

# Subjective expected utility on the basis of ordered categories

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## Abstract

This paper shows that subjective expected utility can be obtained using primitives that are much poorer than a preference relation on the set of acts. Our primitives only involve the fact that an act can be judged either “attractive”, “neutral” or “unattractive”. These categories may be interpreted as denoting the position of an act vis-à-vis a status quo. We give conditions implying that there are a utility function on the set of consequences and a probability distribution on the set of states such that attractive (resp. unattractive) acts have a subjective expected utility that is above (resp. below) some threshold. The numerical representation that is obtained has strong uniqueness properties. Our derivation uses results in conjoint measurement with ordered categories and, hence, we adopt a framework involving a finite set of states.

**Keywords:** Subjective Expected Utility, Ordered categories, Conjoint measurement.

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# 1 Introduction

In spite of the large amount of experimental evidence showing its limited ability to explain the behavior of many subjects, Subjective Expected Utility (SEU) remains the focal point of most works in decision under uncertainty. This is surely due to a rather unique combination of simplicity, analytical tractability and normative appeal.

Four main routes have been followed to obtain behavioral foundations for SEU (Wakker, 1989a). The first one works with a finite set of states and a finite set of consequences and uses separation techniques to ensure that the resulting equalities and inequalities will not be contradictory (see Shapiro, 1979). As in the case of conjoint measurement (see Scott, 1964 or Krantz et al., 1971, Ch. 9), this technique leads to complex conditions that are not easy to test and interpret. The resulting numerical representation does not have strong uniqueness properties. The second route was opened by Savage (1954). It makes no hypothesis on the set of consequences but requires a rich set of states. It leads to relatively simple conditions. The obtained numerical representation has strong uniqueness properties (for a recent advance along this line, see Abdellaoui and Wakker, 2005). The third route is a kind of dual to the second one: it imposes richness on the set of consequences, while working with a finite set of states. Early contributions of this type include Gul (1992), Nakamura (1990) and Wakker (1984, 1989b). Recent advances along this line are surveyed and consolidated in Köbberling and Wakker (2003, 2004) and Wakker and Zank (1999). As the second one, this route leads to simple conditions together with strong uniqueness properties. It uses conditions that are easily compared with the ones used in conjoint measurement (see Krantz et al., 1971, Ch. 6 or Wakker, 1989a, Ch. 3) since, in this framework, acts can be viewed as elements of a homogeneous Cartesian product. A fourth route includes “lotteries” using “objective probabilities” in the analysis. It leads to relatively simpler results than the preceding two approaches. The price to pay for this simplicity is a richer framework that is often seen as less “pure” than frameworks refusing the introduction of objective probabilities. This approach was pioneered by Anscombe and Aumann (1963). It leads to simple conditions together with strong uniqueness properties (a recent development along this line is Sarin and Wakker (1997) who formalize an approach that was common in the early days of Decision Analysis see Raiffa, 1968). The last three approaches have also been used to analyze models extending SEU, such as Rank Dependent Utility (Gilboa, 1987, Wakker, 1989a) or Cumulative Prospect Theory (Tversky and Kahneman, 1992, Wakker and Tversky, 1993). Recent reviews of the field of decision making under uncertainty are Gilboa (2009) and Wakker (2009). This paper will only be concerned with SEU.

In the four approaches considered above, the primitives consist in a (well behaved) preference relation on the set of acts. Given any two acts, the decision

maker is supposed to be in position to compare them in terms of strict preference or indifference. With the last three approaches, i.e., the ones leading to strong uniqueness results, the construction of the numerical representation involves building “standard sequences” (Krantz et al., 1971, Ch. 2). This clearly implies working with several indifference curves (see, e.g., Wakker, 1989a, Fig. 3.5.2, p. 54).

The central originality of this paper will be to work with much poorer primitives. For any act, we only expect the decision maker (DM) to be in position to tell us if she finds it “attractive”, “neutral” or “unattractive”. An attractive (resp. unattractive) act may be interpreted as an act that the DM is (resp. not) willing to accept given her current situation, i.e., an attractive act is felt preferable to the status quo. We work with a finite set of states and a rich set of consequences as in the third route mentioned above. We give conditions implying that this set of attractive (resp. unattractive) acts consists of all acts having a SEU that is above (resp. below) some threshold. The numerical representation will have strong uniqueness properties. This gives SEU alternative behavioral foundations that are based on weak primitives and use conditions that are not much more complex than the ones used in conjunction with the classical primitives. Whereas the usual primitives for SEU lead to work with many (usually, infinitely many) indifference curves, our framework only allows to work with a single indifference curve that lies at the frontier between attractive and unattractive acts. Indeed, all attractive (or unattractive) acts are not supposed to be equally desirable.

This paper is not the first one in decision theory to work with ordered partitions instead of preference relations. The first move in this direction was made by Vind (1991) (see also Vind, 2003) in a rather abstract setting that has immediate application to conjoint measurement. This work was later developed in Bouyssou and Marchant (2009, 2010). While these papers were mainly concerned with additive representations, Goldstein (1991) studied decomposable numerical representations on the basis of such primitives. His work was later developed in Bouyssou and Marchant (2007a,b) and Słowiński et al. (2002). In the area of decision making under risk, Nakamura (2004) has analyzed various models using similar premisses. In particular, he gives Expected Utility à la von Neumann-Morgenstern foundations that are similar to the ones sought here for SEU.

Technically, our results will rely on the analysis in Bouyssou and Marchant (2009) of additive representations of ordered partitions in the context of conjoint measurement. We will add extra conditions that are motivated by the use of a homogeneous Cartesian product, a model that naturally arises when working in decision making under uncertainty with a finite set of states.

The paper is organized as follows. Section 2 introduces our setting and notation. Section 3 presents the conditions used in this paper. Our main results are stated in Section 4. Section 5 concludes. Most proofs are relegated in Appendix.

## 2 The setting

### 2.1 Notation

We adopt a classical setting for decision under uncertainty with a finite number of states and we mainly follow the terminology and notation used in Wakker (1989a). Let  $\Gamma = \{\alpha, \beta, \gamma, \dots\}$  be a set of *consequences*. The set of *states* is  $N = \{1, 2, \dots, n\}$ . It is understood that the elements of  $N$  are exhaustive and mutually exclusive: one and only one state will turn out to be true. An *act* is a function from  $N$  to  $\Gamma$ . The set of all acts is denoted by  $\mathcal{X} = \{a, b, c, \dots\} = \Gamma^N$ . It will prove convenient to identify the set of acts with the homogeneous Cartesian product  $\Gamma^n$ . Hence, the act  $a \in \mathcal{X}$  will often be written as  $(a_1, a_2, \dots, a_n)$ .

Let  $E \subseteq N$  and  $a, b \in \mathcal{X}$ . We denote by  $(a_E, b_{-E})$  the act  $c \in \mathcal{X}$  such that  $c_i = a_i$ , for all  $i \in E$ , and  $c_i = b_i$ , for all  $i \in N \setminus E$ . We will also write that  $a_E \in \Gamma_E$  and  $b_{-E} \in \Gamma_{-E}$ , abusing notation. Similarly  $(\alpha_E, b_{-E})$  will denote the act  $d \in \mathcal{X}$  such that  $d_i = \alpha_i \in \Gamma_i$ , for all  $i \in E$  and  $d_i = b_i$ , for all  $i \in N \setminus E$ . When sets contain few elements, we often omit braces around them and write, e.g.,  $(a_i, b_{-i})$ ,  $(\alpha_{ij}, b_{-ij})$  or  $(\alpha_i, a_j, b_{-ij})$ . This should cause no confusion.

### 2.2 Primitives

The traditional primitives in decision making under uncertainty consist in a binary relation<sup>1</sup>  $\succsim$  on  $\mathcal{X}$ . We use here a threefold ordered partition of  $\mathcal{X}$ . We suppose that acts in  $\mathcal{X}$  are presented to a DM. For each of these acts, she will specify whether she finds it “attractive”, “neutral” or “unattractive”. This process defines a *threefold ordered partition*  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  of the set  $\mathcal{X}$  (note that we abuse terminology here since, at this stage, we do not require each of  $\mathcal{A}$ ,  $\mathcal{F}$  and  $\mathcal{U}$  to be nonempty). Acts in  $\mathcal{A}$  are judged Attractive. Acts in  $\mathcal{U}$  are judged Unattractive. Acts in  $\mathcal{F}$  lie at the Frontier between attractive and unattractive acts. A suggestive, but by no means compulsory, interpretation of our setting is that attractive (resp. unattractive) acts are the acts that are judged strictly better (resp. worse) than the status quo. We often write  $\mathcal{AF}$  instead of  $\mathcal{A} \cup \mathcal{F}$  and  $\mathcal{FU}$  instead of  $\mathcal{F} \cup \mathcal{U}$ . Note that a binary relation  $\succsim$  on  $\mathcal{X}$  induces *many* threefold ordered partitions of this type. Indeed, to each act  $a \in \mathcal{X}$ , we may associate the following threefold partition  $\mathcal{A}^a = \{b \in \mathcal{X} : b \succ a\}$ ,  $\mathcal{F}^a = \{b \in \mathcal{X} : b \sim a\}$  and  $\mathcal{U}^a = \{b \in \mathcal{X} : a \succ b\}$ .

The three categories in  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  are *ordered*. All acts in  $\mathcal{A}$  are preferable to all acts in  $\mathcal{F}$  and the latter are preferable to all acts in  $\mathcal{U}$ . It is important to

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<sup>1</sup>In all what follows, whenever the symbol  $\succsim$  is used to denote a binary relation, it will be understood that  $\succ$  denotes its asymmetric part and  $\sim$  its symmetric part. Similar conventions hold when  $\succsim$  is subscripted or superscripted. A weak order is a complete and transitive binary relation.

notice that acts belonging to  $\mathcal{A}$  are not “equally” preferable. Some of them may be quite attractive while others may only be slightly better than the status quo. A similar remark holds for  $\mathcal{U}$ .

We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is *non-degenerate* if both  $\mathcal{A}$  and  $\mathcal{U}$  are nonempty. We say that a state  $i \in N$  is *influential* for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  if there are  $\alpha, \beta \in \Gamma$  and  $a \in \mathcal{X}$  such that  $(\alpha_i, a_{-i})$  and  $(\beta_i, a_{-i})$  do not belong to the same category in  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . A state that is not influential has no impact on the partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  and, thus, may be suppressed from  $N$ . Hence, we will suppose that all states are influential. As explained in Bouyssou and Marchand (2009), the analysis of the case of two states requires techniques that are quite different and much simpler than the ones used here (this case does not lead to strong uniqueness results). Finally, imposing that the partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate is a mild non-triviality condition. This explains that the following assumption is maintained throughout this paper.

**Assumption 1**

*There are at least three states. All states are influential. The ordered partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate.*

### 2.3 Model

We analyze a model in which all attractive (resp. unattractive) acts have a subjective expected utility that is above (resp. below) some threshold. This model involves a real-valued function  $u$  on  $\Gamma$  and nonnegative real numbers  $p_i$ ,  $i \in N$ , that add up to one. The function  $u$  is interpreted as a utility function and the number  $p_i$  as the subjective probability of state  $i \in N$ . Hence  $\sum_{i=1}^n p_i u(a_i)$  is interpreted as the subjective expected utility of act  $a \in \mathcal{X}$ . Our model is such that, for all  $a \in \mathcal{X}$ ,

$$a \in \left\{ \begin{array}{c} \mathcal{A} \\ \mathcal{F} \\ \mathcal{U} \end{array} \right\} \Leftrightarrow \sum_{i=1}^n p_i u(a_i) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0. \tag{1}$$

Indeed, it is clear that it is not restrictive to suppose that the threshold separating attractive and unattractive acts is set to 0. When the set  $\Gamma$  is endowed with a topology, we might additionally require that the function  $u$  is continuous w.r.t. this topology. We will do so below.

Note that Assumption 1 implies that all states are influential. This clearly implies that, for all  $i \in N$ ,  $p_i > 0$ .

### 3 Axioms

Our first condition is a variant of a condition used in Vind (1991) that has been modified to cope with the case of a homogeneous Cartesian product.

**A1** Inter-State 1-Linearity w.r.t.  $\mathcal{A}$  or r-1-Linearity- $\mathcal{A}$  for short  
For all  $i, j \in N$ , all  $\alpha, \beta \in \Gamma$  and all  $a, b \in \mathcal{X}$ ,

$$\left. \begin{array}{l} (\alpha_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (\beta_j, b_{-j}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, a_{-i}) \in \mathcal{A} \\ \text{or} \\ (\alpha_j, b_{-j}) \in \mathcal{A}. \end{array} \right.$$

It is simple to check that A1 is necessary for model (1). Indeed, its violation easily leads to  $p_i u(\alpha) > p_i u(\beta)$  and  $p_j u(\beta) > p_j u(\alpha)$ , which is contradictory.

The intuitive idea behind this condition (and the other linearity conditions that we will present below) is that consequences can be ordered. We define the binary relation  $\succsim^{\mathcal{A}}$  on  $\Gamma$  letting, for all  $\alpha, \beta \in \Gamma$ ,

$$\alpha \succsim^{\mathcal{A}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathcal{A} \Rightarrow (\alpha_i, a_{-i}) \in \mathcal{A}, \text{ for all } i \in N \text{ and all } a \in \mathcal{X}]. \quad (2)$$

It is simple to check that  $\succsim^{\mathcal{A}}$  is always reflexive and transitive. It will be a weak order as soon as it is complete. The following lemma shows that  $\succsim^{\mathcal{A}}$  is complete iff  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A1 (r-1-Linearity- $\mathcal{A}$ ).

**Lemma 1**

The relation  $\succsim^{\mathcal{A}}$  is complete iff  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A1 (r-1-Linearity- $\mathcal{A}$ ).

**PROOF**

Necessity. Suppose that  $(\alpha_i, a_{-i}) \in \mathcal{A}$  and  $(\beta_j, b_{-j}) \in \mathcal{A}$ . Because  $\succsim^{\mathcal{A}}$  is complete, we have either  $\alpha \succsim^{\mathcal{A}} \beta$  or  $\beta \succsim^{\mathcal{A}} \alpha$ . In the first case  $(\beta_j, b_{-j}) \in \mathcal{A}$  and  $\alpha \succsim^{\mathcal{A}} \beta$  imply  $(\alpha_j, b_{-j}) \in \mathcal{A}$ . In the second case,  $(\alpha_i, a_{-i}) \in \mathcal{A}$  and  $\beta \succsim^{\mathcal{A}} \alpha$  imply  $(\beta_i, a_{-i}) \in \mathcal{A}$ . Hence,  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies r-1-Linearity- $\mathcal{A}$ .

Sufficiency. Suppose that  $\text{Not}[\alpha \succsim^{\mathcal{A}} \beta]$ . This implies that, for some  $i \in N$  and some  $a \in \mathcal{X}$ ,  $(\beta_i, a_{-i}) \in \mathcal{A}$  and  $(\alpha_i, a_{-i}) \notin \mathcal{A}$ . Similarly,  $\text{Not}[\beta \succsim^{\mathcal{A}} \alpha]$  implies that, for some  $j \in N$  and some  $b \in \mathcal{X}$ ,  $(\alpha_j, b_{-j}) \in \mathcal{A}$  and  $(\beta_j, b_{-j}) \notin \mathcal{A}$ . Using r-1-Linearity- $\mathcal{A}$ ,  $(\beta_i, a_{-i}) \in \mathcal{A}$  and  $(\alpha_j, b_{-j}) \in \mathcal{A}$  imply either  $(\alpha_i, a_{-i}) \in \mathcal{A}$  or  $(\beta_j, b_{-j}) \in \mathcal{A}$ , a contradiction.  $\square$

Let us define, for each  $i \in N$ , the binary relation  $\succsim_i^{\mathcal{A}}$  on  $\Gamma$  letting, for all  $\alpha, \beta \in \Gamma$ ,

$$\alpha \succsim_i^{\mathcal{A}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathcal{A} \Rightarrow (\alpha_i, a_{-i}) \in \mathcal{A}, \text{ for all } a \in \mathcal{X}]. \quad (3)$$

We obviously have that  $\succsim^{\mathcal{A}} = \bigcap_{i=1}^n \succsim_i^{\mathcal{A}}$ . When r-1-Linearity- $\mathcal{A}$  holds, all relations  $\succsim_i^{\mathcal{A}}$  are complete and are compatible.

Under A1 (r-1-Linearity- $\mathcal{A}$ ), we have a weak order  $\succsim^{\mathcal{A}}$  on  $\Gamma$ . The set  $\Gamma$  is endowed with the order topology generated by  $\succsim^{\mathcal{A}}$ . The set  $\mathcal{X}$ , viewed as  $\Gamma^n$  is endowed with the product topology. This will allow us to introduce our main structural assumption<sup>2</sup>, the definition of which presupposes that  $\succsim^{\mathcal{A}}$  is a weak order or, equivalently, that A1 (r-1-Linearity- $\mathcal{A}$ ) holds. It is clearly not necessary for model (1).

**A2** Connectedness in the order topology

*When  $\succsim^{\mathcal{A}}$  is a weak order, the set  $\Gamma$  is connected in the order topology generated by  $\succsim^{\mathcal{A}}$ .*

Our next condition will be necessary if the function  $u$  on  $\Gamma$  is required to be continuous w.r.t. the topology on  $\Gamma$  introduced above.

**A3** Openness

*The sets  $\mathcal{A}$  and  $\mathcal{U}$  are open in the product topology on  $\mathcal{X}$ .*

Our next two conditions will bring category  $\mathcal{F}$  into the picture. The first of them will lead to the conclusion that the ordering of the elements of  $\Gamma$  via the relation  $\succsim^{\mathcal{A}}$  will not be contradicted if elements in  $\mathcal{F}$  are taken into account.

**A4** Inter-State 1-Linearity w.r.t. to  $\mathcal{F}$  or r-1-Linearity- $\mathcal{F}$  for short

*For all  $i, j \in N$ , all  $\alpha, \beta \in \Gamma$  and all  $a, b \in \mathcal{X}$ ,*

$$\left. \begin{array}{l} (\alpha_i, a_{-i}) \in \mathcal{F} \\ \text{and} \\ (\beta_j, b_{-j}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, a_{-i}) \in \mathcal{AF} \\ \text{or} \\ (\alpha_j, b_{-j}) \in \mathcal{AF}. \end{array} \right.$$

Condition A4 is clearly necessary for model (1) since its violation easily leads to  $p_i u(\alpha) > p_i u(\beta)$  and  $p_j u(\beta) > p_j u(\alpha)$ .

Our next condition makes clear the particular role played by category  $\mathcal{F}$  in model (1): it lies at the frontier between categories  $\mathcal{A}$  and  $\mathcal{U}$ .

**A5** Inter-State 1-Thinness or r-1-Thinness for short

*For all  $i, j \in N$ , all  $a, b \in \mathcal{X}$  and all  $\alpha, \beta \in \Gamma$ ,*

$$\left. \begin{array}{l} (\alpha_i, a_{-i}) \in \mathcal{F} \\ \text{and} \\ (\beta_i, a_{-i}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\alpha_j, b_{-j}) \in \mathcal{A} \Leftrightarrow (\beta_j, b_{-j}) \in \mathcal{A} \\ (\alpha_j, b_{-j}) \in \mathcal{F} \Leftrightarrow (\beta_j, b_{-j}) \in \mathcal{F} \\ (\alpha_j, b_{-j}) \in \mathcal{U} \Leftrightarrow (\beta_j, b_{-j}) \in \mathcal{U}. \end{array} \right.$$

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<sup>2</sup> Here, we slightly deviate from Bouyssou and Marchant (2009) who require connectedness w.r.t. to the weak order  $\succsim$  introduced later. The two conditions will nevertheless prove to be equivalent. The present condition allows to make an easier relation between our results of the ones in Vind (1991).



The necessity of A5 for model (1) follows from the fact that  $(\alpha_i, a_{-i}) \in \mathcal{F}$  and  $(\beta_i, a_{-i}) \in \mathcal{F}$  imply that  $p_i u(\alpha) = p_i u(\beta)$ , so that, since Assumption 1 implies  $p_i > 0$ ,  $u(\alpha) = u(\beta)$ .

Define the binary relation  $\succsim^{\mathcal{AF}}$  on  $\Gamma$  letting, for all  $\alpha, \beta \in \Gamma$ ,

$$\alpha \succsim^{\mathcal{AF}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathcal{AF} \Rightarrow (\alpha_i, a_{-i}) \in \mathcal{AF}, \text{ for all } i \in N \text{ and all } a \in \mathcal{X}]. \quad (4)$$

Again, it is simple to check that  $\succsim^{\mathcal{AF}}$  is always reflexive and transitive. We will later show that our conditions will imply that  $\succsim^{\mathcal{AF}}$  is complete and that  $\succsim^{\mathcal{AF}}$  and  $\succsim^{\mathcal{A}}$  are not contradictory.

For each,  $i \in N$ , let us also define the relation  $\succsim_i^{\mathcal{AF}}$  on  $\Gamma$  letting, for all  $\alpha, \beta \in \Gamma$ ,

$$\alpha \succsim_i^{\mathcal{AF}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathcal{AF} \Rightarrow (\alpha_i, a_{-i}) \in \mathcal{AF}, \text{ for all } a \in \mathcal{X}]. \quad (5)$$

We obviously have that  $\succsim^{\mathcal{AF}} = \bigcap_{i=1}^n \succsim_i^{\mathcal{AF}}$ .

Let  $i \in N$ . Define  $\Delta_i = \{\beta \in \Gamma : \alpha \succ_i^{\mathcal{A}} \beta \succ_i^{\mathcal{A}} \gamma \text{ for some } \alpha, \gamma \in \Gamma\}$  and  $\Delta = \prod_{i \in N} \Delta_i$ .

Our next condition is inspired by a condition called Standard Sequence Invariance in Krantz et al. (1971, p. 304).

#### A6 Standard Sequence Invariance

For all  $i, j \in N$ , all  $\alpha, \beta, \gamma \in \Delta_i \cap \Delta_j$ , all  $a, b, c, d \in \mathcal{X}$ ,

$$\left. \begin{array}{l} (\alpha_j, a_i, a_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\gamma_j, b_i, b_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\beta_i, c_j, d_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\beta_i, d_j, c_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\alpha_i, c_j, c_{-ij}) \in \mathcal{AF} \quad \text{or} \\ (\gamma_i, d_j, d_{-ij}) \in \mathcal{AF} \quad \text{or} \\ (\beta_j, a_i, b_{-ij}) \in \mathcal{AF} \quad \text{or} \\ (\beta_j, b_i, a_{-ij}) \in \mathcal{AF}. \end{array} \right.$$

Condition A6 is necessary for model (1). Indeed, suppose it is violated. We easily obtain:

$$\begin{aligned} p_j u(\alpha) + p_i u(a_i) &> p_j u(\beta) + p_i u(b_i), \\ p_j u(\gamma) + p_i u(b_i) &> p_j u(\beta) + p_i u(a_i), \\ p_i u(\beta) + p_j u(c_j) &> p_i u(\gamma) + p_j u(d_j), \\ p_i u(\beta) + p_j u(d_j) &> p_i u(\alpha) + p_j u(c_j). \end{aligned}$$

Since  $p_i, p_j > 0$ , the first two equations imply that  $u(\alpha) - u(\beta) > u(\beta) - u(\gamma)$  while the last two imply  $u(\alpha) - u(\beta) < u(\beta) - u(\gamma)$ , a contradiction. Combined with our other conditions, this axiom will imply that, if  $\beta$  is “halfway” between  $\alpha$  and  $\gamma$  in terms of  $u$  as revealed in state  $i \in N$ , the same must be true in all other states. Observe that the stronger condition obtained replacing all terms  $\mathcal{AF}$  in the conclusions by  $\mathcal{A}$  would also be necessary for model (1). Call this stronger condition A'6.

Condition A1 (r-1-Linearity- $\mathcal{A}$ ) only deals with what happens to acts when consequences are modified on a single state. This is not enough for our purposes. The following condition imposes restrictions on what happens when acts are modified on two states. It will only be invoked when  $n \geq 4$ .

**A7** Intra-State 2-Linearity w.r.t.  $\mathcal{A}$  or a-2-Linearity- $\mathcal{A}$  for short  
*For all  $i, j \in N$ , all  $\alpha, \beta, \gamma, \delta \in \Gamma$  and all  $a, b \in \mathcal{X}$*

$$\left. \begin{array}{l} (\alpha_i, \beta_j, a_{-ij}) \in \mathcal{A} \\ \text{and} \\ (\gamma_i, \delta_j, b_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\gamma_i, \delta_j, a_{-ij}) \in \mathcal{A} \\ \text{or} \\ (\alpha_i, \beta_j, b_{-ij}) \in \mathcal{A}. \end{array} \right.$$

Condition A7 is necessary for model (1), since its violation easily leads to  $p_i u(\alpha) + p_j u(\beta) > p_i u(\gamma) + p_j u(\delta)$  and  $p_i u(\gamma) + p_j u(\delta) > p_i u(\alpha) + p_j u(\beta)$ .

Our final condition will only be invoked when  $n = 3$ . As discussed in Bouyssou and Marchant (2009), it plays the role of the Double Cancellation condition (a strengthening of the Thomsen condition) used in conjoint measurement with two attributes (Krantz et al., 1971, p. 250).

**A8** Double Cancellation

*For all  $i, j \in N$ , all  $\alpha, \beta, \gamma \in \Gamma$  and all  $a, b \in \mathcal{X}$ ,*

$$\left. \begin{array}{l} (\alpha_i, \beta_j, a_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\beta_i, \gamma_j, b_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\gamma_i, \alpha_j, c_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, \alpha_j, a_{-ij}) \in \mathcal{AF} \quad \text{or} \\ (\gamma_i, \beta_j, b_{-ij}) \in \mathcal{AF} \quad \text{or} \\ (\alpha_i, \gamma_j, c_{-ij}) \in \mathcal{AF}. \end{array} \right.$$

Condition A8 is necessary for model (1). Indeed, its violation easily implies that:

$$\begin{aligned} p_i u(\alpha) + p_j u(\beta) &> p_i u(\beta) + p_j u(\alpha), \\ p_i u(\beta) + p_j u(\gamma) &> p_i u(\gamma) + p_j u(\beta), \\ p_i u(\gamma) + p_j u(\alpha) &> p_i u(\alpha) + p_j u(\gamma). \end{aligned}$$

Summing these three equations leads to  $0 > 0$ . Observe that the stronger condition obtained replacing all terms  $\mathcal{AF}$  in the conclusions by  $\mathcal{A}$  would also be necessary for model (1). Call this stronger condition A'8.

## 4 Results

The main result of this paper can be stated as follows.

**Theorem 1**

Consider a threefold ordered partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  of  $\mathcal{X}$  such that Assumption 1 holds. Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A1 (*r-1-Linearity- $\mathcal{A}$* ), A2 (*Connectedness in the order topology*), A3 (*Openness*), A4 (*r-1-Linearity- $\mathcal{F}$* ), A5 (*r-1-Thinness*), and A6 (*Standard sequence invariance*). If  $n \geq 4$  suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A7 (*a-2-Linearity- $\mathcal{A}$* ). If  $n = 3$  suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A8 (*Double Cancellation*). Then there are a continuous real-valued function  $u$  on  $\Gamma$  and  $n$  strictly positive real numbers  $p_1, p_2, \dots, p_n$  adding up to 1 such that (1) holds.

The numbers  $p_1, p_2, \dots, p_n$  are unique. The function  $u$  is unique up to a multiplication by a strictly positive constant.

The proof of Theorem 1 is in Appendix. Its logic is as follows. We will first show that all conditions in Theorem 1 except A6 (*Standard sequence invariance*) imply the conditions in Bouyssou and Marchant (2009, Th. 1) (when  $n \geq 4$ , one may also use the results in Vind, 1991), so that, for each  $i \in N$  there is a real-valued functions  $v_i$  on  $\Gamma_i$  such that, for all  $a \in \mathcal{X}$ ,

$$a \in \mathcal{A} \Leftrightarrow \sum_{i=1}^n v_i(a_i) > 0,$$

$$a \in \mathcal{F} \Leftrightarrow \sum_{i=1}^n v_i(a_i) = 0.$$

We then use condition A6 (*Standard sequence invariance*) to show that all functions  $v_i$  must be identical up to a positive linear transformation. This will complete the proof. The logic of the proof is therefore similar to the one used in Wakker (1989a, Ch. 4) to characterize the classical SEU model with a finite set of states based on results characterizing the additive value function model in conjoint measurement. The appendix gives a variant of Theorem 1 in which the elements of  $\mathcal{F}$  are dealt with implicitly as the elements of  $\text{Cl}(\mathcal{A}) \setminus \mathcal{A}$ , where  $\text{Cl}(\mathcal{A})$  denotes the (topological) closure of the set  $\mathcal{A}$ .

## 5 Discussion

This paper has analyzed decision making under uncertainty replacing the traditional primitives consisting in a preference relation on the set of acts by a threefold ordered partition of this set. A possible interpretation of this setting is that each act is positioned vis-à-vis a status quo. Within a framework involving a finite set of states and a rich set of consequences, we have given conditions on a threefold ordered partition ensuring that it can be represented in such a way that all attractive (resp. unattractive) acts have a subjective expected utility that is above

(resp. below) some threshold. The obtained representation has strong uniqueness properties. This gives SEU alternative behavioral foundations based on primitives that are poorer than the ones traditionally considered in the literature and using conditions that are reasonably simple.

The line of research consisting in replacing a preference relation by an ordered partition to analyze classical models is still quite open. In the field of decision making under uncertainty, at least three questions would deserve further analyses.

The first important, but apparently difficult, question is to study models extending SEU such as Rank Dependent Utility (Gilboa, 1987, Wakker, 1989a) or Cumulative Prospect Theory (Tversky and Kahneman, 1992, Wakker and Tversky, 1993) using primitives such as the ones used here.

The second question is to devise conditions that would be simpler and more elegant than the ones presented here. The reference point here are the results of Wakker (1984, 1989a). He derives SEU, with a finite set of states, on the basis of a result in conjoint measurement characterizing additive representations, modifying the classical Triple Cancellation condition to his “Cardinal Coordinate Independence” condition, while keeping all other conditions unchanged. Our analysis is closely related to his. However, compared to the conditions used in Bouyssou and Marchant (2009) to obtain an additive representation of an ordered partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  in a conjoint measurement setting, we have modified several conditions (A1 (r-1-Linearity- $\mathcal{A}$ ), A4 (r-1-Linearity- $\mathcal{F}$ ), and A5 (r-1-Thinness)) and added A6 (Standard sequence invariance). This seems to call for further study.

The third question is empirical. We have obtained behavioral foundations for SEU using uncommon primitives. The numerous empirical studies on the validity of SEU have, to the best of our knowledge, all used the classical primitives consisting in a preference relation on the set of acts. It would therefore be important to know whether the adoption of our weaker primitives has an influence on the empirical validity of SEU. Indeed, it might well happen that “classical violations” of SEU have less impact when working with weaker primitives.

## Appendix: Proofs

### A Preliminary lemmas

Remember first that Lemma 1 has shown that the relation  $\succsim^{\mathcal{A}}$  (as defined by (2)) is complete iff A1 (r-1-Linearity- $\mathcal{A}$ ) holds. The next lemma says that, under the conditions of Theorem 1, restricted solvability holds.

**Lemma 2**

*Suppose that the conditions of Theorem 1 hold. Let  $I \subseteq N$  and  $x, a, b \in \mathcal{X}$ . If  $(x_I, a_{-I}) \in \mathcal{A}$  and  $(x_I, b_{-I}) \in \mathcal{U}$ , then  $(x_I, c_{-I}) \in \mathcal{F}$ , for some  $c \in \mathcal{X}$ .*

PROOF

Let  $A(x_I) = \{y_{-I} \in \Gamma_{-I} : (x_I, y_{-I}) \in \mathcal{A}\}$ . Similarly, let  $U(x_I) = \{z_{-I} \in \Gamma_{-I} : (x_I, z_{-I}) \in \mathcal{U}\}$ . By hypothesis, both  $A(x_I)$  and  $U(x_I)$  are nonempty. They are clearly disjoint. Using A3 (Openness), we know that both sets are open in the product topology on  $\Gamma_{-I}$  (see, e.g., Wakker, 1989a, Lemma 0.2.1, p. 12). Using A2 (Connectedness in the order topology), the set  $\Gamma_{-I}$  is connected. Hence, the sets  $A(x_I)$  and  $U(x_I)$  cannot partition  $\Gamma_{-I}$ , so that there is  $c \in \mathcal{X}$  such that  $(x_I, c_{-I}) \in \mathcal{F}$ .  $\square$

The next three lemmas show that the ordering of the elements of  $\Gamma$  via the relation  $\succ^{\mathcal{A}}$  is not contradicted if the frontier  $\mathcal{F}$  is taken into account.

**Lemma 3**

Suppose that the conditions of Theorem 1 hold. We have, for all  $i, j \in N$ , all  $\alpha, \beta \in \Gamma$  and all  $a, b \in \mathcal{X}$ ,

$$\left. \begin{array}{l} (\alpha_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (\beta_j, b_{-j}) \in \mathcal{AF} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, a_{-i}) \in \mathcal{A} \\ \text{or} \\ (\alpha_j, b_{-j}) \in \mathcal{AF}. \end{array} \right.$$

PROOF

If  $(\beta_j, b_{-j}) \in \mathcal{A}$ , the conclusion is obvious using A1 (r-1-Linearity- $\mathcal{A}$ ). Hence, let us suppose that  $(\beta_j, b_{-j}) \in \mathcal{F}$  and, in contradiction with the thesis, that  $(\beta_i, a_{-i}) \in \mathcal{FU}$  and  $(\alpha_j, b_{-j}) \in \mathcal{U}$ .

We distinguish three cases.

1. Suppose that there is  $c \in \mathcal{X}$  such that  $(\alpha_i, c_{-i}) \in \mathcal{F}$ . Using A4 (r-1-Linearity- $\mathcal{F}$ ),  $(\alpha_i, c_{-i}) \in \mathcal{F}$  and  $(\beta_j, b_{-j}) \in \mathcal{F}$  imply either  $(\beta_i, c_{-i}) \in \mathcal{AF}$  or  $(\alpha_j, b_{-j}) \in \mathcal{AF}$ . Because,  $(\alpha_j, b_{-j}) \in \mathcal{U}$ , we must have  $(\beta_i, c_{-i}) \in \mathcal{AF}$ .  
Suppose first that  $(\beta_i, c_{-i}) \in \mathcal{F}$ . Using A5 (r-1-Thinness),  $(\alpha_i, c_{-i}) \in \mathcal{F}$ ,  $(\alpha_j, b_{-j}) \in \mathcal{U}$ , and  $(\beta_j, b_{-j}) \in \mathcal{F}$  leads to a contradiction. Hence, we must have  $(\beta_i, c_{-i}) \in \mathcal{A}$ . Using,  $(\alpha_i, a_{-i}) \in \mathcal{A}$ , A1 (r-1-Linearity- $\mathcal{A}$ ) implies either  $(\alpha_i, c_{-i}) \in \mathcal{A}$  or  $(\beta_i, a_{-i}) \in \mathcal{A}$ , a contradiction.
2. Suppose that there is  $c \in \mathcal{X}$  such that  $(\alpha_i, c_{-i}) \in \mathcal{U}$ . Since  $(\alpha_i, a_{-i}) \in \mathcal{A}$ , Lemma 2 implies that  $(\alpha_i, d_{-i}) \in \mathcal{F}$ , for some  $d \in \mathcal{X}$ . The reasoning used above therefore leads to a contradiction.
3. Suppose that, for all  $c \in \mathcal{X}$ , we have  $(\alpha_i, c_{-i}) \in \mathcal{A}$ . We distinguish several cases.
  - (a) Suppose that there is  $d \in \mathcal{X}$  such that  $(\alpha_j, d_{-j}) \in \mathcal{F}$ . Using A4 (r-1-Linearity- $\mathcal{F}$ ),  $(\alpha_j, d_{-j}) \in \mathcal{F}$  and  $(\beta_j, b_{-j}) \in \mathcal{F}$  imply either  $(\beta_j, d_{-j}) \in$

$\mathcal{AF}$  or  $(\alpha_j, b_{-j}) \in \mathcal{AF}$ . Because we know that  $(\alpha_j, b_{-j}) \in \mathcal{U}$ , we must have  $(\beta_j, d_{-j}) \in \mathcal{AF}$ .

Suppose that  $(\beta_j, d_{-j}) \in \mathcal{F}$ . Using A5 (r-1-Thinness), we obtain a contradiction since we have  $(\beta_j, d_{-j}) \in \mathcal{F}$ ,  $(\alpha_j, d_{-j}) \in \mathcal{F}$ ,  $(\alpha_i, a_{-i}) \in \mathcal{A}$ , and  $(\beta_i, a_{-i}) \in \mathcal{FU}$ .

Suppose that  $(\beta_j, d_{-j}) \in \mathcal{A}$ . Using A1 (r-1-Linearity- $\mathcal{A}$ ),  $(\beta_j, d_{-j}) \in \mathcal{A}$  and  $(\alpha_i, a_{-i}) \in \mathcal{A}$  imply either  $(\alpha_j, d_{-j}) \in \mathcal{A}$  or  $(\beta_i, a_{-i}) \in \mathcal{A}$ . This is contradictory since we know that  $(\alpha_j, d_{-j}) \in \mathcal{F}$  and  $(\beta_i, a_{-i}) \in \mathcal{FU}$ .

Notice that if there are  $d', d'' \in \mathcal{X}$  such that  $(\alpha_j, d'_{-j}) \in \mathcal{A}$  and  $(\alpha_j, d''_{-j}) \in \mathcal{U}$ , Lemma 2 implies that  $(\alpha_j, d_{-j}) \in \mathcal{F}$ , for some  $d \in \mathcal{X}$ . Hence, we only have two remaining cases to examine.

- (b) Suppose that for all  $d \in \mathcal{X}$ , we have  $(\alpha_j, d_{-j}) \in \mathcal{A}$ . This is contradictory since we know that  $(\alpha_j, b_{-j}) \in \mathcal{U}$ .
- (c) Suppose finally that for all  $d \in \mathcal{X}$ , we have  $(\alpha_j, d_{-j}) \in \mathcal{U}$ . This implies that  $(\alpha_j, \alpha_i, c_{-ij}) \in \mathcal{U}$ . This contradicts the fact that  $(\alpha_i, c_{-i}) \in \mathcal{A}$ , for all  $c \in \mathcal{X}$ .  $\square$

#### Lemma 4

Suppose that the conditions of Theorem 1 hold. The relation  $\succsim^{\mathcal{AF}}$  defined by (4) is complete.

#### PROOF

Suppose that  $\text{Not}[\alpha \succsim^{\mathcal{AF}} \beta]$ . This implies that, for some  $i \in N$  and some  $a \in \mathcal{X}$ ,

$$[(\beta_i, a_{-i}) \in \mathcal{AF} \text{ and } (\alpha_i, a_{-i}) \notin \mathcal{AF}] \quad (6a)$$

Similarly,  $\text{Not}[\beta \succsim^{\mathcal{AF}} \alpha]$  implies that, for some  $j \in N$  and some  $b \in \mathcal{X}$ ,

$$[(\alpha_i, b_{-i}) \in \mathcal{AF} \text{ and } (\beta_i, b_{-i}) \notin \mathcal{AF}] \quad (6b)$$

If both  $(\beta_i, a_{-i})$  and  $(\alpha_i, b_{-i})$  are in  $\mathcal{A}$ , we have a violation of A1 (r-1-Linearity- $\mathcal{A}$ ). Similarly, if both  $(\beta_i, a_{-i})$  and  $(\alpha_i, b_{-i})$  are in  $\mathcal{F}$ , we have a violation of A4 (r-1-Linearity- $\mathcal{F}$ ). If one of  $(\beta_i, a_{-i})$  and  $(\alpha_i, b_{-i})$  belongs to  $\mathcal{A}$  and the other belongs to  $\mathcal{F}$ , using Lemma 3 leads to a contradiction.  $\square$

#### Lemma 5

Suppose that the conditions of Theorem 1 hold. The relation  $\succsim = \succsim^{\mathcal{A}} \cap \succsim^{\mathcal{AF}}$  is complete.

#### PROOF

Using Lemmas 1 and 4, we know that both  $\succsim^{\mathcal{A}}$  and  $\succsim^{\mathcal{AF}}$  are complete. Hence, the relation  $\succsim$  will be incomplete if there are  $\alpha, \beta \in \Gamma$  such that  $\alpha \succ^{\mathcal{A}} \beta$  and  $\beta \succ^{\mathcal{AF}} \alpha$ . This implies that, for some  $a, b \in \mathcal{X}$  and some  $i, j \in N$ ,  $(\alpha_i, a_{-i}) \in \mathcal{A}$ ,  $(\beta_i, a_{-i}) \notin \mathcal{A}$ ,  $(\beta_j, b_{-j}) \in \mathcal{AF}$ ,  $(\alpha_j, b_{-j}) \notin \mathcal{AF}$ . Using Lemma 3 leads to a contradiction.  $\square$

For later use, we define, on each  $i \in N$ , the relation  $\succsim_i = \succsim_i^{\mathcal{A}} \cap \succsim_i^{\mathcal{F}}$ . We obviously have  $\succsim = \bigcap_{i=1}^n \succsim_i$ . The above lemma implies that all relations  $\succsim_i$  are complete and are never contradictory.

Our final lemma in this section says that the ordered partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  reacts as expected, i.e., monotonically, to changes governed by the relation  $\succsim^{\mathcal{A}}$ .

**Lemma 6**

*Suppose that the conditions of Theorem 1 hold. Then, For all  $i \in N$ , all  $\alpha, \beta \in \Gamma$  and all  $a \in \mathcal{X}$ ,*

$$\begin{aligned} (\alpha_i, a_{-i}) \in \mathcal{A} \text{ and } \beta \succsim^{\mathcal{A}} \alpha &\Rightarrow (\beta_i, a_{-i}) \in \mathcal{A}, \\ (\alpha_i, a_{-i}) \in \mathcal{F} \text{ and } \beta \succ^{\mathcal{A}} \alpha &\Rightarrow (\beta_i, a_{-i}) \in \mathcal{A}, \\ (\alpha_i, a_{-i}) \in \mathcal{F} \text{ and } \alpha \succ^{\mathcal{A}} \beta &\Rightarrow (\beta_i, a_{-i}) \in \mathcal{U}. \end{aligned}$$

**PROOF**

The first part follows from the definition of  $\succsim^{\mathcal{A}}$ . For the second part, observe that, since  $\succsim$  is complete,  $\beta \succ^{\mathcal{A}} \alpha$  implies  $\beta \succsim^{\mathcal{A}\mathcal{F}} \alpha$ . Hence, we know that  $(\beta_i, a_{-i}) \in \mathcal{A}\mathcal{F}$ . But supposing that  $(\beta_i, a_{-i}) \in \mathcal{F}$  together with  $(\alpha_i, a_{-i}) \in \mathcal{F}$  is incompatible with  $\beta \succ^{\mathcal{A}} \alpha$  because of A5 (r-1-Thinness). Suppose now that  $(\alpha_i, a_{-i}) \in \mathcal{F}$  and  $\alpha \succ^{\mathcal{A}} \beta$ . In view of the first two parts, it is impossible that  $(\beta_i, a_{-i}) \in \mathcal{A}\mathcal{F}$ . Hence, we must have  $(\beta_i, a_{-i}) \in \mathcal{U}$ .  $\square$

## B The unbounded case

We first deal with the case in which  $\Gamma$  is unbounded, i.e., we suppose that the following assumption holds.

**Assumption 2**

*For all  $\gamma \in \Gamma$ , there are  $\alpha, \beta \in \Gamma$  such that  $\alpha \succ^{\mathcal{A}} \gamma \succ^{\mathcal{A}} \beta$ .*

We will prove the following proposition that is identical to Theorem 1, except that Assumption 2 is required.

**Proposition 1**

*Consider a threefold ordered partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  of  $\mathcal{X}$  such that Assumptions 1 and 2 hold. Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A1 (r-1-Linearity- $\mathcal{A}$ ), A2 (Connectedness in the order topology), A3 (Openness), A4 (r-1-Linearity- $\mathcal{F}$ ), A5 (r-1-Thinness), and A6 (Standard sequence invariance). If  $n \geq 4$  suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A7 (a-2-Linearity- $\mathcal{A}$ ). If  $n = 3$  suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies A8 (Double Cancellation). Then there are a continuous real-valued function  $u$  on  $\Gamma$  and  $n$  strictly positive real numbers  $p_1, p_2, \dots, p_n$  adding up to 1 such that (1) holds.*

*The numbers  $p_1, p_2, \dots, p_n$  are unique. The function  $u$  is unique up to a multiplication by a strictly positive constant.*

## B.1 Lemmas

The next two lemmas show that, when  $\Gamma$  is unbounded, the order topology generated by  $\succ^{\mathcal{A}}$  is identical to the order topology generated by  $\succ$ .

### Lemma 7

Suppose that the conditions of Proposition 1 hold. For all  $i \in N$ , we have  $\succ_i^{\mathcal{A}} = \succ_i^{\mathcal{AF}} = \succ_i$ .

PROOF

Let  $i \in N$ . Since we know from Lemma 5 that  $\succ_i$  is complete, there are two cases to examine. Suppose first that, for some  $\alpha, \beta \in \Gamma$ , we have  $\beta \succ_i^{\mathcal{AF}} \alpha$  and  $\alpha \sim_i^{\mathcal{A}} \beta$ . Because  $\beta \succ_i^{\mathcal{AF}} \alpha$ , we know that  $(\beta_i, a_{-i}) \in \mathcal{AF}$  and  $(\alpha_i, a_{-i}) \in \mathcal{U}$ , for some  $a \in \mathcal{X}$ . Because  $\alpha \sim_i^{\mathcal{A}} \beta$ , we must have  $(\beta_i, a_{-i}) \in \mathcal{F}$ . Let  $j \neq i$ . Using the fact that  $\Gamma$  is unbounded and connected and that  $\mathcal{U}$  is open, we can find  $\gamma \in \Gamma$  such that  $\gamma \succ^{\mathcal{A}} a_j$  and  $(\alpha_i, \gamma_j, a_{-ij}) \in \mathcal{U}$ . Using Lemma 6, we obtain  $(\beta_i, \gamma_j, a_{-ij}) \in \mathcal{A}$ . This violates  $\alpha \sim_i^{\mathcal{A}} \beta$ .

Suppose now that, for some  $\alpha, \beta \in \Gamma$ , we have  $\alpha \sim_i^{\mathcal{AF}} \beta$  and  $\beta \succ_i^{\mathcal{A}} \alpha$ . Because  $\beta \succ_i^{\mathcal{A}} \alpha$ , we know that  $(\beta_i, a_{-i}) \in \mathcal{A}$  and  $(\alpha_i, a_{-i}) \in \mathcal{FU}$ , for some  $a \in \mathcal{X}$ . Because  $\alpha \sim_i^{\mathcal{AF}} \beta$ , we must have  $(\alpha_i, a_{-i}) \in \mathcal{F}$ . Let  $j \neq i$ . Using the fact that  $\Gamma$  is unbounded and connected and that  $\mathcal{A}$  is open, we can find  $\gamma \in \Gamma$  such that  $a_j \succ^{\mathcal{A}} \gamma$  and  $(\beta_i, \gamma_j, a_{-ij}) \in \mathcal{A}$ . Using Lemma 6, we obtain  $(\alpha_i, \gamma_j, a_{-ij}) \in \mathcal{U}$ . This violates  $\alpha \sim_i^{\mathcal{AF}} \beta$ .  $\square$

### Lemma 8

Under the conditions of Proposition 1, we have  $\succ^{\mathcal{A}} = \succ^{\mathcal{AF}} = \succ$ .

PROOF

Immediate from Lemma 7.  $\square$

### Lemma 9

For all  $i \in N$ , the set  $\Gamma$  is connected in the order topology generated by  $\succ_i$ .

PROOF

By definition, the relation  $\succ = \succ^{\mathcal{A}}$  refines all the  $\succ_i$ . Hence, we know that  $\alpha \succ_i \beta \Rightarrow \alpha \succ \beta$ , which implies that all open sets for the order topology generated by  $\succ_i$  are also open in the order topology generated by  $\succ$ . Because we know that  $\Gamma$  is connected in the order topology generated by  $\succ^{\mathcal{A}} = \succ$ , it must also be connected in the order topology generated by  $\succ_i$ .  $\square$



## B.2 Existence of an additive representation

The next lemma shows that there is an additive representation on a suitably chosen subset of  $\mathcal{X}$ . Remember that we have defined  $\Delta_i = \{\beta \in \Gamma : \alpha \succ_i^{\mathcal{A}} \beta \succ_i^{\mathcal{A}} \gamma \text{ for some } \alpha, \gamma \in \Gamma\}$  and  $\Delta = \prod_{i \in N} \Delta_i$ .

### Lemma 10

Assume the conditions of Proposition 1. There are then  $n$  continuous (w.r.t. the order topology induced by  $\succ_i^{\mathcal{A}}$ ) real-valued mappings  $v_i : \Delta_i \rightarrow \mathbb{R}$  such that, for all  $x \in \Delta$ ,

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow \sum_{i \in N} v_i(x_i) > 0, \\ x \in \mathcal{U} &\Leftrightarrow \sum_{i \in N} v_i(x_i) < 0. \end{aligned} \tag{7}$$

Under the above conditions,  $\langle u_i \rangle_{i \in N}$  and  $\langle v_i \rangle_{i \in N}$  are two additive representations of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  in model (7) iff there are real numbers  $\omega_1, \omega_2, \dots, \omega_n, \kappa$  with  $\kappa > 0$  and  $\sum_{i=1}^n \omega_i = 0$  such that for all  $\alpha \in \Gamma$  and all  $i \in N$ ,  $v_i(\alpha) = \kappa u_i(\alpha) + \omega_i$ .

### PROOF

By construction, for each  $i \in N$ , the set  $\Delta_i$  is unbounded w.r.t.  $\succ_i$ . Lemma 9 has shown that, for each  $i \in N$ , the set  $\Gamma$  is connected in the order topology generated by  $\succ_i$ . Given the construction of the set  $\Delta_i \subseteq \Gamma$  this implies that  $\Delta_i$  is connected in the order topology generated by  $\succ_i$ . The conclusion therefore follows from Bouyssou and Marchant (2009, Proposition 1).  $\square$

Let  $I \subseteq N$  be a proper nonempty subset of  $N$ . Let us call  $I$ -Linearity- $\mathcal{A}$ , the following condition:

$$\left. \begin{array}{l} (a_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (b_I, b_{-I}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (b_I, a_{-I}) \in \mathcal{A} \\ \text{or} \\ (a_I, b_{-I}) \in \mathcal{A}. \end{array} \right.$$

Let us call  $I$ -Linearity- $\mathcal{AF}$ , the following condition:

$$\left. \begin{array}{l} (a_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (b_I, b_{-I}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (b_I, a_{-I}) \in \mathcal{A} \\ \text{or} \\ (a_I, b_{-I}) \in \mathcal{AF}. \end{array} \right.$$

Similarly, call  $I$ -Linearity- $\mathcal{F}$  the following condition

$$\left. \begin{array}{l} (a_I, a_{-I}) \in \mathcal{F} \\ \text{and} \\ (b_I, b_{-I}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (b_I, a_{-I}) \in \mathcal{AF} \\ \text{or} \\ (a_I, b_{-I}) \in \mathcal{AF}. \end{array} \right.$$

Call  $I$ -Thinness the following condition

$$\left. \begin{array}{l} (a_I, c_{-I}) \in \mathcal{F} \\ \text{and} \\ (b_I, c_{-I}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (a_I, d_{-I}) \in \mathcal{A} \Leftrightarrow (b_I, d_{-I}) \in \mathcal{A} \\ (a_I, d_{-I}) \in \mathcal{F} \Leftrightarrow (b_I, d_{-I}) \in \mathcal{F} \\ (a_I, d_{-I}) \in \mathcal{U} \Leftrightarrow (b_I, d_{-I}) \in \mathcal{U}. \end{array} \right.$$

Lemma 10 implies that for all proper nonempty subset  $I$  of  $N$ , the partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  restricted to  $\Delta$  satisfies  $I$ -Linearity- $\mathcal{A}$ ,  $I$ -Linearity- $\mathcal{F}$ ,  $I$ -Linearity- $\mathcal{AF}$  and  $I$ -Thinness. Hence, the analogues of Lemmas 1, 4, 5, 6 will hold for  $I$ : there is a weak order  $\succsim_I^{\mathcal{A}}$  on  $\prod_{i \in I} \Delta_i$  and the partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is monotonic w.r.t.  $\succsim_I^{\mathcal{A}}$ .

### B.3 Additional lemmas

The following lemma shows that, under the conditions of Proposition 1, there is at least one state  $\ell \in N$  such that  $\Delta_\ell = \Gamma$ .

#### Lemma 11

Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies the conditions of Proposition 1.

1. For all  $\alpha \in \Delta_i$ ,  $(\alpha_i, x_{-i}) \in \mathcal{F}$ , for some  $x \in \mathcal{X}$ .
2. For all  $\gamma, \delta, \varepsilon, \zeta \in \Gamma$  and for all  $i \in N$ ,  $\gamma \succ_i^{\mathcal{A}} \delta \sim_i^{\mathcal{A}} \varepsilon \succ_i^{\mathcal{A}} \zeta$  implies  $\delta \sim^{\mathcal{A}} \varepsilon$ .
3. For all  $i \in N$ , the restrictions of  $\succsim_i^{\mathcal{A}}$  and  $\succsim^{\mathcal{A}}$  to  $\Delta_i$  are identical,
4. For all  $i \in N$   $\Delta_i$  is an “open interval” of  $\Gamma$  w.r.t.  $\succsim^{\mathcal{A}}$ , i.e., for all  $\alpha, \beta \in \Delta_i$  and all  $\gamma \in \Gamma$ ,  $\beta \succ^{\mathcal{A}} \gamma \succ^{\mathcal{A}} \alpha$  implies  $\gamma \in \Delta_i$ .
5. There is a state, say  $j \in N$ , and  $\alpha^j \in \Gamma$  such that, for all  $\beta \in \Gamma$  with  $\alpha^j \succsim^{\mathcal{A}} \beta$ , we have  $\beta \in \Delta_j$ .
6. There is a state, say  $k \in N$  and  $\alpha^k \in \Gamma$  such that, for all  $\beta \in \Gamma$  with  $\beta \succsim^{\mathcal{A}} \alpha^k$ , we have  $\beta \in \Delta_k$ .
7. There is  $\ell \in N$  such that  $\Delta_\ell = \Gamma$ .

#### PROOF

1. If  $(\alpha_i, y_{-i}) \in \mathcal{A}$ , for all  $y \in \mathcal{X}$ , this implies that  $\alpha \succsim_i^{\mathcal{A}} \beta$ , for all  $\beta \in \Gamma$ . This contradicts  $\alpha \in \Delta_i$ . Similarly,  $(\alpha_i, y_{-i}) \in \mathcal{U}$ , for all  $y \in \mathcal{X}$ , we have  $\beta \succsim_i^{\mathcal{A}} \alpha$ , for all  $\beta \in \Gamma$ , contradicting  $\alpha \in \Delta_i$ . Therefore, there are  $y, z \in \mathcal{X}$  such that  $(\alpha_i, y_{-i}) \in \mathcal{AF}$  and  $(\alpha_i, z_{-i}) \in \mathcal{FU}$ . If  $(\alpha_i, y_{-i}) \in \mathcal{F}$  or  $(\alpha_i, z_{-i}) \in \mathcal{F}$ , there is nothing to prove. Otherwise, the use of Lemma 2 also leads to the desired conclusion.

2. Our assumptions clearly imply that  $\delta \in \Delta_i$ . So, using (1), there is  $x \in \mathcal{X}$  such that  $(\delta_i, x_{-i}) \in \mathcal{F}$ . This, combined with  $\delta \sim_i^{\mathcal{A}} \varepsilon$  implies  $(\varepsilon_i, x_{-i}) \in \mathcal{F}$ . Using A5 (r-1-Thinness), we find  $\delta \sim_j^{\mathcal{A}} \varepsilon$ , for all  $j \in N$ , so that  $\delta \sim^{\mathcal{A}} \varepsilon$ .

3. Consider any  $\alpha, \beta \in \Delta_i$ . Since  $\succsim^{\mathcal{A}} = \bigcap_{j \in N} \succsim_j^{\mathcal{A}}$ , we know that  $\alpha \succsim^{\mathcal{A}} \beta$  implies  $\alpha \succsim_i^{\mathcal{A}} \beta$ . Because of (2),  $\alpha \sim_i^{\mathcal{A}} \beta$  implies  $\alpha \sim^{\mathcal{A}} \beta$ . Hence,  $\alpha \succsim^{\mathcal{A}} \beta$  iff  $\alpha \succsim_i^{\mathcal{A}} \beta$ .

4. Consider  $\alpha, \beta \in \Delta_i$  and  $\gamma \in \Gamma$  such that  $\beta \succ^{\mathcal{A}} \gamma \succ^{\mathcal{A}} \alpha$ . Suppose, contrary to the thesis, that  $\gamma \notin \Delta_i$ . Using (3),  $\beta \succ_i^{\mathcal{A}} \alpha$ . Since  $\gamma \notin \Delta_i$ , we must have  $\alpha \succ_i^{\mathcal{A}} \gamma$  or  $\gamma \succ_i^{\mathcal{A}} \beta$ . But this contradicts the fact that  $\beta \succ^{\mathcal{A}} \gamma \succ^{\mathcal{A}} \alpha$ .

5. Choose a state  $i_1 \in N$ . If  $i_1$  is like in (5), then there is nothing to prove. Otherwise, using (4), there is  $\alpha \in \Gamma$  such that, for all  $\beta \in \Gamma$  such that  $\alpha \succ^{\mathcal{A}} \beta$ , we have  $\beta \notin \Delta_{i_1}$ . Because of the unboundedness of  $\Gamma$ , there are then  $\gamma, \delta \in \Gamma$  such that  $\alpha \succ^{\mathcal{A}} \delta \succ^{\mathcal{A}} \gamma$ . This implies the existence of a state  $i_2 \in N$  different from  $i_1$  such that  $\delta \succ_{i_2}^{\mathcal{A}} \gamma$ . If  $i_2$  is like in (5), then the proof is done. Otherwise, using (4), there is  $\alpha' \in \Gamma$  such that  $\gamma \succ^{\mathcal{A}} \alpha'$  such that, for all  $\beta' \in \Gamma$  such that  $\alpha' \succ^{\mathcal{A}} \beta'$ , we have  $\beta' \notin \Delta_{i_2}$ . Because of the unboundedness of  $\Gamma$ , there are then  $\gamma', \delta' \in \Gamma$  such that  $\alpha' \succ^{\mathcal{A}} \delta' \succ^{\mathcal{A}} \gamma'$ . This implies the existence of a state  $i_3 \in N$  different from  $i_1$  and  $i_2$  such that  $\delta' \succ_{i_3}^{\mathcal{A}} \gamma'$ . We then proceed in the same way with  $i_3, i_4, \dots, i_n$ . If none of these states are like in (5), then there is  $\alpha^*$  such that, for all  $\beta \in \Gamma$  such that  $\alpha^* \succ^{\mathcal{A}} \beta$ , we have  $\beta \notin \bigcup_{i \in N} \Delta_i$ . Because of the unboundedness of  $\Gamma$  and the definition of  $\Delta_i$ , this is clearly impossible. So, there is at least one state  $j \in N$  like in (5). The proof of (6) is similar.

7. Let  $j_b$  be a state as in (5) and  $j_t$  as in (6). They necessarily exist. If  $j_b = j_t$ , then  $\Gamma = \Delta_{j_b}$ . Suppose then  $j_b \neq j_t$ . Suppose also  $\Delta_{j_b} \neq \Gamma \neq \Delta_{j_t}$ . There are then  $\alpha, \beta$  such that  $\alpha \in \Delta_{j_b}$ ,  $\alpha \notin \Delta_{j_t}$ ,  $\beta \in \Delta_{j_t}$ ,  $\beta \notin \Delta_{j_b}$ . Choose now  $\alpha' \in \Gamma$  such that  $\alpha \succ^{\mathcal{A}} \alpha'$ . Of course,  $\alpha' \notin \Delta_{j_t}$  and  $\alpha' \sim_{j_t}^{\mathcal{A}} \alpha$ . Suppose there is  $x_{-j_t}$  such that  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{F}$ . Then,  $\alpha' \sim_i^{\mathcal{A}} \alpha$  implies  $(\alpha'_{j_t}, x_{-j_t}) \in \mathcal{F}$ . Then, using A5 (r-1-Thinness),  $\alpha' \sim_i^{\mathcal{A}} \alpha$  for all  $i \in N$ , contradicting  $\alpha \succ^{\mathcal{A}} \alpha'$ . Hence,  $(\alpha_{j_t}, x_{-j_t}) \notin \mathcal{F}$ , for all  $x_{-j_t}$ . In other words, either  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{A}$ , for all  $x_{-j_t}$  or  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{U}$ , for all  $x_{-j_t}$ . Similarly, we have either  $(\beta_{j_b}, x_{-j_b}) \in \mathcal{A}$ , for all  $x_{-j_b}$  or  $(\beta_{j_b}, x_{-j_b}) \in \mathcal{U}$ , for all  $x_{-j_b}$ . Since  $\alpha \in \Delta_{j_b}$ , there is  $a_{-j_b}$  such that  $(\alpha_{j_b}, a_{-j_b}) \in \mathcal{F}$ . Similarly, there is  $b_{-j_t}$  such that  $(\beta_{j_t}, b_{-j_t}) \in \mathcal{F}$ . Hence, four cases arise.

- a)  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{A}$ , for all  $x_{-j_t}$  and  $(\beta_{j_b}, x_{-j_b}) \in \mathcal{U}$ , for all  $x_{-j_b}$ . From the first part, we derive  $(\alpha_{j_t}, \beta_{j_b}, x_{-j_t j_b}) \in \mathcal{A}$ , for all  $x_{-j_t j_b}$ . A contradiction.
- b)  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{U}$ , for all  $x_{-j_t}$  and  $(\beta_{j_b}, x_{-j_b}) \in \mathcal{A}$ , for all  $x_{-j_b}$ . This case is handled like the previous one.
- c)  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{A}$ , for all  $x_{-j_t}$  and  $(\beta_{j_b}, x_{-j_b}) \in \mathcal{A}$ , for all  $x_{-j_b}$ . Since  $\succsim^{\mathcal{A}}$  is complete, we have  $\alpha \succsim^{\mathcal{A}} \beta$  or  $\beta \succsim^{\mathcal{A}} \alpha$ . If  $\beta \succsim^{\mathcal{A}} \alpha$ , then  $(\beta_{j_t}, b_{-j_t}) \in \mathcal{F}$  implies

$(\alpha_{j_t}, b_{-j_t}) \in \mathcal{F}\mathcal{U}$ , a contradiction. If  $\alpha \succ^{\mathcal{A}} \beta$ , then  $(\alpha_{j_b}, a_{-j_b}) \in \mathcal{F}$  implies  $(\beta_{j_b}, a_{-j_b}) \in \mathcal{F}\mathcal{U}$ , a contradiction.

d)  $(\alpha_{j_t}, x_{-j_t}) \in \mathcal{U}$ , for all  $x_{-j_t}$  and  $(\beta_{j_b}, x_{-j_b}) \in \mathcal{U}$ , for all  $x_{-j_b}$ . This case is handled like the previous one.

Since all the above four cases are contradictory, we have shown that one of  $\Delta_{j_b}$  or  $\Delta_{j_t}$  must be equal to  $\Gamma$ .  $\square$

The next lemma explores the implications of A6 (Standard sequence invariance), in presence of the other conditions of Proposition 1.

**Lemma 12**

Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies the conditions of Proposition 1. Then, for all  $i, j \in N$ , all  $\alpha, \beta, \gamma \in \Delta_i \cap \Delta_j$ , and all  $a, b, c, d \in \mathcal{X}$ ,

$$\left. \begin{array}{l} (\alpha_i, a_j, a_{-ij}) \in \mathcal{F}, \quad (\beta_i, b_j, a_{-ij}) \in \mathcal{F}, \\ (\beta_i, a_j, b_{-ij}) \in \mathcal{F}, \quad (\gamma_i, b_j, b_{-ij}) \in \mathcal{F}, \\ (\alpha_j, c_i, c_{-ij}) \in \mathcal{F}, \quad (\beta_j, d_i, c_{-ij}) \in \mathcal{F}, \\ (\beta_j, c_i, d_{-ij}) \in \mathcal{F}, \end{array} \right\} \Rightarrow (\gamma_j, d_i, d_{-ij}) \in \mathcal{F}.$$

**PROOF**

Suppose, in contradiction with the thesis, that we have:

$$\begin{array}{llll} (\alpha_i, a_j, a_{-ij}) \in \mathcal{F} & (8a) & (\beta_i, b_j, a_{-ij}) \in \mathcal{F} & (8e) \\ (\beta_i, a_j, b_{-ij}) \in \mathcal{F} & (8b) & (\gamma_i, b_j, b_{-ij}) \in \mathcal{F} & (8f) \\ (\alpha_j, c_i, c_{-ij}) \in \mathcal{F} & (8c) & (\beta_j, d_i, c_{-ij}) \in \mathcal{F} & (8g) \\ (\beta_j, c_i, d_{-ij}) \in \mathcal{F} & (8d) & (\gamma_j, d_i, d_{-ij}) \in \mathcal{A}. & (8h) \end{array}$$

Let us show that this leads to a contradiction (the case  $(\gamma_j, d_i, d_{-ij}) \in \mathcal{U}$  is dealt with in a similar way).

Suppose first that  $\alpha \sim^{\mathcal{A}} \beta$ . Using (8a) and (8b), we obtain that  $a_{-ij} \sim_{-ij}^{\mathcal{A}} b_{-ij}$ . Using (8e) and (8f), we obtain that  $\beta \sim^{\mathcal{A}} \gamma$ . Using (8c) and (8d), we obtain that  $c_{-ij} \sim_{-ij}^{\mathcal{A}} d_{-ij}$ . This is contradictory in view of (8g) and (8h). Hence, we must have either  $\alpha \succ^{\mathcal{A}} \beta$  or  $\beta \succ^{\mathcal{A}} \alpha$ .

Suppose that  $\alpha \succ^{\mathcal{A}} \beta$ , the other case being similar. Using (8a) and (8b), we obtain that  $b_{-ij} \succ_{-ij}^{\mathcal{A}} a_{-ij}$ . Using (8e) and (8f), we obtain that  $\beta \succ^{\mathcal{A}} \gamma$ . Using (8h), by connectedness, there are  $\delta \in \Gamma$  and  $r \in \mathcal{X}$  such that:

$$(\delta_j, r_i, r_{-ij}) \in \mathcal{A}, \tag{9a}$$

with  $\gamma \succ^{\mathcal{A}} \delta$ ,  $d_i \succ^{\mathcal{A}} r_i$  and  $d_{-ij} \succ_{-ij}^{\mathcal{A}} r_{-ij}$ .

Using strict monotonicity, (8f) and  $\gamma \succ^{\mathcal{A}} \delta$  imply that

$$(\delta_i, b_j, b_{-ij}) \in \mathcal{U}. \quad (9b)$$

Similarly, (8g) and  $d_i \succ^{\mathcal{A}} r_i$  imply

$$(\beta_j, r_i, c_{-ij}) \in \mathcal{U}. \quad (9c)$$

Finally, (8d) and  $d_{-ij} \succ_{-ij}^{\mathcal{A}} r_{-ij}$  imply

$$(\beta_j, c_i, r_{-ij}) \in \mathcal{U}. \quad (9d)$$

From (9b), using connectedness, there are  $b' \in \mathcal{X}$  satisfying  $b'_j \succ^{\mathcal{A}} b_j$  and  $b'_{-ij} \succ_{-ij}^{\mathcal{A}} b_{-ij}$  such that

$$(\delta_i, b'_j, b'_{-ij}) \in \mathcal{U}. \quad (9e)$$

Starting with (9d), using connectedness, there is  $c' \in \mathcal{X}$  satisfying  $c'_i \succ^{\mathcal{A}} c_i$  such that

$$(\beta_j, c'_i, r_{-ij}) \in \mathcal{U}. \quad (9f)$$

From (8c) and  $c'_i \succ^{\mathcal{A}} c_i$ , we obtain

$$(\alpha_j, c'_i, c_{-ij}) \in \mathcal{A}. \quad (9g)$$

Using (8b) and  $b'_{-ij} \succ_{-ij}^{\mathcal{A}} b_{-ij}$ , we obtain

$$(\beta_i, a_j, b'_{-ij}) \in \mathcal{A}. \quad (9h)$$

Using (8e) and  $b'_j \succ^{\mathcal{A}} b_j$ , we obtain

$$(\beta_i, b'_j, a_{-ij}) \in \mathcal{A}. \quad (9i)$$

By connectedness, there is  $a' \in \mathcal{X}$  such that  $a_j \succ^{\mathcal{A}} a'_j$  such that, from (9h),

$$(\beta_i, a'_j, b'_{-ij}) \in \mathcal{A}. \quad (9j)$$

Using (8a) and  $a_j \succ^{\mathcal{A}} a'_j$ , we obtain

$$(\alpha_i, a'_j, a_{-ij}) \in \mathcal{U}. \quad (9k)$$

Combining (9g), (9a), (9j), and (9i), with (9k), (9e), (9f), and (9c) leads to a violation of A6 (Standard sequence invariance).  $\square$

## B.4 Existence of the SEU representation

We begin with a lemma stating that an SEU representation exists on  $\Delta$ .

### Lemma 13

Assume the conditions of Proposition 1. There is a continuous real-valued mapping  $u : \Gamma \rightarrow \mathbb{R}$  and  $n$  strictly positive real numbers  $p_1, p_2, \dots, p_n$  such that  $\sum_{i \in N} p_i = 1$  and, for all  $x \in \Delta$ ,

$$x \in \mathcal{A} \Leftrightarrow \sum_{i \in N} p_i u(x_i) > 0,$$

$$x \in \mathcal{U} \Leftrightarrow \sum_{i \in N} p_i u(x_i) < 0.$$

### PROOF

Suppose without loss of generality, in view of Lemma 11.7, that  $\Delta_k = \Gamma$ . We are going to prove that, for every  $i \neq k$ , the mapping  $v_i$  in Lemma 10 is an affine transformation of  $v_k$  restricted to  $\Delta_i$ .

Choose any  $i \in N$  ( $i \neq k$ ) and  $\mu \in \Delta_i$ . By Lemma 11.1, there is necessarily  $r \in \mathcal{X}$  such that  $(\mu_i, r_{-i}) \in \mathcal{F}$ . Choose now  $\nu \in \Delta_i$  such that  $\nu \succ^{\mathcal{A}} \mu$  and choose it “so close” to  $\mu$  (in terms of  $v_i$ ) that (i)  $(\nu_i, \lambda_k, r_{-ik}) \in \mathcal{F}$  for some  $\lambda \in \Gamma$  and (ii)  $(\nu_i, r_k, s_{-ik}) \in \mathcal{F}$  for some  $s \in \mathcal{X}$ .

Because  $\Delta_k = \Gamma$  and by Lemma 11.1, there is  $r' \in \mathcal{X}$  such that  $(\mu_k, r'_{-k}) \in \mathcal{F}$ . Choose now  $\nu' \in \Gamma$  such that  $\nu \succ^{\mathcal{A}} \nu' \succ^{\mathcal{A}} \mu$  and choose it “so close” to  $\mu$  (in terms of  $v_k$ ) that (i)  $(\nu'_k, \lambda'_i, r'_{-ik}) \in \mathcal{F}$  for some  $\lambda' \in \Gamma$  and (ii)  $(\nu'_k, r'_i, s'_{-ik}) \in \mathcal{F}$  for some  $s' \in \mathcal{X}$ .

So, the “distance” between  $\nu'$  and  $\mu$  is “small” w.r.t. states  $i$  and  $k$ . Hence, for all  $\alpha, \gamma \in [\mu, \nu'] = \{\delta \in \Gamma : \nu' \succ^{\mathcal{A}} \delta \succ^{\mathcal{A}} \mu\}$ , there is  $\beta \in [\mu, \nu']$ ,  $x, y, z, w \in \mathcal{X}$  such that

$$\begin{aligned} (\alpha_i, x_k, x_{-ik}) &\in \mathcal{F}, & (\alpha_k, z_i, z_{-ik}) &\in \mathcal{F}, \\ (\beta_i, y_k, x_{-ik}) &\in \mathcal{F}, & (\beta_j, w_i, z_{-ik}) &\in \mathcal{F}, \\ (\beta_i, x_k, y_{-ik}) &\in \mathcal{F}, & (\beta_j, z_i, w_{-ik}) &\in \mathcal{F}, \\ (\gamma_i, y_k, y_{-ik}) &\in \mathcal{F}. \end{aligned}$$

Indeed, choose  $\beta$  such that  $v_i(\alpha) - v_i(\beta) = v_i(\beta) - v_i(\gamma)$ . So,  $\beta$  also belongs to  $[\mu, \nu']$ . Because of the way we chose  $\mu$  and  $\nu'$ , there is clearly  $x, y \in \mathcal{X}$  such that  $(\alpha_i, x_k, x_{-ik}) \in \mathcal{F}$ ,  $(\beta_i, y_k, x_{-ik}) \in \mathcal{F}$  and  $(\beta_i, x_k, y_{-ik}) \in \mathcal{F}$ . Because  $v_i(\alpha) - v_i(\beta) = v_i(\beta) - v_i(\gamma)$ , we also know (using Lemma 10) that  $(\gamma_i, y_k, y_{-ik}) \in \mathcal{F}$ .

Because the distance between  $\alpha$  and  $\beta$  is also “small” on state  $k$ , there is also  $z, w \in \mathcal{X}$  such that  $(\alpha_k, z_i, z_{-ik}) \in \mathcal{F}$ ,  $(\beta_k, w_i, z_{-ik}) \in \mathcal{F}$  and  $(\beta_k, z_i, w_{-ik}) \in \mathcal{F}$ .

Using Lemma 12, we find  $(\gamma_k, w_i, w_{-ik}) \in \mathcal{F}$ . Hence,  $v_i(\alpha) - v_i(\beta) = v_i(\beta) - v_i(\gamma)$  implies  $v_k(\alpha) - v_k(\beta) = v_k(\beta) - v_k(\gamma)$ . Since  $v_k$  and  $v_i$  are two representations of the same weak order, we know that  $v_k = \phi_i(v_i)$  for some increasing

transformation  $\phi_i$ . Letting  $v_i(\alpha) = x_a, v_i(\beta) = x_b, v_i(\gamma) = x_c$ , we obtain

$$x_a - x_b = x_b - x_c \Rightarrow \phi_i(x_a) - \phi_i(x_b) = \phi_i(x_b) - \phi_i(x_c).$$

This can also be written as

$$\phi_i\left(\frac{x_a + x_c}{2}\right) = \frac{\phi_i(x_a) + \phi_i(x_c)}{2}.$$

This is Jensen's equality. It holds for all  $x_a, x_c \in [v_i(\mu), v_i(\nu')]$ . There are therefore some real numbers  $\tau_{i,\mu,\nu'}$  and  $\sigma_{i,\mu,\nu'} > 0$  such that  $\phi_i(a) = \sigma_{i,\mu,\nu'}x_a + \tau_{i,\mu,\nu'}$  for all  $x_a \in [v_i(\mu), v_i(\nu')]$  (Aczél, 1966, Ch. 2).

We can then replicate the same reasoning starting from any other value in  $\Delta_i$  instead of  $\mu$ . Hence,  $\phi_i$  is not only locally affine but also globally. In other words, there are  $\tau_i$  and  $\sigma_i > 0$  such that  $\phi_i(x_a) = \sigma_i x_a + \tau_i$  for all  $x_a \in v_i(\Delta_i)$ . Consequently, for all  $i \in N$ , we have  $v_k = \sigma_i v_i + \tau_i$ . Of course,  $\sigma_k = 1$  and  $\tau_k = 0$ .

Let us define  $u = v_k - \sum_{i \in N} \tau_i / n$  and  $p_i = (1/\sigma_i) / \sum_{j \in N} (1/\sigma_j)$ . Then  $\sum_{j \in N} p_j = 1$  and

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow \sum_{j \in N} p_j u(x_j) > 0, \\ x \in \mathcal{U} &\Leftrightarrow \sum_{j \in N} p_j u(x_j) < 0, \end{aligned}$$

for all  $x \in \Delta$ . □

The next lemma extends the representation to all  $\mathcal{X}$ , which will complete the proof of the existence part of Proposition 1.

**Lemma 14**

*Assume the conditions of Proposition 1. The mapping  $u$  and the numbers  $(p_i)_{i \in N}$  of Lemma 13 give a representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  in model (1).*

*The numbers  $p_i$  are unique. The function  $u$  is unique up to a multiplication by a strictly positive constant.*

**PROOF**

Consider an act  $x \in \mathcal{X} \setminus \Delta$ . There is necessarily  $i \in N$  such that  $x_i \notin \Delta_i$ . We then have (a)  $x_i \succ_i^{\mathcal{A}} \alpha$ , for all  $\alpha \in \Delta_i$ , or (b)  $\alpha \succ_i^{\mathcal{A}} x_i$ , for all  $\alpha \in \Delta_i$ .

We handle only (a) because (b) is symmetric. If  $x \in \mathcal{F}$ , then  $(\beta_i, x_{-i}) \in \mathcal{F}$ , for all  $\beta \in \Gamma$  such that  $\beta \succ_i^{\mathcal{A}} x_i$ . Otherwise we would have  $\beta \succ_i^{\mathcal{A}} x_i$  and, hence,  $x_i$  would belong to  $\Delta_i$ . This and A5 (r-1-Thinness) imply that  $\beta \sim_j^{\mathcal{A}} x_i$  for all  $j \in N$  and all  $\beta \in \Gamma$  such that  $\beta \succ_i^{\mathcal{A}} x_i$ . In other words,  $x_i \succ_i^{\mathcal{A}} \gamma$  for all  $\gamma \in \Gamma$ . This contradicts the fact that  $\Gamma$  is unbounded. Consequently,  $x \notin \mathcal{F}$  and, clearly,  $x \in \mathcal{A}$ . We must now prove that  $\sum_{j \in N} p_j u(x_j) > 0$ . Take any  $\beta \in \Gamma$  such that

$x_i \succ_i^{\mathcal{A}} \beta$ . If  $(\beta_i, x_{-i}) \in \mathcal{AF}$ , then, by Lemma 13,  $p_i u(\beta) + \sum_{j \neq i} p_j u(x_j) \geq 0$  and, since  $u(x_i) > u(\beta)$ ,  $\sum_{j \in N} p_j u(x_j) > 0$ . If  $(\beta_i, x_{-i}) \in \mathcal{U}$ , then, by Lemma 2, there is  $\gamma \in \Gamma$  such that  $(\gamma_i, x_{-i}) \in \mathcal{F}$ . By Lemma 13,  $p_i u(\gamma) + \sum_{j \neq i} p_j u(x_j) = 0$  and, since  $u(x_i) > u(\gamma)$ ,  $\sum_{j \in N} p_j u(x_j) > 0$ .

The mapping  $u$  inherits the uniqueness properties of the mappings  $v_k$ : it is unique up to a positive affine transformation. But, for any  $\beta \neq 0$ , it is clear that the mapping  $u' = u + \beta$  does not represent  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . Hence,  $u$  is actually unique up to a positive linear transformation. Besides, it is also clear that the numbers  $p_i$  are defined up to a multiplication by a positive constant. But, since  $\sum p_i = 1$ , the probabilities  $p_i$  must be unique.  $\square$

## C Proof of Theorem 1

If Assumption 2 holds, then we apply Proposition 1 and we are done. Otherwise, let  $b \in \Gamma$  and  $t \in \Gamma$  respectively denote one of the minimal and one of the maximal elements (when they exist) of the set  $\Gamma$  w.r.t.  $\succ^{\mathcal{A}}$ . It is clear that  $t$  (resp.  $b$ ) is also maximal (resp. minimal) w.r.t.  $\succ_i^{\mathcal{A}}$  for every  $i \in N$ .

We first consider the set  $\Lambda = \{\beta \in \Gamma : \alpha \succ^{\mathcal{A}} \beta \succ^{\mathcal{A}} \gamma, \text{ for some } \alpha, \gamma \in \Gamma\}$ . We can apply Proposition 1 to the restriction of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  to  $\Lambda^n$ . Hence, we can find a continuous representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  on  $\Lambda^n$  in model (1). We now extend this representation to the entire set  $\mathcal{X}$ .

Define  $u(b) = \inf\{u(\alpha) : \alpha \in \Lambda\}$  and  $u(t) = \sup\{u(\alpha) : \alpha \in \Lambda\}$ . This extension clearly preserves continuity and it is the only one to do so. The set  $u(\Gamma)$  will be an interval of  $\mathbb{R}$ .

Let  $x \in \mathcal{X} \setminus \Lambda^n$ . Suppose first that  $x \in \mathcal{A}$ . It is not possible that  $x_i = b$  for all  $i \in N$  (this would clearly violate the fact that  $\mathcal{U} \neq \emptyset$ ). Suppose that  $x_i = b$  for all  $i \in B \subsetneq N$ . For all  $i \notin B$ , we can choose  $\varepsilon > 0$  that is so small that, for all  $i \notin B$ , there is a  $y_i \in \Lambda$  such that  $u(y_i) = u(x_i) - \varepsilon/p_i$ . Consider now  $y \in \mathcal{X}$  such that:

1. for all  $i \notin B$ ,  $u(y_i) = u(x_i) - \varepsilon/p_i$ ,
2. for all  $i \in B$ ,  $u(y_i) = u(b)$ .

It is clear that we can always choose  $\varepsilon$  in such a way that  $y \in \mathcal{A}$ . Consider now the act  $y' \in \mathcal{X}$  that is built as follows:

1. for all  $i \notin B$ ,  $y'_i = y_i$ ,
2. for all  $i \in B$ ,  $y'_i$  is such that  $u(y'_i) = u(b) + \varepsilon/np_i$ .



We clearly have  $y' \in \mathcal{A}$  and, with  $\varepsilon$  adequately small,  $y' \in \Lambda^n$ . We therefore know that  $\sum_{i \in N} p_i u(y'_i) > 0$ . Hence, we have  $\sum_{i \in N} p_i u(x_i) = \sum_{i \notin B} p_i (u(y'_i) + \varepsilon/p_i) + \sum_{i \in B} p_i (u(y'_i) - \varepsilon/n p_i) = \sum_{i \in N} u(y'_i) + (n - n_B)\varepsilon - n_B \varepsilon/n$ , where  $n_B = |B|$ . We know that  $\sum_{i \in N} u(y'_i) > 0$ . By hypothesis, we know that  $1 \leq n_B < n$ . This implies that  $(n - n_B)\varepsilon - n_B \varepsilon/n > 0$ . Hence, we obtain that  $\sum_{i \in N} p_i u(x_i) > 0$ , as required. A similar argument applies if  $x \in \mathcal{U}$ .

Finally, suppose that  $x \in \mathcal{F}$ . It is impossible that  $x_i = b$  for all  $i \in N$  since this would imply  $\mathcal{U} = \emptyset$ . Similarly, it is impossible that  $x_i = t$  for all  $i \in N$  since this would imply  $\mathcal{A} = \emptyset$ . Hence, there are  $i, j \in N$  such that  $t \succ^{\mathcal{A}} x_i$  and  $x_j \succ^{\mathcal{A}} b$ . For all  $z = (x_{-j}, y_j)$  with  $x_j \succ^{\mathcal{A}} y_j$ , we have  $z \in \mathcal{U}$  and we have just seen that this implies  $\sum_{k \in N} p_k u(z_k) < 0$ . Using the continuity of  $u$ , we must have  $\sum_{k \in N} p_k u(x_k) \leq 0$ . Similarly, for all  $w = (x_{-i}, y_i)$  with  $y_i \succ^{\mathcal{A}} x_i$ , we have  $y \in \mathcal{A}$  and we have just seen that this implies  $\sum_{k \in N} p_k u(w_k) > 0$ . Using the continuity of  $u$ , we must have  $\sum_{k \in N} p_k u(x_k) \geq 0$ . Hence, we must have  $\sum_{k \in N} p_k u(x_k) = 0$ .

The uniqueness part follows from the uniqueness part of Proposition 1 combined with the uniqueness of the continuity preserving extension used above.  $\square$

## D Extension: no frontier

### D.1 Model

The conditions used in Theorem 1 refer to category  $\mathcal{F}$ . Because this category is interpreted as a frontier between attractive and unattractive alternatives, conditions involving  $\mathcal{F}$  may be seen as less intuitive and involving more difficult judgments than conditions involving only  $\mathcal{A}$  and  $\mathcal{U}$ . We show here that it is possible to derive SEU without explicitly referring to the category  $\mathcal{F}$ .

We use a twofold partition  $\langle \mathcal{A}, \mathcal{U} \rangle$  of the set  $\mathcal{X}$  of all acts. As above, the set  $\mathcal{A}$  is supposed to contain all attractive alternatives. On the contrary, the set  $\mathcal{U}$  will contain here all alternatives that are not felt to be attractive, without distinguishing between “neutral” and “unattractive alternatives”.

The desired representation will be such that, for all  $a \in \mathcal{X}$ ,

$$a \in \left\{ \begin{array}{c} \mathcal{A} \\ \mathcal{U} \end{array} \right\} \Leftrightarrow \sum_{i=1}^n p_i u(a_i) \left\{ \begin{array}{c} > \\ \leq \end{array} \right\} 0. \quad (10)$$

We say that  $\langle \mathcal{A}, \mathcal{U} \rangle$  is *non-degenerate* if both  $\mathcal{A}$  and  $\mathcal{U}$  are nonempty. We say that a state  $i \in N$  is *influential* for  $\langle \mathcal{A}, \mathcal{U} \rangle$  if there are  $\alpha, \beta \in \Gamma$  and  $a \in \mathcal{X}$  such that  $(\alpha_i, a) \in \mathcal{A}$  and  $(\beta_i, a) \in \mathcal{U}$ .

We suppose throughout the rest of this section that the following assumption, merely restating Assumption 1 in our new setting, holds.

**Assumption 3**

There are at least three states. All states are influent. The ordered partition  $\langle \mathcal{A}, \mathcal{U} \rangle$  is non-degenerate.

**D.2 Axioms**

We will use the following conditions.

**B1** Inter-State 1-Linearity w.r.t.  $\mathcal{A}$  or r-1-Linearity- $\mathcal{A}$  for short

For all  $i, j \in N$ , all  $\alpha, \beta \in \Gamma$  and all  $a, b \in \mathcal{X}$ ,

$$\left. \begin{array}{l} (\alpha_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (\beta_j, b_{-j}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, a_{-i}) \in \mathcal{A} \\ \text{or} \\ (\alpha_j, b_{-j}) \in \mathcal{A}. \end{array} \right.$$

This condition is identical to condition A1 in our new setting. Clearly, Lemma 1 holds here. Hence, under B1 (r-1-Linearity- $\mathcal{A}$ ), the relation  $\succ^{\mathcal{A}}$  is a weak order. Hence, all relations  $\succ_i^{\mathcal{A}}$  are weak orders and are never contradictory.

**B2** Connectedness in the order topology

When  $\succ^{\mathcal{A}}$  is a weak order, the set  $\Gamma$  is connected in the order topology generated by  $\succ^{\mathcal{A}}$ .

This condition is identical to condition A2 (Connectedness in the order topology) in our new setting.

**B3** Openness

The set  $\mathcal{A}$  is open in the product topology on  $\mathcal{X}$  and  $\text{Cl}(\mathcal{A}) \neq \mathcal{X}$ .

This condition adapts condition A3 (Openness) to our new setting. It clearly implies that the set  $\mathcal{U}$  is closed in the product topology on  $\mathcal{X}$ . The last part of the condition forbids to have the degenerate situation in which  $\mathcal{U} = \text{Cl}(\mathcal{A}) \setminus \mathcal{A}$ .

**B4** Standard sequence invariance

For all  $i, j \in N$ , all  $\alpha, \beta, \gamma \in \Delta_i \cap \Delta_j$ , all  $a, b, c, d \in \mathcal{X}$ ,

$$\left. \begin{array}{l} (\alpha_j, a_i, a_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\gamma_j, b_i, b_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\beta_i, c_j, d_{-ij}) \in \mathcal{A} \quad \text{and} \\ (\beta_i, d_j, c_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\alpha_i, c_j, c_{-ij}) \in \mathcal{A} \quad \text{or} \\ (\gamma_i, d_j, d_{-ij}) \in \mathcal{A} \quad \text{or} \\ (\beta_j, a_i, b_{-ij}) \in \mathcal{A} \quad \text{or} \\ (\beta_j, b_i, a_{-ij}) \in \mathcal{A}. \end{array} \right.$$

This condition adapts condition A6 (Standard sequence invariance) by replacing the set  $\mathcal{AF}$  by the set  $\mathcal{A}$ . It is the analogue of A'6 in our new setting. It is clearly stronger than A6. It is nevertheless necessary for model (10) (as well as for model (1)). It will be useful to note the following.

**Lemma 15**

*If a twofold partition satisfies B4 (Standard sequence invariance) then it satisfies B1 (r-1-Linearity- $\mathcal{A}$ ).*

**PROOF**

Take, in the expression of B4,  $\gamma = \alpha$ ,  $a_i = b_i$ ,  $a_{-ij} = b_{-ij}$ ,  $c_j = d_j$  and  $c_{-ij} = d_{-ij}$ . B4 implies that

$$\left. \begin{array}{l} (\alpha_j, a_i, a_{-ij}) \in \mathcal{A} \text{ and} \\ (\alpha_j, a_i, a_{-ij}) \in \mathcal{A} \text{ and} \\ (\beta_i, c_j, c_{-ij}) \in \mathcal{A} \text{ and} \\ (\beta_i, c_j, c_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\alpha_i, c_j, c_{-ij}) \in \mathcal{A} \text{ or} \\ (\alpha_i, c_j, c_{-ij}) \in \mathcal{A} \text{ or} \\ (\beta_j, a_i, a_{-ij}) \in \mathcal{A} \text{ or} \\ (\beta_j, a_i, a_{-ij}) \in \mathcal{A}, \end{array} \right.$$

which is B1. □

**B5 Intra-State 2-Linearity w.r.t.  $\mathcal{A}$  or a-2-Linearity- $\mathcal{A}$  for short**

*For all  $i, j \in N$ , all  $\alpha, \beta, \gamma, \delta \in \Gamma$  and all  $a, b \in \mathcal{X}$*

$$\left. \begin{array}{l} (\alpha_i, \beta_j, a_{-ij}) \in \mathcal{A} \\ \text{and} \\ (\gamma_i, \delta_j, b_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\gamma_i, \delta_j, a_{-ij}) \in \mathcal{A} \\ \text{or} \\ (\alpha_i, \beta_j, b_{-ij}) \in \mathcal{A}. \end{array} \right.$$

This condition is identical to condition A7 in our new setting.

**B6 Double Cancellation**

*For all  $i, j \in N$ , all  $\alpha, \beta, \gamma \in \Gamma$  and all  $a, b \in \mathcal{X}$ ,*

$$\left. \begin{array}{l} (\alpha_i, \beta_j, a_{-ij}) \in \mathcal{A} \text{ and} \\ (\beta_i, \gamma_j, b_{-ij}) \in \mathcal{A} \text{ and} \\ (\gamma_i, \alpha_j, c_{-ij}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, \alpha_j, a_{-ij}) \in \mathcal{A} \text{ or} \\ (\gamma_i, \beta_j, b_{-ij}) \in \mathcal{A} \text{ or} \\ (\alpha_i, \gamma_j, c_{-ij}) \in \mathcal{A}. \end{array} \right.$$

This condition adapts condition A8 (Double Cancellation) by replacing the set  $\mathcal{AF}$  by the set  $\mathcal{A}$ . It is the analogue of A'8 in our new setting. It is clearly stronger than A8. It is nevertheless necessary for model (10) (as well as for model (1)).

### D.3 Result

Our main result for the case without frontier is as follows.

#### Theorem 2

Consider a twofold ordered partition  $\langle \mathcal{A}, \mathcal{U} \rangle$  of  $\mathcal{X}$  such that Assumption 3 holds. Suppose that  $\langle \mathcal{A}, \mathcal{U} \rangle$  satisfies  $B_4$  (Standard sequence invariance),  $B_2$  (Connectedness in the order topology), and  $B_3$  (Openness). If  $n \geq 4$  suppose that  $\langle \mathcal{A}, \mathcal{U} \rangle$  satisfies  $B_5$  (a-2-Linearity- $\mathcal{A}$ ). If  $n = 3$  suppose that  $\langle \mathcal{A}, \mathcal{U} \rangle$  satisfies  $B_6$  (Double Cancellation). Then there are a continuous real-valued function  $u$  on  $\Gamma$  and  $n$  strictly positive numbers  $p_1, p_2, \dots, p_n$  adding up to 1 such that (10) holds.

The numbers  $p_1, p_2, \dots, p_n$  are unique. The function  $u$  is unique up to a multiplication by a strictly positive constant.

Compared to Theorem 1, the above result completely dispenses with the frontier and uses less conditions.

The proof of Theorem 2 consists in showing that the induced threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$ , in which the set  $\mathcal{U} \cap \text{Cl}(\mathcal{A})$  plays the role of category  $\mathcal{F}$ , satisfies the conditions of Theorem 1.

### D.4 Proof in the unbounded case

We first use the following additional assumption that is the analogue of Assumption 2 in our new setting.

#### Assumption 4

For all  $\gamma \in \Gamma$ , there are  $\alpha, \beta \in \Gamma$  such that  $\alpha \succ^{\mathcal{A}} \gamma \succ^{\mathcal{A}} \beta$ .

We will prove the following.

#### Proposition 2

Consider a twofold ordered partition  $\langle \mathcal{A}, \mathcal{U} \rangle$  of  $\mathcal{X}$  such that Assumptions 3 and 4 hold. Suppose that  $\langle \mathcal{A}, \mathcal{U} \rangle$  satisfies  $B_4$  (Standard sequence invariance),  $B_2$  (Connectedness in the order topology), and  $B_3$  (Openness). If  $n \geq 4$  suppose that  $\langle \mathcal{A}, \mathcal{U} \rangle$  satisfies  $B_5$  (a-2-Linearity- $\mathcal{A}$ ). If  $n = 3$  suppose that  $\langle \mathcal{A}, \mathcal{U} \rangle$  satisfies  $B_6$  (Double Cancellation). Then there are a continuous real-valued function  $u$  on  $\Gamma$  and  $n$  strictly positive real numbers  $p_1, p_2, \dots, p_n$  adding up to 1 such that (10) holds.

The numbers  $p_1, p_2, \dots, p_n$  are unique. The function  $u$  is unique up to a multiplication by a strictly positive constant.

The proof goes through a number of lemmas. Remember first from Lemma 15 that  $B_1$  (r-1-Linearity- $\mathcal{A}$ ) holds.

#### Lemma 16

The induced threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies  $A_3$  (Openness).

PROOF

By hypothesis the set  $\mathcal{A}$  is open. The set  $\text{Cl}(\mathcal{A})$  is closed by construction. Hence the set  $\mathcal{X} \setminus \text{Cl}(\mathcal{A}) = \mathcal{U} \setminus \text{Cl}(\mathcal{A})$  is open.  $\square$

**Lemma 17**

Let  $i \in N$ ,  $\alpha, \beta \in \Gamma$  and  $a, b \in \mathcal{X}$ . Under the conditions of Proposition 2, the threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  is such that, for all  $(\alpha_i, a_{-i}) \in \mathcal{A}$  and  $(\beta_i, a_{-i}) \in \mathcal{U} \setminus \text{Cl}(\mathcal{A})$ , then  $(\gamma_i, a_{-i}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$ , for some  $\gamma \in \Gamma$ .

PROOF

For all  $i \in N$  and all  $a \in \mathcal{X}$ , define  $U(a_{-i}) = \{\beta \in \Gamma : (\beta_i, a_{-i}) \in \mathcal{U}\}$ ,  $A(a_{-i}) = \{\alpha \in \Gamma : (\alpha_i, a_{-i}) \in \mathcal{A}\}$  and  $\bar{A}(a_{-i}) = \{\alpha \in \Gamma : (\alpha_i, a_{-i}) \in \text{Cl}(\mathcal{A})\}$ . By construction,  $U(a_{-i}) \cap A(a_{-i}) = \emptyset$  and  $U(a_{-i}) \cup A(a_{-i}) = \Gamma$ .

Suppose that  $(x_i, a_{-i}) \in \mathcal{U}$  and  $(y_i, a_{-i}) \in \mathcal{A}$ . This implies  $U(a_{-i}) \neq \emptyset$ ,  $A(a_{-i}) \neq \emptyset$ . Using B3 (Openness), it is clear that  $A(a_{-i})$  is open so that, using B2 (Connectedness in the order topology),  $U(a_{-i})$  is closed. The set  $\bar{A}(a_{-i}) \neq \emptyset$  is clearly closed. If  $\bar{A}(a_{-i}) \cap U(a_{-i}) = \emptyset$ , we have build a partition of  $\Gamma$  into two closed sets, violating B2 (Connectedness in the order topology). Hence we must have  $\bar{A}(a_{-i}) \cap U(a_{-i}) \neq \emptyset$ . Taking any  $\gamma \in \bar{A}(a_{-i}) \cap U(a_{-i})$ , we obtain  $(\gamma_i, a_{-i}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$ .  $\square$

**Lemma 18**

Under the conditions of Proposition 2, the threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies condition A5 (*r-1-Thinness*).

PROOF

Suppose  $(\alpha_i, a_{-i}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$  and  $(\beta_i, a_{-i}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$ . The set  $U(a_{-i}) = \{\beta \in \Gamma : (\beta_i, a_{-i}) \in \mathcal{U}\}$  is nonempty. We have shown above that it is closed. Since  $\beta \succ^A \gamma$  implies  $\beta \succ_i^A \gamma$ , it is clear that  $\beta \succ^A \gamma$  implies  $\gamma \in U(a_{-i})$ . Hence the set  $U(a_{-i})$  is closed and unbounded below. Hence, there is  $\delta(a_{-i}) \in \Gamma$  such that,  $U(a_{-i}) = \{\beta \in \Gamma : \delta(a_{-i}) \lesssim^A \beta\}$ . This implies that we have  $A(a_{-i}) = \{\alpha \in \Gamma : \alpha \succ^A \delta(a_{-i})\}$ , so that  $\bar{A}(a_{-i}) = \{\alpha \in \Gamma : \alpha \lesssim^A \delta(a_{-i})\}$ . Therefore,  $(\alpha, a_{-i}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$  and  $(\beta, a_{-i}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$  imply  $\alpha \sim^A \beta$ . This implies that  $\alpha \sim_j^A \beta$ , for all  $j \in N$ . Hence, for all  $j \in N$  and all  $b \in \mathcal{X}$ , we have  $(\alpha_j, b_{-j}) \in \mathcal{A}$  iff  $(\beta_j, b_{-j}) \in \mathcal{A}$ , for some  $b \in \mathcal{X}$ .

Suppose now that  $(\alpha_j, b_{-j}) \in \mathcal{U} \cap \text{Cl}(\mathcal{A})$  and  $(\beta_j, b_{-j}) \notin \mathcal{U} \cap \text{Cl}(\mathcal{A})$ . This implies that  $\bar{A}(b_{-j}) \neq \emptyset$  and  $U(b_{-j}) \neq \emptyset$ . Using the same reasoning as above, we must have  $\alpha \lesssim^A \delta(b_{-j})$ . Because  $\alpha \sim^A \beta$ , we obtain  $\beta \lesssim^A \delta(b_{-j})$ , so that  $(\beta_j, b_{-j}) \in \text{Cl}(\mathcal{A})$ , a contradiction.  $\square$

**Lemma 19**

Under the conditions of Proposition 2, the threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies condition A4.

PROOF

Suppose that we have

$$\begin{aligned} (\alpha_i, a_{-i}) &\in \mathcal{U} \cap \text{Cl}(\mathcal{A}), & (\beta_i, a_{-i}) &\in \mathcal{U} \setminus \text{Cl}(\mathcal{A}), \\ (\beta_j, b_{-j}) &\in \mathcal{U} \cap \text{Cl}(\mathcal{A}), & (\alpha_j, b_{-j}) &\in \mathcal{U} \setminus \text{Cl}(\mathcal{A}). \end{aligned}$$

The first line implies that  $\overline{A}(a_{-i}) \neq \emptyset$  and  $U(a_{-i}) \neq \emptyset$ . Using the same reasoning as above, this implies that  $\alpha \succsim^{\mathcal{A}} \delta(a_{-i})$  and  $\delta(a_{-i}) \succ^{\mathcal{A}} \beta$ . Similarly, the second line implies  $\overline{A}(b_{-j}) \neq \emptyset$  and  $U(b_{-j}) \neq \emptyset$ , so that  $\beta \succsim^{\mathcal{A}} \delta(b_{-j})$  and  $\delta(b_{-j}) \succ^{\mathcal{A}} \alpha$ , a contradiction.  $\square$

It is easy to check that with the above lemmas at hand, we can use Lemmas 3, 4, 5 and 6 on  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$ .

**Lemma 20**

*Under the conditions of Proposition 2, the threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies condition A8 (Double Cancellation).*

PROOF

Obvious since B6 implies A8.  $\square$

**Lemma 21**

*Under the conditions of Proposition 2, the threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies condition A6 (Standard sequence invariance).*

PROOF

Obvious since B4 implies A6.  $\square$

PROOF OF PROPOSITION 2

Under the conditions of Proposition 2, it is clear that the induced threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  will satisfy Assumptions 1 and 2. Furthermore, conditions A1, A2, A3, A6, If  $n \geq 4$ , it is clear that  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies A7. If  $n = 3$ , Lemma 20 shows that A8 holds.

Using Lemmas 18 and 19, we know that  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$  satisfies A5 and A4.

Hence, we may apply Proposition 1 to the induced threefold partition  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$ . This clearly gives a representation of  $\langle \mathcal{A}, \mathcal{U} \rangle$  in model (10). The uniqueness properties follow from those of Proposition 1.  $\square$

## D.5 Proof of Theorem 2

We must now extend Proposition 2 to cope with potential violations of Assumption 3. We proceed exactly as we did above to go from Proposition 1 to Theorem 1. We omit details since the proof consists in rephrasing the above proof with obvious changes from  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  to  $\langle \mathcal{A}, \mathcal{U} \cap \text{Cl}(\mathcal{A}), \mathcal{U} \setminus \text{Cl}(\mathcal{A}) \rangle$ .  $\square$

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