
<td>int =~</td>
</tr>
<tr>
<td>t1             1.000</td>
</tr>
<tr>
<td>t2             1.000</td>
</tr>
<tr>
<td>t3             1.000</td>
</tr>
<tr>
<td>t4             1.000</td>
</tr>
</tbody>
</table>
```
### slope

- **t1**: 0.000
- **t2**: 1.000
- **t3**: 2.000
- **t4**: 3.000

### Covariances

|          | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| int ~ ~  |          |         |         |         |
| slope    | 0.627    | 0.069   | 9.129   | 0.000   |

### Intercepts

|          | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| int      | 0.617    | 0.077   | 8.029   | 0.000   |
| slope    | 1.005    | 0.042   | 24.013  | 0.000   |
| .t1      | 0.000    |         |         |         |
| .t2      | 0.000    |         |         |         |
| .t3      | 0.000    |         |         |         |
| .t4      | 0.000    |         |         |         |

### Variances

|          | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| int      | 1.928    | 0.169   | 11.439  | 0.000   |
| slope    | 0.576    | 0.050   | 11.540  | 0.000   |
| .t1      | (v1) 0.622 | 0.031   | 20.000  | 0.000   |
| .t2      | (v1) 0.622 | 0.031   | 20.000  | 0.000   |
| .t3      | (v1) 0.622 | 0.031   | 20.000  | 0.000   |
| .t4      | (v1) 0.622 | 0.031   | 20.000  | 0.000   |
R code: using lmer

> # wide to long
> id <- rep(1:400, each = 4)
> score <- lav_matrix_vecr(Demo.growth[,1:4])
> time <- rep(0:3, times = 400)
> growth.long <- data.frame(id = id, score = score, time = time)
> head(growth.long)

    id  score time
 1    1 1.725645  0
 2    1 2.142401  1
 3    1 2.773172  2
 4    1 2.515956  3
 5    2 -1.984160  0
 6    2 -4.400603  1

> library(lme4)
> fit.lmer <- lmer(score ~ 1 + time + (1 + time | id), data = growth.long, +    REML = FALSE)
> summary(fit.lmer, correlation = FALSE)

Linear mixed model fit by maximum likelihood  ['lmerMod']
Formula: score ~ 1 + time + (1 + time | id)
    Data: growth.long
### AIC BIC logLik deviance df.resid
5523.7 5556.0 -2755.9 5511.7 1594

**Scaled residuals:**

<table>
<thead>
<tr>
<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.62395</td>
<td>-0.51865</td>
<td>-0.00867</td>
<td>0.51881</td>
<td>2.83705</td>
</tr>
</tbody>
</table>

**Random effects:**

<table>
<thead>
<tr>
<th>Groups</th>
<th>Name</th>
<th>Variance</th>
<th>Std.Dev.</th>
<th>Corr</th>
</tr>
</thead>
<tbody>
<tr>
<td>id</td>
<td>(Intercept)</td>
<td>1.9279</td>
<td>1.3885</td>
<td></td>
</tr>
<tr>
<td></td>
<td>time</td>
<td>0.5765</td>
<td>0.7592</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>Residual</td>
<td>0.6223</td>
<td>0.7889</td>
<td></td>
</tr>
</tbody>
</table>

**Number of obs: 1600, groups: id, 400**

**Fixed effects:**

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Std. Error</th>
<th>t value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>0.61716</td>
<td>0.07687</td>
</tr>
<tr>
<td>time</td>
<td>1.00519</td>
<td>0.04186</td>
</tr>
</tbody>
</table>
example 2: 1-factor model, cluster size = 3

- model in the multilevel SEM framework:
multilevel SEM syntax

> longData <- read.table("FCovRIcovWB.dat")
> names(longData) <- c("y1", "y2","y3", "y4", "x", "clus")
> model.long <- ' + level: 1 +   fw =~ y1 + y2 + y3 + y4 + level: 2 +   fb =~ y1 + y2 + y3 + y4 + y1 =~ 0*y1 + y2 =~ 0*y2 + y3 =~ 0*y3 + y4 =~ 0*y4 + ' > fit.long <- sem(model.long, data = longData, cluster = "clus", + fixed.x = FALSE) > summary(fit.long)

lavaan (0.6-1.1173) converged normally after 28 iterations

  Number of observations 1200
  Number of clusters [clus] 400

  Estimator ML
  Model Fit Test Statistic 6.432
  Degrees of freedom 8
  P-value (Chi-square) 0.599
Parameter Estimates:

Information Observed
Observed information based on Hessian
Standard Errors Standard

Level 1 [within]:

Latent Variables:

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| fw =~  |          |         |         |         |
| y1    | 1.000    |         |         |         |
| y2    | 1.039    | 0.059   | 17.552  | 0.000   |
| y3    | 0.942    | 0.052   | 18.136  | 0.000   |
| y4    | 0.985    | 0.058   | 17.024  | 0.000   |

Intercepts:

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| .y1   | 0.000    |         |         |         |
| .y2   | 0.000    |         |         |         |
| .y3   | 0.000    |         |         |         |
| .y4   | 0.000    |         |         |         |
| fw    | 0.000    |         |         |         |

Variances:

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| .y1   | 0.491    | 0.027   | 17.880  | 0.000   |
Level 2 [clus]:

Latent Variables:

|        | Estimate | Std.Err | z-value | P(>|z|) |
|--------|----------|---------|---------|---------|
| fb =~  |          |         |         |         |
| y1     | 1.000    |         |         |         |
| y2     | 0.886    | 0.098   | 9.087   | 0.000   |
| y3     | 0.977    | 0.095   | 10.328  | 0.000   |
| y4     | 0.871    | 0.098   | 8.847   | 0.000   |

Intercepts:

|        | Estimate | Std.Err | z-value | P(>|z|) |
|--------|----------|---------|---------|---------|
| .y1    | -0.040   | 0.038   | -1.045  | 0.296   |
| .y2    | -0.049   | 0.037   | -1.335  | 0.182   |
| .y3    | -0.034   | 0.037   | -0.906  | 0.365   |
| .y4    | -0.034   | 0.037   | -0.926  | 0.354   |
| fb     | 0.000    |         |         |         |

Variances:

<p>|        | Estimate | Std.Err | z-value | P(&gt;|z|) |
|--------|----------|---------|---------|---------|
| .y1    | 0.000    |         |         |         |
| .y2    | 0.000    |         |         |         |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>.y3</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.y4</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>fb</td>
<td>0.241</td>
<td>0.046</td>
<td>5.294</td>
</tr>
</tbody>
</table>
wide-format syntax

```r
> wideData <- matrix(lav_matrix_vecr(longData[,1:5]), 400, 15, byrow = TRUE)
> wideData <- as.data.frame(wideData)
> names(wideData) <- paste(rep(c("y1","y2","y3","y4","x"), 3),
+ rep(1:3, each = 5), sep = ".")
> model.wide <- '
+    # WITHIN #
+
+    # within factors, common loadings, common (zero) means, common variance
+    fw1 =~ 1*y1.1 + lw2*y2.1 + lw3*y3.1 + lw4*y4.1
+    fw2 =~ 1*y1.2 + lw2*y2.2 + lw3*y3.2 + lw4*y4.2
+    fw3 =~ 1*y1.3 + lw2*y2.3 + lw3*y3.3 + lw4*y4.3
+    fw1 ~~ fvw*fw1
+    fw2 ~~ fvw*fw2
+    fw3 ~~ fvw*fw3
+
+    # uncorrelated fw1, fw2, fw3
+    fw1 ~~ 0*fw2 + 0*fw3; fw2 ~~ 0*fw3
+
+    # within intercepts (fixed to zero)
+    y1.1 + y2.1 + y3.1 + y4.1 ~ 0*1
+    y1.2 + y2.2 + y3.2 + y4.2 ~ 0*1
+    y1.3 + y2.3 + y3.3 + y4.3 ~ 0*1
+
+    # common residual variances
+    y1.1 ~~ rw1*y1.1; y1.2 ~~ rw1*y1.2; y1.3 ~~ rw1*y1.3
+    y2.1 ~~ rw2*y2.1; y2.2 ~~ rw2*y2.2; y2.3 ~~ rw2*y2.3
```
+ y3.1 ~~ rw3*y3.1; y3.2 ~~ rw3*y3.2; y3.3 ~~ rw3*y3.3
+ y4.1 ~~ rw4*y4.1; y4.2 ~~ rw4*y4.2; y4.3 ~~ rw4*y4.3
+
+ # BETWEEN #
+
+ # between version of y1,y2,y3,y4
+ by1 =~ 1*y1.1 + 1*y1.2 + 1*y1.3
+ by2 =~ 1*y2.1 + 1*y2.2 + 1*y2.3
+ by3 =~ 1*y3.1 + 1*y3.2 + 1*y3.3
+ by4 =~ 1*y4.1 + 1*y4.2 + 1*y4.3
+
+ # between intercepts
+ by1 + by2 + by3 + by4 ~ 1
+
+ # optional: zero residual variances
+ by1 ~~ 0*by1; by2 ~~ 0*by2; by3 ~~ 0*by3; by4 ~~ 0*by4
+
+ # between factor
+ fb =~ by1 + by2 + by3 + by4
+
+ # not correlated with the within lvs
+ fb ~~ 0*fw1 + 0*fw2 + 0*fw3
+
+ '
> fit.wide <- sem(model.wide, data = wideData, information = "observed")
> summary(fit.wide)

lavaan (0.6-1.1173) converged normally after 27 iterations
Number of observations 400

Estimator ML
Model Fit Test Statistic 69.728
Degrees of freedom 74
P-value (Chi-square) 0.619

Parameter Estimates:

Information
Observed information based on Hessian
Standard Errors Standard

Latent Variables:

| Latent Variable | Estimate | Std.Err | z-value | P(>|z|) |
|-----------------|----------|---------|---------|---------|
| fw1 =~          |          |         |         |         |
| y1.1            | 1.000    |         |         |         |
| y2.1 (lw2)      | 1.039    | 0.059   | 17.552  | 0.000   |
| y3.1 (lw3)      | 0.942    | 0.052   | 18.136  | 0.000   |
| y4.1 (lw4)      | 0.985    | 0.058   | 17.024  | 0.000   |
| fw2 =~          |          |         |         |         |
| y1.2            | 1.000    |         |         |         |
| y2.2 (lw2)      | 1.039    | 0.059   | 17.552  | 0.000   |
| y3.2 (lw3)      | 0.942    | 0.052   | 18.136  | 0.000   |
| y4.2 (lw4)      | 0.985    | 0.058   | 17.024  | 0.000   |
| fw3 =~          |          |         |         |         |
| y1.3            | 1.000    |         |         |         |
\[
\begin{align*}
y_{2.3} & \sim (lw2) & 1.039 & 0.059 & 17.552 & 0.000 \\
y_{3.3} & \sim (lw3) & 0.942 & 0.052 & 18.136 & 0.000 \\
y_{4.3} & \sim (lw4) & 0.985 & 0.058 & 17.024 & 0.000 \\
y_{1.1} & \sim 1.000 \\
y_{1.2} & \sim 1.000 \\
y_{1.3} & \sim 1.000 \\
y_{2.1} & \sim 1.000 \\
y_{2.2} & \sim 1.000 \\
y_{2.3} & \sim 1.000 \\
y_{3.1} & \sim 1.000 \\
y_{3.2} & \sim 1.000 \\
y_{3.3} & \sim 1.000 \\
y_{4.1} & \sim 1.000 \\
y_{4.2} & \sim 1.000 \\
y_{4.3} & \sim 1.000 \\
b_{1} & \sim 1.000 \\
b_{2} & \sim 0.886 & 0.098 & 9.087 & 0.000 \\
b_{3} & \sim 0.977 & 0.095 & 10.328 & 0.000 \\
b_{4} & \sim 0.871 & 0.098 & 8.847 & 0.000 \\
\end{align*}
\]
\[
\begin{align*}
fw2 & \quad 0.000 \\
fw3 & \quad 0.000 \\
fw2 \sim & \quad 0.000 \\
fw3 & \quad 0.000 \\
fw1 \sim & \quad 0.000 \\
fb & \quad 0.000 \\
fw2 \sim & \quad 0.000 \\
fb & \quad 0.000 \\
fw3 \sim & \quad 0.000 \\
fb & \quad 0.000 \\
\end{align*}
\]

Intercepts:

|       | Estimate | Std. Err | z-value | P(>|z|) |
|-------|----------|----------|----------|---------|
| .y1.1 | 0.000    |          |          |         |
| .y2.1 | 0.000    |          |          |         |
| .y3.1 | 0.000    |          |          |         |
| .y4.1 | 0.000    |          |          |         |
| .y1.2 | 0.000    |          |          |         |
| .y2.2 | 0.000    |          |          |         |
| .y3.2 | 0.000    |          |          |         |
| .y4.2 | 0.000    |          |          |         |
| .y1.3 | 0.000    |          |          |         |
| .y2.3 | 0.000    |          |          |         |
| .y3.3 | 0.000    |          |          |         |
| .y4.3 | 0.000    |          |          |         |
| by1   | -0.040   | 0.038    | -1.045   | 0.296   |
| by2   | -0.049   | 0.037    | -1.335   | 0.182   |
| by3   | -0.034   | 0.037    | -0.906   | 0.365   |
|     | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| by4 | -0.034   | 0.037   | -0.926  | 0.354   |
| fw1 | 0.000    |         |         |         |
| fw2 | 0.000    |         |         |         |
| fw3 | 0.000    |         |         |         |
| fb  | 0.000    |         |         |         |

**Variances:**

|     | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| fw1 (fvw) | 0.558 | 0.051 | 10.910 | 0.000 |
| fw2 (fvw) | 0.558 | 0.051 | 10.910 | 0.000 |
| fw3 (fvw) | 0.558 | 0.051 | 10.910 | 0.000 |
| y1.1 (rw1) | 0.491 | 0.027 | 17.880 | 0.000 |
| y1.2 (rw1) | 0.491 | 0.027 | 17.880 | 0.000 |
| y1.3 (rw1) | 0.491 | 0.027 | 17.880 | 0.000 |
| y2.1 (rw2) | 0.473 | 0.028 | 16.995 | 0.000 |
| y2.2 (rw2) | 0.473 | 0.028 | 16.995 | 0.000 |
| y2.3 (rw2) | 0.473 | 0.028 | 16.995 | 0.000 |
| y3.1 (rw3) | 0.481 | 0.026 | 18.443 | 0.000 |
| y3.2 (rw3) | 0.481 | 0.026 | 18.443 | 0.000 |
| y3.3 (rw3) | 0.481 | 0.026 | 18.443 | 0.000 |
| y4.1 (rw4) | 0.521 | 0.028 | 18.506 | 0.000 |
| y4.2 (rw4) | 0.521 | 0.028 | 18.506 | 0.000 |
| y4.3 (rw4) | 0.521 | 0.028 | 18.506 | 0.000 |
| by1 | 0.000    |         |         |         |
| by2 | 0.000    |         |         |         |
| by3 | 0.000    |         |         |         |
| by4 | 0.000    |         |         |         |
| fb  | 0.241    | 0.046   | 5.294   | 0.000   |
(optional) wide-format syntax saturated model

> model.sat <- ' 
+ # WITHIN # 
+ 
+ # common variances 
+ y1.1 ~ vw1*y1.1; y1.2 ~ vw1*y1.2; y1.3 ~ vw1*y1.3 
+ y2.1 ~ vw2*y2.1; y2.2 ~ vw2*y2.2; y2.3 ~ vw2*y2.3 
+ y3.1 ~ vw3*y3.1; y3.2 ~ vw3*y3.2; y3.3 ~ vw3*y3.3 
+ y4.1 ~ vw4*y4.1; y4.2 ~ vw4*y4.2; y4.3 ~ vw4*y4.3
+ 
+ # common covariances 
+ y1.1 ~ cw12*y2.1 + cw13*y3.1 + cw14*y4.1; y2.1 ~ cw23*y3.1 + cw24*y4.1; y3 
+ y1.2 ~ cw12*y2.2 + cw13*y3.2 + cw14*y4.2; y2.2 ~ cw23*y3.2 + cw24*y4.2; y3 
+ y1.3 ~ cw12*y2.3 + cw13*y3.3 + cw14*y4.3; y2.3 ~ cw23*y3.3 + cw24*y4.3; y3 
+ 
+ # within means (fixed to zero) 
+ y1.1 + y2.1 + y3.1 + y4.1 ~ 0*1 
+ y1.2 + y2.2 + y3.2 + y4.2 ~ 0*1 
+ y1.3 + y2.3 + y3.3 + y4.3 ~ 0*1 
+ 
+ # BETWEEN # 
+ 
+ # between version of y1,y2,y3,y4 
+ by1 =~ 1*y1.1 + 1*y1.2 + 1*y1.3 
+ by2 =~ 1*y2.1 + 1*y2.2 + 1*y2.3 
+ by3 =~ 1*y3.1 + 1*y3.2 + 1*y3.3 
+ by4 =~ 1*y4.1 + 1*y4.2 + 1*y4.3
+  # between intercepts
+  by1 + by2 + by3 + by4 ~ 1
+  
+  # between variances
+  by1 ~~ by1; by2 ~~ by2; by3 ~~ by3; by4 ~~ by4
+  
+  # between covariances
+  by1 ~~ by2 + by3 + by4
+  by2 ~~ by3 + by4
+  by3 ~~ by4
+ ' 

> fit.sat <- sem(model.sat, data = wideData)
> lavTestLRT(fit.sat, fit.wide)

Chi Square Difference Test

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>AIC</th>
<th>BIC</th>
<th>Chisq</th>
<th>Chisq diff</th>
<th>Df diff</th>
<th>Pr(&gt;Chisq)</th>
</tr>
</thead>
<tbody>
<tr>
<td>fit.sat</td>
<td>66</td>
<td>12565</td>
<td>12660</td>
<td>62.948</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fit.wide</td>
<td>74</td>
<td>12555</td>
<td>12619</td>
<td>69.728</td>
<td>6.7799</td>
<td>8</td>
<td>0.5606</td>
</tr>
</tbody>
</table>
3.2 The ‘survey’ (design-based) approach

- literature:


- only if all variables (and constructs) are at the within-level only

- we treat the clustering as a (sampling) nuisance

- the parameter estimates are ‘aggregated’: they consistently estimate parameters aggregated over any clusters and strata and no explicit modeling of the effects of clusters and strata is involved

- standard errors are design-based

- allows for incorporation of clustering, stratification, unequal probability weights, finite population correction, and multiple imputation
setting up a two-factor model in lavaan with clustered data

> library(lavaan.survey)

> # step 1:
> # fit model ignoring clustering using estimator = "MLM"
> model <- ' 
+ numeric =~ wordlist + cards + matrices 
+ perception =~ figures + animals + occupats 
+ ' 
> fit.naive <- sem(model, data = FamIQData, std.lv = TRUE, 
+ estimator = "MLM")

> # step 2:
> # create a survey design object with family clustering
> survey.design <- svydesign(ids = ~family, prob = ~1, data = FamIQData)

> # step 3:
> # refit, taking survey.design into account
> fit.survey <- lavaan.survey(lavaan.fit = fit.naive, 
+ survey.design = survey.design)
> summary(fit.survey)

lavaan (0.6-1.1173) converged normally after 33 iterations

Number of observations 399
Estimator	ML	Robust
Model Fit Test Statistic	9.786	7.671
Degrees of freedom	8	8
P-value (Chi-square)	0.280	0.466
Scaling correction factor
for the Satorra-Bentler correction

Parameter Estimates:

Information
Information saturated (h1) model Structured
Standard Errors Robust.sem

Latent Variables:

| Latent Variable | Estimate | Std.Err | z-value | P(>|z|) |
|-----------------|----------|---------|---------|---------|
| numeric =~      |          |         |         |         |
| wordlist        | 4.346    | 0.338   | 12.873  | 0.000   |
| cards           | 4.286    | 0.303   | 14.147  | 0.000   |
| matrices        | 4.000    | 0.262   | 15.248  | 0.000   |
| perception =~   |          |         |         |         |
| figures         | 4.104    | 0.258   | 15.881  | 0.000   |
| animals         | 4.446    | 0.254   | 17.501  | 0.000   |
| occupats        | 4.298    | 0.292   | 14.720  | 0.000   |

Covariances:

| Latent Variable | Estimate | Std.Err | z-value | P(>|z|) |
|-----------------|----------|---------|---------|---------|
| numeric ~ perception | 0.676    | 0.041   | 16.405  | 0.000   |
### Intercepts:

|                | Estimate | Std.Err | z-value | P(>|z|) |
|----------------|----------|---------|---------|---------|
| .wordlist      | 29.947   | 0.449   | 66.671  | 0.000   |
| .cards         | 29.845   | 0.448   | 66.552  | 0.000   |
| .matrices      | 29.734   | 0.437   | 68.069  | 0.000   |
| .figures       | 30.075   | 0.457   | 65.786  | 0.000   |
| .animals       | 30.108   | 0.451   | 66.815  | 0.000   |
| .occupats      | 30.008   | 0.481   | 62.438  | 0.000   |
| numeric        | 0.000    |         |         |         |
| perception     | 0.000    |         |         |         |

### Variances:

|                | Estimate | Std.Err | z-value | P(>|z|) |
|----------------|----------|---------|---------|---------|
| .wordlist      | 7.262    | 0.898   | 8.087   | 0.000   |
| .cards         | 6.801    | 0.812   | 8.381   | 0.000   |
| .matrices      | 8.382    | 0.747   | 11.225  | 0.000   |
| .figures       | 9.260    | 0.839   | 11.034  | 0.000   |
| .animals       | 5.433    | 0.648   | 8.385   | 0.000   |
| .occupats      | 6.923    | 0.830   | 8.343   | 0.000   |
| numeric        | 1.000    |         |         |         |
| perception     | 1.000    |         |         |         |
3.3 MUthén ML (MUML) estimator using multiple group SEM

sample statistics

- the total covariance (at the population level) can be decomposed as
  \[ \text{Cov}(y) = \Sigma_T = \Sigma_W + \Sigma_B \]

- we can decompose the sample data in a similar way:
  \[ S_T = S_W + S_B \]

- it is tempting to fit a within-cluster model using \( S_W \), and a between-cluster model using \( S_B \), but unfortunately, we can not use \( S_W \) as an estimator of \( \Sigma_W \) and \( S_B \) as an estimator of \( \Sigma_B \)

- if we have balanced data (same number of observations in each cluster), we can estimate \( \Sigma_W \) by the pooled within-clusters covariance matrix:
  \[ \hat{\Sigma}_W = S_{PW} = \frac{\sum_{j=1}^{J} \sum_{i=1}^{N_j} (y_{ji} - \bar{y}_j)(y_{ji} - \bar{y}_j)'}{N - J} \]
where \( N = \sum_{j=1}^{J} N_j \) is the total sample size

- the ‘scaled’ between-clusters covariance matrix can be estimated by

\[
S^*_B = \frac{\sum_{j=1}^{J} N_j (\overline{y} - \overline{y}_j)(\overline{y} - \overline{y}_j)'}{J - 1}
\]

- note: \( S^*_B \) is not an estimate of \( \Sigma_B \) (even in the balanced case)
- it can be shown that \( S^*_B \) is an estimate of the sum of \( \Sigma_W \) and \( \Sigma_B \) where the latter term is scaled by the common cluster size \( s = N_g \):

\[
S^*_B = \Sigma_W + s \cdot \Sigma_B
\]

- Muthén (1989, 1990, 1994) suggested to fit a two-level SEM model by using a conventional multigroup SEM analysis based on \( S_{PW} \) and \( S^*_B \)
- in the balanced case, this results in ML estimates; in the unbalanced case this is called MUML
the MUML setup

• the within part of the model is specified both for the within ‘group’ and the between ‘group’ with equality constraints to ensure that the same within structure is estimated for both groups

• the between part of the model is only specified for the between ‘group’ but with a scale factor of $\sqrt{s}$ hard-wired in the lambda matrix

• although tricky to set up, this results in ML estimates for a two-level SEM model in the balanced case

• if the data is unbalanced, we replace $s$ by

$$ s^* = \left[ N^2 - \sum_{j=1}^{J} n_j^2 \right] / (N(J - 1)) $$
## sample statistics in R

```r
> library(lavaan)
> Data <- read.table("hoxdata1.dat",header = FALSE)
> ov.names <- c("y1","y2","y3","y4","y5","y6")
> names(Data) <- c(ov.names, "cluster")
> cluster.size <- as.integer(table(Data$cluster))
> nClusters <- length(cluster.size)
> N <- sum(cluster.size)
> # between sample statistics
> DataB <- with(Data, aggregate(Data[,ov.names],
+ by = list(cluster), FUN = mean))[,ov.names]
> # weighted mean
> B.mean <- colSums(DataB * cluster.size/N)
> # weighted cov
> DataBc <- as.matrix(sweep(DataB, 2, STATS=B.mean))
> B.cov <- crossprod(DataBc * cluster.size, DataBc) / (nClusters - 1)
> # within
> cluster.idx <- as.integer(as.factor(Data[,"cluster"]))
> DataW <- Data[,ov.names] - DataB[cluster.idx,]
> W.mean <- colMeans(DataW)
> W.cov <- cov(DataW) * (N - 1) / (N - nClusters)
> # s (average sample size)
> s <- (N^2 - sum(cluster.size^2)) * 1/(N * (nClusters - 1))
> # 20
> # B.cov.star
> B.cov.star <- (B.cov - W.cov)/s
```
setting up the syntax for MULM in lavaan

```r
> model <- ' 
+  ## within model 
+  
+  # factor loadings 
+  f1 =~ c(11,11)*y1 + c(12,12)*y2 + c(13,13)*y3 
+  f2 =~ c(14,14)*y4 + c(15,15)*y5 + c(16,16)*y6 
+  
+  # residual variances 
+  y1 ~~ c(1,1)*y1; y2 ~~ c(1,1)*y2; y3 ~~ c(1,1)*y3 
+  y4 ~~ c(1,1)*y4; y5 ~~ c(1,1)*y5; y6 ~~ c(1,1)*y6 
+  
+  # factor variances/covariances 
+  f1 ~~ c(1,1)*f1; f2 ~~ c(1,1)*f2; f1 ~~ c(1,1)*f2 
+  
+  # means 
+  # y1 + y2 + y3 + y4 + y5 + y6 ~ 0*1 
+  # f1 + f2 ~ 0*1 
+  
+  ## between model 
+  
+  y1b =~ c(0,4.472136)*y1 
+  y2b =~ c(0,4.472136)*y2 
+  y3b =~ c(0,4.472136)*y3 
+  y4b =~ c(0,4.472136)*y4 
+  y5b =~ c(0,4.472136)*y5 
+  y6b =~ c(0,4.472136)*y6 
```
\[
\text{fb} = \sim c(0, \text{NA}) \ast y_{1b} + c(0, \text{NA}) \ast y_{2b} + c(0, \text{NA}) \ast y_{3b} + \\
c(0, \text{NA}) \ast y_{4b} + c(0, \text{NA}) \ast y_{5b} + c(0, \text{NA}) \ast y_{6b}
\]

# residuals
\[
y_{1b} \sim c(0, \text{NA}) \ast y_{1b}; y_{2b} \sim c(0, \text{NA}) \ast y_{2b}; y_{3b} \sim c(0, \text{NA}) \ast y_{3b}
\]
\[
y_{4b} \sim c(0, \text{NA}) \ast y_{4b}; y_{5b} \sim c(0, \text{NA}) \ast y_{5b}; y_{6b} \sim c(0, \text{NA}) \ast y_{6b}
\]

# fb variance
\[
\text{fb} \sim c(0, 1) \ast \text{fb}
\]

# means
\[
y_{1b} \sim c(0, \text{NA}) \ast 1
\]
\[
y_{2b} \sim c(0, \text{NA}) \ast 1
\]
\[
y_{3b} \sim c(0, \text{NA}) \ast 1
\]
\[
y_{4b} \sim c(0, \text{NA}) \ast 1
\]
\[
y_{5b} \sim c(0, \text{NA}) \ast 1
\]
\[
y_{6b} \sim c(0, \text{NA}) \ast 1
\]

\[
\text{fb} \sim c(0, 0) \ast 1
\]

> fit <- lavaan(model, sample.cov = list(within=W.cov, between=B.cov),
+               sample.cov.rescale = FALSE,
+               sample.mean = list(W.mean, B.mean),
+               sample.nobs = list(N-nClusters, nClusters), std.lv = TRUE)
> summary(fit)

lavaan (0.6-1.1173) converged normally after 69 iterations
Number of observations per group
within  1900
between 100

Estimator  ML
Model Fit Test Statistic  10.763
Degrees of freedom  23
P-value (Chi-square)  0.985

Chi-square for each group:
within 2.668
between 8.096

Parameter Estimates:

<table>
<thead>
<tr>
<th>Information</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Information saturated (h1) model</td>
<td>Structured</td>
</tr>
<tr>
<td>Standard Errors</td>
<td>Standard</td>
</tr>
</tbody>
</table>

Group 1 [within]:

Latent Variables:

| Estimate | Std.Err | z-value | P(>|z|) |
|----------|---------|---------|---------|
| f1 =~ y1  | 0.300   | 0.016   | 18.507  | 0.000   |
|      | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| y2  | 0.397    | 0.019   | 20.955  | 0.000   |
| y3  | 0.494    | 0.022   | 22.267  | 0.000   |
| f2  | ~        |         |         |         |
| y4  | 0.305    | 0.016   | 19.001  | 0.000   |
| y5  | 0.382    | 0.019   | 20.394  | 0.000   |
| y6  | 0.527    | 0.022   | 23.579  | 0.000   |
| y1b | ~        |         |         |         |
| y1  | 0.000    |         |         |         |
| y2b | ~        |         |         |         |
| y2  | 0.000    |         |         |         |
| y3b | ~        |         |         |         |
| y3  | 0.000    |         |         |         |
| y4b | ~        |         |         |         |
| y4  | 0.000    |         |         |         |
| y5b | ~        |         |         |         |
| y5  | 0.000    |         |         |         |
| y6b | ~        |         |         |         |
| y6  | 0.000    |         |         |         |
| fb  | ~        |         |         |         |
| y1b | 0.000    |         |         |         |
| y2b | 0.000    |         |         |         |
| y3b | 0.000    |         |         |         |
| y4b | 0.000    |         |         |         |
| y5b | 0.000    |         |         |         |
| y6b | 0.000    |         |         |         |

Covariances:

|      | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|

Yves Rosseel  Multilevel Structural Equation Modeling  with lavaan
\f1 \sim^\sim
\begin{align*}
\f2 & (v12) & -0.013 & 0.034 & -0.384 & 0.701 \\
\end{align*}

Intercepts:

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| y1b | 0.000    |         |         |         |
| y2b | 0.000    |         |         |         |
| y3b | 0.000    |         |         |         |
| y4b | 0.000    |         |         |         |
| y5b | 0.000    |         |         |         |
| y6b | 0.000    |         |         |         |
| fb  | 0.000    |         |         |         |
| .y1 | 0.000    |         |         |         |
| .y2 | 0.000    |         |         |         |
| .y3 | 0.000    |         |         |         |
| .y4 | 0.000    |         |         |         |
| .y5 | 0.000    |         |         |         |
| .y6 | 0.000    |         |         |         |
| f1  | 0.000    |         |         |         |
| f2  | 0.000    |         |         |         |

Variances:

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| .y1 (r1) | 0.251 | 0.010 | 24.509 | 0.000 |
| .y2 (r2) | 0.247 | 0.014 | 18.292 | 0.000 |
| .y3 (r3) | 0.262 | 0.019 | 13.884 | 0.000 |
| .y4 (r4) | 0.243 | 0.010 | 24.187 | 0.000 |
| .y5 (r5) | 0.281 | 0.013 | 20.996 | 0.000 |
| Latent Variable | Estimate | Std.Err | z-value | P(>|z|) |
|----------------|----------|---------|---------|---------|
| f1 =~           |          |         |         |         |
| y1 (11)        | 0.300    | 0.016   | 18.507  | 0.000   |
| y2 (12)        | 0.397    | 0.019   | 20.955  | 0.000   |
| y3 (13)        | 0.494    | 0.022   | 22.267  | 0.000   |
| f2 =~           |          |         |         |         |
| y4 (14)        | 0.305    | 0.016   | 19.001  | 0.000   |
| y5 (15)        | 0.382    | 0.019   | 20.394  | 0.000   |
| y6 (16)        | 0.527    | 0.022   | 23.579  | 0.000   |

Group 2 [between]:

<table>
<thead>
<tr>
<th>Latent Variable</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>y1b =~</td>
<td>4.472</td>
</tr>
<tr>
<td>y1</td>
<td>4.472</td>
</tr>
<tr>
<td>y2b =~</td>
<td>4.472</td>
</tr>
<tr>
<td>y2</td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{align*}
y_{3b} &= y_3 \\
y_{4b} &= y_4 \\
y_{5b} &= y_5 \\
y_{6b} &= y_6 \\
f_b &= f_1 \\
\end{align*}
\]

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| y_{1b} | 0.468    | 0.071   | 6.603   | 0.000   |
| y_{2b} | 0.449    | 0.070   | 6.384   | 0.000   |
| y_{3b} | 0.260    | 0.065   | 3.986   | 0.000   |
| y_{4b} | 0.530    | 0.069   | 7.671   | 0.000   |
| y_{5b} | 0.425    | 0.069   | 6.123   | 0.000   |
| y_{6b} | 0.282    | 0.065   | 4.368   | 0.000   |

\textbf{Covariances:}

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| f_1   | -0.013   | 0.034   | -0.384  | 0.701   |
| f_2   | (v_{12}) |         |         |         |

\textbf{Intercepts:}

|       | Estimate | Std.Err | z-value | P(>|z|) |
|-------|----------|---------|---------|---------|
| y_{1b} | -0.003   | 0.071   | -0.038  | 0.969   |
| y_{2b} | -0.000   | 0.070   | -0.007  | 0.995   |
| y_{3b} | 0.014    | 0.061   | 0.236   | 0.813   |
| y_{4b} | -0.004   | 0.071   | -0.062  | 0.950   |
| y_{5b} | 0.006    | 0.068   | 0.087   | 0.930   |
| Variance | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| y6b      | 0.002    | 0.061   | 0.026   | 0.979   |
| fb       | 0.000    |         |         |         |
| .y1      | 0.000    |         |         |         |
| .y2      | 0.000    |         |         |         |
| .y3      | 0.000    |         |         |         |
| .y4      | 0.000    |         |         |         |
| .y5      | 0.000    |         |         |         |
| .y6      | 0.000    |         |         |         |
| f1       | 0.000    |         |         |         |
| f2       | 0.000    |         |         |         |

Variances:

| Variance | Estimate | Std.Err | z-value | P(>|z|) |
|----------|----------|---------|---------|---------|
| .y1      | 0.251    | 0.010   | 24.509  | 0.000   |
| .y2      | 0.247    | 0.014   | 18.292  | 0.000   |
| .y3      | 0.262    | 0.019   | 13.884  | 0.000   |
| .y4      | 0.243    | 0.010   | 24.187  | 0.000   |
| .y5      | 0.281    | 0.013   | 20.996  | 0.000   |
| .y6      | 0.216    | 0.020   | 10.811  | 0.000   |
| f1       | 1.000    |         |         |         |
| f2       | 1.000    |         |         |         |
| y1b      | 0.264    | 0.050   | 5.291   | 0.000   |
| y2b      | 0.266    | 0.049   | 5.383   | 0.000   |
| y3b      | 0.277    | 0.045   | 6.171   | 0.000   |
| y4b      | 0.205    | 0.047   | 4.356   | 0.000   |
| y5b      | 0.267    | 0.048   | 5.525   | 0.000   |
| y6b      | 0.266    | 0.044   | 6.102   | 0.000   |
| fb       | 1.000    |         |         |         |
same model using ML

> model.long <- '  
+   level: 1  
+     f1 =~ y1 + y2 + y3  
+     f2 =~ y4 + y5 + y6  
+   level: 2  
+     f1 =~ y1 + y2 + y3  
+     f2 =~ y4 + y5 + y6  
+ '  
> fit.long <- sem(model.long, data = Data, cluster = "cluster",  
+                  std.lv = TRUE)  
> summary(fit.long)

lavaan (0.6-1.1173) converged normally after 47 iterations

<table>
<thead>
<tr>
<th>Information</th>
<th>Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of observations</td>
<td>2000</td>
</tr>
<tr>
<td>Number of clusters [cluster]</td>
<td>100</td>
</tr>
<tr>
<td>Estimator</td>
<td>ML</td>
</tr>
<tr>
<td>Model Fit Test Statistic</td>
<td>10.413</td>
</tr>
<tr>
<td>Degrees of freedom</td>
<td>16</td>
</tr>
<tr>
<td>P-value (Chi-square)</td>
<td>0.844</td>
</tr>
</tbody>
</table>

Parameter Estimates:
Level 1 [within]:

Latent Variables:

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| f1 =~ |          |         |         |         |
| y1  | 0.300    | 0.016   | 18.501  | 0.000   |
| y2  | 0.397    | 0.019   | 20.949  | 0.000   |
| y3  | 0.494    | 0.022   | 22.260  | 0.000   |
| f2 =~ |          |         |         |         |
| y4  | 0.304    | 0.016   | 18.995  | 0.000   |
| y5  | 0.382    | 0.019   | 20.382  | 0.000   |
| y6  | 0.527    | 0.022   | 23.556  | 0.000   |

Covariances:

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| f1 ~ f2 | -0.013  | 0.034   | -0.374  | 0.709   |

Intercepts:

|   | Estimate | Std.Err | z-value | P(>|z|) |
|---|----------|---------|---------|---------|
| .y1 | 0.000    |         |         |         |
| .y2 | 0.000    |         |         |         |
| .y3 | 0.000    |         |         |         |
| .y4 | 0.000    |         |         |         |
| .y5 | 0.000    |         |         |         |
### Variances:

|     | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| .y6 | 0.000    |         |         |         |
| f1  | 0.000    |         |         |         |
| f2  | 0.000    |         |         |         |

### Level 2 [cluster]:

#### Latent Variables:

|     | Estimate | Std.Err | z-value | P(>|z|) |
|-----|----------|---------|---------|---------|
| f1  =~ |          |         |         |         |
| y1  | 0.475    | 0.074   | 6.452   | 0.000   |
| y2  | 0.457    | 0.073   | 6.300   | 0.000   |
| y3  | 0.260    | 0.066   | 3.946   | 0.000   |
| f2  =~ |          |         |         |         |
| y4  | 0.539    | 0.072   | 7.468   | 0.000   |
| y5  | 0.430    | 0.070   | 6.124   | 0.000   |
| y6  | 0.278    | 0.066   | 4.226   | 0.000   |
### Covariances:

|        | Estimate | Std.Err | z-value | P(>|z|) |
|--------|----------|---------|---------|---------|
| f1 ~ f2 | 0.944    | 0.094   | 10.054  | 0.000   |

### Intercepts:

|        | Estimate | Std.Err | z-value | P(>|z|) |
|--------|----------|---------|---------|---------|
| .y1    | -0.012   | 0.070   | -0.172  | 0.863   |
| .y2    | -0.002   | 0.070   | -0.029  | 0.976   |
| .y3    | 0.064    | 0.061   | 1.062   | 0.288   |
| .y4    | -0.020   | 0.071   | -0.280  | 0.779   |
| .y5    | 0.027    | 0.068   | 0.393   | 0.695   |
| .y6    | 0.007    | 0.061   | 0.116   | 0.908   |
| f1     | 0.000    |         |         |         |
| f2     | 0.000    |         |         |         |

### Variances:

|        | Estimate | Std.Err | z-value | P(>|z|) |
|--------|----------|---------|---------|---------|
| .y1    | 0.252    | 0.053   | 4.763   | 0.000   |
| .y2    | 0.254    | 0.051   | 4.941   | 0.000   |
| .y3    | 0.274    | 0.045   | 6.123   | 0.000   |
| .y4    | 0.191    | 0.052   | 3.657   | 0.000   |
| .y5    | 0.258    | 0.049   | 5.315   | 0.000   |
| .y6    | 0.265    | 0.044   | 6.028   | 0.000   |
| f1     | 1.000    |         |         |         |
| f2     | 1.000    |         |         |         |
4 Last slide

- be careful with a small number of clusters (may lead to biased results)


- topics not discussed in this workshop:
  - construct reliability in the multilevel setting
  - mediation and moderation
  - random slopes
  - categorical outcomes
  - missing data
  - the gllamm framework

- when will lavaan 0.6 be officially released?