,
1

A measurement-theoretic axiomatization of trapezoidal membership functions

Thierry Marchant

***Abstract*— In many applications of fuzzy set theory, the membership of an object is not defined directly. One of its attributes (like height, age, weight, ...) is first mapped to a real number (often by means of a physical instrument) and a parametric function then maps this real number to a membership degree in some fuzzy set (like 'tall', 'old', 'heavy', ...). A very common parametric function is the trapezoidal one. This paper presents some conditions guaranteeing the existence of such a trapezoidal membership function representing the knowledge of an expert. Further experimental research is needed for testing whether these conditions are satisfied by human agents.**

***Index Terms*— Membership, measurement**

I. INTRODUCTION

There are different sources of fuzzines (e.g. [1], [2]). One of them is that some concepts (like 'tall' or 'old') are vague, even in absence of measurement imprecision [3]. For instance, if we know for sure that the person I is exactly 50 years old, it is not clear that I is old. It depends on the context and on our perception of the concept 'old'. So, if we want to capture the knowledge of an expert and if we think his knowledge is fuzzy because of the vagueness of some concepts, we may want to use fuzzy sets.

Some authors (e.g. [4], [5], [6], [7], [8], [9], [10], [11], [12]) have axiomatized different measurement techniques that permit us to represent the membership of some objects by a function that is unique up to some transformations (strictly increasing, positive affine or linear) thereby showing how we can try to measure the membership on some kind of scale (ordinal, interval, ratio).

In these papers, the set X of objects for which we want to measure the membership has no special structure (for example, $X = \{$ Ian, Jin, Khaled $\}$) and the membership function directly maps X in $[0, 1]$, as illustrated in fig. 1. So, even if we measure the membership of these objects (or people) in the fuzzy set 'tall' on an interval scale, we do not know if it is possible to obtain a parametric membership function like the trapezoidal one or any other one. Those papers tell us nothing about the relation between the membership of an individual in the set 'tall' and his height as measured in meters.

But in many applications, contrary to what is done in these theoretical papers, the membership of an object in a fuzzy set is not defined directly: the set X is first mapped into $\mathbb{R}$ (often using a physical instrument). For example, the height of Ian is represented by the real number f(Ian), in meters. Then another mapping—the membership function μ_{tall}—maps each real number (in some range) to a membership degree. For example, f(Ian) is mapped on $\mu_{\text{tall}}(f(\text{Ian}))$. This is illustrated in fig. 2.

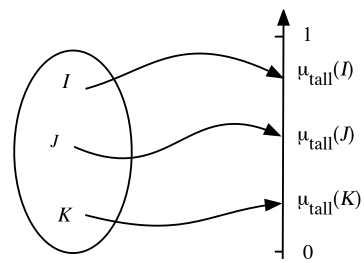

Fig. 1. Direct representation of the membership

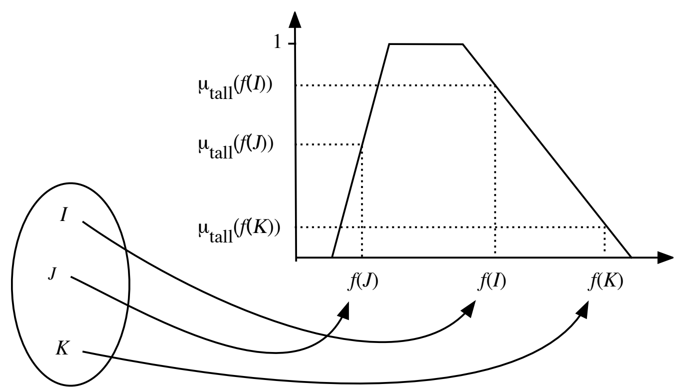

Fig. 2. Indirect representation of the membership

In Section II, I will try to analyze, from a measurement-theoretic viewpoint, the indirect approach (the one depicted in fig. 2). I will not suppose that f is given. Instead, I will start with a structure allowing to construct the representation f (for example the height measured in meters) with some nice uniqueness properties. Then I will introduce another structure allowing to construct the representation μ of the membership in some fuzzy set. Finally, I will present some conditions, linking both structures, that permit to indirectly measure the membership and that yield trapezoidal membership functions. Further experimental research is needed for testing whether these conditions are satisfied by human agents. If this is the case, then we would have a justification of the use of trapezoidal membership functions.

Note that, in order to obtain a trapezoidal membership function, f must be unique up to some transformations and the set of transformations must be a subset of the positive affine transformations. Otherwise, when varying the representation f,

T. Marchant is with Ghent University, B-9000 Ghent, Belgium (e-mail: Thierry.Marchant@UGent.be).

we cannot guarantee that the membership function will remain trapezoidal. This means that f must be an interval or ratio scale. There are different measurement techniques leading to interval scales: extensive measurement, bisection, conjoint measurement, difference measurement, …In this paper, I will use only bisection but I will show in Section III that this is not restrictive.

II. Main result

In this paper, the primitives are

- $X = \{x, y, \dots\}$, the universal set (uncountable),
- $\succsim^*_A$, a binary relation defined on $X \times X$ and representing differences of membership in a fuzzy set A, as perceived by an expert. For instance, $xy \succsim^*_A wz$ means the difference of membership between x and y is at least as large as the difference of membership between w and z,
- $\circ$, a binary operation from $X \times X$ into X. It is a bisection or 'midpoint' operation. For instance, $x \circ y$ represents the midpoint between x and y, obtained by means of a physical device or by asking an expert,
- $\succsim$, a binary relation on X.

The primitives are empirically observable. They are not explained nor defined by the theory. In particular, the fuzzy set A has no mathematical structure or property. It is just an expression in ordinary language (e.g. 'tall') that can be seen as a fuzzy set.

I now recall some classical conditions that guarantee the existence of a numerical representation μ_A of the relation $\succsim^*_A$, unique up to positive affine transformations. These conditions are those axiomatizing algebraic difference structures [13].

A 1: Weak Ordering of $\succsim^*_A$. The relation $\succsim^*_A$ on $X \times X$ is a weak order, i.e. is transitive and complete

A 2: Reversal. For all w, x, y, z in X,
$xy \succsim^*_A wz \Leftrightarrow zw \succsim^*_A yx$.

A 3: Monotonicity of $\succsim^*_A$. For all x, x', y, y', z, z' in X,
$xy \succsim^*_A x'y'$ and $yz \succsim^*_A y'z' \Rightarrow xz \succsim^*_A x'z'$.

A 4: Solvability of $\succsim^*_A$. For all w, x, y, z in X,
$xy \succsim^*_A wz \succsim^*_A xx \Rightarrow \exists z', z'' : xz' \sim^*_A wz \sim^*_A z''y$.

Let us denote the asymmetric (resp. symmetric) part of $\succsim^*_A$ by $\succ^*_A$ (resp. $\sim^*_A$).

A 5: Archimedeanness of $\succsim^*_A$. If $x_1, x_2, \dots, x_i, \dots$ is a strictly bounded standard sequence based on $\succsim^*_A$ (i.e. $x_{i+1}x_i \sim^*_A x_2x_1$ for every x_i, x_{i+1} in the sequence; x_2x_1 not $\sim^*_A xx$; and there exist y, z in X such that $yz \succ^*_A x_ix_1 \succ^*_A zy$ for all x_i in the sequence), then it is finite.

Definition 1: The structure $\langle X, \succsim^*_A \rangle$ is an algebraic difference structure iff it satisfies Weak Ordering of $\succsim^*_A$ (A1), Reversal (A2), Monotonicity of $\succsim^*_A$ (A3), Solvability of $\succsim^*_A$ (A4) and Archimedeanness of $\succsim^*_A$ (A5).

I now recall some classical conditions that guarantee the existence of a numerical representation f of the relation $\succsim$, also unique up to positive affine transformations. These conditions are those axiomatizing bisymmetric structures [13].

A 6: Weak Ordering of $\succsim$. The relation $\succsim$ on X is a weak order.

A 7: Monotonicity of $\circ$. For all $x, y, z \in X$,
$x \succsim y$ iff $x \circ z \succsim y \circ z$ iff $z \circ x \succsim z \circ y$.

Let us denote the asymmetric (resp. symmetric) part of $\succsim$ by $\succ$ (resp. $\sim$).

A 8: Bisymmetry. For all x, y, z, w in X,
$(x \circ y) \circ (z \circ w) \sim (x \circ z) \circ (y \circ w)$.

A 9: Restricted Solvability of $\circ$. If $y' \circ z \succsim x \succsim y'' \circ z$ (or $z \circ y' \succsim x \succsim z \circ y''$), then there exists y such that $y \circ z \sim x$ (or $z \circ y \sim x$).

A 10: Archimedeanness of $\circ$. Every strictly bounded standard sequence based on $\circ$ is finite, where $\{x_i : x_i \in X, i \in \mathbb{N}\}$ is a standard sequence based on $\circ$ iff there exist $y, z \in X$ such that $y \succ z$ and, for all $i, i+1 \in \mathbb{N}$, $x_i \circ y \sim x_{i+1} \circ z$ or, for all $i, i+1 \in \mathbb{N}$, $y \circ x_i \sim z \circ x_{i+1}$.

Definition 2: The structure $\langle X, \succsim, \circ \rangle$ is a bisymmetric structure iff it satisfies Weak Ordering of $\succsim$ (A6), Monotonicity of $\circ$ (A7), Bisymmetry (A8), Restricted Solvability of $\circ$ (A9) and Archimedeanness of $\circ$ (A10).

Note that the conditions imposed on $\circ$ do not force us to interpret $\circ$ as a bisection. It could be the concatenation of an extensive structure. But we will later impose another condition that makes sense only if $\circ$ is a bisection.

We now introduce some new notation and two new conditions that will make it possible to construct a trapezoidal membership function.

Let $\succsim_A$ be a binary relation on X defined by $x \succsim_A y$ iff $xy \succsim^*_A xx$. If $\langle X, \succsim^*_A \rangle$ is an algebraic difference structure, then $\succsim_A$ is a weak order. Let $T(A) = \{x \in X : x \succsim_A y \ \forall y \in X\}$ (the set of elements with maximal membership in A) and $B(A) = \{x \in X : y \succsim_A x \ \forall y \in X\}$ (the set of elements with minimal membership in A). These two sets may be empty (think e.g. of the fuzzy set of infinitely old people). Let $L_A = \{x \in X : z \precsim x \precsim y \ \forall y \in T(A)$ and $\forall z \in B(A)$ with $z \prec y\}$ and $R_A = \{x \in X : z \precsim x \precsim y \ \forall z \in T(A)$ and $\forall y \in B(A)$ with $z \prec y\}$. The elements in L_A correspond to the increasing part of the membership function while those in R_A correspond to the decreasing part. The sets L_A and R_A can be empty (e.g. if we have degenerate fuzzy sets, as in Section IV).

A 11: Quasi-Convexity. There are $x_1, x_2, x_3, x_4 \in X$, with $x_1 \prec x_2 \prec x_3 \prec x_4$, such that:

- $x \in B(A)$ iff $x \precsim x_1$ or $x \succsim x_4$;
- $x \in T(A)$ iff $x_2 \precsim x \precsim x_3$;
- $x_1 \prec x \precsim y \prec x_2$ implies $y \succsim_A x$;
- $x_3 \prec x \precsim y \prec x_4$ implies $x \succsim_A y$.

Remark that Quasi-Convexity implies that $T(A)$ and $B(A)$ are not empty.

Quasi-Convexity is a very mild condition. It just says that, when moving from small to large elements (w.r.t. $\succsim$), the membership is first minimal then increases, reaches a maximum, decreases and reaches again the same minimum. The next condition is much stronger: it imposes a very strict consistency or compatibility between the bisymmetric structure (often measured with a physical instrument) and the algebraic difference structure (based on the knowledge of the expert).

A 12: Consistency. For all $x, y \in L_A$, $x(x \circ y) \sim^*_A (x \circ y)y$. The same holds for all $x, y \in R_A$.

Note that this condition makes sense only if $\circ$ is a bisection.

We denote by $X/\sim$ the set of equivalence classes on X under $\sim$.

Theorem 1: Let the structures $\langle X, \succsim_A^* \rangle$ and $\langle X, \succsim, \circ \rangle$ be, respectively, an algebraic difference structure (definition 1) and a bisymmetric structure (definition 2) with $X/\sim$ uncountable. If, in addition, $\langle X, \succsim_A^*, \succsim, \circ \rangle$ satisfies Quasi-Convexity (A11) and Consistency (A12), then there exist $f_A : X \mapsto \mathbb{R}$ and $\mu_A : X \mapsto [0,1]$ such that

$$\mu_A(x) - \mu_A(y) \geq \mu_A(z) - \mu_A(w) \Leftrightarrow xy \succsim_A^* zw, \quad \forall x,y,z,w \in X, \tag{1}$$

$$\mu_A(x) = 0 \ \forall x \in B(A) \text{ and } \mu_A(x) = 1 \ \forall x \in T(A), \tag{2}$$

$$f(x) \geq f(y) \Leftrightarrow x \succsim y, \ \forall x,y \in X, \tag{3}$$

$$f(x \circ y) = \frac{f(x)+f(y)}{2} \quad \forall x,y \in X, \tag{4}$$

$$\mu_A(x) = a_A^L f(x) + b_A^L \quad \forall x,y \in L_A, \tag{5}$$

and

$$\mu_A(x) = a_A^R f(x) + b_A^R \quad \forall x,y \in R_A, \tag{6}$$

with $a_A^L > 0$ and $a_A^R < 0$.

The function μ_A is unique. The functions μ_A and f' also satisfy (1–6) iff there are real numbers $p > 0$ and q such that $f' = pf + q$. We then have $a'_L = a_L/p$, $b'_L = b_L(1 - a_L/p)$, $a'_R = a_R/p$ and $b'_R = b_R(1 - a_R/p)$.

Note that all conditions are necessary except Solvability of $\succsim_A^*$ and Restricted Solvability of $\circ$.

Proof: By Theorem 2 of [13, p.151], there exists a real-valued function μ'_A on X such that

$$\mu'_A(x) - \mu'_A(y) \geq \mu'_A(z) - \mu'_A(w) \Leftrightarrow xy \succsim_A^* zw.$$

This mapping is unique up to a positive affine transformation. By Quasi-Convexity, there are $x' \in B(A)$ and $x'' \in T(A)$. Hence, $\mu'_A(x') = \inf_{x\in X} \mu'_A(x)$ and $\mu'_A(x'') = \sup_{x\in X} \mu'_A(x)$. We can therefore find a positive affine transformation μ_A of μ'_A such that $\mu_A(x') = 0$ and $\mu_A(x'') = 1$. We then have $\mu_A(x) = 0 \ \forall x \in B(A)$ and $\mu_A(x) = 1 \ \forall x \in T(A)$. This mapping is unique.

By Theorem 10 of [13, p.295], there exists a real-valued function f on X and real numbers $\beta > 0$, $\gamma > 0$ and δ such that $f(x) \geq f(y)$ iff $x \succsim y$ and

$$f(x \circ y) = \beta f(x) + \gamma f(y) + \delta.$$

This mapping is unique up to a positive affine transformation. By Quasi-Convexity, for all $x,y \in L_A$, $f(x) \geq f(y)$ iff $x \succsim y$ iff $xy \succsim_A^* xx$ iff $\mu_A(x) - \mu_A(y) \geq 0$ iff $\mu_A(x) \geq \mu_A(y)$. So, there exists a strictly increasing mapping $\phi_A^L : \mathbb{R} \mapsto \mathbb{R}$ such that $\mu_A(x) = \phi_A^L[f(x)]$ for all $x \in L_A$. Similarly, there exists a strictly increasing mapping $\phi_A^R : \mathbb{R} \mapsto \mathbb{R}$ such that $\mu_A(x) = \phi_A^R[f(x)]$ for all $x \in R_A$.

By Consistency, for all $x,y \in L_A$,

$$\mu_A(x) - \mu_A(x \circ y) = \mu_A(x \circ y) - \mu_A(y).$$

So,

$$\phi_A^L[f(x)] - \phi_A^L[f(x \circ y)] = \phi_A^L[f(x \circ y)] - \phi_A^L[f(y)]$$

and

$$\phi_A^L[f(x)] - \phi_A^L[\beta f(x) + \gamma f(y) + \delta] = \phi_A^L[\beta f(x) + \gamma f(y) + \delta] - \phi_A^L[f(y)].$$

If we reorder the terms, we obtain

$$\phi_A^L[f(x)] + \phi_A^L[f(y)] = 2\phi_A^L[\beta f(x) + \gamma f(y) + \delta].$$

The left-hand side of this functional equation is symmetric in x and y. Therefore, $\beta = \gamma$. If we set $x = y$, we find

$$2\phi_A^L[f(x)] = 2\phi_A^L[\beta f(x) + \gamma f(x) + \delta].$$

Hence $\delta = 0$ and $\beta + \gamma = 1$. So, $\beta = \gamma = 1/2$ and, finally,

$$f(x \circ y) = \frac{f(x)+f(y)}{2}. \tag{7}$$

and

$$\frac{\phi_A^L[f(x)] + \phi_A^L[f(y)]}{2} = \phi_A^L\left[\frac{f(x)+f(y)}{2}\right], \forall x,y \in L_A. \tag{8}$$

For any $x,y \in L_A$ with $x \succ y$, by (7), we have $x \succ x \circ y \succ y$. So, the relation $\succsim$ on L_A is dense, in the sense that, for any $x \succ y$, there is $z : x \succ z \succ y$. Because, in addition, $X/\sim$ is uncountable, $f(L_A)$ is a closed interval of $\mathbb{R}$. We can therefore rewrite (8) as

$$\frac{\phi_A^L(r) + \phi_A^L(s)}{2} = \phi_A^L\left(\frac{r+s}{2}\right), \tag{9}$$

$\forall r,s \in [f(x_1), f(x_2)]$. This is the well-known Jensen's functional equation [14, p. 44] and its unique solution (because ϕ_A^L is strictly increasing) is $\phi_A^L(r) = a_A^L r + b_A^L$ for all r in $[f(x_1), f(x_2)]$, where $a_A^L > 0$ and b_A^L are real constants or, equivalently, $\mu_A(x) = \phi_A^L(f(x)) = a_A^L f(x) + b_A^L$ for all $x \in L_A$. Because $\mu_A(x_1) = \phi_A^L(f(x_1)) = 0$ and $\mu_A(x_2) = \phi_A^L(f(x_2)) = 1$, we find

$$a_A^L = \frac{1}{f(x_2) - f(x_1)} \quad \text{and} \quad b_A^L = \frac{f(x_1)}{f(x_1) - f(x_2)}.$$

We can follow the same reasoning with R_A and we obtain $\mu_A(x) = \phi_A^R(f(x)) = a_A^R f(x) + b_A^R$ for all $x \in R_A$ and

$$a_A^R = \frac{1}{f(x_3) - f(x_4)} \quad \text{and} \quad b_A^R = \frac{f(x_4)}{f(x_4) - f(x_3)}.$$

■

III. OTHER STRUCTURES

We now look at cases where no bisection operation is used in the construction of f.

Suppose that we want to measure the membership of different persons in the fuzzy set 'tall'. Very often, we will first measure the length of these persons and then map each length on a membership degree. The standard technique for measuring length is extensive measurement [13] and is based on a concatenation operation usually denoted by $\oplus$. Such an operation cannot satisfy Consistency. But we can define an operation $\circ$ by means of $x \circ y = z$ iff $z \oplus z = x \oplus y$. It is easy to see that, if $\oplus$ satisfies all conditions [13, Th. 1, p. 74] guaranteeing the existence of an additive representation—thus unique up to a positive linear transformation—then $\circ$

satisfies all conditions of Definition 2 and, hence, a variant of Theorem 1 for extensive measurement will hold.

If difference measurement is used for constructing f, based on some relation $\succsim'$ defined on $X \times X$, then we can define a bisection operation on X by means of $x \circ y = z$ iff $xz \sim' zy$. If $\langle X, \succsim' \rangle$ is an algebraic difference structure (different of course of $\langle X, \succsim_A^* \rangle$), then $\circ$ will satisfy all conditions of Definition 2 and, hence, a variant of Theorem 1 for difference measurement will hold.

Let us look at a last case. Suppose we have a multi-attributed set of objects $X \times X_1 \times \ldots \times X_n$ and a relation $\succsim''$ on this set. If conjoint measurement [13] is used for constructing f on X, then we can again define a bisection operation on X but now by means of $x \circ y = z$ iff $(x, x_1, \ldots, x_n) \sim'' (z, y_1, \ldots, y_n)$ and $(z, x_1, \ldots, x_n) \sim'' (y, y_1, \ldots, y_n)$. Here also, if $\succsim''$ satisfies all conditions [13, Th. 2, p. 257] guaranteeing the existence of an additive representation—thus unique up to a positive affine transformation—then $\circ$ will satisfy all conditions of Definition 2 and, hence, a variant of Theorem 1 for conjoint measurement will hold.

The same kind of reasoning can be applied to many other measurement techniques leading to interval or ratio scales.

IV. Other membership functions

In this paper, we have focused on the trapezoidal membership function because it is very popular but, following the same approach as above, it should be possible, at least in principle, to axiomatize other kinds of membership functions. We mention, for instance, two other forms that can be easily axiomatized: the left-degenerate and right-degenerate trapezoidal functions, depicted in fig. 3. It is easy to see that replacing Quasi-Convexity in Theorem 1 by Left- or Right-Degenerate Quasi-Convexity (defined below) yields an axiomatization of these two membership functions.

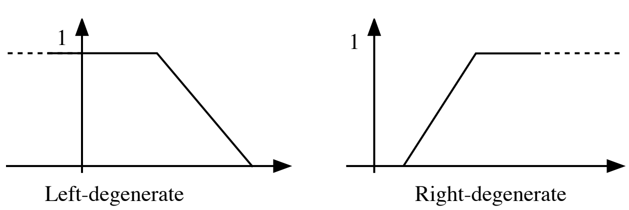

Fig. 3. The left- and right-degenerate trapezoidal functions

A 13: Left-Degenerate Quasi-Convexity. There are $x_3, x_4 \in X$, with $x_3 \prec x_4$, such that:

- $x \in B(A)$ iff $x \succsim x_4$;
- $x \in T(A)$ iff $x \precsim x_3$;
- $x_3 \prec x \precsim y \prec x_4$ implies $x \succsim_A y$.

A 14: Right-Degenerate Quasi-Convexity. There are $x_1, x_2 \in X$, with $x_1 \prec x_2$, such that:

- $x \in B(A)$ iff $x \precsim x_1$;
- $x \in T(A)$ iff $x \succsim x_2$;
- $x_1 \prec x \precsim y \prec x_2$ implies $y \succsim_A x$.

V. Conclusion

A lot of representation theorems in measurement theory have constructive proofs: they prove the existence of the desired representation by showing how to construct it. These theorems are therefore interesting not only for the theoretician but also for the practitioner. Facing a measurement problem, he can follow the proof for constructing the representation he needs. In our case, Theorem 1 is of little interest for the practitioner. If we want to construct a trapezoidal membership function for the height, the technique is extremely simple: we just have to ask the expert what are the heights corresponding to x_1, x_2, x_3 and x_4 in the expression of Quasi-Convexity. We then draw straight lines between these points and we are done.

But this technique will work—in the sense that it will yield a membership function faithfully representing the knowledge of the expert—only under some conditions: those presented in Theorem 1. And here lies the interest of this theorem. It shows that the use of trapezoidal membership functions relies on a lot of assumptions, some of them being strong.

- The conditions grouped under the label *bisymmetric structure* are often satisfied if f results from measurement by means of physical instruments (like clocks, thermometers, balances, $\ldots$). But if f is based on the answers of an expert to some questions, it is not obvious that his answers will form a bisymmetric structure. Nevertheless, in most applications, f results from physical measurement. Hence, this is not a crucial point.
- On the contrary, the mapping μ_A, representing the perception of the membership by an expert, is usually not based on physical measurement but on the answers of an expert. The conditions grouped under the label *algebraic difference structure* must therefore be satisfied by his answers. Today, we do not know if people (or some of them) have perceptions compatible with an algebraic difference structure. This should be investigated experimentally.
- Quasi-Convexity and Consistency make the link between the bisymmetric and the algebraic difference structure. Hence they must be satisfied by the answers of the expert. The validity of these condition must also be investigated experimentally.

Last remark: suppose all conditions of Theorem 1 hold not only for $\langle X, \succsim_A^*, \succ, \circ \rangle$ but also for a similar structure $\langle X, \succsim_B^*, \succ', \circ' \rangle$ where B is another fuzzy set. We can then measure the membership of the elements of X in A and in B by means of two trapezoidal membership functions. Suppose now we want to find the membership of an element, say x, in $A \cap B$. The usual way to do this is to choose a t-norm T and then to compute $T(\mu_A(x), \mu_B(x))$. But which t-norm? Very often, when the membership is measured on an ordinal scale, the minimum is chosen, for meaningfulness reasons [15]. When the membership is measured on an interval, ratio or absolute scale (like in our case), some other t-norms are often used (e.g. the Łukasiewicz t-norm). But [11] has shown that, in the framework of measurement theory, no t-norm except the minimum can model the intersection even when measuring the membership on an interval, ratio or absolute scale or even when we have a measurement-theoretic sound trapezoidal membership function. Only the minimum can be used and the conditions guaranteeing that it can adequately model the intersection have been presented in [6]. A symmetric remark holds of course for the union and the maximum.

REFERENCES

[1] D. Dubois and H. Prade, “The three semantics of fuzzy sets,” *Fuzzy Sets and Systems*, vol. 90, pp. 141–150, 1997.

[2] E. Hisdal, “Infinite-valued logic based on two-valued logic and probability. part 1.2: Different sources of fuzziness.” *International Journal of Man-Machine Studies*, vol. 25, no. 2, pp. 113–138, 1986.

[3] H.-J. Zimmermann and P. Zysno, “Quantifying vagueness in decision models,” *European Journal of Operational Research*, vol. 22, pp. 148–158, 1985.

[4] A. M. Norwich and I. B. Türksen, “The fundamental measurement of fuzziness,” in *Fuzzy Sets and Possibility Theory : Recent Developments*, R. R. Yager, Ed. Pergamon, 1982, pp. 49–60.

[5] I. B. Türksen, “Measurement of membership functions and their acquisition,” *Fuzzy Sets and Systems*, vol. 40, pp. 5–38, 1991.

[6] P. Bollmann-Sdorra, S. K. M. Wong, and Y. Y. Yao, “A measure-theoretic axiomatization of fuzzy sets,” *Fuzzy Sets and Systems*, vol. 60, pp. 295–307, 1993.

[7] T. Bilgiç and I. B. Türksen, “Measurement-theoretic justification of connectives in fuzzy set theory,” *Fuzzy Sets and Systems*, vol. 76, pp. 289–308, 1995.

[8] T. Bilgiç, “The archimedean assumption in fuzzy set theory,” in *Proceedings of NAFIPS’96 June 19-22, 1996, University of California, Berkeley*, 1996.

[9] T. Bilgiç and I. B. Türksen, “Measurement-theoretic frameworks for fuzzy set theory,” in *Fuzzy Logic in Artificial Intelligence : towards Intelligent Systems*, ser. Lecture Notes in Artificial Intelligence 1188, T.P. Martin and A. Ralsecu, Eds. Springer, 1997, pp. 252–265.

[10] ——, “Measurement of membership functions: theoretical and empirical work.” in *Fundamental of Fuzzy Sets*, ser. The Handbooks of Fuzzy Sets, Vol. 7, D. Dubois and H. Prade, Eds. Kluwer Academic Publishers, 1999, pp. 195–232.

[11] T. Marchant, “The measurement of membership by comparisons,” *Fuzzy Sets and Systems*, vol. 148, pp. 157–177, 2004.

[12] ——, “The measurement of membership by subjective ratio estimation,” *Fuzzy Sets and Systems*, vol. 148, pp. 179–199, 2004.

[13] D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky, *Foundations of measurement: Additive and polynomial representations*. London: Academic Press, 1971.

[14] J. Aczél, *Lectures on functional equations and their applications*. Academic Press, 1966.

[15] F. S.. Roberts, *Measurement theory, with applications to Decision Making, Utility and the Social Sciences*. Addison-Wesley, 1979.

Thierry Marchant is Professor of Statistics at Ghent University. He received his his Ph.D. in Engineering from Université Libre de Bruxelles. His interests lie in the axiomatic foundations of Fuzzy Sets, Social Choice Theory and Multiple Criteria Decision Support.

Figure captions

LIST OF FIGURES