# The measurement of membership by subjective ratio estimation

T. Marchant,

*Ghent University, Dunantlaan 1, 9000 Ghent, Belgium*

**Abstract**

When we perform algebraic operations on membership degrees, we must be sure that they are compatible with the kind of scale on which these degrees are measured. For example, we will not add (in a weighted average) membership degrees that are measured on an ordinal scale. An important question is therefore how to determine on which kind of scale we measure the membership.

There are several techniques for measuring membership. In this paper, we present some of them, based on Stevens' technique of ratio estimation, and we characterize them, thereby providing a sound way to determine the level of measurement. We also discuss the problem of the representation of the union or intersection by a t-conorm or t-norm.

*Key words:* Membership, measurement

## 1 Introduction

In statistics, it is well-known that the arithmetic mean or the variance of a random variable $U$ in a sample should be used only if $U$ is measured at least on an interval scale (interval, ratio or absolute). Similarly, if we want to use the variation coefficient ($s_u/\overline{u}$), $U$ must be measured at least on a ratio scale (ratio or absolute). Of course, this also applies to any kind of number on which we want to perform an algebraic operation and, in particular, to membership degrees. If we want to perform any algebraic operation on membership degrees, we need be sure that the membership degrees are on the right type of scale. But how do we know on what scale our membership degrees are measured? If

*Email address:* `thierry.marchant@UGent.be` (T. Marchant).

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we just ask an expert "What are the membership degrees of $x$, $y$ and $z$ in the fuzzy set $A$?" and he answers "0.8, 0.4 and 0.2," we cannot say anything about the scale. We do not know how to interpret these numbers. Does 0.8 indicate a membership twice as large as 0.4? Or does 0.8 just indicate a membership larger than 0.4? The ratio 0.8/0.4 is equal to 0.4/0.2. Is this a coincidence or does it mean that the ratio (not necessarily 2) between the membership of $x$ and that of $y$ is the same as the ratio between the membership of $y$ and that of $z$?

If we want to be able to say something about the properties of membership degrees, we must ask questions to the expert about the relations between the membership of different elements. For example:

(1) Is the ratio between the membership of $x$ and that of $y$ in $A$ larger than or equal to the ratio between the membership of $y$ and that of $z$ in $A$?
(2) Is the difference between the membership of $x$ and that of $y$ in $A$ larger than or equal to the difference between the membership of $y$ and that of $z$ in $A$?
(3) Is the membership of $x$ in $A \cap B$ larger than or equal to that of $x$ in $A$?
(4) What is the ratio between the membership of $x$ and that of $y$ in $A$?

If we ask many such questions, it is then sometimes possible to use some measurement techniques in order to arrive at membership degrees with some particular properties and, eventually, on a particular type of scale. Measurement theory is the theory that analyzes these measurement techniques (Krantz et al., 1971; Roberts, 1979). A few papers have already looked at the measurement problems arising when measuring the membership of some objects in fuzzy sets and at some techniques that can be used in that context (Norwich and Türksen, 1982; French, 1984, 1987; Türksen, 1991; Bollmann-Sdorra et al., 1993; Bilgiç and Türksen, 1995; Bilgiç, 1996; Bilgiç and Türksen, 1997, 2000). All these papers consider techniques based on questions involving comparisons (like items 1–3 in the preceding enumeration). Their results are summarized, analyzed and extended in Marchant (2002).

In the present paper, we follow a different route and consider measurement techniques based on answers to questions like item 4 in the preceding enumeration (for a similar approach in different domains, see Luce, 2002; Narens, 2002). We ask an expert his perception of the ratio between the membership of two elements $x$ and $y$ in a given fuzzy set. This ratio is of course subjective: it reflects *his* knowledge, *his* experience. We will then try to construct membership degrees that reflect in some sense (to be defined) his knowledge and, in particular, the answers to our questions. Because the way the membership degrees must reflect the answers of the expert is not defined a priori, we have a lot of freedom. For example, if an expert says that the ratio between the membership of $x$ and that of $y$ in $A$ is 2, we are not forced to choose

$\mu_A(x) = 2\mu_A(y)$. Before going further, it is important to make a clear distinction between, on the one hand, the ratio between the membership of $x$ and $y$, as perceived by the expert, and, on the other hand, the ratio between the membership *degrees* of $x$ and $y$, constructed so as to reflect the ratios given by the expert.

In measurement theory, the elements on which a measurement technique is based, are called primitives. In this paper, the primitives are

- $\mathcal{F} = \{A, B, C, \ldots\}$, a set whose elements can be interpreted as all fuzzy sets that are relevant in a particular context,
- $X = \{x, y, \ldots\}$ of any cardinality, the universal set,
- $X_A$, the subset of all elements of $X$ that belong at least to some extent to $A$ and
- a mapping $\rho_A : X \times X \mapsto \mathbb{R}^+ \cup \{\infty\} : (x, y) \mapsto \rho_A(x|y)$, defined for all $A$ in $\mathcal{F}$ and all $x$, $y$ in $X$. If $\rho_A(x|y) = a$, we interpret this as "the ratio between the membership of $x$ in $A$ and the membership of $y$ in $A$ is equal to $a$." The numbers $\rho_A(x|y)$ are called *subjective ratios*. The expert is told that $\rho_A(x|x) = 1$. The subjective ratio $\rho_A(x|y)$ is conventionally set at $\infty$ when $y$ does not belong at all to $A$ and $x$ belongs to some extent to $A$. In that case, we also fix $\rho_A(y|x) = 0$. The values 0 and $\infty$ are used only in those two cases.

The primitives are empirically observable. They are not explained nor defined by the theory. In particular, the fuzzy set $A$ has no mathematical structure or property. It is just an expression in ordinary language (e.g. 'Old') that can be seen as a fuzzy set.

In the rest of the paper, we will present different measurement techniques, study the conditions (often called axioms) under which they can be used and analyze the uniqueness of the representations obtained by means of these techniques.

In the next two sections, we present a simple and intuitive measurement technique, where only the ordering of the subjective ratios is reflected by the membership degrees. The conditions characterizing this technique are so weak that it can probably be used in many instances but it leads to weak representations. We then turn (Section 4) to another very intuitive but much more demanding technique: we construct membership degrees whose ratios are equal to the subjective ratios given by the expert. Contrary to the previous sections, the conditions are here very strong: probably too strong. In the rest of the paper we then analyze some measurement techniques lying between these two extremes. Section 5 presents a classical measurement technique called difference measurement. The next section builds on Section 5 and analyzes the consequence of an invariance condition. In Sections 5 and 6, we use two solvability conditions that can look unattractive. Sections 7 shows that these two conditions are extremely natural if $X$ is a continuum, as in many applications. The last two sections are devoted to the extension of the previous results to the measurement of the membership of several objects in several fuzzy sets, some of them being eventually equal to the union or intersection of some others.

## 2 Ordinal measurement, with an arbitrary reference

In this section, we choose an arbitrary reference $z$ and we construct a measure $\mu_A$ on $X$ that reflects the ordering given by $\rho_A(\cdot|z)$.

**Theorem 1** *Given the structure $\langle X, A, \rho_A \rangle$ and $z$ in $X$, there exists $\mu_A : X \mapsto [0,1]$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall x, y \in X. \tag{1}$$

*Let $\mu'_A$ be a strictly increasing transformation of $\mu_A$. Then $\mu'_A$ also satisfies (1). Furthermore, if $\mu''_A$ also satisfies (1), then it is a strictly increasing transformation of $\mu_A$.*

**Proof.** Define

$$\mu_A = \frac{\rho_A(\cdot|z)}{\rho_A(\cdot|z) + 1}.$$

It satisfies (1). The rest of the proof is easy. □

Note that $\mu_A$ depends on the choice of the reference $z$.

In most applications, the membership degrees are supposed to be between 0 and 1 and it is assumed that the two extreme values are observed for some objects. In the next result, we analyze the consequences of such an assumption.

**Theorem 2** *Given the structure $\langle X, A, \rho_A \rangle$ and $z$ in $X$, there are $u$ and $l$ such that $\rho_A(u|z) \geq \rho_A(w|z) \geq \rho_A(l|z)$ for all $w$ in $X$ if and only if there exists $\mu_A : X \mapsto [0,1]$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall x, y \in X \tag{2}$$

*and such that $\mu_A(x) = 1$ for some $x$ and $\mu_A(y) = 0$ for some $y$. Let $\mu'_A$ be a strictly increasing transformation of $\mu_A$. Then $\mu'_A$ also satisfies (2). Furthermore, if $\mu''_A$ also satisfies (2), then it is a strictly increasing transformation of $\mu_A$.*

Obviously, $\mu'_A$ and $\mu''_A$ are not necessarily in $[0,1]$. Note that $\mu_A(x) = 1$ does not need to be interpreted as "$x$ belongs fully to $A$." It just means that, among all elements of $X$, $x$ has the highest membership in $A$.

## 3 Ordinal measurement, without reference

In this section, we look for a representation $\mu_A$ that is independent of the choice of the reference, as it is (implicitely) assumed in most applications. This is not always possible and the following condition characterizes the cases where such a representation exists.

**A 1** Reference Independence. $\rho_A(x|z) \geq \rho_A(y|z) \Leftrightarrow \rho_A(x|w) \geq \rho_A(y|w)$, *for all $x$ and $y$ in $X$ and all $z$ and $w$ in $X_A$ .*

**Theorem 3** *The structure $\langle X, A, X_A, \rho_A \rangle$ satisfies Reference Independence (A1) if and only if there exists $\mu_A : X \mapsto [0,1]$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall x, y, z \in X, \forall z \in X_A. \tag{3}$$

*Let $\mu'_A$ be a strictly increasing transformation of $\mu_A$. Then $\mu'_A$ also satisfies (3). Furthermore, if $\mu''_A$ also satisfies (3), then it is a strictly increasing transformation of $\mu_A$.*

**Proof.** The necessity of Reference Independence is obvious. Suppose now that Reference Independence holds. Let us choose $w$ in $X_A$ and define

$$\mu_A = \frac{\rho_A(\cdot|w)}{\rho_A(\cdot|w) + 1}.$$

We will prove that $\mu_A$ satisfies (3). We have

$$\begin{aligned} \mu_A(x) \geq \mu_A(y) &\Leftrightarrow \frac{\rho_A(x|w)}{\rho_A(x|w)+1} \geq \frac{\rho_A(y|w)}{\rho_A(y|w)+1} \quad \text{(by construction)} \\ &\Leftrightarrow \rho_A(x|w) \geq \rho_A(y|w) \\ &\Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall z \in X_A \quad \text{(by Ref. Ind.).} \end{aligned}$$ □

In Sections 2 and 3, we have taken into account only the ordering of the subjective ratios. The representation reflects that ordering and nothing else. We may then consider that the membership degrees are on ordinal scales. But they are cases where we might wish to reflect more than the ordering. For example, if an expert says that $\rho_A(x|y) = 1.1$ and $\rho_A(x|z) = 10$, we know that the ordering of the memberships is $x$ then $y$ then $z$ (from high to low membership) but we may also consider that the huge difference between 1.1 and 10 means something. We may then try to construct membership degrees (a mapping $\mu_A$) that not only reflects the ordering but also the relations (for example the ratios) between the subjective ratios. This is what we do in the next section.

## 4 A multiplicative representation

In this section, we look for a representation $\mu_A$ such that ratios of $\mu_A$ have a interpretation. In particular, $\mu_A(x)/\mu_A(y) = \rho_A(x|y)$, i.e. the ratio of the membership degrees is precisely equal to the subjective ratio given by the expert. In order to characterize it, we need the following condition.

**A 2** Multiplicative Property. $\rho_A(x|z) = \rho_A(x|y)\rho_A(y|z)\ \forall x, y, z \in X$.

**Theorem 4** *The structure $\langle X, A, X_A, \rho_A\rangle$ satisfies Reference Independence (A1) and the Multiplicative Property (A2) if and only if there exists $\mu_A : X \mapsto [0,1]$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall x, y \in X, \forall z \in X_A \tag{4}$$

*and*

$$\mu_A(x)/\mu_A(y) = \rho_A(x|y) \quad \forall x, y \in X. \tag{5}$$

*Let $\mu'_A = p\mu_A$ with $p$ a positive real number. Then $\mu'_A$ also satisfies (4) and (5). Furthermore, if $\mu''_A$ also satisfies (4) and (5), then there is a positive real number $q$ such that $\mu''_A = q\mu_A$.*

Because of the strong uniqueness result (the only possible transformation of the representation is a multiplication by a constant), we can consider that the membership degrees obtained by means of the technique characterized in this theorem are on a ratio scale.

**Proof.** Reference Independence is obviously necessary for the desired representation. We now show the necessity of the Multiplicative Property. For all $x, y$ and $z$, we have

$$\rho_A(x|z) = \frac{\mu_A(x)}{\mu_A(z)} = \frac{\mu_A(x)}{\mu_A(y)}\frac{\mu_A(y)}{\mu_A(z)} = \rho_A(x|y)\rho_A(y|z).$$

We now turn to the sufficiency of the conditions. Choose arbitrarily an element $w$ in $X_A$ and define

$$\mu_A = \frac{\rho_A(\cdot|w)}{\kappa},$$

where $\kappa = max_{y\in X}\{\rho_A(y|w)\}$. Because of Reference Independence, the representation $\mu_A$ obviously satisfies (4) and is in $[0,1]$. We now prove that it also satisfies (5).

$$\begin{aligned} \mu_A(x) &= \rho_A(x|w)/\kappa \text{ (by definition)} \\ &= \rho_A(x|y)\rho_A(y|w)/\kappa \text{ (by the Mult. Prop.)} \\ &= \rho_A(x|y)\mu_A(y) \ \text{ (by definition).} \end{aligned}$$

Let us discuss now the uniqueness. Clearly, if $\mu_A$ satisfies (4) and (5), then $p\mu_A$ also satisfies (4) and (5). Suppose now that $\mu'_A$ satisfies (4) and (5). Because $\mu_A(x) = \rho_A(x|y)\mu_A(y)$ and $\mu'_A(x) = \rho_A(x|y)\mu'_A(y)$ for any $y$, we have

$$\frac{\mu_A(x)}{\mu'_A(x)} = \frac{\mu_A(y)}{\mu'_A(y)}.$$

So, $\mu'_A = q\mu_A$ for some positive and real $q$. □

The multiplicative property is a very strong one. Even though we have no experimental data on this, we conjecture that an expert will seldom give subjective ratios satisfying this condition. We will therefore try to use a less restrictive model.

## 5 Difference Measurement

In this section, we look for a representation $\mu_A$ such that ratios of $\mu_A$ have an interpretation, but weaker than in the multiplicative representation. The basic idea in this section is to construct (standard) sequences of elements that are equally spaced, i.e. such that two consecutive elements are always in the same subjective ratio. This sequence will then be used as a yard stick to measure the membership. Suppose that $x$ perfectly belongs to $A$. If $x$ is the first element of a standard sequence, then, by making the sequence as long as we want, we can "reach" any element of $X_A$ provided the following conditions are satisfied.

**A 3** Reversal. *For all $w, x, y, z$ in $X_A$,*

$$\rho_A(x|y) \geq \rho_A(w|z) \Leftrightarrow \rho_A(y|x) \leq \rho_A(z|w).$$

**A 4** Weak Monotonicity. *For all $x, x', y, y', z, z'$ in $X_A$,*

$$\rho_A(x|y) \geq \rho_A(x'|y') \quad \text{and} \quad \rho_A(y|z) \geq \rho_A(y'|z') \Rightarrow \rho_A(x|z) \geq \rho_A(x'|z').$$

**A 5** Solvability. *For all $w, x, y, z$ in $X_A$,*

$$\rho_A(x|y) \geq \rho_A(w|z) \geq 1 \Rightarrow \exists z', z'' : \rho_A(x|z') = \rho_A(w|z) = \rho_A(z''|y).$$

**A 6** Archimedeanness. *If $x_1, x_2, \ldots, x_i, \ldots$ is a strictly bounded standard sequence (i.e. $\rho_A(x_{i+1}|x_i) = \rho_A(x_2|x_1)$ for every $x_i, x_{i+1}$ in the sequence; $\rho_A(x_2|x_1) \neq 1$; and there exist $y, z$ in $X_A$ such that $\rho_A(y|z) > \rho_A(x_i|x_1) > \rho_A(z|y)$ for all $x_i$ in the sequence), then it is finite.*

**Theorem 5** *If the structure $\langle X, A, X_A, \rho_A \rangle$ satisfies Reversal (A3), Weak Monotonicity (A4), Solvability (A5) and Archimedeanness (A6), then there* *exist* $V_A : \mathbb{R}^{+,\infty} \mapsto \mathbb{R}^{+,\infty}$, *strictly increasing and* $\mu_A : X \mapsto \mathbb{R}^+$ *such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall x, y \in X, \forall z \in X_A, \tag{6}$$

*and*

$$\mu_A(x)/\mu_A(y) = V_A[\rho_A(x|y)]\ \ \forall x, y \in X. \tag{7}$$

*The functions* $\mu'_A$ *and* $V'_A$ *also satisfy (6) and (7) iff there are real positive numbers* $p$ *and* $q$ *such that* $\mu_A = p\mu_A^q$ *and* $V'_A = V_A^q$.

*Moreover, if there is* $u$ *such that* $\rho_A(u|x) \geq 1$ *for all* $x$ *in* $X$*, then we can choose* $\mu_A$ *such that* $\mu_A : X \mapsto [0, 1]$ *with* $\mu_A(u) = 1$.

The kind of uniqueness obtained in Theorem 5 is typical of interval scales because it involves two parameters. The term log-interval scale is sometimes used for such scales (Krantz et al., 1971).

Only Reversal and Weak Monotonicity are necessary conditions in this theorem.

**Proof.** Let $\dot{\geq}$ be a binary relation on $X_A \times X_A$ defined by $xy \dot{\geq} wz$ iff $\rho_A(x|y) \geq \rho_A(w|z)$. It is easy to check that the structure $\langle X \times X, \dot{\geq} \rangle$ is an algebraic-difference structure as defined in (Krantz et al., 1971, p.151). So, by Theorem 2 on p.151 of the same book, there is $\phi : X_A \mapsto \mathbb{R}$ such that

$$\phi(x) - \phi(y) \geq \phi(w) - \phi(z) \Leftrightarrow xy \dot{\geq} wz \Leftrightarrow \rho_A(x|y) \geq \rho_A(w|z). \tag{8}$$

Because the exponential function is strictly increasing, we have

$$\exp(\phi(x) - \phi(y)) \geq \exp(\phi(w) - \phi(z)) \Leftrightarrow \rho_A(x|y) \geq \rho_A(w|z)$$

and, letting

$$\mu_A(x) = \begin{cases} e^{\phi(x)} & \text{if } x \in X_A \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\mu_A(x)/\mu_A(y) \geq \mu_A(w)/\mu_A(z) \Leftrightarrow \rho_A(x|y) \geq \rho_A(w|z),$$

with $\mu_A(x) > 0$ for all $x \in X_A$. This last equivalence implies that, for $\mu_A$ given, there is $V_A$, strictly increasing and unique, such that $\mu_A(x)/\mu_A(y) = V_A[\rho_A(x|y)]$. We now discuss the unicity of $\mu_A$. By Theorem 2 in (Krantz et al., 1971, p.151), $\phi'$ verifies (8) iff there are $q > 0$ and $s$ such that $\phi' = q\phi + s$. After exponentiation, we obtain $\exp(\phi') = \mu'_A = p\mu_A^q$, with $p = e^s$.

We now prove the last part of the theorem. Suppose $\rho_A(u|x) \geq 1$ for all $x$ in $X$. Let $\kappa = \mu_A(u)$. Obviously, $\kappa \geq \mu_A(x), \forall x \in X$. Define $\mu'_A = \mu_A/\kappa$. We then have $\mu'_A(x) \in [0,1], \forall x \in X$ and $\mu'_A(u) = 1$. □

In the following example, we show that $X$ needs not be infinite in Theorem 5.

**Example 1** *Suppose* $X = \{0, 1/4, 1/2, 3/4, 1\}$ *and*

$$\rho_A(x|y) = \begin{cases} 2 & \textit{if } x > y = 0, \\ 1 & \textit{if } x = y = 0, \\ 1 & \textit{if } x, y > 0, \\ 1/2 & \textit{if } y > x = 0. \end{cases} \tag{9}$$

*This structure satisfies all axioms of Theorem 5.*

Archimedeanness is not implied by the other conditions of Theorem 5, even if $X$ is infinite, as shown in the following example.

**Example 2** *Suppose* $X = [0,1]$ *and* $\rho_A(x|y)$ *is defined as in (9). This structure satisfies Reversal, Weak Monotonicity and Solvability but not Archimedeanness. Consider the following infinite standard sequence:* $\{1/2, 1/4, 1/8 \ldots\}$ $= \{x_1, x_2, \ldots, x_i, \ldots\}$, *with* $x_i = 1/2^i$. *It is bounded because* $\rho_A(0|1) = 1/2 < \rho_A(x_i|x_1) = 1 < \rho_A(1|0) = 2$ *for all* $x_i$ *in the sequence.*

Note that, when $X$ is finite, Archimedeanness is trivially satisfied because there is no infinite standard sequence. This is why Archimedeanness is not a necessary condition.

In the representation characterized by Theorem 5, no attention is paid to the value of the subjective ratios but only to their ordering. Any other mapping $\rho_A$ with the same ordering yields the same representation $\mu_A$ but a different $V$.

Note that, although we only spoke of ratios in this section, its title is 'difference measurement'. This is not a mistake. In fact the technique to construct a representation based on differences is the same as the one based on ratios. The difference is that, with differences, you naturally get an additive representation whereas, with ratios, you get a multiplicative one. But this difference is only superficial because, if you have an additive representation, then you also have a multiplicative one, and vice versa. To go from one to the other just take the exponential or the logarithm of your representation. So, there is no essential difference between both and the name *difference measurement* is standard for both. That is why we chose that name for this section. For a deeper discussion on this point, see Krantz et al. (1971), in the section on difference measurement.

## 6 Difference Measurement and Multiplicative Invariance

The Multiplicative Property is very strong in two different ways. It not only implies that the subjective ratio $\rho_A(x|y)$ given by the expert must be precisely equal to a function of $\rho_A(x|z)$ and $\rho_A(z|y)$ but, in addition, this function must be the product of $\rho_A(x|z)$ and $\rho_A(z|y)$. In Theorem 5, we relaxed both constraints. This led us to using only the ordering of the subjective ratios, so that the subjective ratio $\rho_A(x|y)$ was no longer a function of $\rho_A(x|z)$ and $\rho_A(z|y)$. In the present section, we will relax only the second constraint: the function will not necessarily be the product. In order to achieve this, we will need to use again the value of the subjective ratios and not just their ordering.

We will need the following conditions.

**A 7** Multiplicative Invariance.

$$\left.\begin{array}{l}\rho_A(u|v) = \lambda\rho_A(x|y)\\ \rho_A(v|w) = \lambda\rho_A(y|z)\end{array}\right\} \Rightarrow \rho_A(u|w) = \lambda^2\rho_A(x|z).$$

**A 8** Solvability 2. *For any $r, r' > 0$ there are $x, y$ and $z$ such that $r = \rho_A(x|y)$, $r' = \rho_A(y|z)$.*

**Theorem 6** *If the structure $\langle X, A, X_A, \rho_A\rangle$ satisfies Reversal (A3), Weak Monotonicity (A4), Solvability (A5), Solvability 2 (A8), Archimedeanness (A6) and Multiplicative Invariance (A7), then there exists $\mu_A : X \mapsto \mathbb{R}^+$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z),\ \forall x, y \in X, \forall z \in X_A \tag{10}$$

*and*

$$\mu_A(x)/\mu_A(y) = \alpha[\rho_A(x|y)]^\beta \quad \forall x, y \in X, \tag{11}$$

*with $\alpha, \beta > 0$. The function $\mu'_A$ and the numbers $\alpha'$ and $\beta'$ also satisfy (10) and (11) iff there are real positive numbers $p$ and $q$ such that $\mu'_A = p\mu_A^q$, $\alpha' = \alpha^q$ and $\beta' = q\beta$.*

*Moreover, if there is $u$ such that $\rho_A(u|x) \geq 1$ for all $x$ in $X$, then we can choose $\mu_A$ such that $\mu_A : X \mapsto [0, 1]$ with $\mu_A(u) = 1$.*

The uniqueness result tells us that, under the conditions of Theorem 6, we can consider the membership degrees as on a log-interval scale, just like in Theorem 5. The advantage of this result, compared to Theorem 5, is that we now have an analytic form for $V$. It can therefore be estimated from a small number of observations.

Only Reversal, Weak Monotonicity and Multiplicative Invariance are necessary conditions in this theorem.

**Proof.** By Solvability 2, for any $\lambda > 0$, $r, s > 0$ there are $x, y, z, u, v$ and $w$ such that $r = \rho_A(u|v) = \lambda\rho_A(x|y)$ and $s = \rho_A(v|w) = \lambda\rho_A(y|z)$. So, by Multiplicative Invariance, $\rho_A(u|w) = \lambda^2\rho_A(x|z)$. Thanks to Theorem 5, we know that there is $\mu_A$ and $V_A$ such that (10) holds and $\mu_A(u) = V_A[\rho_A(u|v)]\mu_A(v)$. Therefore, $V_A[\rho_A(u|w)] = V_A[\rho_A(u|v)]V_A[\rho_A(v|w)]$. So,

$$V_A[\lambda^2\rho_A(x|z)] = V_A\left[\lambda^2 V_A^{-1}\left(V_A[\rho_A(x|y)]V_A[\rho_A(y|z)]\right)\right] = V_A[\lambda\rho_A(x|y)]V_A[\lambda\rho_A(y|z)].$$

So, we have come to the functional equation

$$V_A\left(\lambda^2 V_A^{-1}[V_A(a)V_A(b)]\right) = V_A(\lambda a)V_A(\lambda b),\ \forall a, b, \lambda > 0. \tag{12}$$

Keeping $\lambda$ fixed, with the notations $H(a) = V_A(\lambda a)$, $G(a) = V_A(\lambda^2 a)$, $V_A(a) = u$ and $V_A(b) = v$, we obtain

$$G\left[V_A^{-1}(uv)\right] = H[V_A^{-1}(u)]H[V_A^{-1}(v)].$$

The general solution to this Pexider equation is well-known to be (Aczél, 1966)

$$H[V_A^{-1}(u)] = pu^k\ ;\ G[V_A^{-1}(u)] = p^2u^k.$$

Letting $\lambda$ vary again, we have

$$H[V_A^{-1}(u)] = V_A(\lambda a) = p(\lambda)V_A(a)^{k(\lambda)} \tag{13}$$

and

$$G[V_A^{-1}(u)] = V_A(\lambda^2 a) = p(\lambda)^2 V_A(a)^{k(\lambda)} \tag{14}$$

There are now two cases.

(1) $k$ is identically 1. Then $V_A(\lambda a) = p(\lambda)V_A(a)$, for all $\lambda > 0$. The unique solution of this Pexider equation is well-known to be

$$V_A(a) = qa^\delta, \tag{15}$$

where $q$ and $\delta$ are positive constants.

(2) $k$ is not identically 1. Then $V_A(\lambda a) = p(\lambda)V_A(a)^{k(\lambda)}$, for all $\lambda > 0$. Taking the logarithm on both sides, $\ln V_A(\lambda a) = \ln p(\lambda) + k(\lambda)\ln V_A(a)$. Introduce the notation $\lambda = e^m$, $a = e^n$, $W(u) = \ln V_A(e^u)$, $R(u) = \ln p(e^u)$ and $S(u) = k(e^u)$. We obtain

$$W(m+n) = R(m) + S(m)W(n),\quad m, n \in \mathbb{R}. \tag{16}$$

Given its definition, $S$ cannot be identically 1 because, by assumption, $k$ is not identically 1. Therefore, the unique solution of (16) is given by

$$W(m) = \alpha e^{\beta m} + d, \tag{17}$$

with $\alpha, \beta \neq 0$ (Aczél, 1966, p.150). Hence, $V_A(a) = t\exp(\alpha a^\beta)$, with $t > 0$. If we replace $V_A$ in (12), we find that $\beta$ must be equal to 0 and, so, $V_A$ is constant. But this is not possible. So, the only solution of (12) is the power function defined by (15).

The last part of the theorem is proved as in Theorem 5. $\square$

## 7 When $X$ is a continuum

So far, we considered a set $X$ without any particular structure. For example, $X = \{A\text{nn}, B\text{arbara}, C\text{aroline}\}$ and we are interested by the membership of $A, B$ and $C$ in the fuzzy set "Old".

But in many applications, the set $X$ of elements whose we want to measure the membership in a fuzzy set is a continuum: it has a topological structure similar to that of $\langle \mathbb{R}^+, \geq \rangle$. For example we want to determine the membership of any age (any strictly positive real number) in the fuzzy set "Old". In this section, we try to exploit this structure in the formulation of our axioms. More specifically, we will drop the solvability conditions and replace them by other conditions linked to the structure of $X$.

We say that $\langle X, \trianglerighteq \rangle$ is a *continuum* (Cantor, 1895; Narens, 1985) if $\langle X, \trianglerighteq \rangle$ is isomorphic to $\langle \mathbb{R}, \geq \rangle$, that is there exists a one-to-one mapping $\phi : X \mapsto \mathbb{R}$ such that

$$x \trianglerighteq y \Leftrightarrow \phi(x) \geq \phi(y).$$

In such a case, it is reasonable in many applications to assume that the subjective ratios given by the expert vary continuously. Formally,

**A 9** Continuity. *$\langle X, \trianglerighteq \rangle$ is a continuum and $\rho_A(x|y)$ is a continuous function of $x$ and $y$.*

We can now prove a lemma that will soon prove useful.

**Lemma 1** *If the structure $\langle X, \trianglerighteq, A, \rho_A \rangle$ is Continous, then $\langle X, A, \rho_A \rangle$ satisfies Solvability.*

**Proof.** Suppose $\rho_A(x|y) \geq \rho_A(w|z) \geq 1$. We know that $\rho_A(x|x) = 1$. So, by Continuity, there is $z'$ such that $\rho_A(x|z') = \rho_A(w|z)$. Similarly, $\rho_A(y|y) = 1$. By Continuity again, there is $z''$ such that $\rho_A(z''|y) = \rho_A(w|z)$. $\square$

We can thus restate an equivalent of Theorem 5 for continua, without Solvability.

**Theorem 7** *If the structure $\langle X, \trianglerighteq, A, X_A, \rho_A \rangle$ satisfies Reversal (A3), Weak Monotonicity (A4), Continuity (A9) and Archimedeanness (A6), then there exists $V_A : \mathbb{R}^{+,\infty} \mapsto \mathbb{R}^{+,\infty}$, strictly increasing and $\mu_A : X \mapsto \mathbb{R}^+$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z), \;\; \forall x, y \in X, \forall z \in X_A, \tag{18}$$

*and*

$$\mu_A(x)/\mu_A(y) = V_A[\rho_A(x|y)] \;\; \forall x, y \in X. \tag{19}$$

*The function $\mu'_A$ and $V'_A$ also satisfy (18) and (19) iff there are real positive numbers $p$ and $q$ such that $\mu'_A = p\mu_A^q$ and $V'_A = V_A^q$.*

*Moreover, if there is $u$ such that $\rho_A(u|x) \geq 1$ for all $x$ in $X$, then we can choose $\mu_A$ such that $\mu_A : X \mapsto [0, 1]$ with $\mu_A(u) = 1$.*

We are now ready to restate Theorem 6 without Solvabilty nor Solvability 2.

**Theorem 8** *Suppose $\emptyset \neq X_A \subsetneq X$. If the structure $\langle X, \trianglerighteq, A, X_A, \rho_A \rangle$ satisfies Reversal (A3), Weak Monotonicity (A4), Continuity (A9), Archimedeanness (A6) and Multiplicative Invariance (A7), then there exists $\mu_A : X \mapsto \mathbb{R}^+$ such that*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z), \;\; \forall x, y \in X, \forall z \in X_A \tag{20}$$

*and*

$$\mu_A(x)/\mu_A(y) = \alpha[\rho_A(x|y)]^\beta \;\; \forall x, y \in X, \tag{21}$$

*with $\alpha, \beta > 0$. The function $\mu'_A$ and the numbers $\alpha'$ and $\beta'$ also satisfy (20) and (21) iff there are real positive numbers $p$ and $q$ such that $\mu'_A = p\mu_A^q$, $\alpha' = \alpha^q$ and $\beta' = q\beta$.*

*Moreover, if there is $u$ such that $\rho_A(u|x) \geq 1$ for all $x$ in $X$, then we can choose $\mu_A$ such that $\mu_A : X \mapsto [0, 1]$ with $\mu_A(u) = 1$.*

**Proof.** By Theorem 7, there exists $\mu_A : X_A \mapsto \mathbb{R}^+$ and $V_A : \mathbb{R}^{+,\infty} \mapsto \mathbb{R}^{+,\infty}$ such that $\mu_A(x) = V_A[\rho_A(x|y)]\mu_A(y)$. Because $\mu_A(x) = V_A[\rho_A(x|x)]\mu_A(x)$, we obtain $V_A(1) = 1$. Because $V_A$ is strictly increasing, $x \geq 1 \Leftrightarrow V_A(x) \geq 1$. Note also that $1 = V_A[\rho_A(x|x)] = V_A[\rho_A(x|y)]V_A[\rho_A(y|x)]$. We now prove that $\langle X_A, \trianglerighteq, A, \rho_A \rangle$ satisfies Solvability 2. There are four cases we need to consider:

(1) $r \geq 1, r' \geq 1$. Because $\rho_A$ is continuous, $\rho_A(x|x) = 1$ and $\rho_A(x|w) = \infty$ for $w \notin X_A$, there exists $y$ such that $\rho_A(x|y) = r \geq 1$. Because $\rho_A$ is continuous, $\rho_A(y|y) = 1$ and $\rho_A(y|w) = \infty$ for $w \notin X_A$, there exists $z$ such that $\rho_A(y|z) = r' \geq 1$.

(2) If $r \geq 1, r' < 1$, then two cases can arise.
- Suppose $V_A(r)V_A(r') \leq 1$. Because $\rho_A$ is continuous and $\rho_A(w|w') = 0$ for $w \notin X_A$, $w' \in X_A$, there exists $y, z$ such that $\rho_A(y|z) = r'$. Using the representation of Theorem 7, we obtain $V_A[\rho_A(z|y)] = 1/V_A[\rho_A(y|z)] = 1/V_A(r') \geq V_A(r)$. So, $\rho_A(z|y) \geq r$. Because $\rho_A(y|y) = 1$ and by Continuity, there exists $x$ such that $\rho_A(x|y) = r$.

- Suppose $V_A(r)V_A(r') > 1$. As above we can find $x$ and $y$ such that $\rho_A(x|y) = r$. Using the representation of Theorem 7, we obtain $V_A[\rho_A(y|x)] = 1/V_A[\rho_A(x|y)] = 1/V_A(r) < V_A(r')$. So, $\rho_A(y|x) < r'$. But we also have $r' < 1 = \rho_A(y|y)$. Hence, by Continuity, there is $z$ such that $\rho_A(y|z) = r'$.

(3) $r < 1, r' \geq 1$. As above, we can find $x$ and $y$ such that $\rho_A(x|y) = r$. Because $\rho_A(y|y) = 1$ and by Continuity, there is $z$ such that $\rho_A(y|z) = r'$.

(4) $r < 1, r' < 1$. By Continuity, we can find $y$ and $z$ such that $\rho_A(y|z) = r'$. We have $\rho_A(y|y) = 1$ and $\rho_A(w|y) = 0$, for $w \notin X_A$. So, by Continuity, there is $x$ such that $\rho_A(x|y) = r$.

So, we have proven that Solvability 2 holds. By Lemma 1, Solvability also holds. Hence, we can apply Theorem 6 and the proof is complete. □

The two results in this section show that Solvability and Solvability 2, in many cases, are very reasonable structural conditions.

## 8 Measuring the membership in different fuzzy sets simultaneously

In the introduction, we said that one of our primitives was $\mathcal{F} = \{A, B, \ldots\}$, the set of all fuzzy sets that are relevant in a particular context. But so far, we worked only with one fuzzy set, namely $A$. Of course, the analysis also applies to $B$, $C$ and so on. We can ask subjective ratios to an expert for the membership of $x$ and $y$ in $A$ or in $B$, etc. In doing so, we can construct different membership degrees functions $\mu_A$, $\mu_B$, etc. One for each fuzzy set in $\mathcal{F}$.

Because these representations for the membership in different set are constructed separately, they are not related to each other and we may not, in general, claim that $x$ belongs more to $A$ than to $B$ because $\mu_A(x) > \mu_B(x)$. The following example illustrates this point.

**Example 3** *Suppose $X = \{w, x, y, z\}$, $X_A = \{w, x, y\}$, $X_B = \{x, y, z\}$ and the subjective ratios are given by Table 1. We have two structures satisfying*

| $\rho_A$ | $w$ | $x$ | $y$ |
|---|---|---|---|
| $w$ | 1 | 2 | 4 |
| $x$ | 1/2 | 1 | 2 |
| $y$ | 1/4 | 1/2 | 1 |

| $\rho_B$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 1 | 1/2 | 2 |
| $y$ | 2 | 1 | 8 |
| $z$ | 1/2 | 1/8 | 1 |

Table 1
The subjective ratios of Example 3

*all axioms of Theorem 6. So, we can construct two representations as in The-*

*orem 6. Table 2 shows two equally valid pairs of representations. For the first one, $V_A(s) = s$ and $V_B(s) = 4s^2$ whereas for the second one, $V'_A(s) = s^2$, $V'_B(s) = \sqrt{2s}$. If we compare $\mu_A(x)$ to $\mu_B(x)$, we are tempted to conclude that $x$ belongs more to $A$ than to $B$ but if we compare $\mu'_A(x)$ to $\mu'_B(x)$, we are led to the opposite conclusion. In fact, we have no way to make the comparison. We did not collect the data that would allow us to draw such a conclusion.*

| | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $\mu_A$ | 1 | 1/2 | 1/4 | 0 |
| $\mu_B$ | 0 | 1/16 | 1 | 1/256 |

| | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $\mu'_A$ | 1/2 | 1/8 | 1/32 | 0 |
| $\mu'_B$ | 0 | 1/2 | 1 | 1/4 |

Table 2
Two different representations for Example 3

In the next pages, we will therefore try to see what we need in order to make comparisons across sets. The first piece of information we can ask to the expert is, for each fuzzy set $A$ in $\mathcal{F}$, the subset of elements fully belonging to $A$, denoted by $X^A$. The following example illustrates what we can do with this new primitive.

**Example 4** *Suppose all data are as in Example 3 plus we have $X^A = \{w\}$ and $X^B = \{y\}$. We can impose, in addition to (6) and (7), that $\mu_A(x) = 1$, $\forall x \in X^A, \forall A \in \mathcal{F}$. This is a kind of normalization of the scales and can be considered as very reasonable. Unfortunately, this is not enough, as shown in Table 3, where $V_A(s) = s$, $V_B(s) = 4s^2$,$V'_A(s) = s^2$ and $V'_B(s) = \sqrt{2s}$ With the first representation, we are tempted to conclude that $y$ belongs more to $A$ than $x$ to $B$ whereas, with the second one, we are led to the opposite conclusion. We are still not able to make comparisons across sets for the objects that are not in $X_A$ nor in $X^A$, i.e. the most interesting objects.*

| | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $\mu_A$ | 1 | 1/2 | 1/4 | 0 |
| $\mu_B$ | 0 | 1/16 | 1 | 1/256 |

| | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $\mu'_A$ | 1 | 1/4 | 1/16 | 0 |
| $\mu'_B$ | 0 | 1/2 | 1 | 1/4 |

Table 3
Two different representations for Example 4

A possible way to solve this problem is to ask the expert to make more complex estimations. For example, "what is the ratio between the membership of $x$ in $A$ and the membership of $y$ in $B$?" (for all $x, y \in X, A, B \in \mathcal{F}$). That would give us data that, if consistent, would permit us to construct two (or more if there are more fuzzy sets) mappings $\mu_A$ and $\mu_B$ such that comparisons between them would be meaningful. But there is a better way: asking questions of the form "Is the membership of $x$ in $A$ larger than the membership of $y$ in $B$?" This information is much more easy to get than complex ratios. In fact, it amounts to asking if the subjective ratio between the membership of $x$ in $A$

and the membership of $y$ in $B$ is larger than 1 (instead of asking its precise value). It turns out that it is also sufficient to construct a representation for the membership degrees.

More formally, we have to include one more primitive in our framework: the binary relation $\succsim$ on $(X \times A)$ to be interpreted as follows: $(x, A) \succsim (y, B)$ means "the membership of $x$ in $A$ is at least as large as the membership of $y$ in B." The relation $\succsim$ needs to be consistent in some sense with the subjective ratios, as expressed by the following condition.

**A 10** Condition U. *For all $A, B \in \mathcal{F}$, for all $x, y, z, w \in X_A, x', y', z', w' \in X_B$, if $(x, A) \sim (x', B)$, $(z, A) \sim (z', B)$ and $(w, A) \sim (w', B)$, then*

$$(y, A) \succsim (y', B) \quad \textit{iff} \quad \frac{\ln \frac{\rho_A(x|y)}{\rho_A(x|z)}}{\ln \frac{\rho_A(x|w)}{\rho_A(x|z)}} \leq \frac{\ln \frac{\rho_B(x'|y')}{\rho_B(x'|z')}}{\ln \frac{\rho_B(x'|w')}{\rho_B(x'|z')}}.$$

This condition is difficult to interpret. We will discuss this further.

**Theorem 9** *Suppose that, for all $A, B$ in $\mathcal{F}$, there is $x_{AB}, w_{AB}, z_{AB} \in X_A$ and $x_{BA}, w_{BA}, z_{BA} \in X_B$ such that $(x_{AB}, A) \sim (x_{BA}, B)$, $(z_{AB}, A) \sim (z_{BA}, B)$, $(w_{AB}, A) \sim (w_{BA}, B)$ and $(x, A) \succ (w, A) \succ (z, A)$. If the structure $\langle X, \mathcal{F}, \succsim, (X_A, X^A, \rho_A)_{A \in \mathcal{F}} \rangle$ satisfies U (A10), and, for all $A$ in $\mathcal{F}$, Reversal (A3), Weak Monotonicity (A4), Solvability (A5), Solvability 2 (A8), Archimedeanness (A6) and Multiplicative Invariance (A7), then, for all $A \in \mathcal{F}$ there exists $\mu_A : X \mapsto \mathbb{R}^+$ such that, $\forall x, y \in X, \forall z \in X_A, \forall A, B \in \mathcal{F}$,*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z), \tag{22}$$

$$\mu_A(x) \geq \mu_B(y) \Leftrightarrow (x, A) \succsim (y, B), \tag{23}$$

*and*

$$\mu_A(x)/\mu_A(y) = \alpha_A[\rho_A(x|y)]^{\beta_A}, \tag{24}$$

*with $\alpha_A, \beta_A > 0$. The function $\mu'_A$ and the numbers $\alpha'_A, \beta'_A$ also satisfy (22) and (24) iff there are real positive numbers $p$ and $q$ such that, for all $A \in \mathcal{F}$, $\mu'_A = p\mu_A^q$, $\alpha'_A = \alpha_A^q$ and $\beta'_A = q\beta_A$.*

*Moreover, if $(x, A) \sim (y, B)$ for all $A, B \in \mathcal{F}$, $x \in X^A$ and $y \in X^B$, then for every $A$ in $\mathcal{F}$, we can choose $\mu_A$ so that $\mu_A(x) = 1$ for all $x$ in $X^A$.*

**Proof.** By Theorem 6, for each $A$ in $\mathcal{F}$ there is $\mu_A : X \mapsto \mathbb{R}^+$ and $\alpha_A, \beta_A > 0$ such that

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z), \ \forall x, y \in X, \forall z \in X_A$$

and

$$\mu_A(x) = \alpha_A[\rho_A(x|y)]^{\beta_A}\mu_A(y) \ \ \forall x, y \in X.$$

Because $\mu_A, \alpha_A$ and $\beta_A$ are not unique we can choose them in such a way that

$$\mu_A(x_{AB}) = a \tag{25}$$
$$\mu_A(z_{AB}) = b \tag{26}$$
$$\mu_A(w_{AB}) = c \tag{27}$$

where $a, b$ and $c$ are some arbitrary positive real numbers. In order to do so, we divide (25) by (26) and we obtain

$$\alpha_A \rho_A(x_{AB}|z_{AB})^{\beta_A} = a/b. \tag{28}$$

Similarly, we divide (25) by (27).

$$\alpha_A \rho_A(x_{AB}|w_{AB})^{\beta_A} = a/c. \tag{29}$$

We now divide (28) by (29) and we find

$$\left(\frac{\rho_A(x_{AB}|z_{AB})}{\rho_A(x_{AB}|w_{AB})}\right)^{\beta_A} = c/a.$$

From this equation, we find the value of $\beta_A$. We replace in (28) or in (29) and we find the value of $\alpha_A$. Because $(x_{AB}, A) \sim (x_{BA}, B)$, $(z_{AB}, A) \sim (z_{BA}, B)$ and $(w_{AB}, A) \sim (w_{BA}, B)$ we want also that $\mu_B(x_{BA}) = a$, $\mu_B(z_{BA}) = b$ and $\mu_B(w_{BA}) = c$. Following the same resaoning as above, we find

$$\alpha_B \rho_B(x_{BA}|z_{BA})^{\beta_B} = a/b$$

and

$$\left(\frac{\rho_B(x_{BA}|z_{BA})}{\rho_B(x_{BA}|w_{BA})}\right)^{\beta_B} = c/a.$$

From this system of two equations, we can find the values of $\alpha_B$ and $\beta_B$. Using the representation that we just obtained, we can compute $\mu_B(x)$ for any $x$ in $X$. In particular for $x_{BC}$, $z_{BC}$ and $w_{BC}$.Because $(x_{BC}, B) \sim (x_{CB}, C)$, $(z_{BC}, B) \sim (z_{CB}, C)$ and $(w_{BC}, B) \sim (w_{CB}, C)$ we want also that $\mu_B(x_{BC}) = \mu_C(x_{CB})$, $\mu_B(z_{BC}) = \mu_C(z_{CB})$ and $\mu_B(w_{BC}) = \mu_C(w_{CB})$. Following the same reasoning as above, we find

$$\alpha_C \rho_C(x_{CB}|z_{CB})^{\beta_C} = \frac{\mu_B(x_{BC})}{\mu_B(z_{BC})}$$

and

$$\left(\frac{\rho_C(x_{CB}|z_{CB})}{\rho_C(x_{CB}|w_{CB})}\right)^{\beta_C} = \frac{\mu_B(x_{BC})}{\mu_B(w_{BC})}.$$

Note that this system of two equations has only two unknowns, $\alpha_C$ and $\beta_C$, because $\mu_B$ has been fixed at the previous step. So, we can find the values of $\alpha_C$ and $\beta_C$. Using these values, we then compute $\mu_C(x_{CD})$, $\mu_C(z_{CD})$ and $\mu_C(w_{CD})$ that we use in turn to find $\alpha_D$ and $\beta_D$ and so on.

We now have a representation of our structure and, by construction, we know that it gives the right values for $\mu$ when we want to compare the membership of $x_{AB}$ in $A$ to that of $x_{BA}$ in $B$, for any $A, B$. But we have to check that the values of $\mu$ are also the right ones when we compare the membership of any $x$ in $A$ to that of any $y$ in $B$, for any $A, B$. More precisely, we have to check that $\mu_A(x) \geq \mu_B(y)$ if and only if $(x, A) \succsim (y, B)$. In the next lines, we will show that it is so, provided that condition U holds.

$$
\begin{aligned}
\mu_A(x) \geq \mu_B(y) &\Leftrightarrow \frac{\mu_A(x_{AB})}{\mu_A(x)} \leq \frac{\mu_B(x_{BA})}{\mu_B(y)} \\
&\Leftrightarrow \alpha_A \rho_A(x_{AB}|x)^{\beta_A} \leq \alpha_B \rho_B(x_{BA}|y)^{\beta_B} \\
&\Leftrightarrow \frac{\rho_A(x_{AB}|x)^{\beta_A}}{\rho_A(x_{AB}|w_{AB})^{\beta_A}} \leq \frac{\rho_B(x_{BA}|y)^{\beta_B}}{\rho_B(x_{BA}|w_{BA})^{\beta_B}} \\
&\Leftrightarrow \beta_A \ln \frac{\rho_A(x_{AB}|x)}{\rho_A(x_{AB}|w_{AB})} \leq \beta_B \ln \frac{\rho_B(x_{BA}|y)}{\rho_B(x_{BA}|w_{BA})} \\
&\Leftrightarrow \frac{\beta_A}{\beta_B} \leq \frac{\ln \frac{\rho_B(x_{BA}|y)}{\rho_B(x_{BA}|w_{BA})}}{\ln \frac{\rho_A(x_{AB}|x)}{\rho_A(x_{AB}|w_{AB})}}.
\end{aligned}
$$

Because $(x_{AB}, A) \sim (x_{BA}, B)$ and $(z_{AB}, A) \sim (z_{BA}, B)$, we have

$$
\frac{\beta_A}{\beta_B} = \frac{\ln \frac{\rho_B(x_{BA}|w_{BA})}{\rho_B(x_{BA}|z_{BA})}}{\ln \frac{\rho_A(x_{AB}|w_{AB})}{\rho_A(x_{AB}|z_{AB})}}.
$$

So,

$$
\begin{aligned}
\mu_A(x) \geq \mu_B(y) &\Leftrightarrow \frac{\ln \frac{\rho_B(x_{BA}|w_{BA})}{\rho_B(x_{BA}|z_{BA})}}{\ln \frac{\rho_A(x_{AB}|w_{AB})}{\rho_A(x_{AB}|z_{AB})}} \leq \frac{\ln \frac{\rho_B(x_{BA}|y)}{\rho_B(x_{BA}|w_{BA})}}{\ln \frac{\rho_A(x_{AB}|x)}{\rho_A(x_{AB}|w_{AB})}} \\
&\Leftrightarrow (x, A) \succsim (y, B).
\end{aligned}
$$

□

A major problem with Theorem 9 is that condition U is not easily interpretable. That is why we now present two other conditions, much more appealing but also stronger.

**A 11** Strong Consistency. *For all $A, B \in \mathcal{F}$, for all $x, y \in X_A, x', y' \in X_B$, if $(x, A) \sim (x', B)$, then*

$$
(y, A) \succsim (y', B) \quad \textit{iff} \quad \rho_A(x|y) \leq \rho_B(x'|y').
$$

This condition imposes that the expert uses the same subjective scale when he compares the membership of two objects in $A$ or in $B$.

**A 12** Consistency. *For all $A, B \in \mathcal{F}$, for all $x, y, w \in X_A, x', y', w' \in X_B$, if $(x, A) \sim (x', B)$ and $(w, A) \sim (w', B)$, then*

$$(y, A) \succsim (y', B) \quad \textit{iff} \quad \frac{\rho_A(x|y)}{\rho_A(x|w)} \leq \frac{\rho_B(x'|y')}{\rho_B(x'|w')}.$$

This condition does not impose that the scale be the same but they must be homothetic. It is easy to check that Strong Consistency implies Consistency which, in turn, implies condition U and, so, we have the following corollary.

**Corollary 1** *Theorem 9 holds when condition U is replaced by Consistency or Strong Consistency.*

Condition U, Consistency and Strong Consistency are not equivalent, as shown in the following examples.

**Example 5** *Suppose $X = X_A = X_B = \{w, x, y, z\}$, $(x, A) \sim (x, B)$, $(y, A) \sim (y, B)$, $(z, A) \sim (z, B)$, $(w, A) \sim (w, B)$ and the subjective ratios are given by Table 4. It is simple to check that all conditions of Theorem 9 are satisfied,*

| $\rho_A$ | $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|---|
| $x$ | 1 | 2 | 4 | 8 |
| $y$ | 1/2 | 1 | 2 | 4 |
| $z$ | 1/4 | 1/2 | 1 | 2 |
| $w$ | 1/8 | 1/4 | 1/2 | 1 |

| $\rho_B$ | $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|---|
| $x$ | 1 | 4 | 16 | 64 |
| $y$ | 1/4 | 1 | 4 | 16 |
| $z$ | 1/16 | 1/4 | 1 | 4 |
| $w$ | 1/64 | 1/16 | 1/4 | 1 |

Table 4
The subjective ratios of Example 5

*including condition U, but not Consistency nor Strong Consistency.*

**Example 6** *Suppose $X = X_A = X_B = \{x, y, z\}$, $(x, A) \sim (x, B)$, $(y, A) \sim (y, B)$, $(z, A) \sim (z, B)$ and the subjective ratios are given by Table 5. It is*

| $\rho_A$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 1 | 2 | 4 |
| $y$ | 1/2 | 1 | 2 |
| $z$ | 1/4 | 1/2 | 1 |

| $\rho_B$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 2 | 4 | 8 |
| $y$ | 1 | 2 | 4 |
| $z$ | 1/2 | 1 | 2 |

Table 5
The subjective ratios of Example 6

*simple to check that all conditions of Theorem 9 are satisfied, plus Consistency but not Strong Consistency.*

## 9 Modelling the union and the intersection

So far, the fuzzy sets in $\mathcal{F}$ were just abstract objects and no relation was supposed to hold among them. But it would be interesting to consider elements of $\mathcal{F}$ of the form $A \cap B$ or $A \cup B$, where $\cap$ or $\cup$ are not the (fuzzy) set-theoretic operators but their subjective and empirical equivalents; they are primitives of our theory. For example, we can ask an expert "What is the ratio between the membership of John and that of Ann in the set *'Old and Intelligent'* ?" The answers of the expert ($\rho_{A\cap B}(x|y)$ for all $x, y \in X$) would permit us to construct a membership function $\mu_{A\cap B}$ representing the subjective ratios given by the expert. Similarly, we can also construct a membership function $\mu_{A\cup B}$.

Instead of constructing the membership function $\mu_{A\cap B}$ from the subjective ratios $\rho_{A\cap B}(x|y)$, it is tempting to compute it from $\mu_A$ and $\mu_B$, using, as it is usual in fuzzy sets theory, a t-norm. For example, $\mu_{A\cap B} = \min(\mu_A, \mu_B)$ or $\mu_{A\cap B} = \max(0, \mu_A + \mu_B - 1)$. But this will make sense only if $\mu_{A\cap B}$, computed via a t-norm, is a representation of (or is compatible with) the subjective ratios $\rho_{A\cap B}(x|y)$ given by the expert for all $x$ and $y$. This section is devoted to the analysis of the conditions that make this derivation of $\mu_{A\cap B}$ from $\mu_A$ and $\mu_B$ compatible with the observed subjective ratios.

First we show why $\cap$ cannot be represented by the Łukasiewicz t-norm. Suppose we ask an expert whether John belongs more to the set *'Old'* or *'Old and Old'*. It is almost certain that the expert will answer that the membership in both sets is the same. If he does not, we may wonder if he understands the question. So, we have the relation $\mu_{A\cap A}(x) = \mu_A(x)$, for any $x$ in $X$. This is definitely not representable by means of the Łukasiewicz t-norm. In fact, this example shows why any non-idempotent t-norm will not work. So, we are left with the only idempotent t-norm: the minimum. We must now find the conditions we need to add to those of Theorem 9 so that the subjective ratios given by the expert and the relation $\succsim$ can be represented by some membership function and the minimum. These conditions were first presented by Bollmann-Sdorra et al. (1993) in a context where only $\succsim$ is observed but not the subjective ratios. They can also be found in Marchant (2002). Because these conditions involve only comparisons of the form $(x, A) \succsim (x, B)$ we state them in terms of the relation $\succsim_x$ defined by

$$A \succsim_x B \Leftrightarrow (x, A) \succsim (x, B). \tag{30}$$

**A 13** Order of Operations. *For all $A, B$ in $\mathcal{F}$, $A \cup B \succsim_x A \cap B$.*

**A 14** Weak Commutativity. *For all $A, B$ in $\mathcal{F}$,*

$$A \cup B \sim_x B \cup A \text{ and}$$

$$A \cap B \sim_x B \cap A.$$

**A 15** Weak Associativity. *For all $A, B, C$ in $\mathcal{F}$,*

$$A \cup (B \cup C) \sim_x (A \cup B) \cup C \text{ and}$$

$$A \cap (B \cap C) \sim_x (A \cap B) \cap C.$$

**A 16** Weak Absorption. *For all $A, B$ in $\mathcal{F}$,*

$$A \sim_x A \cap (A \cup B) \text{ and}$$

$$A \sim_x A \cup (A \cap B).$$

**A 17** Weak Right Monotonicity. *For all $A, B, C$ in $\mathcal{F}$,*

$$A \succsim_x B \text{ implies } A \cap C \succsim_x B \cap C \text{ and}$$

$$A \succsim_x B \text{ implies } A \cup C \succsim_x B \cup C.$$

Thanks to these conditions, we can formulate our last result.

**Theorem 10** *Suppose all conditions of Theorem 9 are fulfilled. Then, for all $x$ in $X$, the structures $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ derived according to (30) satisfy Order of Operations (A13), Weak Commutativity (A14), Weak Associativity (A15), Weak Absorption (A16) and Weak Monotonicity (A17) if and only if, for all $A \in \mathcal{F}$ there exists $\mu_A : X \mapsto \mathbb{R}^+$ such that, $\forall x, y \in X, \forall z \in X_A, \forall A, B \in \mathcal{F}$,*

$$\mu_A(x) \geq \mu_A(y) \Leftrightarrow \rho_A(x|z) \geq \rho_A(y|z), \tag{31}$$

$$\mu_A(x) \geq \mu_B(y) \Leftrightarrow (x, A) \succsim (y, B), \tag{32}$$

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)), \tag{33}$$

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \tag{34}$$

*and*

$$\mu_A(x)/\mu_A(y) = \alpha_A[\rho_A(x|y)]^{\beta_A}, \tag{35}$$

*with $\alpha_A, \beta_A > 0$. The function $\mu'_A$ and the numbers $\alpha'_A, \beta'_A$ also satisfy (31) and (35) iff there are real positive numbers $p$ and $q$ such that, for all $A \in \mathcal{F}$, $\mu'_A = p\mu_A^q$, $\alpha'_A = \alpha_A^q$ and $\beta'_A = q\beta_A$.*

*Moreover, if $(x, A) \sim (y, B)$ for all $A, B \in \mathcal{F}$, $x \in X^A$ and $y \in X^B$, then for every $A$ in $\mathcal{F}$, we can choose $\mu_A$ so that $\mu_A(x) = 1$ for all $x$ in $X^A$.*

**Proof.** Because all conditions of Theorem 9 are fulfilled, we have a representation satisfying (31), (32) and (35). From (32), it is clear that $\succsim$ is a weak order (a complete and transitive relation). We can therefore apply Theorem 4 of Marchant (2002), saying that there are numerical representations for the membership satisfying (33) and (34). According to the same theorem, these representations are unique up to a strictly increasing transformation. We can therefore choose them equal to the representations derived from Theorem 9. □

In Sections 8 and 9, we have chosen to build on Theorem 6 because we think it is the most interesting one. But we could of course use the conditions of Bollmann-Sdorra et al. (1993) in conjunction with any theorem presented in this paper. This would yield about a dozen of new theorems. We do not present these theorems because it would make this paper much too long and, to some extent, repetitive.

## 10 Conclusion

In Sections 2 to 8, we have considered several alternative numerical representations for the membership of some objects in fuzzy sets. For each kind of representation, we have shown under which conditions it is possible to construct membership degrees representing the subjective ratios given by an expert. Each result was accompanyied by a uniqueness result saying how unique the representation is. Based on these uniqueness results, we can say which technique leads to an ordinal, an interval or a ratio scale.

But in Section 9, we have shown that, even if an interval scale is necessary for the use of many t-norms (for example the Łukasiewicz t-norm), it is not always sufficient. If an algebraic operation is used for representing an empirically observable operation on the primitives, then the algebraic operation must have properties that reflect those of the empirical operation. For example, because the union is almost certainly idempotent, the algebraic operation used to represent the union must also be idempotent, independently of the scale, of the level of measurement. It is therefore important, when choosing a particular algebraic operation, to consider not only the kind of scale but also the properties of the empirical operation and those of the algebraic one.

Of course, not all algebraic operations are meant to represent empirical ones. For example, if two experts give you estimations of the membership degree of $x$ in a fuzzy set $A$ (directly or using a measurement technique), and if you want to aggregate these two estimations into one membership degree, you might want to use a uninorm or some kind of aggregation operator. But this operation would not reflect any empirically observable operation. There does not exist, or no expert can give you, an aggregated membership degree. The aggregated membership degree is something you define but cannot observe. In such a case, it seems then that only the level of measurement is important.

It is important to remember that the conditions guaranteeing that some measurement technique can be used are conditions on the empirically observable primitives. Therefore, whether we actually may or not use such techniques must be empirically checked. We must verify that an expert is able to give subjective ratios satisfying the conditions characterizing the representation we want to use.

We finally indicate some directions for future research.

- In Marchant (2002) and in the present paper, we have discussed the measurement of membership. But in the fuzzy literature, we do not have only fuzzy sets, we also have fuzzy logic, where truth values play a role similar to membership degrees. Some of our results can probably be easily transposed to fuzzy logic but perhaps not all of them. An important problem with fuzzy sets that has no equivalent in fuzzy logic is the problem of commensurability of the membership degrees in two different sets. Using for example the technique of Theorem 5, we can construct $\mu_A$ and $\mu_B$ but not be able to compare $\mu_A(x)$ to $\mu_B(x)$. In fuzzy logic, such a problem does probably not arise. Other examples of domains where our results can eventually be transposed are fuzzy preference modelling and possibility theory.
- In Marchant (2002) and in the present paper, besides the measurement of the membership degrees, we also addressed the problem of the representation of the union, the intersection and the complementation. There might be other interesting empirical operations. For example, the implication in fuzzy logic.
- In Marchant (2002), we did not pay any attention to the structure of $X$ but only to statements (in the form of comparisons) about the membership of the elements of $X$ in some sets. In the present paper, except in Section 7, we did not look either at the structure of $X$. Though we think it would be extremely useful to have a measurement technique whose output would be a membership function analytically defined on $X$. This is of course only possible if we take into account the structure of $X$.

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