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CHAPTER 18

Growth Curve Analysis in Contemporary Psychological Research

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PREFACE

The term *growth curve* was originally used to describe a graphic display of the physical stature (e.g., the height or weight) of an individual over consecutive ages. Growth curves have unique features: (a) The same entities are repeatedly observed, (b) the same procedures of measurement and scaling of observations are used, and (c) the timing of the observations is known. The term *growth curve analysis* denotes the processes of describing, testing hypotheses, and making scientific inferences about the growth and change patterns in a wide range of time-related phenomena. In this sense, growth curve analyses are a specific form of the larger set of developmental and longitudinal research methods, but the unique features of growth data permit unique kinds of analyses.

Contemporary methods of growth curve analysis are considered here. Of course, the techniques to analyze growth data are among the most widely studied and well-developed mathematical and statistical techniques in all scientific research—growth curve analyses have roots in the 17th- and 18th-century calculus of Newton and probability of Pascal—but this chapter is concerned with more recent historical developments. Techniques for the analysis of growth curves were initiated in the physical sciences and were more fully

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developed in the biological sciences, where they were used in studies of the size and health of plants, animals, and humans. In the behavioral sciences, growth curve analyses have routinely been applied to a wide range of phenomena—from experimental learning curves, to the growth and decline of intellectual abilities and academic achievements, to changes in other psychological traits over the full life span.

These formal models for the analysis of growth curves have been developed in many different substantive domains, but all share a common goal—to examine and uncover a fundamental set of regularity conditions, or basic functions, responsible for the manifest growth and change. The goals of these models were organized in terms of five "objectives of longitudinal research" and described by Baltes and Nesselroade (1979, pp. 21–27) using the following enumeration:

1. The direct identification of *intra-* (within-) individual change
2. The direct identification of *inter-* (between-) individual differences in intra-individual change
3. The analysis of *interrelationships* in change
4. The analysis of *causes* (determinants) of intra-individual change
5. The analysis of *causes* (determinants) of interindividual differences in intra-individual change

In this chapter, growth curve analyses are related to these objectives of longitudinal research. In current statistical methodology, *intra-individual* is termed *within-person* and *interindividual* is termed *between-person*, but these remain the essential goals of most longitudinal data analyses (e.g., Campbell, 1988; McArdle & Bell, 2000).

This chapter is organized into the following sections: (a) an introduction to growth curves, (b) linear models of growth, (c) multiple groups in growth curve models, (d) aspects of dynamic theory for growth models, and (e) multiple variables in growth curve analyses. The chapter then concludes with a discussion of future issues raised by the current growth models. In all sections we try to present historical perspective to illustrate different kinds of mathematical and statistical issues for the analyses of these data.

The growth curve models are presented in basic algebraic detail, but this presentation is not intended to be overly technical. Instead, we focus on the mathematical formulation, statistical estimation, and substantive interpretation of *latent growth curve* analyses. This focus allows us to show a range of new models and examine why some classical data analysis problems, such as the calculation of difference scores or the unreliability of errors of measurements, are no longer impediments to development research. Other related techniques such as *time-series* and *dynamical systems* analyses are briefly discussed in the later sections of this chapter. All numerical results are based on a single set of data (the longitudinal data of Figure 18.6), and available computer software for these analyses is described. We use these illustrations to highlight both the benefits and limitations of contemporary growth curve analyses.

INTRODUCTION

Classical Growth Curve Applications

The collection of growth curve data is not a new topic. The first measurements classified as growth curve data appear to have been collected by the French Count de Montbeillard (~1759) and consist of semiannual measurements on the growth of the height of his son over the course of nearly 18 years; these data are plotted as the upper curve of Figure 18.1. As Scammon (1927) reported, "It will be noted that the curve shows the typical four phases which most modern students have observed in the postnatal growth in stature of man, and which are characteristic of the growth of so many parts of the body" (p. 331). The first analysis of these data, by the naturalist Buffon (~1799), "should be given full credit for the discovery of seasonal differences in growth a full hundred years before the modern investigation of this work" (p. 334). The lower growth curve in Figure 18.1 is based on group averages of physical growth obtained by Variot (~1908). As Scammon (1927) suggests, "it is interesting to note that, while the absolute values of the two series are quite different, the general

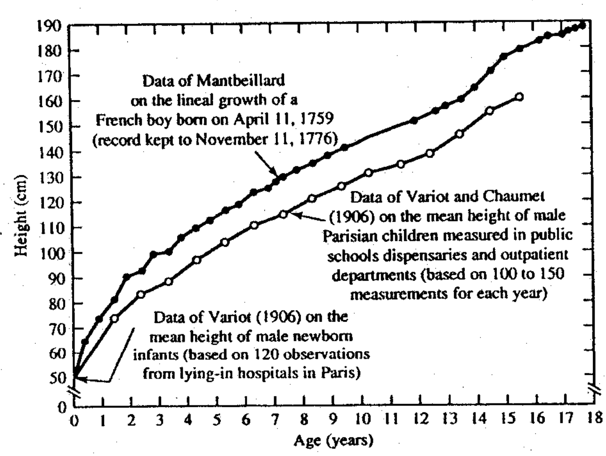

Figure 18.1 The initial growth curves of human height data from Scammon (1927, p. 334); the vertical (y) axis represents the height in cm and the horizontal (x) axis represents the age in years from birth.

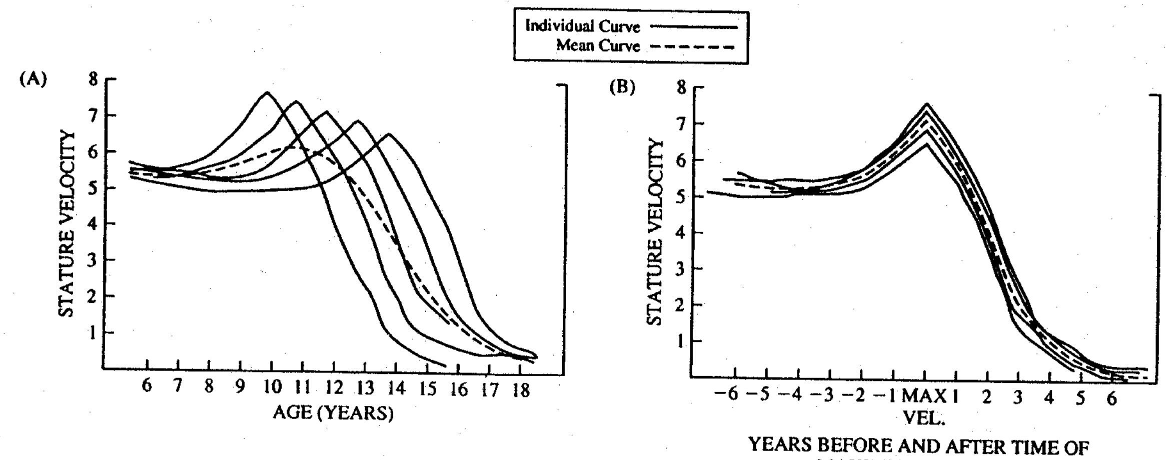

Figure 18.2 Alternative velocity curves of physical stature from Tanner (1960, p. 22); (a) The change in height as a function of the change in time (y) versus the age in years (x); (b) the same curves (y) plotted around the time of maximum change in scores (x) for each individual.

form of the curve is essentially the same in both instances" (p. 335).

These early growth curves were precursors to the collection of an enormous body of biological data on growth and change. More recent illustrations come from the important work of Tanner and his colleagues (1955, 1960). In the individual plots of height in Figure 18.2, (a) the *velocity* (or *rate of change*) is plotted at each age, and (b) these curves are plotted against their own *highest peak velocity*. This display demonstrated two interesting features of physical growth: (a) Persons who start growth at the earliest ages also attain the greatest height, but (b) all individuals share a remarkably similar shape in the "adolescent growth spurt." This relationship between chronological time and what has been called *biological time* remains an important substantive issue.

Experimental psychologists have routinely collected different kinds of growth curves. Among the first here were the classical *forgetting curves* collected by Ebbinghaus (~1880), and this introduced the use of quantitative methods in the study of learning and memory and stimulated many experimental data collections. Other classic examples are found in the animal learning curve experiments of Thorndike (~1911), in which *trial-and-error learning* was defined by decreasing response time, and the lack of smooth function over trials was considered error. Thorndike used these growth (or decline) curves to illustrate several classical principles of learning, including the *law of exercise* and the *law of effect* (for review, see Garrett, 1951; Estes, 1959). Other classic examples are found in the *acquisition curves* presented by Estes (1959) and reproduced here in Figure 18.3. The data collected here (i.e., the dots) were measured over the same animals (rats) working for consistent reward in a free operant Skinner box (a T-maze learning experiment), and the four plots show different aspects of the behaviors (i.e., responses, reinforcements, trials, time). These figures also show how the average probabilities and changes in probabilities were well predicted using mathematical models from statistical learning theory (Estes, 1959). The current emphasis on formal models for growth and change has obvious roots in this kind of experimental research.

Differential psychologists have also contributed growth data in many different substantive areas. One good example of this tradition is given in the plots of Figure 18.4 (from work of Bayley, 1956). Individual growth curves of mental abilities from birth to age 25 are plotted for a selected set of boys and girls from the well-known Berkeley Growth Study. Because mental ability was not easily measured in exactly the same way at each age, these individual curves were created by adjusting the means and standard deviations of different mental ability tests (i.e., Stanford-Binet, Terman-McNemar) at different ages into a common metric. As Bayley says,

> They are not in "absolute" units, but they do give a general picture of growth relative to the status of this group at 16 years. These curves, too, are less regular than the height curves, but perhaps no less regular than the weight curves. One gets the impression both of differences in rates of maturing and of differences in inherent capacity." (p. 66)

This application of "linked" measurement scales created a novel set of growth data, raised many issues about the comparability of measurement over time, and permitted the use of

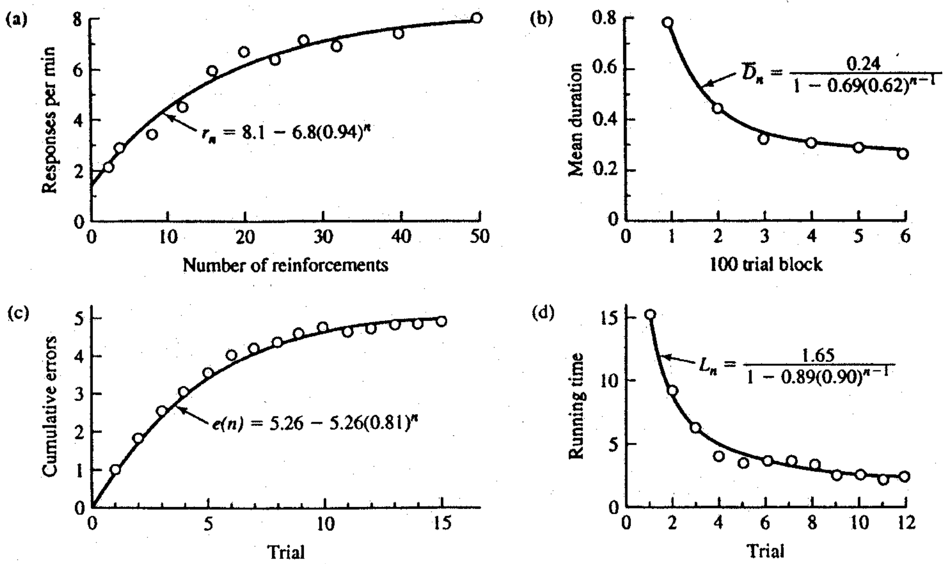

Figure 18.3 Selected acquisition curves of memory from statistical learning theory by Estes (1959).

growth curve analyses initially derived in other scientific areas.

Early work in biological research was directed at characterizing the parallel properties of different growth variables. Models were originally developed to deal with the size of two different organs, and early 19th-century work was used by Huxley (~1924, 1932) to form a classical *alometric* model—two variables having a constant ratio of growth rates throughout the growth period—and many physical processes were found to grow in parallel, or in an ordered time-sequence. A good example is found in the multivariate research of Tanner (1955): Figure 18.5 is a plot of growth and change in four physical variables that were found to follow a fundamental pattern over time (i.e., a relatively invariant

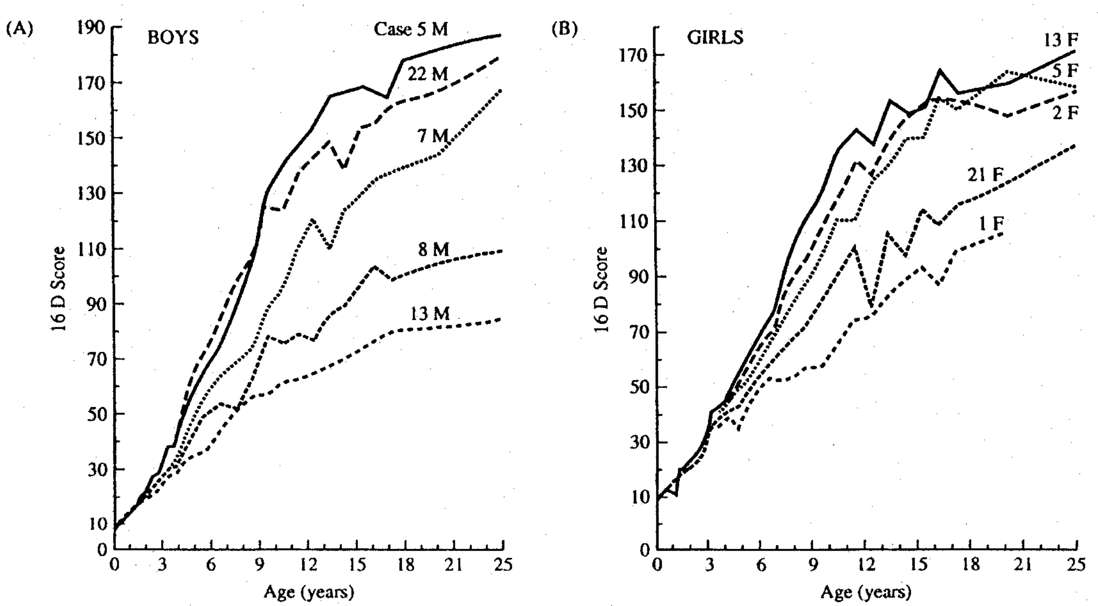

Figure 18.4 Growth curves of intellectual abilities in selected boys and girls from the Berkeley Growth Studies of Bayley (1956, p. 67); age 16 D scores (y) plotted as a function of age at measurement (x) for (a) five boys and (b) five girls.

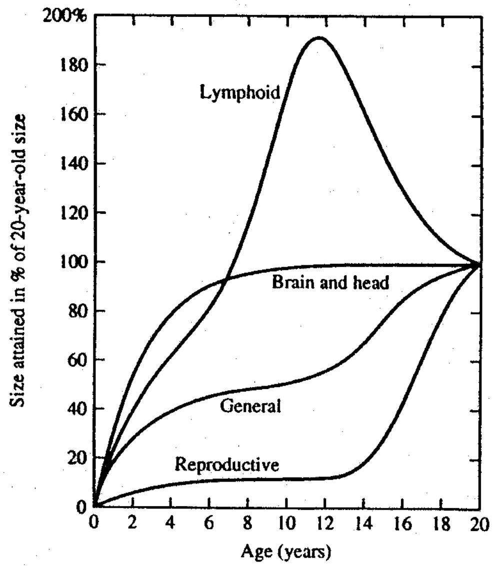

Figure 18.5 Growth curves of tissues and different parts of the body from Tanner (1955).

time-based sequence within an individual). As a result, these physical variables were thought to be indicators of some fundamental time-based dynamic processes. These basic multivariate findings, and questions about the underlying dynamics of multiple growth processes, are still key features of current research.

Classical Growth Curve Analyses

Techniques for the analysis of growth curve analyses are not novel. A classical paper by Wishart (1938) was one of the first to deal with these growth curve analysis problems in an exploratory and empirical fashion. Here, Wishart extended the classical analysis of variance (ANOVA) models to form a linear growth model with group and individual differences. Wishart also showed how power polynomials could be used to better fit the curvature apparent in growth data. The individual growth curve (consisting of $t = 1, T$ occasions) is summarized into a small set of linear orthogonal polynomial coefficients based on a power-series of time $(t, t^2, t^3, \dots t^p)$ describing the general nonlinear shape of the growth curve. In Wishart's models, the basic shape of each individual's curve could be captured with a small number of fixed parameters and random variance components, and the average of the individual parameters could represent the group growth curve (see Cohen & Cohen, 1983; Joosens & Brems-Heynes, 1975).

More complex forms of mathematical and statistical analyses were developed to deal with growth curve data. In his initial growth curve analyses, Ebbinghaus (~1880) described his forgetting curves using a form of the classic *exponential growth model* (see Figure 18.3) in which the rate of change is defined as a linear function of the percentage of initial size (e.g., compound interest). The Velhurst (~1839) curve of population growth, an S-shaped *logistic curve,* was used by Pearl (~1925) for many forms of cognitive growth. In related work, Thurstone (~1919) found that a *hyperbolic curve* of learning best fit the norms of many different tests; Peters (~1930) advocated an *ogival curve* of growth in ideational learning; and Ettlinger (~1926) and Valentine (~1930) demonstrated the relationships among these functions (see Bock & Thissen, 1980; Seber & Wild, 1989).

A popular model for physical growth was initially presented by Gompertz (~1825), who described the *derivative* (instantaneous rate of change) of the growth curve in terms of two exponential accumulations of different rates toward different asymptotes. This flexible model was studied by Winsor (~1932), used by Medwar (~1940) to study the growth of chicken hearts, and used by Deming (~1957) for human physical growth. Another popular growth model was introduced by von Bertalanffy (1938, 1957) and proposed that the individual's change in a physical variable (e.g., weight) was the direct result of the difference in opposing forces of anabolism and catabolism. Although the exact relationship among these forces was not known, von Bertalanffy used a fixed *alliometric* value (of $\gamma = 2/3$) based on prior research.

In related work on nonlinear growth models, Richards (1959) criticized and expanded the original von Bertalanffy model by demonstrating how all prior models can be seen as specific solutions of a "family" of deterministic differential equations (i.e., specific restrictions led to the exponential, logistic, Gompertz, and von Bertalanffy equations). This work was extended by Nelder (1961) and Sandland and MacGhilcrest (1978; for reviews, see Sieber & Wild, 1989; Zeger & Harlow, 1987). Attempts to fit a single growth model to observations over a wide range of ages with a minimal set of parameters led researchers to combine aspects of other models. A recent expansion based on the logistic model was developed by Preece and Baines (1978), who suggested that all previous models could be written as a derivative based on some predefined function of time and some asymptotic value. This kind of model is related to the *partial adjustment model* used in sociometrics (e.g., Coleman, 1968; Tuma & Hannan, 1984), and has proven useful in recent studies of physical growth (see Hauspie, Lindgren, Tanner, & Chrzastek-Spruch, 1991).

More complex linear (and nonlinear) models have been used to represent growth. Some of these share the common feature of a piecewise model applied to different age or time segments. These kinds of segmented or composite models have also been a mainstay of nonlinear modeling. One of the first truly nonlinear composite forms was the *Jenss curve* (or *normal exponential*), in which a linear part (to fit the early rapid-growth phase) was added at a particular age to an exponential part (to fit negative acceleration of the later slowing-down phase) by Jenss and Bayley (~1937). A more complex composite model, the sum of multiple logistic curves, was suggested by Robertson (~1908) and Burt (~1937), but was not fully developed and made practical until Bock and his colleagues did so (see Bock, 1991; Bock & Thissen, 1980; Bock et al., 1973). These composite models allowed for different dynamics at different ages and represent a practically important innovation.

The logic of fitting model segments was also apparent in more recent extensions of Wishart's (1938) polynomial model. One model, based on the summations of latent curves, was proposed simultaneously by both Rao (1958) and Tucker (1958, 1966). In the early descriptions of this model, principal components analysis of the raw growth data led to the sum of a small number of unspecified linear functions. In the interpretation of these components, the shapes of the latent curves are determined by the component loadings, and the individual curve parameters are the component scores. The summation of latent curves has roots in the classical work of Fourier (~1822), but the principal components representation included individual differences. These kinds of linear growth models can offer a relatively parsimonious organization of individual differences, and we highlight these models in later applications.

This brief historical perspective demonstrates that there are many different approaches to the analysis of growth curve data. We find a tendency to introduce more general and flexible forms of growth models, but these models are often complex and each model has slightly different theoretical and practical features. One common feature that does emerge is that most growth models can be written explicitly as a set of dynamic change equations, and we return to this issue later in the chapter. Also, we consistently find efforts made to relate the growth parameters to biologically or psychologically meaningful concepts—this is a difficult but most useful goal for any growth curve analysis.

Contemporary Issues in Statistical Data Analysis

Additional kinds of growth curve analyses are presented in the next few sections. These models include classical linear and nonlinear models as well as some newer models adapted from multivariate analyses. Most of these growth models are designed to deal with the practical issues involving (a) alternative models of change, (b) unequal intervals, (c) unequal numbers of persons in different groups, (d) nonrandom attrition, (e) the altering of measures over time, and (f) multiple outcomes.

This contemporary, model-based description of change can be used to clarify some problems inherent in *observed rates of change*. The potential confounds in difference scores have been a key concern of previous methods using observed change scores or rate-of-change scores (e.g., Beriter, 1963; Burr & Nesselroade, 1992; Cronbach & Furby, 1970; Rogosa & Willett, 1985; Willett, 1990). This research has shown that when observed rates are used as outcomes in standard regression analyses, the results can be biased by several factors, including residual error, measurement error, regression to the mean, and regression from the mean (e.g., Allison, 1990; Nesselroade & Bartsch, 1977; Nesselroade & Cable, 1974; Nesselroade, Siegler, & Baltes, 1980; Raykov, 1999; Williams & Zimmerman, 1996). These problems can be severe when using standard linear regression with time-dependent variables (e.g., Boker & McArdle, 1995; Hamagami & McArdle, 2000).

One of the key reasons we present the contemporary modeling approach is to move beyond these classical problems. Modern statistical procedures have been developed to minimize some of these problems by fitting the model of an *implied trajectory over time* directly to the observed scores. Alternative mathematical forms of growth can be considered using different statistical restrictions. From such formal assumptions we can write the set of expectations for the means, variances, and covariances for all observed scores, and use these expectations to identify, estimate, and examine the goodness-of-fit of latent variable models representing change over time. Most of these models discussed here are based on fitting observed raw-score longitudinal growth data to a theoretical model using likelihood-based techniques (as in Little & Rubin, 1987; McArdle & Bell, 2000). In general, we find it convenient to *describe* the data using the observed change scores (defined as $\Delta Y_n / \Delta t$), but we make *inferences* about the underlying growth processes by directly estimating parameters of the latent change scores (defined as $\Delta y_n / \Delta t$).

In a recent and important innovation, Meredith and Tisak (1990) showed how the *Tuckerized curve models* (so named in recognition of Tucker's contributions) could be represented and fitted using structural equation modeling of common factors. These growth modeling results were important because this made it possible to represent a wide range of alternative growth models. This work also led to interest in methodological and substantive studies of growth processes using structural equation modeling techniques (McArdle, 1986, 1997; McArdle & Anderson, 1990; McArdle & Epstein, 1987;

McArdle & Hamagami, 1991, 1992). These latent growth models have since been expanded upon and used by many others (Duncan & Duncan, 1995; McArdle & Woodcock, 1997; Metha & West, 2000; B. O. Muthen & Curran, 1997; Willett & Sayer, 1994). The contemporary basis of latent growth curve analyses can also be found in the recent developments of *multilevel models* (Bryk & Raudenbush, 1987, 1992; Goldstein, 1995) or *mixed-effects models* (Littell, Miliken, Stoup, & Wolfinger, 1996; Singer, 1999). Perhaps most important is that the work by Browne and du Toit (1991) showed how the nonlinear dynamic models could be part of this same framework (see Cudeck & du Toit, 2001; McArdle & Hamagami, 1996, 2001; Pinherio & Bates, 2000). For these reasons, the term *latent growth models* seems appropriate for any technique that describes the underlying growth in terms of latent changes using the classical assumptions (e.g., independence of residual errors).

The model-based fitting of structural assumptions about the group and individual differences holds the key to later substantive interpretations. These theoretical restrictions may not hold exactly in the examination of real data, and this leads to the general issues of model testing and goodness-of-fit. Recent research has also produced a variety of new statistical and computational procedures for the analysis of latent growth curves, and their unique features are somewhat difficult to isolate. This means that the likelihood-based approach to the estimation and fitting of growth curve analyses can be accomplished using several widely available computer packages (e.g., SAS: Littell et al., 1996, Singer, 1998, and Verneke & Molenberghs, 2000; SPlus: Pinherio & Bates, 2000; MIXREG: Hedecker & Gibbons, 1996, 1997). A few available computer programs (e.g., Mx: Neale, Boker, Xie, & Maes, 1998; AMOS: Arbuckle & Wotke, 1999, and Mplus: L. K. Muthen & Muthen, 1998), can be used to estimate the parameters of all analyses described herein.

The Bradway-McArdle Longitudinal Growth Data

To illustrate many of the issues and models in this chapter, we use some longitudinal growth data in Figure 18.6. These are age-plots of data from a recent study of intellectual abilities—the *Bradway-McArdle Longitudinal* study (see McArdle & Hamagami, 1996; McArdle, Hamagami, Meredith, & Bradway, in press). The persons in this study were first measured in 1931, when they were aged 2 to 7 years, as part of the larger standardization sample of the Stanford-Binet test ($N = 212$). They were measured again about 10 years later by Katherine P. Bradway as part of her doctoral dissertation in 1944 ($N = 138$). Many of these same persons were measured twice more by Bradway as adults at average ages of 30 and 42 using the Wechsler Adult Intelligence Scales (WAIS, $N = 111$; for further details, see Bradway & Thompson, 1962; Kangas & Bradway, 1971). About half ($n = 55$) of the adolescents tested in 1944 were measured again in 1984 at ages 55 to 57, and between 1993 and 1997 at ages ranging from 64 to 72; 34 were tested in 1993 through 1997 on the WAIS (McArdle, Hamagami, et al., 2001).

These plots illustrate further complexity that needs to be dealt with in longitudinal growth curve analyses. The first plot (Panel A of Figure 18.6) gives individual growth curve

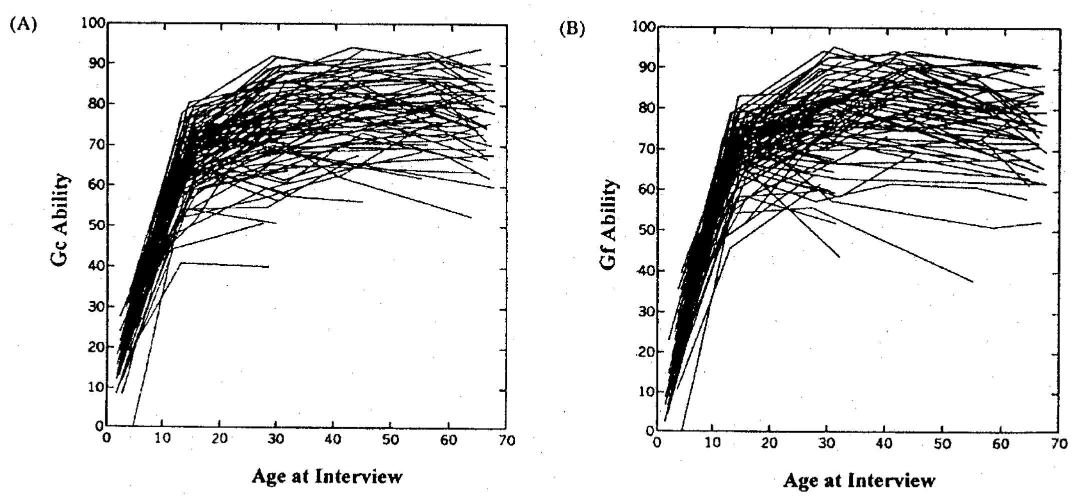

Figure 18.6 Growth curves of verbal (Gc) and nonverbal (Gf) abilities in complete and incomplete data from the Bradway Longitudinal Growth Study (see McArdle, Hamagami, Bradway & Meredith, 2001); Rasch scaled scores (y) plotted as a function of age at measurement (x) for (a) $N = 29$ participants with complete data and (b) $N = 82$ participants with incomplete data.

data for verbal ability (Rasch scaled) at each age at testing for $n = 29$ individuals who were measured at each time of testing, and for the $n = 82$ persons who were measured at some (but not all) ages of testing. The second plot of Figure 18.6 (Panel B) is a similar plot for data from nonverbal measurements. The comparison of Panels A and B is informative, and leads to important practical issues in subject recruitment and attrition in longitudinal studies. Although not depicted here, multiple variables from the Stanford-Binet and the WAIS have been repeatedly measured, including separate measures of *verbal* (or *knowledge*) ability, and of nonverbal (or *reasoning*) ability (for details, see McArdle, Hamagami, Horn, & Bradway, 2002).

Table 18.1 is a listing of numerical information from this study to be used in subsequent examples of growth curve analyses. The overall subject participation is listed in Panel A of Table 18.1, and here we can see the nearly continual loss of participants over time. The means and standard deviations for two composite variables are listed in Panel B, and here we find early increases followed by less change in the later years. The correlations of these measures over six occasions are listed in Panel C, and here we find a complex pattern of results, some correlations suggesting high stability of individual differences (e.g., $r > .9$) and others suggesting low stability ($r < .1$). The summary information presented in Panels B and C is limited to those $n = 29$ participants with complete data at all six time points of measurement, but information on $N = 111$ available through adulthood is used in the growth curve examples to follow.

As with any data-oriented study, the information in this data set has some clear limitations (e.g., Pinneau, 1961). Among these, the participants are all from one birth cohort (~1928), in the same geographical area (San Francisco), of one ethnicity (Caucasian), and come from volunteer families with above-average socioeconomic status; moreover, most of them score above average on most cognitive tasks. Whereas

TABLE 18.1 Description of the Bradway-McArdle Longitudinal Study Data

A. Subject Ascertainment History

Category	Time 1 Age 2–7 *N* (%)	Time 2 Age 12–17 *N* (%)	Time 3 Age 28–32 *N* (%)	Time 4 Age 40–43 *N* (%)	Time 5 Age 55–58 *N* (%)	Time 6 Age 63–66 *N* (%)
Tested	212 (100.)	138 (65.)	111 (80.)	48 (43.)	53 (48.)	51 (46.)
Inaccessible	0 (0.)	0 (0.)	0 (0.)	7 (6.)	5 (5.)	6 (5.)
Deceased	0 (0.)	0 (0.)	0 (0.)	2 (2.)	9 (8.)	19 (17)
Refused testing	0 (0.)	0 (0.)	0 (0.)	7 (6.)	1 (1.)	12 (11.)
Not located	0 (0.)	74 (35.)	27 (20.)	47 (42.)	43 (39.)	23 (21.)

B. Means and Standard Deviations ($N = 29$)

Variables	Time 1 Age 4	Time 2 Age 14	Time 3 Age 30	Time 4 Age 42	Time 5 Age 57	Time 6 Age 65
Nonverbal mean	25.55	70.40	80.06	82.99	80.60	78.64
(nonverbal S.D.)	(12.61)	(5.89)	(7.87)	(7.84)	(7.53)	(7.80)
Verbal means	22.22	65.84	75.65	78.76	80.70	77.97
(verbal S.D.)	(8.80)	(7.37)	(9.20)	(8.23)	(7.86)	(7.59)

C. Correlations of Nonverbal and Verbal Scores ($N = 29$)

	NV_4	NV_{14}	NV_{30}	NV_{42}	NV_{57}	NV_{65}	V_4	V_{14}	V_{30}	V_{42}	V_{57}	V_{65}
NV_4	1.00											
NV_{14}	.12	1.00										
NV_{30}	−.10	.37	1.00									
NV_{42}	−.04	.19	.81	1.00								
NV_{57}	−.02	.20	.85	.82	1.00							
NV_{65}	.02	.25	.78	.85	.83	1.00						
V_4	.92	.16	−.03	−.08	.03	.02	1.00					
V_{14}	.28	.68	.18	.05	.03	.16	.36	1.00				
V_{30}	−.02	.25	.56	.45	.41	.57	.09	.43	1.00			
V_{42}	.07	.26	.53	.37	.42	.50	.24	.27	.83	1.00		
V_{57}	−.01	.21	.50	.37	.36	.44	.15	.38	.91	.89	1.00	
V_{65}	.02	.24	.52	.46	.41	.56	.10	.35	.85	.77	.90	1.00

the longitudinal age span and the number of measures taken are large, the number of occasions of measurement was limited by practical concerns (e.g., cooperation, fatigue, and practice effects). The benefits and limitations of these classic longitudinal data make it possible to examine both the benefits and limitations of the new models for the growth and change discussed in this chapter.

THE BASIC STRUCTURE OF GROWTH MODELS

Growth Models of Within-Person Changes

Growth curve data are characterized as having multiple observations based on *longitudinal* or *repeated measures.* Assume we observe variable Y at multiple occasions (in brackets, $t = 1$ to T) on some persons (in subscripts, $n = 1$ to N), and we write

$$Y[t]_n = y_{0,n} + A[t] y_{s,n} + e[t]_n \qquad (18.1)$$

where the y_0 are scores representing an individual's initial level (e.g., intercept); the y_s are scores representing the individual *linear change over time* (e.g., slopes); the set of $A[t]$ are termed *basis weights,* which define the timing or shape of the change over time for the group (e.g., age at testing); and the $e[t]$ are error scores at each measurement.

The latent-change model is constant *within* an individual but it is not assumed to be the same *between* individuals (with subscripts n). The unobserved variables that presumably do not change over time are written in lowercase (y_0, y_s) are similar to the predicted (i.e., nonerror) scores in a standard regression equation. We can write

$$y_{0,n} = \mu_0 + e_{0,n} \quad \text{and} \quad y_{s,n} = \mu_s + e_{s,n}, \qquad (18.2)$$

where the group means (μ_0, μ_s) are fixed effects for the intercept and the slopes and the new scores are deviations (e_0, e_s) around these means. We can define additional features of these scores using standard expected value ($E\{y\}$) notation. First we presume the means of all deviations scores are zero (i.e., $E\{e\} = 0$). Next, we define the nonzero variance and covariance terms as

$$E\{e_0, e_0\} = \sigma_0^2, \quad E\{e_s, e_s\} = \sigma_s^2, \quad E\{e_0, e_s\} = \sigma_{0s}, \quad \text{and}$$
$$E\{e[t], e[t]\} = \sigma_e^2, \qquad (18.3)$$

so these individual differences around the means are termed *random effects* (σ_0^2, σ_s^2, σ_{0s}). In many applications we assume only one random error variance (σ_e^2) at all occasions of measurement. As in classical regression analyses, the validity of the interpretations is are limited by the most basic model assumptions—for example, linearity, additivity, independence of residuals, independence of other effects, no interactions, and so on.

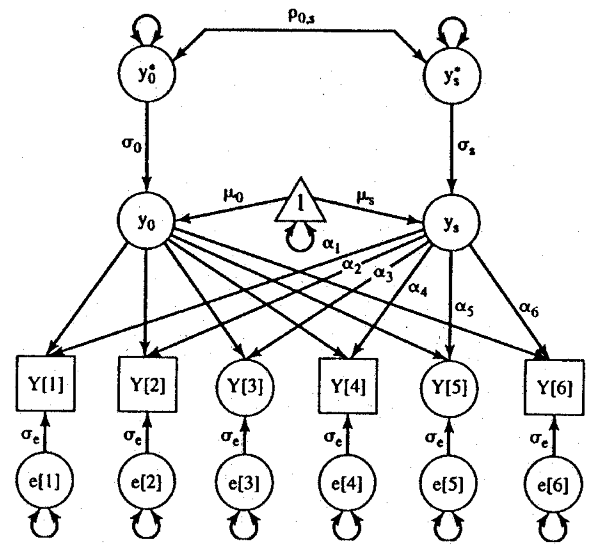

Figure 18.7 The basic latent growth structural model as a path diagram from McArdle & Epstein (1987) and McArdle & Hamagami (1992).

In order to clarify growth models, we can use a *path diagram* such as the one displayed as Figure 18.7. These kinds of diagrams were originally only used with regression models, but more recently have been used in the context of growth and change (e.g., see McArdle, 1986; McArdle & Aber, 1990; Wright, 1921). In this representation the observed variables are drawn as squares, the unobserved variables are drawn as circles, and the implied unit constant (i.e., scores of 1 before the intercept parameter in Equation 18.1) is included as a triangle. Model parameters representing fixed or group coefficients are drawn as one-headed arrows, while random or individual features are drawn as two-headed arrows. The observed variables ($Y[t]$) are seen to be produced by latent intercepts (y_0) with unit weights, by the latent slopes (y_s) with weights ($A[t] = [\alpha[1], \alpha[2], \ldots \alpha[T]]$), and by an individual error term ($e[t]$).

Following Equations 18.2 and 18.3, the initial level and slopes are often assumed to be random variables with fixed means (μ_0, μ_s) but random variances (σ_0^2, σ_s^2) and covariances ($\sigma_{0,s}$). The standard deviations (σ_0, σ_s) are sometimes drawn in the picture to permit the direct representation of the covariances as scaled correlations ($\rho_{0,s}$). The error terms are assumed to be distributed with a mean of zero, a single variance (σ_e^2), and no correlation with any of the other latent scores (further statistical tests may assume these errors follow a normal distribution as well). These formal structural assumptions distinguish these latent growth models from the other kinds of analyses of growth data.

Considering Alternative Growth Models

As in any form of data analysis, a growth model can be evaluated only in relation to other possibilities. A first set of alternative models might be based on simplifications of the previous model parameters. In this kind of *trajectory equation*, the $Y[t]$ is formed for each group and individual from the $A[t]$ basis coefficients. These coefficients also determine the metric or scaling and interpretation of these scores, so alterations of $A[t]$ can lead to many different models.

As a simple example, suppose we require all $A[t] = \mathbf{0}$, and effectively eliminate all slope parameters. This leads to a simple additive model

$$Y[t]_{\mathrm{n}} = y_{0,\mathrm{n}} + e[t]_{\mathrm{n}}, \tag{18.4}$$

where only the intercept y_0 and the $e[t]$ error terms are included. As later shown, this model is termed a *baseline* or *no-growth alternative* because it is consistent with observations only where there is no change over time in the means, variances, or correlations.

Other simple growth curve analyses are based on simple mathematical functions, and the fitting of a straight line to a set of measures is a standard procedure in scientific research. So, as a next example, let us assume there are $T = 4$ time points and we have set the basis $A[t] = [0,\ 1,\ 2,\ 3]$. Following Equation 18.1, this leads to a set of linear equations where

$$\begin{aligned} Y[1]_{\mathrm{n}} &= y_{0,\mathrm{n}} + 0y_{s,\mathrm{n}} + e[1]_{\mathrm{n}},\\ Y[2]_{\mathrm{n}} &= y_{0,\mathrm{n}} + 1y_{s,\mathrm{n}} + e[2]_{\mathrm{n}},\\ Y[3]_{\mathrm{n}} &= y_{0,\mathrm{n}} + 2y_{s,\mathrm{n}} + e[3]_{\mathrm{n}}, \quad \text{and}\\ Y[4]_{\mathrm{n}} &= y_{0,\mathrm{n}} + 3y_{s,\mathrm{n}} + e[4]_{\mathrm{n}}. \end{aligned} \tag{18.5}$$

At the first time point the specific coefficient $a[1] = 0$, so the slope term drops out of the expression and the score at the first time point is composed of only the intercept plus an error. At the second time point, $a[2] = 1$, so the score is the sum of the intercept (y_0) plus a change over time (y_s) plus a new error score ($e[2]$). At the third time point, $a[3] = 2$, so the score is the sum of the intercept (y_0) plus 2 times the prior change over time ($2y_s$) plus a new error ($e[3]$). At the fourth time point, $a[4] = 3$, so the score is the sum of the intercept (y_0) plus three times the prior change over time ($3y_s$) plus a new error ($e[4]$). Each additional score would add another weighted change and a new error term. The basic interpretation would change only slightly if we altered the linear basis to be $A[t] = [1, 2, 3, 4]$, because now the intercept (where $t = 0$) is presumably prior to the first time point. A different change of the linear basis to be $A[t] = [0.00, 0.33, 0.67, 1.00]$, would have the effect of shifting the units of the slope to units to be a proportion of the entire range of time but we would still be considering straight-line change.

In contrast, other alterations of the basis coefficients can alter the interpretation of the shape of the changes. For example, if we redefine $A[t] = [1, 2, 2, 1]$, then the model does not represent straight-line change—instead, the basis represents a curve that starts up (1 to 2), flattens out (2 to 2), and then goes back down (2 to 1). Other, more complex alterations of the basis will lead to more complex trajectory models.

As in all linear models, the set of loadings ($A[t]$) defines the shape of the group curve over time. In a latent basis model approach (Meredith & Tisak, 1990), we allow the curve basis to take on a shape based on the empirical data. We fit a factor model based on the standard linear model (Equation 18.1) as before, with two common factor scores, an intercept (y_0) with unit loadings, a linear slope (y_s), and independent unique factor scores ($e[t]$); but the factor loadings ($A[t]$) are now estimated from the data. The two common factor scores account for the means and covariances, and the estimated factor loadings each describe a weight or *saturation* of the slope at a specific time of measurement. The $A[t]$ are estimated as factor loadings and have the usual mathematical and statistical identification problems of any factor analysis. This means we fit the latent basis model as

$$\begin{aligned} Y[1]_{\mathrm{n}} &= y_{0,\mathrm{n}} + 0y_{s,\mathrm{n}} + e[1]_{\mathrm{n}},\\ Y[2]_{\mathrm{n}} &= y_{0,\mathrm{n}} + 1y_{s,\mathrm{n}} + e[2]_{\mathrm{n}},\\ Y[3]_{\mathrm{n}} &= y_{0,\mathrm{n}} + \alpha[3]y_{s,\mathrm{n}} + e[3]_{\mathrm{n}}, \quad \text{and}\\ Y[4]_{\mathrm{n}} &= y_{0,\mathrm{n}} + \alpha[4]y_{s,\mathrm{n}} + e[4]_{\mathrm{n}}. \end{aligned} \tag{18.6}$$

In the typical case, at least one entry of the $A[t]$ will be fixed as, say, $a[1] = 1$, to provide a reference point for the other model parameters. If a nonzero covariance (σ_{0s}) among common factors is allowed, then two fixed values (e.g., $a[1] = 0$ and $a[2] = 1$), can be used to distinguish the factor scores and assure overall model identification (as in McArdle & Cattell, 1994). The other parameters are allowed to be freely estimated (e.g., Greek notation for the estimated parameters $\alpha[3]$ and $\alpha[4]$), so we obtain what should be an *optimal shape* for the group curve. Change from any one time to another ($\Delta y_n/\Delta t$) is a function of the slope score (y_s) and the change in the factor loadings ($\Delta A[t]$).

We may now consider a variety of more complex models. One simple version of a quadratic polynomial growth model can be written as

$$Y[t]_n = y_{0n} + A[t]y_{1n} + \frac{1}{2}A[t]^2 y_{2n} + e[t]_n, \quad (18.7)$$

where the $A[t]$ are fixed at known values, and a new component (y_2) is introduced to represent the change in the change (i.e., the *acceleration*). This implies the expected growth curve may turn direction at least once in a nonlinear (i.e., parabolic) fashion. The additional latent score (y_2) is allowed to have a mean (μ_2) and a variance (σ_2^2) and to be correlated with the other latent scores ($\rho_{0,2}$, $\rho_{1,2}$). Any set of growth data might require a second-order (quadratic), third-order (cubic), or even higher order polynomial model. In each of these alternatives, however, more complexity is added because any pth-order model includes p latent means, $p+1$ latent variances and $p(p-1)/2$ covariance terms for the group and individual differences across all observations.

A variety of other growth models can now be studied using this general linear framework. For example, the linear polynomial model (Equation 18.7) could be fitted with orthogonal polynomial constraints, or in an alternative form (e.g., Stimpson, Carmines, & Zeller, 1978), or even with a latent basis (i.e., $\alpha[t]$ and $\frac{1}{2}\alpha[t]^2$). Also, in each model listed previously, it is possible to add assumptions about the structure of the relationships among the residual terms ($e[t]$). We can consider specific-factor terms and consider alternative mechanisms for their construction (e.g., autoregressive, increasing over time, etc.). These *structured residual models* are valuable in statistical efforts to improve the precision, fit, and forecasts of the model, but they do not provide the substantive information we use here (but see Cnaan, Laird, & Slasor, 1997; Littell et al., 1996).

Expectations and Estimation in Linear Growth Models

The parameters of any growth model lead to a set of expectations for the observed data, and these expectations will be used in subsequent model fitting. The previous assumptions can be combined to form the expected trajectories over time. This can be calculated from the *algebra of expectations* (with sums of average cross-products symbolized as $E\{YX'\}$) or from the tracing rules of path analysis (see McArdle & Aber, 1990; Wright, 1934). Using either approach, the observed mean at any occasion can be written in terms of the linear model parameters as

$$\mu_{Y[t]} = E\{Y[t]1'\} = \mu_0 + A[t]\mu_s, \quad (18.8)$$

(where the constant vector **1** is again used). This implies the mean at any time ($\mu_{Y[t]}$) is the initial-level mean (μ_0) plus the slope mean (μ_s) weighted by the specific basis coefficient ($A[t]$) that is either fixed or estimated. This also implies that changes in the basis weights determine all changes in the mean trajectory.

The expectation of the observed score variance at any occasion can be written as

$$\sigma^2_{Y[t]} = E\{(Y[t] - \mu_{Y[t]})^2\} = \sigma_0^2 + \sigma^2_{y[t]} + \sigma_e^2$$
$$= \sigma_0^2 + \left(A[t]\sigma_s^2 A[t] + A[t]\sigma_{0s} + \sigma_{0s}A[t]\right) + \sigma_e^2. \quad (18.9)$$

This implies the observed variance at any time ($\sigma^2_{Y[t]}$) is the sum of the initial-level variance (σ_0^2) plus the variance of the latent changes ($\sigma^2_{y[t]}$; with lowercase y) plus the error variance (σ_e^2). Again we find changes in the basis weights account for all the changes in the variance over time. Following this same logic, we can write the expected values for the covariances among the same variable at two occasions, $Y[i]$ and $Y[j]$, as

$$\sigma_{Y[i,j]} = E\{(Y[i] - \mu_{Y[i]})(Y[j] - \mu_{Y[j]})\} = \sigma_0^2 + \sigma_{y[i,j]}$$
$$= \sigma_0^2 + \left(A[i]\sigma_s^2 A[j] + A[i]\sigma_{0s} + \sigma_{0s}A[j]\right). \quad (18.10)$$

This implies the observed covariance at any time ($\sigma_{Y[i,j]}$) is the sum of the initial-level variance (σ_0^2) plus the covariance of the latent changes ($\sigma_{y[i,j]}$); changes in the basis weights account for all changes in the covariances over time. Each of these Equations 18.8 through 18.10 can be traced in the diagram (e.g., Equation 18.9 is from any $Y[t]$ back to itself).

These growth model expectations are useful because they can be compared to the observed growth statistics for the estimation of model parameters and the evaluation of goodness-of-fit. Whereas the summary statistics form the basis of the expectations, recent computational techniques can be used to estimate the model parameters directly from the entire collection of raw data. Following standard theory in this area (e.g., Lange, Westlake, & Spence, 1976; Lindsey, 1993), the *multivariate normal model* for an observed vector $Y[t]$ is used to define the *maximum likelihood estimates* (*MLEs*) of the parameters, and a single numerical value termed the *model* *likelihood* (L) can be calculated to index the *misfit* of the model expectations to the observed data.

Assuming we have one or more alternative models (see next section), we can compare these models using the differences in log-likelihood ($\Delta L = L_1 - L_2$) and the difference in the numbers of parameters estimated ($\Delta NP = NP_1 - NP_2$). Under standard normal theory assumptions about the distribution of the errors, we can compare model differences to a chi-square distribution ($\Delta L \sim \chi^2$, $\Delta NP \sim df$) and determine the accuracy (i.e., significance) of our comparison. To index the multivariate effect sizes, we can calculate a noncentrality index and provide the statistical power ($P = 1 - \beta$) for all likelihood-based comparisons (e.g., based on $\alpha = .01$ test size). These likelihood-based calculations can answer basic questions phrased as *To what degree do the data conform to the model expectations?* We can also use this same likelihood approach to answer more complex questions, such as *Which is the best model for our data* and *Do the fit statistics indicate that the same dynamic patterns exist in different sub-groups?* Although we do not need to make a rigorous use of probability tests, we do provide information to calculate alternative indices of fit, including test statistics for perfect or close fit (e.g., Browne & Cudeck, 1983; Burnham & Anderson, 1998; McArdle, Prescott, Hamagami, & Horn, 1998).

The resulting parameter estimates allow us to form expected group growth curves for both the observed and true scores (for details, see McArdle, 1986, 2001; McArdle & Woodcock, 1997). We can also characterize the relative size of these parameters by calculating time-specific ratios of the estimated variances

$$\eta^2_{[t]} = \left(\sigma^2_{Y[t]} - \sigma^2_e\right) / \sigma^2_{Y[t]} = \sigma^2_{y[t]} / \sigma^2_{Y[t]}, \quad \text{and}$$
$$\Delta\eta^2_{[i-j]} = \eta^2_{[i]} - \eta^2_{[j]}. \qquad (18.11)$$

These *growth-reliability* ratios can be useful in investigating the changes in the true score variance ($\sigma^2_{y[t]}$) and changes in the reliability of the variable at different points in time (for examples, see McArdle & Woodcock, 1997; Tisak & Tisak, 1996). These simple formulas also suggest that the parameters of the changes are difficult to consider in isolation—that is, the *variance of the changes* is not equal to the *changes in the variance*. In the same way, the expectations of the observed correlations over time ($\rho_{Y[i,j]}$) can be calculated from the basic expressions (the ratio of Equation 18.10 to a function of Equation 18.9), but the resulting expected correlations are usually a complex ratio of the more fundamental parameters. In many growth models, it is complicated to express patterns of change using only correlations. In general, the growth pattern depends on basic model parameters that may have no isolated interpretation.

Initial Results From Fitting Linear Growth Models

The complete and incomplete data from the six-occasion Bradway-McArdle longitudinal study (Figure 18.6) have been fitted and reported in McArdle & Hamagami (1996) and McArdle et al. (McArdle, Hamagami, et al., in press, 2002). A selected set of these results is presented for illustration here. On a computational note, the standard HLM, MLn, VARCL, MX, and SAS PROC MIXED programs produced similar results for all models with a fixed basis. The models with estimated factor loadings ($A[t]$) were fitted using the general Mx unbalanced raw data option (e.g., the variable length approach) and with SAS PROC NLMIXED and the results are similar. All of these programs follow the same general procedures, so we will consider these as equivalent procedures unless otherwise stated.

The first model (labeled $M0$) was a no-growth model (Equation 18.4) fitted to the nonverbal scores of the Bradway-McArdle data. This simple model was fitted estimated with only three parameters, and we obtained a baseline for fit ($L = 4{,}440$). The parameters estimated include an initial-level mean ($\mu_0 = 46.4$), a small initial-level standard deviation ($\sigma_0 = 0.01$), and a large error deviation ($\sigma_e = 49.8$).

The second model fitted was a linear growth model ($M1$) with a fixed basis (Equation 18.5) and six free parameters. This basis was first formed by using the actual age of the persons at the time of measurement $A[t] = [4, 14, 30, 42, 56, 64]$. Estimates were obtained yielded a fit ($L = 4{,}169$) that represented a clear improvement over the baseline ($\chi^2 = 271$ on $df = 3$) model, and the error variance has been reduced substantially (to $\sigma_e\{M1\} = 18.1$). The resulting parameters lead to a straight line of expected means that increases rapidly over age; $\mu[t] = [45.4, 52.9, 64.7, 74.1, 84.7, 90.6]$. The variance estimates of the intercept and slope parameters were small, so we refit the model with a simpler basis: That is, $A[t] = [(\text{Age}[t] - 4)/56] = [0.00, 0.19, 0.49, 0.73, 1.00, 1.15]$, so the weights are proportional to the range of data between the *early* age of 4 and the *middle* age of 56. This resulted in identical mean expectations, but the latent variances were still too small to interpret.

This latent basis model ($M2$) was fitted next. For the purposes of estimation, the $A[1] = 0$ (at Age = 4) and $A[5] = 1$ (at Age = 56) were fixed (as proportions) but the four other coefficients were estimated from the data. This resulted in a likelihood ($L = 3{,}346$) which is substantially better than the baseline model ($\chi^2 = 1{,}094$ on $df = 7$) and the linear model ($\chi^2 = 823$ on $df = 4$), and the error variance has been

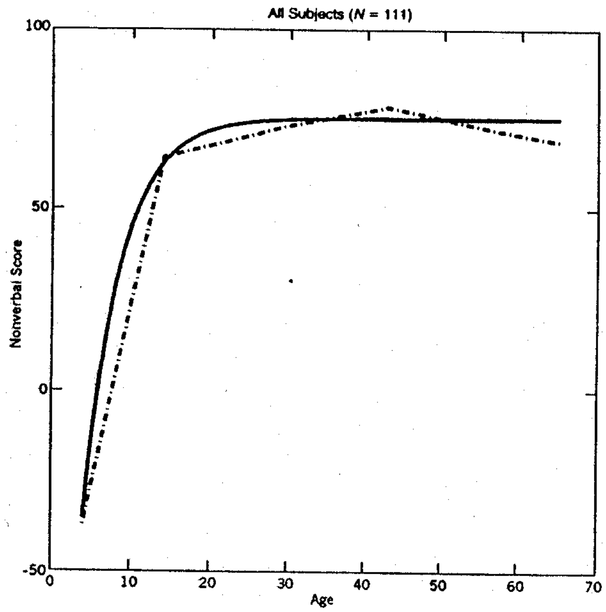

Figure 18.8 Alternative latent growth curve model expectations for the average growth curve of nonverbal abilities fitted to the complete and incomplete data from the Bradway-McArdle data (see Figure 18.7; see McArdle & Hamagami, 1996).

substantially reduced (from $\sigma_e\{M1\} = 18.1$ to $\sigma_e\{M2\} = 5.1$). The estimated basis coefficients were $A[t] = [= 0, 0.93, 1.01, 1.06, = 1, .97]$, and the estimated latent means were $\mu_1 = 23.8$ and $\sigma_s = 52.8$. This leads to a group trajectory $\mu[t] = [23.8, 69.2, 76.6, 80.8, 76.6, 75.0]$ that rises quickly between ages 4 and 14, peaks at age 42, and starts a small decline at ages 56 to 65. This group curve is plotted as a dashed line in Figure 18.8 and it is very similar to the general features of the raw data in Figure 18.6. The individual differences in this model are not seen in the means but in the large variances for the level ($\sigma_0 = 10.1$) and the slope ($\sigma_s = 12.3$) parameters, and the latent level and slope scores have a high correlation ($\rho_{1s} = -0.82$).

The improved fit of this latent basis compared to the linear basis model suggests the need for some form of a nonlinear curve. To explore the addition of fixed higher-order growth components, the quadratic polynomial model ($M3$, Equation 18.10) was fitted to these data using the same procedures. The goodness-of-fit was slightly improved over the linear ($\chi^2 = 7$ on, $df = 4$, $\sigma_e\{M3\} = 18.0$). Although the latent basis ($M2$) and quadratic basis ($M3$) models are not nested, the quadratic model did not seem as useful as the latent basis model did. Also, problems arose in the estimation of all variance terms, so the polynomial approach was not considered further.

ADDING GROUP INFORMATION TO GROWTH CURVE ANALYSES

Latent Path and Mixed-Effects Models

We next consider analyses which include more detailed information about group differences. In the basic growth model (Equations 18.1–18.4), the latent variance terms in the model tell us about the size of the between group differences at each age (Equation 18.11), but this does not tell us the sources of this variation. To further explore the differences between persons, we can expand the basic growth model. Let us assume a variable termed X indicates some measurable characteristic of the person (e.g., sex, educational level, etc.). If we measure this variable at one occasion we might like to examine its influence in the context of a growth model for $Y[t]$. One popular model is written

$$Y[t]_n = y_{0:x,n} + A[t]y_{s:x,n} + \omega X_n + e[t]_n \quad (18.12)$$

where the ω are fixed (group) coefficients with the same-sized effect on the measured $Y[t]$ scores at all occasions, and the X is an independent observed (or assigned) predictor variable. It is useful to recognize that this model implies the latent score change over time is independent of the X variable(s). That is, the other growth parameters ($\mu_{0:x}$, $\mu_{s:x}$, $\sigma_{0:x}$, $\sigma_{s:x}$, $\sigma_{0,s:x}$) are conditional on the expected values of the measured X variable. This use of adjusted growth parameters is popularly represented in the techniques of the analysis of covariance, and the reduction of error variance from one model to the next ($\sigma_e^2 - \sigma_{ex}^2$) is often considered as a way to understand the impact (see Snyders & Boskers, 1995).

An alternative but increasingly popular way to add another variable to a growth model is to write expressions in which the X variable has a direct effect on the individual differences scores of the growth curve. This can be stated as

$$\begin{aligned} Y[t]_n &= y_{0n} + A[t]y_{sn} + e[t]_n \quad \text{with} \\ y_{0n} &= \nu_{00} + \nu_{0x}X_n + e_{0n}, \quad \text{and} \qquad (18.13) \\ y_{sn} &= \nu_{s0} + \nu_{sx}X_n + e_{sn}, \end{aligned}$$

where the regression of the latent variables (y_0, y_s) on X includes intercepts (ν_{00}, ν_{s0}) and slopes (ν_{0x}, ν_{sx}). We can rewrite this model into a compact reduced form,

$$\begin{aligned} Y[t]_n &= [\nu_{00} + \nu_{0x}X_n + e_{0n}] + A[t][\nu_{s0} + \nu_{sx}X_n + e_{sn}] \\ &\quad + e[t]_n \\ &= \nu_{00} + \nu_{0x}X_n + e_{0n} + A[t]\nu_{s0} + A[t]\nu_{sxs}X_n \\ &\quad + A[t]e_{sn} + e[t]_n \end{aligned}$$

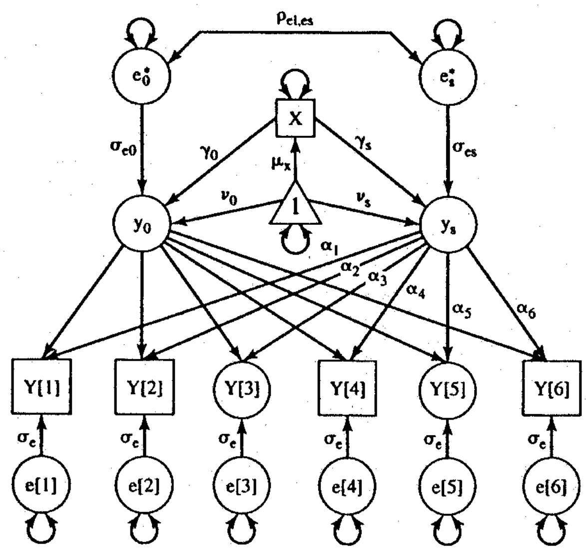

Figure 18.9 Latent growth as a path diagram with mixed-effects or multilevel predictors (from McArdle & Epstein, 1987; McArdle, 1989).

$$= [\nu_{00} + A[t]\nu_{s0} + \nu_{0x}X_n + A[t]\nu_{sx}X_n] + [e_{0n} + A[t]e_{sn} + e[t]_n], \quad (18.14)$$

and this separates the fixed-effects (first four terms) from the random components (last three terms). This model is drawn as a path diagram in Figure 18.9. This diagram is the same as Figure 18.7, except here we have included the X as a predictor of the levels and slope components. This diagram gives the basic idea of external variable models, and other more complex alternatives are considered in later sections.

In this simple latent growth model, as in more complex models to follow, we can always add other predictors X for the intercepts and the slopes because these models are simply latent growth models with "extension variables" (e.g., McArdle & Epstein, 1987). This kind of model (Equation 18.13 or 18.14) can also be seen as having two levels—a first-level equation for the observed scores, and a second-level equation for the intercepts and slopes. For these reasons, such models have been termed *random-coefficients* or *multilevel models, slopes as outcomes,* or *mixed-effects models* (Bryk & Raudenbush, 1987, 1992; Littell et al., 1996). Variations on these models can be compared for goodness-of-fit indices, and we can examine changes in the model variance explained at both the first and second levels (see Snyders & Boskers, 1995). In any terminology, the between-group differences in the within-group changes can be represented by the parameters in the model of Figure 18.9.

Group Differences in Growth Using Multiple Group Models

The previous models used the idea of having a measured variable X characterizing the group differences and then examining the effect of X on the model parameters. However, this method is limiting in a number of important ways. For example, some of the classical forms of growth processes, such as examining different amplitudes and phase shifts (e.g., Figure 18.2) are not easy to account for within the single-group latent growth framework. A more advanced treatment of the group problem model uses concepts derived from multiple-group factor analysis (e.g., Jöreskog & Sörbom, 1999; Honr & McArdle, 1992; McArdle & Cattell, 1994). In these kinds of models, each group, $g = 1$ to G, is assumed to follow some kind of latent growth model, such as

$$Y[t]_n^{(g)} = y_{0,n}^{(g)} + A[t]^{(g)}y_{1,n}^{(g)} + e[t]_n^{(g)} \quad \text{for } g = 1 \text{ to } G, \quad (18.15)$$

with basis parameters $A[t]^{(g)}$ defined by the application. Figure 18.10 gives a path diagram representing several kinds of multiple-group growth models (McArdle, 1991; McArdle & Epstein, 1987; McArdle & Hamagami, 1992). The persons in the groups are assumed to be independent, so this kind of grouping can only be done for observed categorical variables (i.e., sex). The first two groups in Figure 18.10 can be considered as data separated into males or females (or experimentals and controls). Although not necessary, in Figure 18.10 we assume some of the $Y[t]$ occasions were considered incomplete, possibly to represent a collection gathered at unequal intervals of time. In structural modeling diagrams (and programs), the unbalanced data for $Y[3]$ and $Y[5]$ are simply included as latent variables (see McArdle & Aber, 1990). In any case, this multiple-group model now allows us the opportunity to examine a variety of invariance hypotheses.

The multiple-group growth model permits the examination of the presumed invariance of the latent basis functions,

$$A[t]^{(1)} = A[t]^{(2)} = \ldots A[t]^{(g)} = \ldots A[t]^{(G)}. \quad (18.16)$$

The rejection of these constraints (based on χ^2/df) implies that some independent groups have a different basic shape of the growth curve. This is one kind of model that is not easy to represent using standard mixed-effects or multilevel models (Equation 18.13). If a reasonable level of invariance is found, we can further examine a sequence of other group differences. For example, we may examine the equality of the

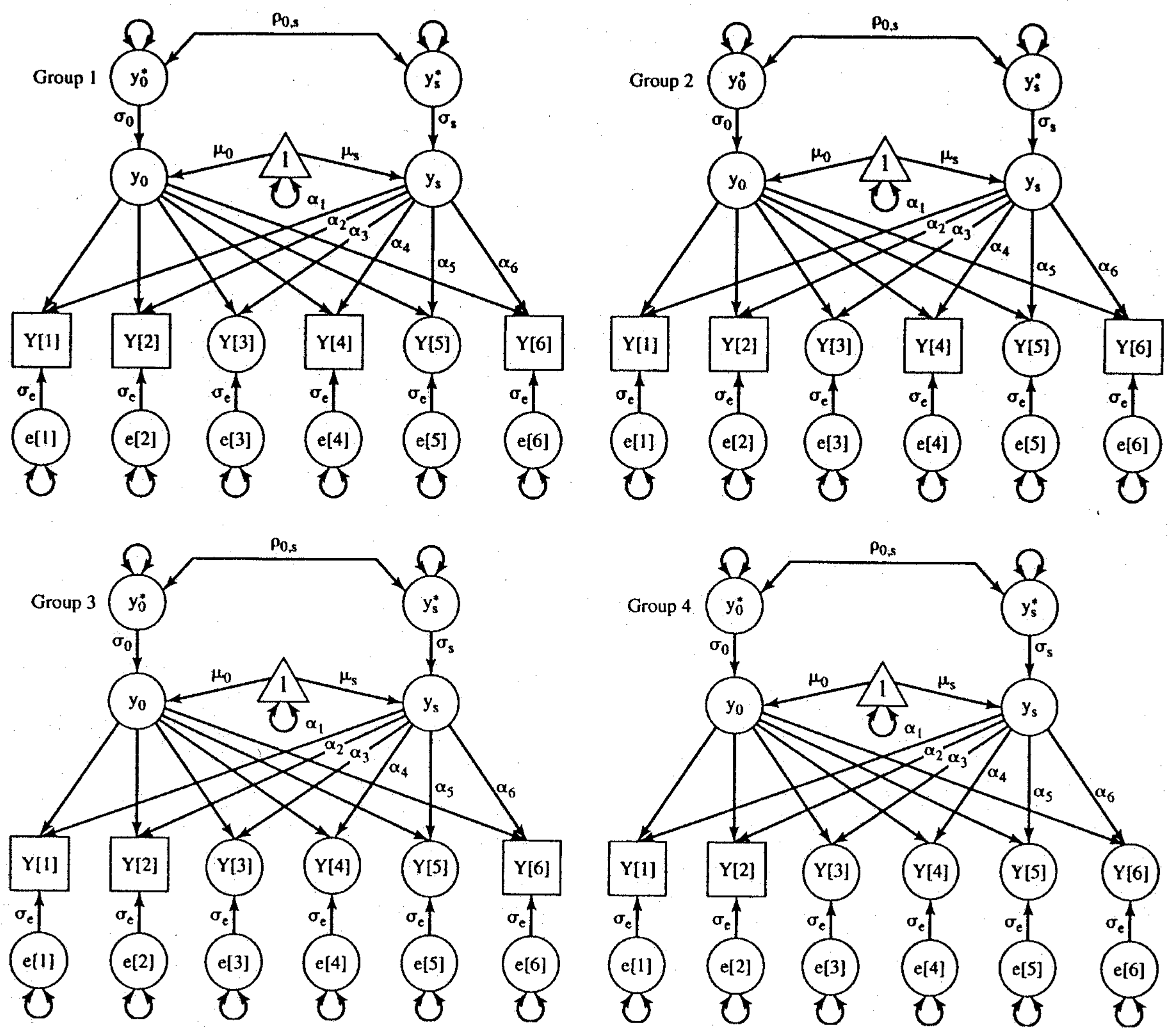

Figure 18.10 A path diagram of a multiple-group latent growth model (Groups 1 and 2) and the inclusion of with patterns of incomplete data (Groups 3 and 4; from McArdle & Hamagami, 1991, 1992).

variances of the latent levels and slopes by writing

$$\sigma_0^{(1)} = \sigma_0^{(2)} = \ldots \sigma_0^{(g)} = \ldots \sigma_0^{(G)} \quad \text{and} \qquad (18.17)$$
$$\sigma_s^{(1)} = \sigma_s^{(2)} = \ldots \sigma_s^{(g)} = \ldots \sigma_s^{(G)}.$$

Other model combinations could include the error deviations ($\sigma_e^{(g)}$), the total slope variance and covariances, and functions of all the other parameters. We may still consider the typical mixed-effects group difference parameters when we examine the invariance of the latent means for initial levels and slopes. If we assume invariance of latent shapes (Equation 18.16) and latent variances (Equation 18.17), we can meaningfully examine

$$\mu_0^{(1)} = \mu_0^{(2)} = \ldots \mu_0^{(g)} = \ldots \mu_0^{(G)} \quad \text{and} \qquad (18.18)$$
$$\mu_s^{(1)} = \mu_s^{(2)} = \ldots \mu_s^{(g)} = \ldots \mu_s^{(G)}.$$

Group differences in the fixed effects can even be coded in the same way as in the typical mixed-effects analyses. Each of these multiple-group hypotheses represent a nonlinearity that may not be possible to examine using a standard mixed-effects approach.

Multiple-group models can be a useful way to express problems of incomplete data. Longitudinal data collections often include different numbers of data points for different people and different variables, and one good way to deal with these kinds of statistical problems is to include multiple-group models that permit different numbers of data points on each person (e.g., Little & Rubin, 1987; McArdle, 1994). The third and fourth groups of Figure 18.10 represent persons with incomplete data on some occasions. In other cases, the data from any one age may not overlap very much with those of another group of another age. In order to uniquely identify and estimate the model parameters from this collection of data (all four groups), all parameters are *forced to be invariant over all groups*. This kind of multiple-group model can be symbolized as

$$Y[t]_n = \sum_{g=1,G} \left(F^{(g)}\{y_{0,n} + y_{1,n}A[t] + e[t]_n\}\right), \quad (18.19)$$

where the $F^{(g)}$ is a binary filter-matrix for each group that defines the pattern of complete (1) and incomplete (0) data entries (for further details, see McArdle & Anderson, 1990; McArdle & Hamagami, 1992). This multiple-group incomplete patterning approach is identical to the statistical models in which we fit structural models to the *raw score information for each person on each variable at each time*. The available information for any subject on any data point (i.e., data at any occasion) is used to build up a likelihood function, and the numerical routine is used to optimize the model parameters with respect to the available data (Neale et al., 1999; Hamagami & McArdle, 2001).

This method assumes the invariance of all growth parameters across different patterns of data is a rigid form of "longitudinal convergence" (after Bell, 1954; see McArdle & Bell, 2000). Although invariance is a reasonable goal in many studies, it is not necessarily a hypothesis that can be tested with all incomplete patterns (McArdle & Anderson, 1990; Miyazaki & Raudenbush, 2000; Willet & Sayer, 1995). One key assumption in our use of these *MLE*-based techniques is that the incomplete data are *missing at random* (*MAR;* Little & Rubin, 1987). This assumption does not require the data to be *missing completely at random* (*MCAR*), but *MAR* does assume there is some observed information that allows us to account for and remove the bias in the model estimates created by the lack of complete data (e.g., Hedecker & Gibbons, 1997; McArdle, 1994; McArdle & Hamagami, 1992). In many cases, this *MAR* assumption is a convenient starting point, and allows us to use all the available information in one analysis. In other cases, invariance of some parameters may fail for a number of reasons and it is important to evaluate the adequacy of this helpful *MAR* assumption whenever possible (e.g., Hedecker & Gibbons, 1997; McArdle, 1994).

Latent Groups Based on Growth-Mixture Models

Another fundamental problem is the discrimination of (a) models of multiple curves for a single group of subjects from (b) models of multiple groups of subjects with different curves. For example, we could have two clusters of people, each with a distinct growth curve, but when we summarize over all the people we end up with poor fit because we need multiple slope factors. One clue to this separation is based on the higher-order distribution of the factor scores—groups are defined by multiple peaked distributions in the latent factor scores. In standard linear structural modeling, these higher-order moments are not immediately accessible, so the multiple-factor versus multiple-group discrimination is not easy. These and other kinds of problems require an a priori definition of the groups before we can effectively use the standard multigroup approach.

These practical problems set the stage for a new and important variation on this multiple-group model—models that test hypotheses about *growth curves between latent groups*. The recent series of models termed *growth mixture models* have been developed for this purpose (L. K. Muthen & Muthen, 1998; Nagin, 1999). In these kinds of analyses, the distribution of the latent parameters is assumed to come from a mixture of two or more overlapping distributions. Current techniques in mixture models have largely been developed under the assumption of a small number of discrete or probabilistic "classes of persons" (e.g., two classes), often based on mixtures of multivariate normal distributions. More formally, we can write this kind of a model as a weighted sum of curves

$$Y[t]_n = \sum_{c=1,C} \left(P\{c_n\} \cdot \{y_{0,n}^{(c)} + A[t]^{(c)} y_{1,n}^{(c)} + e[t]_n^{(c)}\}\right), \quad \text{with} \quad \sum_{c=1,C} (P\{c_n\}) = 1, \quad (18.20)$$

where $P\{c_n\}$ is constrained to sum to unity so that it acts as a probability of class membership for the person in $c = 1$ to C classes.

Using growth-mixture models we can estimate the most likely threshold parameter for each latent distribution (τ_p, for the pth parameter) while simultaneously estimating the separate model parameters for the resulting latent groups. The concept of an unknown or latent grouping can be based on a succession of invariance hypotheses about the growth parameters. We can initially separate latent level means and variances, then separate latent slope means and variances, then both the level and slope, then on the basis loadings, and so on. The resulting maximum likelihood estimates yield a fit that can be compared to the results obtained from more restrictive single class models, so the concept of a mixture distribution of multiple classes can be treated as a hypothesis to be investigated.

In essence, this growth-mixture model provides a test of the invariance of growth model parameters without requiring exact knowledge of the group membership of each individual. It follows that, as we do in standard discriminant or logistic analysis, we can also estimate the probability of assignment of individuals to each class in the mixture, and this estimation of a different kind of latent trait can be a practically useful device. A variety of new program scripts (e.g., Nagin, 1999) and computer programs (e.g., Mplus, by L. K. Muthen & Muthen, 1998) permit this analysis.

Results From Fitting Group Growth Models

We have studied a variety of mixed-effect or multilevel models of the Bradway data. To allow some flexibility here, we used the same latent basis curve model (*M2*) but now we add a few additional variables as predictors. These variables included various aspects of demographic (e.g., gender, educational attainment by age 56, etc.), self-reported health behaviors (e.g., smoking, drinking, physical exercise, etc.), health problems (e.g., general health, illness, medical procedures, etc.), and personality measures (e.g., 16 PF factors). As one example, in a mixed-effects model (see Figure 18.9), we added gender as an effect-coded variable (i.e., females $= -0.5$ and males $= +0.5$). The results obtained for nonverbal scales included the latent basis $A[t] = [= 0, 0.93, 1.01, 1.06, = 1, .97]$ as before. But now, in the same model, we found the males start at slightly lower initial levels ($\nu_{0x} = -0.06$) but had larger positive changes over time ($\nu_{sx} = 0.30$). The addition of gender does not produce large changes in fit ($\chi^2 = 10$ on $df = 4$), so all gender mean differences may be accounted for using the latent variables, but gender does not account for much the variance of the latent scores (.03, .05). To account for more of this variance we proceed using basic principles of multiple regression: In a third model we added educational attainment, in a fourth model we added both gender and education, and in a fifth model we added an interaction of sex and education.

Group differences in the Bradway-McArdle data were also studied using multiple-group growth curves. In a general model the latent means, deviations, and basis shape of the changes were considered different for the males and the females. The key results for males and females show a lack of invariance for the initial basis hypothesis ($A[t]^{(m)} = A[t]^{(f)}$, $\chi^2 = 40$ on $df = 5$). The separate group results show that the females have a higher basis function, and this implies more growth over time (e.g., McArdle & Epstein, 1987). This last result does not deal only with mean differences, but rather includes both mean and covariance differences, and it may be worth pursuing.

Multiple-group growth models have been used in all prior analyses described here to fit the complete and incomplete subsets of the Bradway-McArdle data (Figure 18.6). We compared the numerical results for the complete data (Figure 18.6, Panel A) versus the complete and incomplete data together (Figure 18.6, Panel A plus Panel B), and the parameters remain the same. As a statistical test for parameter invariance over these groups, we calculated from the difference in the model likelihoods, and these differences were trivial ($\chi^2 < 20$ on $df = 20$). This suggests that selective dropout or subject attrition can be considered random with respect to the nonverbal abilities. This last result allows us to combine the complete and incomplete data sets in the hopes for a more accurate, powerful, and unbiased analysis.

In our final set of multiple-group models, we used the latent mixture approach to estimate latent groupings of models results for the nonverbal scores, and some results are graphed in Figure 18.11. The latent growth model using all the data was fitted with free basis coefficients and the same fits as were reported earlier (*M2*). In a first latent mixture model, we allowed the additional possibility of two latent classes ($C = 2$) with different parameters for the latent means and variance but assuming the same growth basis. The two-class growth model (Figure 18.11, Panel A) assumed the same free basis coefficients as previously, smaller latent variances, and an estimated class threshold ($z = 2.48$) separating (a) Class 1 with 92% of the people with high latent means ($\mu_0 = 25$, $\mu_s = 58$), from (b) Class 2 with 8% of the people with lower latent means ($\mu_0 = 16$, $\mu_s = 53$). This two-class model yielded an likelihood that (assuming these two models are nested) represents a substantial change in fit ($\chi^2 = 30$ on $df = 3$). This result suggests that a small group of the Bradway persons may have started at a lower average score with a smaller change. A sequence of parameters were compared under the assumption of two classes, and the final result is presented in Figure 18.11, Panel B. The two-class growth model yielded an estimated class threshold ($z = -0.72$) separating two classes with 33% and 67% of the people. The first class seems to have a higher starting point and lower variability, but the plots of Figure 18.11b seem to show the two curves converge in adulthood. Although this is an interesting possibility, this complete two-class growth-mixture model yielded only a small improvement in fit ($L = -1628$, $\chi^2 = 34$, on $df = 12$), so we conclude that only one class of persons is needed to account for the basic growth curves underlying these data.

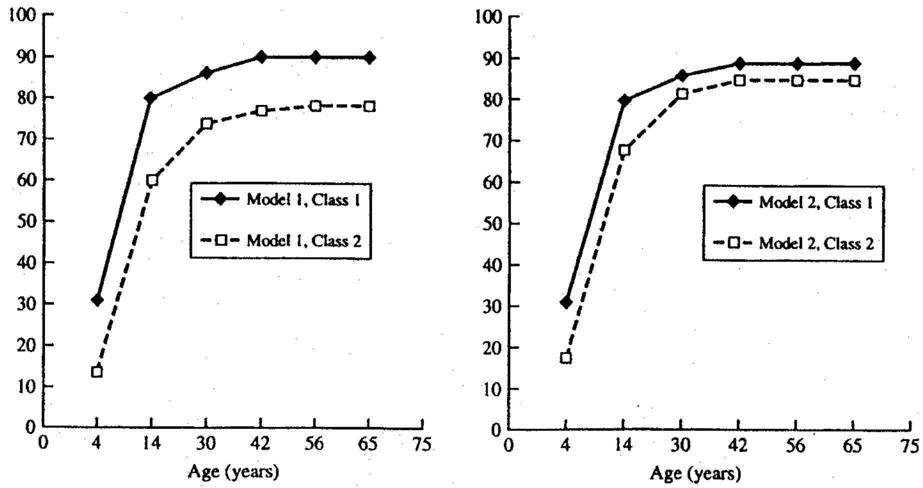

Figure 18.11 Bradway-McArdle verbal score expectations from latent growth-mixture models; Model 1 includes two classes with free means and covariances, and Model 2 is the same with adds four free basis coefficients.

GROWTH CURVE MODELS FROM A DYNAMIC PERSPECTIVE

Growth Models Based on Dynamic Theory

The linear models previously presented can be used to describe a variety of nonlinear shapes, but other models have explicitly included nonlinear functions of the parameters. The development of many of these nonlinear models was based on *differential equations* as an expression of changes as a function of time: that is, dynamic changes. For example, we can write an exponential growth model (see Figure 18.3) as

$$dy/dt = \pi y[t] \quad \text{so} \quad y[t] = y[0] + \exp(-\pi t). \qquad (18.21)$$

where the instantaneous derivative (dy/dt) of the score (y) is a proportional function (π) of the current size of the score ($y[t]$). This change model leads to the integral equation with change over time in the score based on an initial starting point ($y[0]$, sometimes set to zero) with an exponential accumulation (**exp**) based directly on the *growth rate parameter* (π). In classical forms of this model, the rate of change is defined as a linear function of the percentage of initial size (e.g., compound interest).

In contemporary nonlinear model fitting, we can add individual differences to this model in several ways. One approach that is consistent with our previous growth models is to simply rewrite the derivative and integral equations as

$$dy_n/dt = \pi y[t]_n \quad \text{so} \quad Y[t]_n = y_{0,n} + A[t]y_{s,n} + e[t]_n$$
$$\text{with} \quad A[t] = 1 - \exp(-\pi t). \qquad (18.22)$$

In this approach, the classic nonlinear exponential model (Equation 18.21) is now in the form of a latent growth curve *with structured loadings* (as in Browne & duToit, 1991; McArdle & Hamagami, 1996). Individual trajectories start at different initial levels, but then rise or fall in exponential fashion towards some asymptotic values. In this approach, the group curve is based on the latent means and is not based on an averaging of exponential functions (cf. Keats, 1983; Tucker, 1966). This common factor approach allows us to use current computing techniques to examine the empirical fit of this nonlinear model.

A related approach has been used with a form of the von Bertalanffy model,

$$dy_n/dt = (\alpha_n g[t]_n) - (\beta_n - d[t]_n^{\gamma}), \quad \text{so}$$
$$Y[t]_n = y_{0,n} + [\exp(-\alpha t) - \exp(-\beta t)]y_{s,n} + e[t]_n, \quad (18.23)$$

where α = the rate of growth, β = the rate of decline, and γ = some relationship between the two components. In this simplified form (i.e., $\gamma = 1$), there is only one slope (y_s) and one nonlinear set of $A[t]$, but we interpret this as separate growth and decline phases of an underlying continuous latent process. The parameters also yield estimated score peaks ($dy/dt = 0$) and valleys ($d^2y/dt^2 = 0$) with individual differences (e.g., McArdle, Ferrer-Caja, Hamagami, & Woodcock, in press; Simonton, 1989).

Several alternative growth curve models have been developed from dynamic change equations with more parameters. A *logistic curve* can be written as

$$dy_n/dt = \alpha_n y[t]_n(\beta_n - y[t]_n) \quad \text{so}$$
$$Y[t]_n = y_{0,n} + \alpha_n/[1 + \exp\{\beta_n - \gamma_n t\}] + e[t]_n \quad (18.24)$$

with α = the asymptote, β = an influence on the slope (i.e., the slope is $\alpha\beta/4$), and γ = the location of maximum velocity. The expression allows for three individual differences terms with structured loadings (see Browne & du Toit, 1991). A related model is the Gompertz growth curve, written as

$$dy_n/dt = \alpha_n y[t]_n \exp(\beta_n - y[t]_n) \quad \text{so}$$
$$Y[t]_n = y_{0,n} + \alpha_n \exp(-\beta_n \exp\{[t-1]\gamma_n\}) + e[t]_n, \tag{18.25}$$

with α = the asymptote, β = the distance from the asymptote on the first trial, and γ = the rate of change. Browne and duToit (1991) clearly showed how this model could be rewritten as a latent growth curve with structured loadings, including interpretations of individual differences in the rates of growth.

The Preece-Baines family of models start with a derivative based on some predefined function of t ($f\{t\}$) and some asymptotic value ($y[\theta]$). To obtain logistic models (Equation 18.24), the functional form used a proportional distance from the starting point ($f\{t\} = \gamma\{y_{0,n} - y[t]\}$). In other models, this function was the simple rate parameter $f\{t\} = \pi$, so

$$dy_n/dt = \pi\{y[t]_n - y[\theta]_n\} \quad \text{so}$$
$$Y[t]_n = y_{0,n} + (\exp\{-(t-1)\pi\})y[\theta]_n + e[t]_n \tag{18.26}$$

where the amount of change is a function of the distance from the asymptote. This approach allows us to obtain a form of the partial adjustment model of Coleman (1968; McArdle & Hamagami, 1996). These models seem to have practical features for the description of individual changes over long periods of time (see Hauspie et al., 1991; Nesselroade & Boker, 1994).

Growth Curve Models Using Connected Segments

Complex linear and nonlinear models can be used to represent growth. Some models share the common feature of a piecewise analysis applied to different age or time segments—that is, the model considers the possibility that a specific dynamic process does not hold over all time periods. In the simplest cases, we may assume that growth is linear over specific periods of time, and these times are connected by a critical *knot point*—this leads to a conjoined or linear *spline* model (e.g., Bryk & Raudenbush, 1992; Smith, 1979). If we assume one specific cutoff time ($t = C$), we can write

$$\text{if } (t = C), \text{ then } Y[t]_n = y_{0,n} + e[t]_n \quad \text{but}$$
$$\text{if } (t < C), \text{ then } Y[t]_n = y_{0,n} + A1[t]\,y_{1,n} + e[t]_n \quad \text{but}$$
$$\text{if } (t > C), \text{ then } Y[t]_n = y_{0,n} + A2[t]\,y_{2,n} + e[t]_n. \tag{18.27}$$

where the latent growth basis is different before ($A1[t]$) and after ($A2[t]$) the cut point. This piecewise linear model assumes the first component (y_0) is the score at the cutoff, the second component (y_1) is the slope score before the cutoff, and the third component (y_2) is the slope score after the cut point. As before, the fixed effects (means μ_0, μ_1, μ_2) describe the group curve, but the random coefficients (y_0, y_1, y_2) have variances and covariance and account for the individual differences in curves across all observations.

In some growth data sets, it is possible to estimate optimal cut points ($t = C_n$) as an operationally independent random component (see Cudeck, 1996). Unless the cut points are estimated, this model may require a relatively large number of fixed and random parameters to achieve adequate fit. In a recent mixed-effects analysis, Cudeck & du Toit (2001) followed previous work (e.g., Seber & Wild, 1989) and used a "segmented polynomial" nonlinear mixed model based on an individual a *latent transition point* for each individual. This model can be written in our notation as

$$Y[t]_n = y_{0,n} + y_{1,n}A[t] + y_{2,n}(A[t] - y_{3,n})^2 + e[t]_n,$$
$$|A[t] \le y_{3,n} \quad \text{and} \tag{18.28}$$
$$Y[t]_n = y_{0,n} + y_{1,n}A[t] + e[t]_n, \quad |A[t] > y_{3,n},$$

where the parameter y_3 is the value of $A[t]$ when the polynomial of the first phase changes to the linear component of the second phase. Important practical suggestions about fitting multilevel nonlinear curves were presented by Cudeck and DuToit (2001).

These segmented or composite models have also been a mainstay of nonlinear modeling. For example, the segmented logistic model (see Bock, 1975; Bock & Thissen, 1980) can be written as a trajectory where

$$Y[t]_n = \sum_{k=1,K} [\alpha_{k,n}/(1 + \exp\{\beta_{k,n} - \gamma_{k,n}t\})] + e[t]_n \tag{18.29}$$

is the sum of $k = 1$ to K logistic age-segments. Within each segment, α_k = the asymptote, β_k = an influence on the slope (i.e., the slope is $\alpha_k\beta_k/4$), γ_k = the location of maximum velocity, and no intercept is fitted. Within each segment, the rate of growth exhibits early increases, reaches a maximum (peak growth velocity), and decreases towards the asymptote; the final value of one segment is used as the starting value of the next segment. While each segment has a simple logistic curve, the overall curve fitted (e.g., over the full life span) has a particularly complex nonlinear form. These composite models allow for different dynamics at different ages, and this represents an important innovation.

Growth Models Based on Latent Difference Scores

The complexities of fitting and extending the previous dynamic models have limited their practical utility. In recent research we have considered some ways to retain the basic dynamic change interpretations but use conventional analytic techniques. This has led us to recast the previous growth models using *latent difference scores* (see McArdle, 2001). This approach is not identical to that represented by the differential equations considered earlier (e.g., Arminger, 1987; Coleman, 1968), but it offers a practical approximation that can add clear dynamic interpretations to traditional linear growth models.

In the latent difference approach, we first assume we have a pair of observed scores $Y[\mathbf{t}]$ and $Y[\mathbf{t}-\mathbf{1}]$ measured over a defined interval of time ($\Delta\mathbf{t} = \mathbf{1}$), and we write

$$Y[\mathbf{t}]_{\mathbf{n}} = y[\mathbf{t}]_{\mathbf{n}} + e[\mathbf{t}]_{\mathbf{n}}, \quad Y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}} = y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}} + e[\mathbf{t}-\mathbf{1}]_{\mathbf{n}}$$
$$\text{and} \quad y[\mathbf{t}]_{\mathbf{n}} = y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}} + \Delta y[\mathbf{t}]_{\mathbf{n}} \qquad (18.30)$$

with corresponding latent scores $y[\mathbf{t}]$ and $y[\mathbf{t}-\mathbf{1}]$, and error of measurements $e[\mathbf{t}]$ and $e[\mathbf{t}-\mathbf{1}]$. It follows that by simple algebraic rearrangement, we can define

$$y[\mathbf{t}]_{\mathbf{n}} = y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}} + \Delta y[\mathbf{t}]_{\mathbf{n}} \quad \text{so}$$
$$\Delta y[\mathbf{t}]_{\mathbf{n}} = (y[\mathbf{t}]_{\mathbf{n}} - y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}}) \qquad (18.31)$$

where the additional latent variable is directly interpreted as a *latent difference score*. This simple algebraic device allows us to generally define the trajectory equation as

$$Y[\mathbf{t}]_{\mathbf{n}} = y_{0,\mathbf{n}} + \left(\sum_{\mathbf{i}=\mathbf{1},\mathbf{t}} \Delta y[\mathbf{i}]_{\mathbf{n}}\right) + e[\mathbf{t}]_{\mathbf{n}} \qquad (18.32)$$

where the summation ($\sum_{\mathbf{i}=\mathbf{1},\mathbf{t}}$) or accumulation of the latent changes ($\Delta y[\mathbf{t}]$) up to time t is included. In this latent difference score approach, we do not directly define the $A[\mathbf{t}]$ coefficients, but instead we directly define changes as an accumulation of the first differences among latent variables.

This latent difference score ($\Delta y[\mathbf{t}]_{\mathbf{n}}$) of Equation 18.31 is not the same as an observed difference score ($\Delta Y[\mathbf{t}]_{\mathbf{n}}$) because the latent score is considered after the removal of the model-based error component. Although this difference $\Delta y[\mathbf{t}]_{\mathbf{n}}$ is a theoretical score, it has practical value because now we can write any structural model for the latent change scores without immediate concern about the resulting trajectory (as in McArdle, 2001; McArdle & Hamagami, 2001; McArdle & Nesselroade, 1994). For example, Coleman (1968) suggests we write a change model for consecutive time points as

$$\Delta y[\mathbf{t}]_{\mathbf{n}} = \pi(y_{\mathbf{a},\mathbf{n}} - y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}}), \qquad (18.33)$$

where $y_{\mathbf{a}}$ is a latent asymptote score that is constant over time, and the π describes the proportional change based on the current distance from the asymptote (i.e., partial adjustment; see Equation 18.26). A slightly more general change expression model is written as

$$\Delta y[\mathbf{t}]_{\mathbf{n}} = \alpha y_{\mathbf{s},\mathbf{n}} + \beta y[\mathbf{t}-\mathbf{1}]_{\mathbf{n}} \qquad (18.34)$$

where the $y_{\mathbf{s}}$ is a latent slope score that is constant over time, and the α and β are coefficients describing the change. This second expression (Equation 18.34) is more general because we can add restrictions ($\alpha = \pi$, $\beta = -\pi$) and obtain the first expression (Equation 18.33). We refer to this as a *dual change score* (DCS) model because it permits both a systematic constant change (α) and a systematic proportional change (β) over time, and no stochastic residual is added (i.e., $z[\mathbf{t}]$; see McArdle, 2001). This is an interesting linear model because the expectations lead to a mixed-effects model trajectory with a distinct nonlinear form (e.g., $A[\mathbf{t}]$ in Equation 18.22), but the corresponding accumulation of differences (Equation 18.32) remains unchanged.

One advantage of this approach is that this dynamic model can fitted using standard structural modeling software. The structural path diagram in Figure 18.12 illustrates how the latent change score model (Equations 18.30–18.34) can be directly represented using standard longitudinal structural equation models. This set of equations is drawn in Figure 18.12 by using (a) *unit-valued regression weights* among variables by fixed nonzero constraints (as in McArdle & Nesselroade, 1994), (b) a *constant time lag* by using additional latent variables as placeholders (as in Horn & McArdle, 1980), (c) each *latent change score as the focal outcome variable,* and (d) a repetition (by equality constraints) of the α and β structural coefficients. Following the standard linear growth models, we assume the unobserved initial-level component (y_0) has a mean and variance (i.e., μ_0 and σ_0^2), while the error of measurement has mean zero, has constant variance $\sigma_e^2 > 0$), and is uncorrelated with every other component. As in the linear change model of Figure 18.7, the constant change component (y_s) has a nonzero mean (i.e., μ_s, the average of the latent change scores), a nonzero variance (i.e., σ_s^2, the variability of the latent change scores), and a nonzero correlation with the latent initial levels (i.e., ρ_{0s}). As in other latent growth models, the numerical values of the parameters α and β can now be combined to form

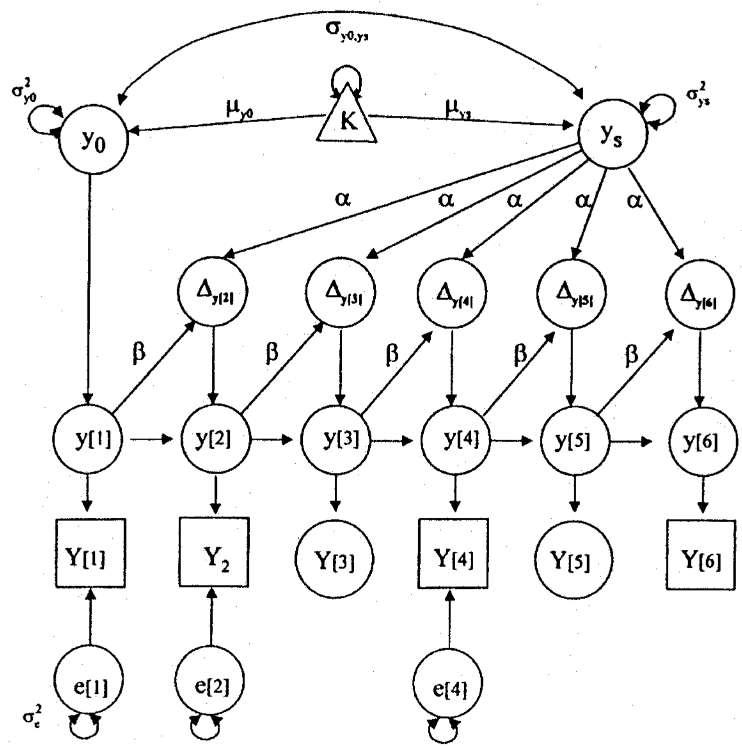

Figure 18.12 A reinterpretation and extension of the latent growth model as a latent difference score structural model, including both additive (α) and proportional (β) change parameters (see McArdle, 2001).

many different kinds of individual and group trajectories over age or time, including the addition of covariates. All these features are apparent without directly writing or specifying a model for the full trajectory over time.

Results From Fitting Dynamic Growth Models

To illustrate some of these dynamic growth models, the Bradway-McArdle Non-Verbal data were fitted (using Mx and NLMIXED). A new model (*M*4) based on the partial adjustment model (as described in Equation 18.26) required four free parameters with individual differences in the initial-level variance (an asymptote) and in the latent slope (the distance from asymptote) parameters. This model requires all loadings to have be an exponential function formed from a single rate parameter (estimated at $\pi = -0.16$), and the resulting expected trajectory is drawn as the solid line in Figure 18.9. In contrast to the shape of previous latent basis model, this is an exponential shape that rises rapidly and then stays fairly constant at the asymptote (or equilibrium point) from age 42 to age 65.

This model fit was not as good as that of the latent basis model, but the difference is relatively small compared to the difference in degrees of freedom ($\chi^2 = 59$ on $df = 3$), the error variance is similar (from $\sigma_e\{M2\} = 5.1$ to $\sigma_e\{M4\} = 6.5$). Unlike the latent basis model, this negative exponential model makes explicit predictions at all ages (e.g., Equation 18.26 for $\mu[22] = 82.7$). A second model was fit allowing individual stochastic differences (random coefficients) in the rate parameter (π_n). The resulting fitted curves show only a small change in the average rate ($\pi = -0.15$), the random variance of these rates is very small ($\sigma_\pi < .01$), and the fit is not much better than that of the simpler partial adjustment model ($\chi^2 = 14$, $df = 4$, $\sigma_e\{M5\} = 5.8$).

The comparison of the latent basis (*M*2) and the partial adjustment models (*M*4 or *M*5) suggests that the decline in nonverbal intellectual abilities by age 65 is relatively small. The expectations from these two models yield only minor departures of the exploratory latent basis model (*M*2) from the partial adjustment model (*M*4). The further comparison of the stochastic adjustment (*M*5) and the partial adjustment (*M*4) model suggests that the same shape of change in nonverbal intellectual abilities call be applied to all persons. Although these analyses illustrate only a limited set of substantive hypotheses about dynamic growth processes, these are key questions in aging research.

An example of the segmented models fitted to the Bradway-McArdle data has been published by Cudeck & DuToit (2001). Using data from persons who had data on at least one of the last three occasions ($N = 74$), these authors fit a nonlinear mixed model based on Equation 18.32 and found an estimated transition age ($\beta_3 = 18.6$, $\sigma_3 = 0.60$) where the polynomial of the first phase changes to the linear decline component ($\beta_1 = -.141$, $\sigma_3 = 0.05$; $\beta_2 = -.571$) of the second phase. The estimated mean response shows a growth curve with rapid increases and gradual decline (after $Age[t] = 18.6$). The variability of these estimated parameters allows for a variety of different curves, and some of these are drawn in Figure 18.13. "Although the trend is decreasing overall, a few individuals actually exhibit increases, while for others the response is essentially constant into old age. . . . The two individuals in Figure (A) had large differences in intercepts, β_{i0} (70.8 versus 91.9); those in Figure (B) had large differences in slopes, β_{i1} (−.32 versus 0.04); those in Figure (C) had large differences in transition age, β_{i3} (14.1 versus 23.6)" (Cudeck & duToit, 2001; p. 13). The addition of individual differences in transition points contributes to our understanding of these growth curves.

Four alternative latent difference score models (Figure 18.12) were fitted to the nonverbal scores (Figure 18.6). To facilitate computer programming (e.g., Mx) the original data were rescaled into 5-year age segments (i.e., 30 to 35, 35 to 40, etc.). A baseline no-change model (NCS) was fitted with only three parameters and the results using this approach were comparable to those of the baseline growth model (*M*0). This was also true for a constant change score (CCS; α only) model, and the result was identical to that of the linear basis model (*M*1). The proportional change model (PCS; β only), not fit earlier, shows a minor improvement in fit ($\chi^2 = 5$ on $df = 1$).

To fit the dual change model (Equation 18.34), the additive slope coefficient was fixed for identification purposes ($\alpha = 1$), but the mean of the slopes was allowed to be free (μ_s). This allowed estimation of the effects for nonverbal with (a) inertial effects ($\beta = -1.38$), (b) initial-level means ($\mu_0 = 32$) at Age = 5, and (c) a linear slope mean ($\mu_s = 81$) for each 5-year period after Age = 5. The goodness-of-fit of the DCS model can be compared to that of every other nested alternative, and these comparisons show the best fit was achieved using this model ($\chi^2 = 785$ on $df = 2$; $\chi^2 = 485$ on $df = 1$; $\chi^2 = 385$ on $df = 1$). From these results we calculate the expected group trajectories and the 5-year latent change accumulation as the combination of Equations 18.32

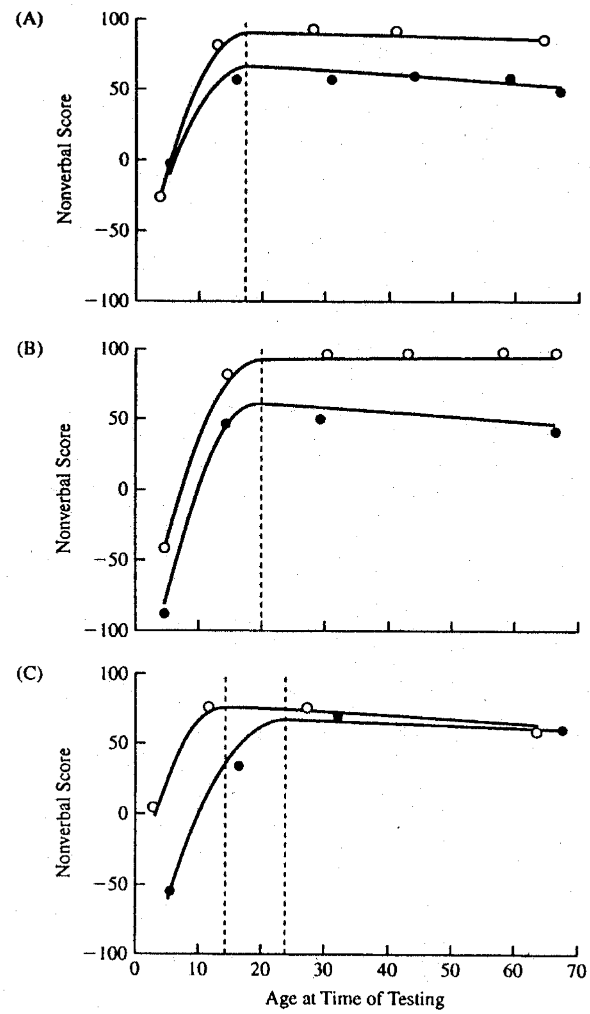

Figure 18.13 Fitted curves for selected individuals from the segmented growth model (from Cudeck & du Toit, 2001).

and 18.34, and we find the expected trajectory over time for the nonverbal variable represented in this way is the same as the previous nonlinear solid line in Figure 18.9 (see Hamagami & McArdle, 2001). This dynamic result is explored more in the next section.

MULTIPLE VARIABLES IN LATENT GROWTH CURVE MODELS

Including Measurement Models Within Latent Growth Analyses

Previous research on growth models for multiple variables has considered the application of standard multivariate models to growth data (e.g., Harris, 1963; Horn, 1972). A parsimonious alternative that has been explored in prior work is the inclusion of a so-called measurement model embedded in these dynamic structural models (for references, see McArdle, 1988; McArdle & Woodcock, 1997). This can be fitted by including common factor scores ($f[t]$), proportionality via factor loadings (λ_y, λ_x), and uniqueness (u_y, u_x). We could write a model as

$$Y[t]_n = \nu_y + \lambda_y f_n + u_{y,n}, \quad \text{and} \qquad X[t]_n = \nu_x + \lambda_x f_n + u_{x,n}, \tag{18.35}$$

so that each score is related to a common factor ($f[t]$) with time-invariant factor loadings (λ_j), unique components (u_j), and scaling intercepts (ν_j). We can then consider whether all latent changes in these observed scores are characterized by the growth parameters of the common factor scores

$$f[t]_n = f_{0,n} + Af[t]f_{s,n} + ef[t]_n. \tag{18.36}$$

This common factor growth model is drawn as a path diagram in Figure 18.14. We can also recast these common factor scores into a latent difference form of

$$f[t]_n = f[t-1]_n + \Delta f[t]_n \quad \text{and} \qquad \Delta f[t]_n = \alpha f f_{s,n} + \beta f f[t-1]_n, \tag{18.37}$$

so that the dynamic features of the common factors are estimated directly (e.g., Figure 18.12).

The expectations from this kind of a model can be seen as *proportional growth curves,* even if the model includes additional variables or factors. If this kind of restrictive model of changes in the factor scores among multiple curves provides a reasonable fit to the data, we have evidence for the dynamic construct validity of the common factor (as in McArdle & Prescott, 1992). To the degree multiple measurements are made, this common factor hypothesis about the change pattern is a strongly rejectable model (e.g., McArdle, Ferrer-Caja et al., in press; McArdle & Woodcock, 1997). In either form (Equation 18.36 or Equation 18.37) this multivariate dynamic model is highly restrictive, so it may serve as a *common cause* baseline that can help guide the appropriate level of analysis (as in McArdle & Goldsmith, 1990; Nesselroade & McArdle, 1997).

One explicit assumption made in all growth models is that the scores are adequate measures of the same construct(s) over all time and ages. This assumption may be evaluated whenever we fit the measurement hypothesis (i.e., is $\Lambda[t] = \Lambda[t+1]$?). It may be useful to examine the assumption of *metric factorial invariance* over occasions without the necessity of a simple structure basis to the measurement model (Horn & McArdle, 1992; McArdle & Cattell, 1994; McArdle & Nesselroade, 1994). However, in long-term

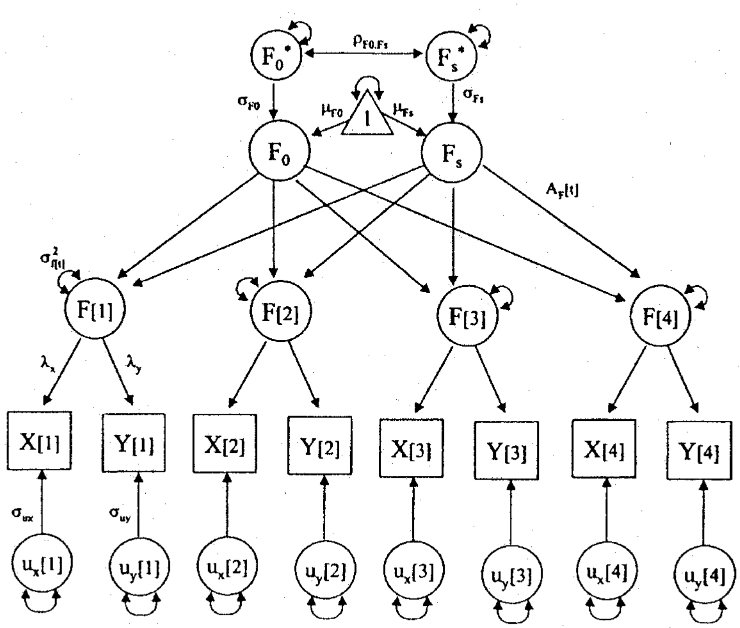

Figure 18.14 A path diagram of a multiple variable measurement model with a latent "curve of factor scores" (McArdle, 1988).

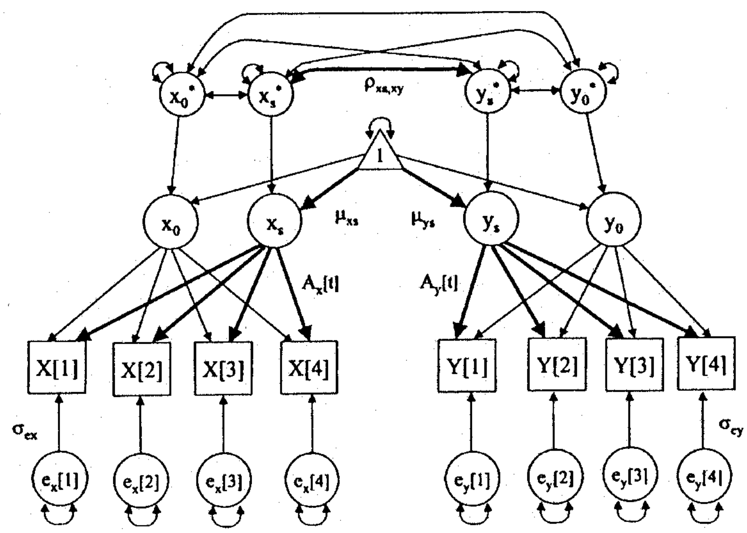

Figure 18.15 A path diagram representing a bivariate latent growth model for multiple variables (from McArdle, 1989).

longitudinal data collections, we often use repeated measures models when different variables measuring the same constructs were used at different ages. The basic requirements of meaningful and age-equivalent measurement models are a key problem in the behavioral sciences, and future research is needed to address these fundamental concerns (see Burr & Nesselroade, 1990; Fischer & Molenaar, 1995).

Modeling Interrelationships Among Growth Curves

The collection of multiple variables at each occasion of measurement leads naturally to questions about relationships among growth processes and multivariate growth models. The early work on this topic led to sophisticated models based on systems of differential equations for the size of multiple variables. In one comprehensive multivariate model, Turner (1978) extended the simple growth principles to more variables, and permitted an examination of biologically important interactions based on the size and sign of the estimated parameters (see Griffiths & Sandland, 1984). Multivariate research in the behavioral sciences has not gone as far yet, and seems to have relied on advanced versions of the linear growth models formalized by Rao (1958), Pothoff and Roy (1964), and Bock (1975).

Some recent structural equation models described in the statistical literature have emphasized the examination of *parallel growth curves,* including the correlation of various components (McArdle, 1988, 1990; Willett & Sayer, 1994). The models fitted here can be represented in latent growth notation for two variables by

$$Y[t]_n = y_{0,n} + A_y[t]\, y_{s,n} + e_y[t]_n \quad \text{and} \qquad (18.38)$$
$$X[t]_n = x_{0,n} + A_x[t]\, x_{s,n} + e_x[t]_n ,$$

where $Y[t]$ and $X[t]$ are two different variables observed over time, there are two basis functions ($A_y[t]$ and $A_x[t]$), and

$$E\{y_0, x_0\} = \sigma_{y0,x0}, \quad E\{y_0, x_s\} = \sigma_{y0,xs}, \qquad (18.39)$$
$$E\{y_s, x_0\} = \sigma_{ys,x0}, \quad \text{and} \quad E\{y_s, x_s\} = \sigma_{ys,xs},$$

and all covariances ($\sigma_{y[i],x[j]}$) are allowed among the common latent variables. A path diagram of this bivariate growth model is presented in Figure 18.15.

This set of structural equations has been used to examine a variety of substantive hypotheses. One hypothesis relies on the equality of the basis coefficients (e.g., $A_y[t] = A_x[t]$) to examine the overall shape of the two curves. Interpretations have also been made about the size and sign of nonzero covariance of initial levels (i.e., $|\sigma_{y0,x0}| > 0$) and *covariance of slopes* (i.e., $|\sigma_{ys,xs}| > 0$), but these interpretations are limited. These random coefficients reflect individual similarities in the way persons start and change over time across different variables, and these are key features for some researchers (e.g., Duncan & Duncan, 1995; Raykov, 1999; Willet & Sayer, 1994). However, it should be noted that this simple relationship is not time-dependent, so it may not fully characterize the interrelationships over time. This might lead us to consider other, more elaborate models for the time-dependent interrelationships among the measures. That is, if we think one of these variables is responsible for the growth in the other, then we might need to fit a related but decidedly different set of models. The next section presents some advanced models used to solve these kinds of problems.

Multivariate Dynamic Models of Determinants of Changes

The previous models use information about the time-dependent nature of the scores, and there are several extension of these models of interest in multivariate growth curve analysis (Arminger, 1987; Nesselroade & Boker, 1994). One of the most basic extensions is the combination of a measurement model with a dual change score model among common factor scores. This kind of model was displayed earlier in Figure 18.14 but can now be extended into Figure 18.15. In other extensions, we may be interested in a combination of several previous models, including parallel growth curves and time-varying covariates.

Suppose a new variable $X[t]$ is measured at multiple occasions and we want to examine its influence in the context of a growth model for $Y[t]$. One popular model used in multilevel and mixed-effects modeling is based on the analysis of covariance (Equation 18.13) with $X[t]$ as a time-varying predictor. In our notation we can write

$$Y[t]_n = y_{0:x n} + A[t] y_{s:x,n} + \delta X[t]_n + e[t]_n \quad (18.40)$$

where the δ are fixed (group) coefficients with the same effect on $Y[t]$ scores at all occasions. In this case the growth parameters ($\mu_{0:x}$, $\mu_{s:x}$, $\sigma_{s:x}$, etc.) are conditional on the expected values of the external $X[t]$ variable. By taking first differences we find that this model implies the true score change over time is

$$\Delta y[t]_n = \Delta A[t] y_{s:x,n} + \delta \Delta X[t]_n, \quad (18.41)$$

so the basis coefficients still reflect changes based on a constant slope ($y_{s:x}$) independent of $X[t]$, and the new coefficient (δ) represents the effect of changes in X (i.e., $\Delta X[t]$) on changes in Y (i.e., $\Delta y[t]$). This time-varying covariate model is relatively easy to implement using available mixed-effects software (e.g., Sliwinski & Buchele, 1999; Sullivan, Rosenbloom, Lim, & Pfefferman, 2000; Verbeke & Molengerghs, 2000; cf. McArdle, Hamagami, et al., in press).

Modeling for multiple variables over time has been considered in the structural modeling literature. For many researchers, the most practical solution is to fit a cross-lagged regression model (see Cook & Campbell, 1977; Rogosa, 1978). This model can be written for latent scores as

$$y[t]_n = \nu_y + \phi_y y[t-1]_n + \delta_{yx} x[t-1]_n + e_y[t]_n, \quad \text{and}$$
$$x[t]_n = \nu_x + \phi_x x[t-1]_n + \delta_{xy} y[t-1]_n + e_x[t]_n, \quad (18.42)$$

where we assume a complementary regression model for each variable with auto-regressions (ϕ_y, ϕ_x) and cross-regressions (δ_{yx}, δ_x) for time-lagged predictors. This model yields a set of first difference equations that are similar to Equation 18.41, where each change model has zero intercept and the lagged changes. The cross-lagged coefficients (δ) are interpreted as the effect of changes (e.g., $\Delta x[t]$) on changes (e.g., $\Delta y[t]$), and form the basis for the critical hypotheses (e.g., $\delta_{yx} > 0$ but $\delta_{xy} = 0$).

The literature on nonlinear dynamic models has also dealt with similar multivariate issues, but clear examples are not easy to find. One dynamic bivariate model based on the partial adjustment concept was proposed by Coleman (1968) and Arminger (1987) using different techniques for estimation. This model can be written in difference score form as a set of simultaneous equations where

$$\Delta y[t]_n = \pi_y (y_{a,n} - y[t-1]_n) \quad \text{with}$$
$$y_{a,n} = \alpha_y + \gamma_{yx} x[t-1]_n,$$

and

$$\Delta x[t]_n = \pi_x (x_{a,n} - x[t-1]_n) \quad \text{with}$$
$$x_{a,n} = \alpha_x + \gamma_{xy} y[t-1]_n. \quad (18.43)$$

In this model we include pairs of latent asymptotes (y_a and x_a), rates of adjustment (π_y and π_x), intercepts (α_y and α_x), and cross-effects (γ_{yx} and γ_{xy}). The partial adjustment system has some features of a multilevel model for intercepts and slopes (Equation 18.13).

Now, following our previous latent difference scores model, we can also write a *bivariate dynamic change score* model as

$$\Delta y[t]_n = \alpha_y y_{s,n} + \beta_y y[t-1]_n + \gamma_{yx} x[t-1]_n, \quad \text{and}$$
$$\Delta x[t]_n = \alpha_x x_{s,n} + \beta_x x[t-1]_n + \gamma_{xy} y[t-1]_n, \quad (18.44)$$

where we assume a complementary dual change score model for each variable. In the first part of each change score we assume a dual change score model represented by parameters α and β. This model also permits a *coupling* parameter (γ_{yx}) representing the time-dependent effect of latent $x[t]$ on $y[t]$, and another coupling parameter (γ_{xy}) representing the time-dependent effect of latent $y[t]$ on $x[t]$. If we restrict the

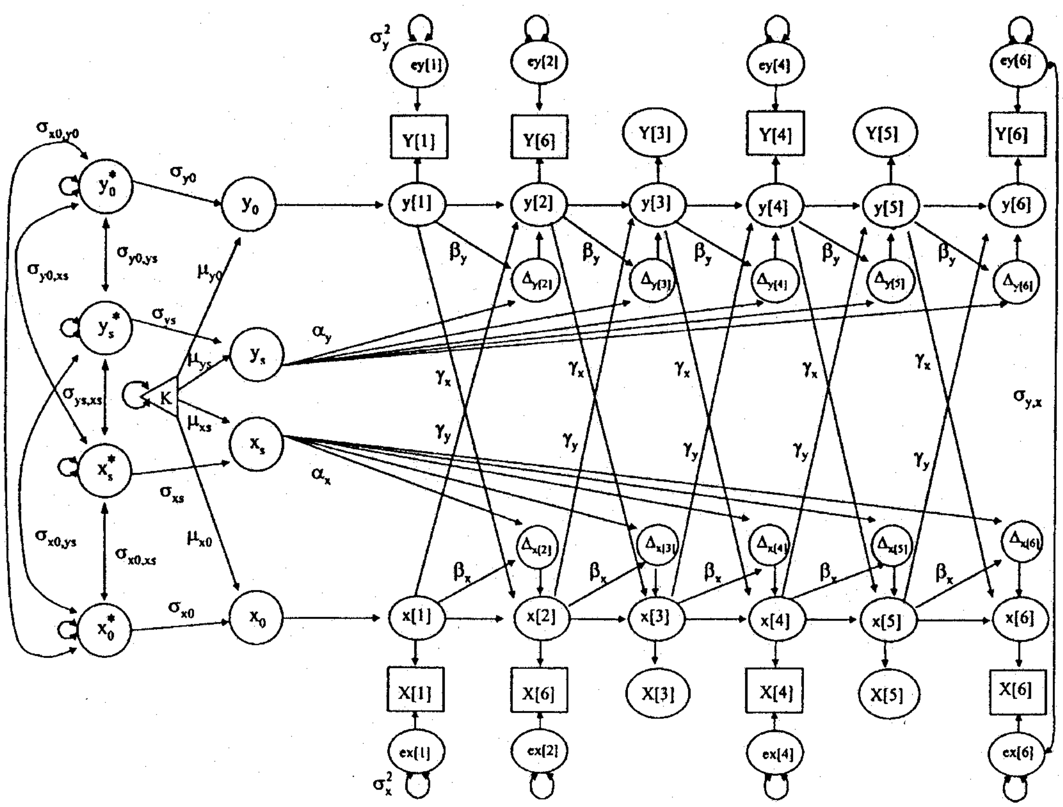

Figure 18.16 A path diagram representing a bivariate latent difference score structural model; each variable is allowed dual changes within variables (α = additive and β = proportional) as well as covariance (σ) and coupling (γ) across variables (from McArdle & Hamagami, 2001).

parameters of Equation 18.44 so that $\alpha = -\beta$, then this model is a reduced form of the partial adjustment system (Equation 18.43). This model is close to the partial adjustment system (Equation 18.43) but is not the same as time-varying covariate models (Equation 18.41) or cross-lagged models (Equation 18.42)—the latent changes in this system of equations have an intercept (α) and the coupling parameters (γ) are direct effects from prior time-varying levels ($x[t-1]$ and $y[t-1]$). Results from these alternative models can be quite different (see McArdle & Hamagami, 2001).

This bivariate dynamic model is described in the path diagram of Figure 18.16. Again the key features of this model include the used of fixed unit values (to define $\Delta y[t]$ and $\Delta x[t]$) and equality constraints (for the α, β, and γ parameters). These latent difference score models can lead to more complex nonlinear trajectory equations (e.g., nonhomogeneous equations) but these can be described simply by writing the respective bases ($A_j[t]$) as the linear accumulation of first differences (Equation 18.31) for each variable.

On a formal basis, however, this bivariate dynamic model of Equation 18.44 permits hypotheses to be formed about (a) parallel growth, (b) covariance among latent components, (c) proportional growth, and (d) dynamic coupling over time. That is to say, in addition to the previous restrictions on the dynamic parameters ($\alpha = 0, \alpha = 1$, and/or $\beta = 0$) we can focus on evaluating models in which one or more of the coupling parameters is restricted (i.e., $\gamma_{yx} = 0$ and/or $\gamma_{xy} = 0$). If only one of these coupling parameters is large and reliable, we may say we have estimated a coupled dynamic system with leading indicators in the presence of growth. To the degree these parameters are zero, we can say we have estimated an uncoupled system. Additional descriptions of the relevant dynamic aspects of these model coefficients, including the stability or instability of long run behaviors, can be evaluated from additional calculations (e.g., eigenvalues and equilibrium formulas; Arminger, 1987; Tuma & Hannan, 1984). Additional information can also come from a visual inspection of the bivariate expectations (after Boker & McArdle, 1995).

By combining some aspects of the previous sections, we can now represent a *group difference dynamic change score* model in at least three different ways. Assume C is a observed vector describing some kind of group differences (e.g., effect or dummy codes, for g = 1 to G groups.). If so, we can consider a model whereby the group contrasts (C_n) have a direct effect on the latent change

$$\Delta y[t]_n/\Delta t = \alpha_y y_{s,n} + \beta_y y[t-\Delta t]_n + \gamma_{yx} x[t-\Delta t]_n + \kappa_y C_n, \quad (18.45)$$

with group coefficient κ_y. Alternatively, we can write a model in which the contrasts have direct effects on the latent slopes

$$\Delta y[t]_n/\Delta t = \alpha_y y_{s,n} + \beta_y y[t-\Delta t]_n + \gamma_{yx} x[t-\Delta t]_n \text{ with } y_{s,n} = \kappa_0 + \kappa_c C_n + e_z[t]_n. \quad (18.46)$$

Finally, we can write a model in which multiple groups (superscripts g) are used to indicate independent group dynamics,

$$\Delta y[t]_n^{(g)}/\Delta t = \alpha_y^{(g)} y[s]_n^{(g)} + \beta_y^{(g)} y[t-\Delta t]_n^{(g)} + \gamma_{yx}^{(g)} x[t-\Delta t]_n^{(g)}. \quad (18.47)$$

In the first model (Equation 18.45), we add the group contrasts as a covariate in the difference model. In the second model (Equation 18.46), we add a multilevel prediction structure of the dynamic slopes. In the third model (Equation 18.47), we indicate a potentially different dynamic parameter for each group. This third model can be fitted and used in the same way as any multiple-group models can (e.g., McArdle & Cattell, 1995; McArdle & Hamagami, 1996).

Results From Fitting Multiple Variable Growth Models

Measurement problems arise in the fitting of any statistical model with longitudinal data, and these issues begin with scaling and metrics. Our first problem with the Bradway-McArdle data comes from the fact that the Stanford-Binet (SB) was the measure administered at early ages (4, 14, 30) and the Wechsler Adult Intelligence Scale (WAIS) was used at the later ages (30, 42, 56, 64). Although these are both measures of intellectual abilities, they are not scored in the same way, and they may measure different intellectual abilities at the same or at different ages. These data were examined using a set of structural equation models with common factors for composite scores from the SB and the WAIS.

The initial structural equation model was based on information from the age 30 data in which both measurements were made, and assumed invariance across all measures at other occasions. In model fitting, the factor loading of the first variable was fixed ($\lambda_y = 1$) to identify the factor scores, and the other loading ($\lambda_x = .84$) was estimated and required to be invariant over all times of measurement. The results quickly showed a single common factor model does not produce a good fit ($\chi^2 = 473$, $df = 34$) even though most of the parameter estimates seem reasonable ($\sigma_x = 1.39$; $\sigma_f = .06$; $\sigma_{sf} = 5.3$). In subsequent analyses, the items in each scale (SB & WAIS) were separated on a theoretical basis—some were considered as verbal items, and these were separated from the items that were considered as nonverbal items in each scale (memory and number items were separated; see Hamagami, 1998; McArdle et al., 2002). The single-factor model was refitted to each new scale, and these models fit much better than before ($\chi^2 = 63$, $df = 32$). At least two separate constructs were needed to reflect the time-sequence information in the interbattery data.

Next we followed the early work of Bayley (1956; see Figure 18.4), and we created longitudinal scores with equal intervals by using some new forms of *item response theory* (IRT) and *latent trait models* (Embretson, 1996; Fisher, 1995; McDonald, 1999). From these analyses, we formed a scoring system or translation table for each construct from the SB and WAIS measures by using IRT calibration (using the MSTEPS program) based on the data from the testing at age 30, in which both the SB & WAIS were administered. These analyses resulted in new and (we hope) age-comparable scales for the verbal and nonverbal items from all occasions (as displayed in Table 18.1 and Figure 18.6).

Several alternative verbal-nonverbal bivariate coupling models were fitted to the data (for details, see McArdle, Hamagami, et al., in press). A first model included all the bivariate change parameters described previously (Equation 18.44). This includes six dynamic coefficients (two each for α, β, γ), four latent means (μ), six latent deviations (σ), and six latent correlations (ρ). This model was fitted with $N = 111$ individuals with at least one point of data and 498 individual data observations, and it yields an overall fit ($L = 7118$) that was different from that of a random baseline ($\chi^2 = 379$ on $df = 16$). The group {and individual} trajectories of the best-fitting model can be written for the verbal ($V[t]$) and nonverbal ($N[t]$) scores in the following way

$$V[t]_n = \mathbf{15.4}\{\pm 1.3\} + \left(\sum_{i=1,t} \Delta V[t]_n\right) + \mathbf{0}\{\pm 4.7\}, \quad \text{and}$$

$$N[t]_n = \mathbf{33.4}\{\pm 7.8\} + \left(\sum_{i=1,t} \Delta N[t]_n\right) + \mathbf{0}\{\pm 11.5\}, \quad \text{with}$$

$$\sigma_{y0x0} = .77, \quad \sigma_{y0xs} = .90, \quad \sigma_{ysx0} = .08, \quad \sigma_{ysxs} = -.05. \quad (18.48)$$

More fundamentally, the respective latent change scores were modeled as

$$\begin{aligned} \Delta V[\mathrm{t}]_{\mathrm{n}} &= \mathbf{-10.1}\{\pm 11.2\} V_{\mathrm{s,n}} + \mathbf{-0.99}\, V[\mathrm{t}-1]_{\mathrm{n}} \\ &\quad + \mathbf{1.02} N[\mathrm{t}-1]_{\mathrm{n}}, \quad \text{and} \\ \Delta N[\mathrm{t}]_{\mathrm{n}} &= \mathbf{34.6}\{\pm 4.3\} N_{\mathrm{s,n}} + \mathbf{-0.28} N[t-1]_{\mathrm{n}} \\ &\quad + \mathbf{-0.16}\, V[\mathrm{t}-1]_{\mathrm{n}} \end{aligned} \tag{18.49}$$

The fitting of a sequence of alternative models suggested some systematic *coupling* across the $V[\mathrm{t}]$ and $N[\mathrm{t}]$ variables. Three additional models were fit to examine whether one or more of the coupling parameters (γ) were different from zero. In the first alternative model, the parameter representing the effect of $N[\mathrm{t}]$ on $\Delta V[\mathrm{t}]$ was fixed to zero ($\gamma_x = 0$), and this led to a notable loss of fit ($\chi^2 = 123$ on $df = 1$). The second alternative assumed no effect from $V[\mathrm{t}]$ on $\Delta N[\mathrm{t}]$ ($\gamma_y = 0$), and this is a much smaller loss of fit ($\chi^2 = 27$ on $df = 1$). Another model was fit in which no coupling was allowed ($\gamma_x = 0$ and $\gamma_y = 0$), and this resulted in a clear loss of fit ($\chi^2 = 126$ on $df = 2$). These results suggest a dynamic system in which *the nonverbal ability is a positive leading indicator of changes on verbal ability, but the negative effect of verbal ability on the nonverbal changes is not as strong*. The parameters listed previously are specific to the time interval chosen (i.e., $\Delta t = 5$), and any calculation of the explained latent variance requires a specific interval of age. These seemingly small differences can accumulate over longer periods of time, however, so the $N[\mathrm{t}]$ is expected to account for an increasing proportion of the variance in $\Delta V[\mathrm{t}]$ over age.

These mathematical results of these kinds of models can be also displayed in the pictorial form of a *vector field plot* of Figure 18.17 (for details, see McArdle, Hamagami, et al., in press). This allows us to write the model expectations in a relatively scale-free form: Any pair of coordinates is a starting point (y_0, x_0), and the directional arrow is a display of the expected pair of 5-year changes (Δy, Δx) from this point. These pictures show an interesting dynamic property—*the change expectations of a dynamic model depend on the starting point*. From this perspective, we can also interpret the positive level-level correlation ($\rho_{y0,x0} = .77$), which describes the placement of the individuals in the vector field, and the small slope-slope correlation ($\rho_{ys,xs} = -.05$), which describes the location of the subsequent scores for individuals in the vector field. In any case, the resulting flow shows a dynamic process in which scores on nonverbal abilities have a tendency to impact score changes on the verbal scores, but there is no notable reverse effect.

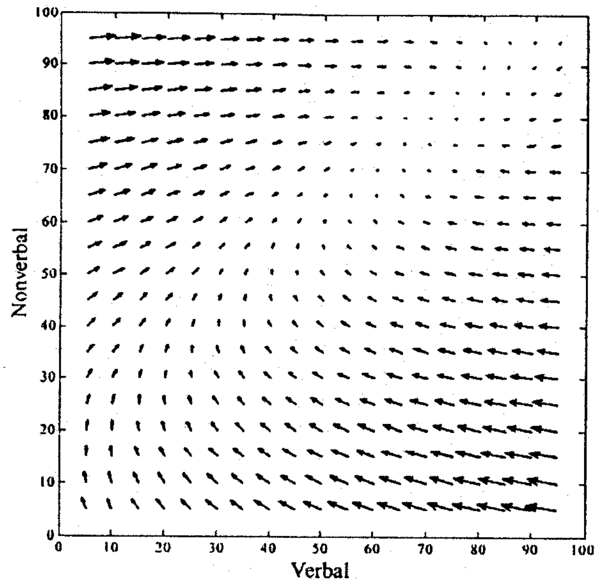

Figure 18.17 A vector field diagram representing the expected trajectories from the bivariate latent difference score model for nonverbal (y-axis) and verbal (x-axis) changes (i.e., each pair of points is a starting point, and the arrow is the directional change over the next 5 years (for details, see McArdle, Hagamani, et al., in press; McArdle et al., 2002).

Additional models were fit to examine a common growth factor model proportionality hypothesis. In this case, the factor model has two indicators at each time, $V[\mathrm{t}]$ and $N[\mathrm{t}]$, and it was combined with the previous dual change model (Equation 18.36). The basic model required only nine parameters in common factor loadings ($\lambda_y = 1$, $\lambda_x = .35$) and common factor dynamic parameters ($\alpha_z = 1$, $\beta_z = .14$, $\mu_{sz} = -0.13$, with no γ) and achieved convergence. However, the fit of this common factor DCS was much worse than that of the bivariate DCS model ($\chi^2 = 1262$ on $df = 11$), and this is additional evidence that separate process models are needed for verbal and nonverbal growth processes.

Differences between various Bradway demographic groups were examined using the multiple-group dynamic growth models (Equation 18.47). First we examined results when the data for males and females were considered separately. Here we found that an overall test of invariance across groups now yielded only a small difference ($\chi^2_{(\mathrm{m+f})} = 21$ on $df = 20$). We also find no difference in the coupling hypothesis across gender groups ($\chi^2_{(\mathrm{m})} = 10$ on $df = 2$). The same kinds of dynamic comparisons were calculated for participants with some college experience (ce) versus those with no college education (nc). Here we find that an overall test of invariance across groups yields another small difference ($\chi^2_{(\mathrm{ce+nc})} = 32$ on $df = 20$). However, when we pursue this result in more detail, we do find a large difference in the coupling hypothesis across these groups—the nonverbal to verbal coupling effect is enhanced in the group with some college education ($\gamma = -.28$, $\chi^2_{(ce)} = 25$ on $df = 1$) even though both groups started at similar initial levels.

The final group model was designed to answer several questions about nonrandom attrition. This was addressed by comparing results for participants with complete six-occasion longitudinal data ($n = 29$) from those with some incomplete data ($n = 82$). The differences in fit due to the assumption of invariance of the dynamic process over data groups is relatively small, but nontrivial ($\chi^2_{(c+i)} = 54$ on $df = 20$). This means we did alter the results by using all available data rather than just that for the persons with all data at all time points. This leaves us with a complex issue that requires further investigation.

FUTURE RESEARCH USING GROWTH CURVE ANALYSES

Future Bases of Growth Curve Analyses

The study of behavioral development and change has come a long way in the past few decades. After many years of debating whether and how to measure and represent change, it became clear that a promising solution to many of the problems of change measurement lay in collecting multiple rather than just two occasions of measurement. There had long been a mystique surrounding longitudinal methods in general, but this became translated into a much more functional approach to the representation and assessment of change. Change could be conceptualized as a function defined across time, rather than being based on a single difference score. This meant that a researcher could gather data on 3, 4, 5, or 10 occasions, often within a very short time frame, and instead of getting bogged down in an array of different kinds of change scores, could think in terms of fitting a curve over the multiple time points to represent the course of change. Moreover, concerns regarding individual differences could be cast in terms of the resemblance between these idealized functions and each person's actual trajectory.

These realizations about how to represent change processes were accelerated by the development of the variety of methods we have been referring to as latent growth curves, mixed-effect and multilevel models, and dynamic systems models. The developments in growth curve analysis have provided a number of key substantive and methodological contributions, as have been referenced previously. These developments can be classified by features of the models themselves: (a) the degree of mathematical specification, (b) the way these statistical models are fitted, and (c) the clarity and substantive meaning of the results. These issues have not been completely resolved, so we end with some comments about each of these topics.

The Mathematical Basis of Growth Curve Analyses

Most current growth curve models can be written in a common symbolic form (Seber & Wild, 1992). That is, a general model for a change in the scores over time (often using derivatives dy/dt or differences $\Delta y/\Delta t$) can be based on some mathematical functional form ($f\{x, t\}$) with unobserved scores ($x[t]$) and with unknown parameters ($A[t]$) to be estimated. Additional forms not discussed here can be included, such as auto-regressive residual structures, Markov chains, and Poisson processes. The list of growth functions described here is not exhaustive, and future extensions to other generalized functions (e.g., Ramsey & Silverman, 1997) and dynamic and chaotic formulations (May, 1997) are likely.

It is now clear that growth curve models of arbitrary complexity can be fitted to any observed trajectory over time (i.e., the integral), and the unknown parameters can be estimated to minimize some statistical function (e.g., weighted least squares, maximum likelihood) using, for example, nonlinear programming. Several different computer programs were used for the growth curve analyses discussed here. For many of the initial analyses, standard SAS programs were used, including PROC MIXED and PROC NLMIXED. The Mx-SEM computer program used herein was based on a simplified matrix approach to model expectations. All of these programs can deal with incomplete data patterns using the likelihood-based incomplete data approach presented earlier. The SEM programming is not as convenient as the mixed-effects program input scripts are, but SEM is far more flexible for programming the dynamic models (McArdle, 2001).

Growth curve modeling as an important step—but only a step—in the long progression toward better and better ways to represent behavioral development and change. Indeed, it is important to keep in mind the limitations as well as the strengths of growth curve modeling. Growth curve analysis per se results in a curve or curves defined over concrete measurement intervals—that is, a particular curve or curves. We have moved this towards a more dynamic representation that is defined across the abstract occasions (t, t + 1, t + 2, etc.) that can be integrated and solved for a particular solution. This kind of *dynamic generalizability* seems every bit as central as the more traditional concerns of subject and variable sampling.

The Statistical Basis of Growth Curve Analyses

The generic statistical approach featured here avoids some problems in older techniques, such as fitting a model to a log-scale, or directly to the velocity, or to the analysis of difference score data. These new techniques make it possible to address the critical problems of forecasting future observations, and further research on Bayesian estimation is a proper focus of additional efforts (for details, see Sieber & Wild, 1989).

The present model-fitting approach also permits a wide range of new possibilities for dealing directly with unbalanced, incomplete, or missing data. In classical work, linear polynomials were used extensively to deal with these kinds of problems (e.g., Joossens & Brems-Heyns, 1975). But the more recent work on linear and nonlinear mixed- and multilevel models indicates that it is possible to estimate growth curves and test hypotheses by collecting only small segments of data on each individual (McArdle, Ferrer-Caja, et al., in press; Pinherio & Bates, 2000; Verbeke & Molenberghs, 2000). These statistical models are being used in many longitudinal studies to deal with self-selection and subject attrition, multivariate changes in dynamic patterns of development, and the trade-offs between statistical power and costs of person and variable sampling. The statistical power questions of the future may not be *How* many *occasions do we need?,* but rather *How* few *occasions are adequate?* and *Which persons should we measure at which occasions?* (McArdle & Bell, 2000; McArdle & Woodcock, 1997).

In much the same way, the issues surrounding goodness-of-fit and the choice of an appropriate model are not simply formal statistical issues (see Burnham & Anderson, 1998). The way we conceptualize the relationships among these variables and the substantive issues involved has a great deal to do with the choice of model fitted. If we think our key variables represent substantively different growth processes, we would fit a specific growth model representing this idea (Figure 18.15). However, if we think our key variables are simply indicators of the same underlying common latent variables, then we would fit a different growth model (Figure 18.14). If we think our variables are growing and have time-lagged features, we would fit another model (Figure 18.16). If we do not know the difference, we might fit all kinds of models, examine the relative goodness-of-fit, and make some decisions about the further experiments needed. Although this exploratory approach is probably not optimal and probably requires extensive cross-validation, it certainly seems better to examine a large variety of possibilities rather than to limit our perspective on theories of growth and change (as in McArdle, 1988).

The multiple-group models presented here challenge the current approaches to an important theoretical area in behavioral science research—the study of group dynamics. Although simpler models are more common in popular usage, they seem to be special cases in a multiple-group dynamic framework. Variations of these models can be used to examine combinations of variables, even in the context of latent classes based on mixture models. In the future, we should not be surprised if our best models are checked against exploratory searches for latent-mixtures within dynamic models (Equation 18.42).

The Substantive Basis of Growth Curve Analyses

Some of the most difficult problems for future work on growth curves do not involve statistical analyses or computer programming, but rather deal with the elusive substantive meaning of the growth model parameters. As it turns out, these issues are not new but are unresolved controversies that have important implications for all other areas (Seber & Wild, 1993):

> "It is customary to say we are 'model-making.' Whether or not our model is biologically meaningful can only be tested by experiments. Here and in subsequent models we share G. F. Gause's View [Gause, 1934, p. 10]: 'There is no doubt that [growth, etc.] is a biological problem, and that it ought to be solved by experimentation and not at the desk of a mathematician. But in order to penetrate deeper into the nature of these phenomena, we must combine the experimental method with the mathematical theory, a possibility which has been created by [brilliant researchers]. The combination of the experimental method with the quantitative theory is in general one of the most powerful tools in the hands of contemporary science.' " (p. xx)

Of course, the growth parameters will only have substantive meaning if the measurements themselves and the changes that can be inferred from these measurements have a clear substantive interpretation and meaning. Thus, the basic requirements of meaningful age-equivalent measurement models are fundamental, and future measurement research is needed to address these concerns (see Fischer & Molenaar, 1995). Some of the multivariate models presented here may turn out to be useful, but these will need to be further extended to a fully dynamic time-dependent form. Empirical information will be needed to judge the utility of any growth curve model.

As students of behavior and behavioral change continue to improve their theoretical formulations, there will be a continued need to further strengthen the stock of available methods.

As the late Joachim Wohlwill (1972, 1991) argued, theory and method are partners eternally locked in a dance, with one of them leading at one time and the other leading at another time—neither partner leads all the time. Growth curve modeling has resulted in significant substantive findings that have further bolstered theories about development and change. We can expect in the not-too-distant future that strengthened theory will request even stronger methods. Until that time, however, the promise and power of these modeling techniques should be exploited.

Given the long history of elegant formulations from mathematics and statistics in this area, it is somewhat humbling to note that major aspects of the most insightful growth curve analyses have been based on careful visual inspection of the growth curves. The insight gained from visual inspection of a set of growth curves is not in dispute now; in fact, obvious visual features should be highlighted and emphasized in future research (e.g., Pinherio & Bates, 2000; Wilkinson, 1999). Much as in the past, the best future growth curve analyses are likely to be the ones we can all see most clearly.

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