The measurement of membership by comparisons

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Abstract

Suppose we want to use a particular algebraic operation (for example the drastic product t-norm) for representing an operation on fuzzy sets (for example the intersection). This algebraic operation will be used with membership degrees, i.e., most of the time, numbers between 0 and 1. If we want that the results of our calculations make sense, we must be sure that the algebraic operation is compatible with the nature of the membership degrees, which is determined by the technique used to measure them. But this technique must in turn be compatible with the structural properties of the knowledge we want to represent. This paper addresses these issues within the framework of measurement theory.

We provide sound theoretical foundations for the measurement of membership on ordinal and interval scales. But we also show that the level of measurement (ordinal, interval or even ratio) is not critical for the choice of a particular algebraic operation. Within measurement theory, whatever the scale, only the max and the min can be used for representing the intersection and the union. We also present some results about the complementation.

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1 Introduction

Suppose we ask the following question to an expert: what is the membership degree of x in the set A? He answers 0.5. Then, what is the membership degree

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of x in B, C and D? He says 0.1, 0.3 and 0.4. If we choose a particular t-conorm, we can compute from his answers the membership degree of x in $A \cup B$ and $C \cup D$. Suppose we choose the Łukasiewicz t-conorm, i.e. $S(a, b) = \min(1, a+b)$. Then, the membership degree of x in $A \cup B$ is 0.6. For $C \cup D$, it is 0.7. We conclude then that x belongs more to $C \cup D$ than to $A \cup B$.

But we know that the membership degrees given by the expert must be taken with a pinch of salt. On another occasion, the same expert might give slightly different answers, for example, 0.5, 0.1, 0.2 and 0.3 for the membership of x in A, B, C and D respectively. After all, these numbers express the same ordering. The problem is that, with these new membership degrees and the Łukasiewicz t-conorm, the membership degree of x in $A \cup B$ is 0.6 and 0.5 in $C \cup D$. We now conclude then that x belongs more to $A \cup B$ than to $C \cup D$ although the information provided by the expert did not really change. He gave the same ordering and we cannot really expect that his answers carry more than an ordering.

It is a well-known problem (see e.g. French, 1984, 1987) and it is not limited to the Łukasiewicz t-conorm or t-norm. Only the max and the min escape this particular problem. But very few authors pay attention to it although it is crucial; it invalidates any inference we can make from a set of membership degrees.

Because the information carried by the answers of the expert is an ordering, we say that the membership degrees are measured on an ordinal scale. And because max and min are only sensitive to the ordering, it is tempting to claim that the max should be used as the t-conorm (and min as the t-norm). We will show that this is not right. Even if the algebraic operators max and min are compatible with ordinal information, if we have an ordering of membership degrees, given by our expert, max and min are not necessarily the right operations for representing the union and the intersection.

In this paper, we will use the framework provided by measurement theory (see e.g. Krantz et al., 1971; Roberts, 1979) for showing which algebraic conditions can represent the union, the intersection or the complementation. We will also provide the conditions under which these algebraic operations are valid representations. We will assume that we can obtain from the expert some information (in the form of statements) that really makes sense for him. Not statements like "the membership of x in A is 0.35," because it is not clear at all how this should be interpreted, but statements like

- The object x belongs more to the set A than to the set B.
- The object x belongs more to the set A than to the set $B \cup C$.
- The object x belongs more to the set A than to the complement of A.
- The difference between the membership of x in A and that of y in A is larger

than the difference between the membership of x in A and that of z in A.

These pieces of information are called primitives. They constitute the empirically observable phenomenon that we want to represent with our measurement model. Note that the primitives are not explained within the model; they are not defined either. They are just structured observations. In particular, $\cup$ is not defined as the formal union of two sets. The expression $A \cup B$ represents whatever the subject or expert understands or thinks at when I tell him '$A \cup B$' or 'A or B'.

In each section, we will assume the existence of different primitives and we will see what kind of measurement technique we can use with those primitives. Thanks to this approach, we will prove different results that can be roughly summarized as follows.

- Having an ordering of the membership degrees is not enough for guaranteeing that max and min can represent the union and the operation (Section 2.1).
- Some reasonable conditions (first presented by Bollmann-Sdorra et al., 1993) can guarantee that max and min can represent the union and the operation (Section 2.2 and 3).
- There are techniques that permit us to measure membership degrees on interval scales or ratio scales. But, even with these techniques, no t-conorm and t-norm except the max and min can represent the union and the intersection (Section 2.3 and 4).
- It is not necessary to have an interval scale in order to use the standard negation "$1 - a$" for representing the complementation of the membership degree a. An ordering plus a mild condition is enough (Section 5).

It is important to remember that these results are derived within the framework of measurement theory and, therefore, are not universal. They hold only when we try to numerically represent the knowledge of an expert. But there are many contexts where we use fuzzy sets without trying to represent an expert's knowledge. For example, many automatic classification systems (based on fuzzy data or not) have as outcome a membership degree for each object in each class or category. These membership degrees do not represent the knowledge of an expert. They are the outcome of an algorithm (eventually based on some definitions) and their properties and meaning is completely determined by that algorithm. Measurement theory has nothing to say about such membership degrees ... unless the algorithm is supposed to mimic the reasoning of an expert.

Before going further with measurement theory, let us mention an alternative approach to the problem of measurement: psychometrics (Crocker and Algina, 1986, for example). It is a part of psychology, heavily relying on probability theory and aimed at measuring psychological characteristics of subjects (intelligence or neuroticism for example) or the (necessarily subjective) perception of a phenomenon by a subject. It could be used to construct a scale for the membership, with some particular properties but, as far as we know, it cannot help us to find a good representation for the union or the intersection. Let us also mention psychophysics (Gescheider, 1997, for a broad introduction), another part of pyschology, whose aim is to measure the perception of physical stimuli (light intensity, sound intensity or pitch, temperature, . . .) by subjects. There are two main schools in psychophysics: one is a psychometric one (e.g. Baird and Noma, 1978), the other is measurement-theoretic (e.g. Falmagne, 1985).

2 The membership of one object in several sets, in presence of a union and an intersection $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$.

In this section, the primitives are $\mathcal{F}$, a set whose elements can be interpreted as all fuzzy sets that are relevant in a particular context, $\succsim_x$, a binary relation on $\mathcal{F}$, $\cup$, a binary operation on $\mathcal{F}$, interpreted as the union of any two elements of $\mathcal{F}$, and $\cap$, another binary operation, interpreted as the intersection of any two elements of $\mathcal{F}$. The statement

$$A \succsim_x B \cup C$$

is interpreted as "the object x belongs at least as much to A as to $B \cup C$." Note that, with these primitives, it is not possible to express the fact that x belongs more to A than y belongs to A.

Several papers have dealt with the measurement of $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$. One is by Bollmann-Sdorra et al. (1993) and the others are by Bilgiç, Norwich and Türksen, in different combinations (Bilgiç, 1996; Bilgiç and Türksen, 1995, 1997, 2000; Norwich and Türksen, 1982; Türksen, 1991). The results presented in Bollmann-Sdorra et al. (1993) yield representations of $\cup$ and $\cap$ by means of the max and min operations. Most of the results presented by Bilgiç, Norwich and Türksen yield representations of $\cup$ by means of the Łukasiewicz t-conorm (or some generalizations), i.e. $S(a, b) = \min(1, a + b)$. I hereafter present the result of Bollmann-Sdorra et al. (1993) and one result typical of those found in the papers by Bilgiç, Norwich and Türksen.

2.1 A first representation theorem

If the relation $\succsim_x$ on $\mathcal{F}$ has the same structure as the relation $\geq$ on the reals and if $\mathcal{F}$ is not "too rich", then it is possible to find a representation for $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ in the real numbers. More formally, we need the following condition.

A 1 Weak Order. *A relation $\dot{\geq}$ on a set $\mathcal{S}$ is a weak order iff it satisfies*

- *Completeness. For all s, t in $\mathcal{S}$, $s \dot{\geq} t$ or $t \dot{\geq} s$.*
- *Transitivity. For all s, t, r in $\mathcal{S}$, $s \dot{\geq} t$ and $t \dot{\geq} r$ implies $s \dot{\geq} r$.*

The following result is then a straightforward application of a classical result in measurement theory (see e.g. Krantz et al., 1971).

Theorem 1 *Consider the structure $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ where $\mathcal{F}$ is a countable non-empty set closed under $\cup$ and $\cap$. The relation $\succsim_x$ on $\mathcal{F}$ satisfies Weak Order (A1) if and only if there exists a real-valued mapping η_x such that $\eta_x(A) \geq \eta_x(B)$ iff $A \succsim_x B$. Moreover, the representation is unique up to a strictly increasing transformation.*

So, under the conditions of this theorem, it is possible to measure the membership degree of x in various fuzzy sets $A, B, \ldots$ on an ordinal scale. I use here the symbol η for the membership function and not the more classical μ because I will use μ when representing the membership degree of various objects $x, y, \ldots$ in one fuzzy set.

It is often stated that, with an ordinal scale, the operations $\cup$ and $\cap$ must be represented by the max and min operators. In other words, the membership of x in $A \cup B$ can be derived from the membership in A and the membership in B by

$$\eta_x(A \cup B) = max(\eta_x(A), \eta_x(B)).$$

Note that Theorem 1 does not say such a thing. In fact, it does not say anything about $\cup$ and $\cap$. The following example shows that measuring the membership degrees on an ordinal scale is not enough to guarantee that $\cup$ and $\cap$ can be represented by max and min.

Example 1 *Let $\mathcal{F} = \{A, B, A \cup B\}$ and $A \cup A = A$, $B \cup B = B$, $A \cap A = A$, $B \cap B = B$, $A \cap B = B$, $(A \cup B) \cup A = A \cup B$, $(A \cup B) \cap A = A$, $(A \cup B) \cup B = A \cup B$, $(A \cup B) \cap B = B$. Let the relation $\succsim_x$ be the weak order $A \cup B \succ_x A \succ_x B$. We can represent this relation on an ordinal scale but it is clear that $\cup$ cannot be represented by the max. The problem is the fact that $A \cup B \succ_x A$. More conditions are needed and are presented in the following section.*

Note that when we look at the example above, we may be tempted to think that B is a subset of A because $A \cap B = B$; but this is just an interpretation, it is not a logical consequence. Do not forget that $\cap$ and $\cup$ do not denote mathematically defined operations. They denote two empirical operations that we observe by asking questions like "Does x belong more to 'A and B' or to

'B' ?" A priori, the operations $\cap$ and $\cup$ do not have any particular properties. We will later see what happens when they satisfy some nice properties.

2.2 The operations $\cup$ *and* $\cap$ *represented by* max *and* min

Bollmann-Sdorra et al. (1993) introduce the following axioms.

A 2 Order of Operations. *For all* A, B *in* $\mathcal{F}$, $A \cup B \succsim_x A \cap B$.

A 3 Weak Commutativity. *For all* A, B *in* $\mathcal{F}$,

$$A \cup B \sim_x B \cup A \text{ and}$$

$$A \cap B \sim_x B \cap A.$$

A 4 Weak Associativity. *For all* A, B, C *in* $\mathcal{F}$,

$$A \cup (B \cup C) \sim_x (A \cup B) \cup C \text{ and}$$

$$A \cap (B \cap C) \sim_x (A \cap B) \cap C.$$

A 5 Weak Absorption. *For all* A, B *in* $\mathcal{F}$,

$$A \sim_x A \cap (A \cup B) \text{ and}$$

$$A \sim_x A \cup (A \cap B).$$

A 6 Weak Right Monotonicity. *For all* A, B, C *in* $\mathcal{F}$,

$$A \succsim_x B \text{ implies } A \cap C \succsim_x B \cap C \text{ and}$$

$$A \succsim_x B \text{ implies } A \cup C \succsim_x B \cup C.$$

They then prove the following theorem (this is Theorem 2 in Bollmann-Sdorra et al. (1993), in a slightly different form) .

Theorem 2 *Consider the structure* $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ *where* $\mathcal{F}$ *is a countable non-empty set closed under* $\cup$ *and* $\cap$ *and the relation* $\succsim_x$ *on* $\mathcal{F}$ *satisfies Weak Order (A1). The structure satisfies Order of Operations (A2), Weak Commutativity (A3), Weak Associativity (A4), Weak Absorption (A5) and Weak Monotonicity (A6) if and only if there exists a real-valued mapping* η_x *such that*

(i) $\eta_x(A) \geq \eta_x(B)$ *iff* $A \succsim_x B$,
(ii) $\eta_x(A \cup B) = \max(\eta_x(A), \eta_x(B))$,
(iii) $\eta_x(A \cap B) = \min(\eta_x(A), \eta_x(B))$.

Moreover, the representation is unique up to a strictly increasing transformation. So, there is also a representation in $[0, 1]$.

With other words, Theorem 2 says the following: under some conditions (Weak Order, . . .), it is possible to measure the membership degree of x in various sets on an ordinal scale. Furthermore, there are algebraic operations (max and min) that can be used with the obtained membership degrees and that model the empirical operations $\cup$ and $\cap$. So, we have with this theorem a scale for the membership degrees and two operations that are compatible with this scale.

Some people may consider the representation obtained in Theorem 2 as a weak one because it is unique up to a strictly increasing transformation and we know other measurement-theoretic results where the representation is unique up to a positive affine transformation (for example, conjoint measurement) or even a positive linear transformation (for example, extensive measurement). But in conjoint or extensive measurement, if we find an additive representation, then we know that there are also multiplicative and many other representations : the representations of the operations are not at all unique. For example, if we apply extensive measurement (Krantz et al., 1971) to the measurement of mass, we find that under some conditions, there exist a representation $u : X \mapsto \mathbb{R}^+$ such that $u(x \oplus y) = u(x) + u(y)$ and $u(x) \geq u(y)$ iff $x \succsim y$, where X is a set of objects with different mass, $\oplus$ is the operation of putting two objects on the same side of a balance and $\succsim$ denotes the relation "heavier than" as observed by putting objects on the opposite sides of a balance. In this case, if we want an additive representation, i.e. $u(x \oplus y) = u(x) + u(y)$, then u is unique up to a positive linear transformation. But we can also find non-additive representations. For example a multiplicative one: we just define $u' = e^u$ and it is easy to check that $u'(x \oplus y) = u(x)u(y)$. In Theorem 2, on the contrary, even if we apply a transformation to the scale, the only possible representation for $\cup$ and $\cap$ are max and min. So the uniqueness of the scale in Theorem 2 may be weak but the uniqueness of the operations representation is much stronger than in most measurement-theoretic results.

The problem with this theorem is that the obtained operations are the max and min and not many fuzzy engineers or fuzzy system designers like these operations because they do not permit any kind of compensation. Yet in many applications, some kind of compensation seems desirable. So, it is natural to look for other conditions that would lead to other operations than the max and min and it is what by Bilgiç, Norwich and Türksen did. I present this in the next subsection.

2.3 The operation $\cup$ represented by the Łukasiewicz t-conorm and t-norm

I present here one result (Case 2 of Theorem 7 of Bilgiç and Türksen (1995)) among those found in the different papers by Bilgiç, Norwich and Türksen. Most of their results are very similar to this one or are generalizations of this one. But all share the same essential characteristic : Archimedeanness. This will be of some interest in our discussion. Here are the conditions used by Bilgiç and Türksen.

A 7 Weak $\cup$-Associativity. *For all A, B, C in $\mathcal{F}$,*

$$A \cup (B \cup C) \sim_x (A \cup B) \cup C.$$

A 8 Weak $\cup$-Monotonicity. *For all A, B, C in $\mathcal{F}$,*

$$A \succsim_x B \Rightarrow A \cup C \succsim_x B \cup C \text{ and } C \cup A \succsim_x C \cup B.$$

A 9 $\cup$-Solvability. *For all A, B in $\mathcal{F}$ such that $A \succ_x B$, there exists C such that $A \sim_x B \cup C$.*

A 10 $\cup$-Positivity. *For all A, B in $\mathcal{F}$, $A \cup B \succsim_x A, B$.*

A 11 $\cup$-Archimedeanness. *For all A, B in $\mathcal{F}$,*

$$\text{if } \bigcup_{i=1}^{n} A \prec_x B \text{ for all } n, \text{ then } A \text{ is an identity for } \cup,$$

that is $A \cup C \sim_x C$ for all C in $\mathcal{F}$.

A 12 $\cup$-Homogeneity. *For all A, B, C in $\mathcal{F}$,*

$$A \succsim_x B \Leftrightarrow A \cup C \succsim_x B \cup C \Leftrightarrow C \cup A \succsim_x C \cup B.$$

Note that $\cup$-Homogeneity implies the absence of a maximal element in $\mathcal{F}$. It can also be shown (Fuchs, 1963, Lemma C, p.163) that, if $\cup$-Homogeneity is not satisfied, then there is a maximal element u in $\mathcal{F}$ in the sense that $u \succsim x, \forall x \in \mathcal{F}$ and $\exists\, y \in \mathcal{F} : u \succ y$.

The representation theorem is then:

Theorem 3 *Consider the structure $\langle \mathcal{F}, \succsim_x, \cup \rangle$ where $\mathcal{F}$ is closed under $\cup$ and the relation $\succsim_x$ on $\mathcal{F}$ satisfies Weak Order (A1). If the structure satisfies Weak $\cup$-Associativity (A7), Weak $\cup$-Monotonicity (A8), $\cup$-Solvability (A9), $\cup$-Positivity (A10) and $\cup$-Archimedeanness (A11) but* NOT *$\cup$-Homogeneity (A12), then there exists a mapping η_x from $\mathcal{F}$ into $[0, 1]$ such that*

(i) *$\eta_x(A) \geq \eta_x(B)$ iff $A \succsim_x B$,*
(ii) *$\eta_x(A \cup B) = \min(1, \eta_x(A) + \eta_x(B))$.*

Some comments are now in order.

(1) Theorem 3 is in fact not exactly Case 2 of Theorem 7 in Bilgiç and Türksen (1995) but Case P1 of Theorem 2 in (Fuchs, 1963, p.165), which was originally proved in Hölder (1901). We do so because the theorem presented in Bilgiç and Türksen (1995) is not completely correct: they also cite Fuchs (1963) as their source but they omitted Positivity and Closedness.

(2) There is another error (or at least ambiguity) in Theorem 7 of Bilgiç and Türksen (1995). They claim that the obtained representation is on an absolute scale. It is indeed so if we impose that the representation be in $[0, 1]$ but we have no good reason to impose that. If we want to obtain an additive representation (this is the essence of Theorem 3), then the value 0 plays a special role but the value 1 can be replaced by any other positive number. So, the representation obtained in Theorem 3 can be multiplied by any positive real number α and we obtain then a representation in $[0, \alpha]$. The algebraic operation corresponding to $\cup$ is then $\min(\alpha, \eta_x(A) + \eta_x(B))$. The representation obtained in Theorem 3 is therefore on a ratio scale.

(3) The Archimedan condition looks very sensible in many different contexts but, here, it is completely inappropriate because $\cup$ is necessarily an idempotent operation, i.e. $A \cup A \sim_x A$, as shown in the following lines. We can expect some experimental violations of idempotency; for example, because of the limited cognitive abilities of the subjects, I would not be surprised if a subject would say

$$(A \cup (B \cap C)) \cup (A \cup (B \cap C)) \not\sim_x (A \cup (B \cap C)).$$

Such a statement is too complex. But if x is a person 1.95m tall, A= *small* and B= *tall*, then it is almost certain that a subject will say "the membership of x in the set *small or small* is the same as in the set *small*." In other words,

$$A \cup A \sim_x A.$$

Similarly, it is extremely unlikely that a subject says

$$A \cup A \cup A \cup \ldots \cup A \succsim_x B.$$

But if we want to empirically validate Theorem 3, we precisely need to observe such statements. These arguments show that Theorem 3, even if it is correct, cannot be applied to the measurement of membership degrees

(4) Bilgiç and Türksen (1995, p.21) give a wrong interpretation of the Archimedean condition. I quote them, with slightly different notations.

Let $A \succsim_x B$ stand for "John is funnier than he is bright." Archimedean axiom asserts that there should be a *finite amount* of "brightness" which, when attributed to John, makes John brighter than he is funny.

This interpretation is misleading. The Archimedean condition doesn't say anything about some changes in John or in x (the symbol that stands for John). It is only about the membership of John in different sets, one of which is an arbitrary long union of A with itself.

(5) It is probably possible to prove a similar theorem using another Archimedean axiom, using a standard sequence of different sets instead of a series of unions of A with itself. But this would not solve the problem: the obtained representation would still violate idempotency and this is not realistic.

(6) In the axiomatic papers about t-norms and t-conorms, idempotency is not always assumed ($S(a,a) = a$). Very often, instead, Archimedeannness is assumed ($S(a,a) > a$). But in that literature, a is a membership degree or a truth value and, when we consider $S(a,a)$, the two a's can be two identical membership degrees in two different fuzzy sets. Therefore, it is not completely unplausible that $S(a,a) > a$.

In the context of measurement, things are different: $A \cup A$ is the union of A with itself. This is why idempotency is unescapable and, therefore, there is no hope of obtaining the Łukasiewicz t-conorm as a representation for $\cup$, in the framework of measurement theory, even with different (but sensible) axioms.

We now turn to a different problem.

3 The membership of several objects in several sets, in presence of a union and an intersection $\langle \mathcal{F} \times X, \succsim, \cup, \cap \rangle$

Suppose we are interested not only by the membership of x in $A, B, \ldots$ but also by the membership of $y, z, \ldots$ in these different sets. Let $X = \{x, y, z, \ldots\}$ be the set of all objects x for which it is makes sense to consider a relation $\succsim_x$ on $\mathcal{F}$. We can then consider different structures similar to those of Section 2, one for each element in X: $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$, $\langle \mathcal{F}, \succsim_y, \cup, \cap \rangle$, $\langle \mathcal{F}, \succsim_z, \cup, \cap \rangle$,...

Under the conditions of Theorem 2, we then know that there is an ordinal representation $\eta_x, \eta_y, \eta_z, \ldots$ for each structure but each representation is independent of the other ones. It is therefore meaningless to make comparisons such as $\eta_x(A) \geq \eta_y(B)$ or even $\eta_x(A) \geq \eta_y(A)$. They do not make sense because we compare numbers that are on independent scales. If we want to make such comparisons, we then need a richer structure. Bollmann-Sdorra et al. (1993) propose to use the following structure: $\langle \mathcal{F} \times X, \succsim, \cup, \cap \rangle$. The relation $\succsim$ is on $\mathcal{F} \times X$ and the statement $(A, x) \succsim (B, y)$ is interpreted as "x belongs at least as much to A as y belongs to B."

From the relation $\succsim$ we can easily derive the relations $\succsim_A$ and $\succsim_x$ by means of the following equivalences :

$$x \succsim_A y \Leftrightarrow (A, x) \succsim (A, y) \text{ and} \tag{1}$$

$$A \succsim_x B \Leftrightarrow (A, x) \succsim (B, x). \tag{2}$$

Obiously, if $\succsim$ is a weak order, then $\succsim_A$ and $\succsim_x$ are weak orders for all x in X and all A in $\mathcal{F}$. As previously mentioned, imposing the conditions of Theorem 2 on each structure $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ is not enough to guarantee the existence of a representation for $\langle \mathcal{F} \times X, \succsim, \cup, \cap \rangle$. One additional condition is needed : the relation $\succsim$ must be a weak order.

Theorem 4 *Consider the structure* $\langle \mathcal{F} \times X, \succsim, \cup, \cap \rangle$ *where the relation* $\succsim$ *on* $\mathcal{F} \times X$ *satisfies Weak Order (A1),* X *is countable and* $\mathcal{F}$ *is a countable non-empty set closed under* $\cup$ *and* $\cap$*. Each structure* $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ *derived according to (2) satisfies Order of Operations (A2), Weak Commutativity (A3), Weak Associativity (A4), Weak Absorption (A5) and Weak Monotonicity (A6) if and only if there exists* $\mu : \mathcal{F} \times X \mapsto \mathbb{R} : (A, x) \mapsto \mu_A(x)$ *such that*

(*i*) $\mu_A(x) \geq \mu_B(y)$ *iff* $(A, x) \succsim (B, y)$,
(*ii*) $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$,
(*iii*) $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$.

Moreover, the representation is unique up to a strictly increasing transformation. So, there is also a representation in $[0, 1]$.

The statement of this theorem is simpler but perfectly equivalent to Theorem 3 in Bollmann-Sdorra et al. (1993). Under the conditions of Theorem 4, it is perfectly licit to make comparisons of membership degrees across sets and elements and we can represent the union and intersection by max and min.

4 Difference measurement

In Sections 2 and 3, we have seen that measuring the membership degrees on an ordinal scale does not imply that we may use the max and min operations to represent $\cup$ and $\cap$. Some additional conditions are needed: these conditions are those of Theorem 2 and 4. In this section, we will see that measuring the membership degrees on an interval scale does not entail the use of "cardinal" operators (operators that use the cardinal properties of the membership degrees).

4.1 The membership of several objects in one set: $\langle X, \succsim_A^* \rangle$

We present here a result found in Norwich and Türksen (1982) and Türksen (1991). It can also be found in Krantz et al. (1971, p.151), under the heading *Algebraic Difference Measurement.*

The relation $\succsim_A^*$ is on X^2 and the statement $(x, y) \succsim_A^* (w, z)$ is interpreted as "the difference between the membership of x and y in A is at least as large as the difference between the membership of w and z in A." In the following definition, we summarize the axioms that we will use in our first representation theorem for a difference measurement structure.

Definition 1 *The structure $\langle S, \dot{\geq} \rangle$ is an algebraic difference structure iff, for all x, y, z, w, x', y', and z' in S and all sequences $x_1, x_2, \ldots, x_i, \ldots$ in S, the following five axioms are satisfied:*

(1) The relation $\dot{\geq}$ on S^2 satisfies Weak Order (A1).
(2) If $(x, y) \dot{\geq} (w, z)$, then $(z, w) \dot{\geq} (y, x)$.
(3) If $(x, y) \dot{\geq} (x', y')$ and $(y, z) \dot{\geq} (y', z')$, then $(x, z) \dot{\geq} (x', z')$.
(4) If $(x, y) \dot{\geq} (z, w) \dot{\geq} (x, x)$, then there exist w', w'' in S such that $(x, w') \doteq (z, w) \doteq (w'', y)$.
(5) If $(x_{i+1}, x_i) \doteq (x_2, x_1)$ for all x_{i+1}, x_i in a sequence, if it is not the case that $(x_2, x_1) \doteq (x_1, x_1)$ and if there exists $w', w'' \in S$ such that $(w', w'') \dot{>} (x_i, x_1) \dot{>} (w'', w')$ for all x_i in the sequence (we say $x_1, x_2, \ldots, x_i, \ldots$ is a strictly bounded standard sequence), then the sequence is finite.

Theorem 5 *Suppose the structure $\langle X, \succsim_A^* \rangle$ is an algebraic difference structure. Then, there is a real-valued mapping μ_A such that*

$$\mu_A(x) - \mu_A(y) \geq \mu_A(w) - \mu_A(z) \text{ iff } (x, y) \succsim_A^* (w, z).$$

Moreover, the representation is unique up to a positive affine transformation, i.e. $\mu'_A = \alpha\mu_A + \beta$ is also a representation.
If we suppose in addition that there are u and e in X such that $(u, e) \succsim_A^* (x, y)$, $\forall x, y \in X$, then there is a representation in $[0, 1]$, i.e. $\mu_A(x) \in [0, 1]$, $\forall x \in X$.

In other words, under the conditions of this theorem, we can measure the membership on an interval scale. But it measures the membership in one single set A. If we want to measure the membership in B, we have to consider another algebraic difference structure and measure it independently of the structure related to A. So, we get two completely unrelated representations and this theorem does not tell anything about the representation of $\cup$ and $\cap$. Therefore, the use of any t-conorm such as max, Łukasiewicz, probabilistic sum, . . . is unsupported by this theorem.

We now turn to a richer structure, in order to measure simultaneously the membership in several sets.

4.2 The membership of several objects in several sets: $\langle \mathcal{F} \times X, \succsim^* \rangle$

The relation $\succsim^*$ is on $(\mathcal{F} \times X)^2$ and the statement $(A, x, B, y) \succsim^* (C, w, D, z)$ is interpreted as "the difference between the membership of x in A and y in B is larger than the difference between the membership of w in C and z in D." Note that such statements are very complex from a cognitive viewpoint and it is not obvious that a subject can make such statements in a consistent way. But we may not a priori reject this measurement technique without conducting first an empirical study.

Theorem 6 *Suppose the structure* $\langle \mathcal{F} \times X, \succsim^* \rangle$ *is an algebraic difference structure. Then, there is a mapping* $\mu : \mathcal{F} \times X \mapsto \mathbb{R} : (A, x) \mapsto \mu_A(x)$ *such that*

$$\mu_A(x) - \mu_B(y) \geq \mu_C(w) - \mu_D(z)$$

$$\textit{iff } (A, x, B, y) \succsim^* (C, w, D, z).$$

Moreover, the representation is unique up to a positive affine transformation, i.e. $\mu' = \alpha\mu + \beta$ *is also a representation.*
If we suppose in addition that there are u, e *in* X *and* A, B *in* $\mathcal{F}$ *such that* $(A, u, B, e) \succsim^* (C, x, D, y)$*, for all* $x, y \in X$ *and all* $C, D \in \mathcal{F}$*, then there is a representation in* $[0, 1]$*, i.e.* $\mu_A(x) \in [0, 1]$*, for all* $x \in X$ *and all* $A \in \mathcal{F}$*.*

So, we now have a common interval scale for the membership degrees of several objects in several sets. But, because this theorem does not tell anything about $\cup$ and $\cap$, we still do not know if we can obtain a numerical representation for these two empirical operations.

Those who are familiar with meaningfulness theory (e.g. Roberts, 1979) might now say that we can use a t-conorm like the Łukasiewicz one because μ is measured on one common interval scale for all sets. In some sense, this is right : a statement like "$\min(1, \mu_A(x)+\mu_B(x)) > \min(1, \mu_A(y)+\mu_B(y))$" is meaningful. If it is true, then it remains true after a positive affine transformation. But the problem is that, even if it is meaningful, we do not know if the Łukasiewicz t-conorm models what we want to model, i.e. the union of two sets. Meaningfulness theory doesn't say a word about this.

In order to obtain a numerical representation for $\cup$ and $\cap$, we still need a richer structure.

4.3 The membership of several objects in several sets, in presence of a intersection and an intersection $\langle \mathcal{F} \times X, \succsim^*, \cup, \cap \rangle$

From the relation $\succsim^*$, it is possible to derive the relation $\succsim$ by means of

$$(A, x) \succsim (B, y) \text{ iff } (A, x, B, y) \succsim^* (A, x, A, x). \tag{3}$$

It is also possible to derive $\succsim_x$ from $\succsim$ by means of (2). In the next theorem, I use the interval representation of Theorem 6 and I add some conditions under which there is a representation for $\cup$ and $\cap$.

Theorem 7 *Suppose the structure $\langle \mathcal{F} \times X, \succsim^* \rangle$ is an algebraic difference structure. Suppose also that the relation $\succsim_x$ derived according to (2) and (3) satisfies Order of Operations (A2), Weak Commutativity (A3), Weak Associativity (A4), Weak Absorption (A5) and Weak Monotonicity (A6) for all $x \in X$. Then, there is a mapping $\mu : \mathcal{F} \times X \mapsto \mathbb{R} : (A, x) \mapsto \mu_A(x)$ as in Theorem 6 and such that*

(i) $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$,
(ii) $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$.

Moreover, the representation is unique up to a positive affine transformation, i.e. $\mu' = \alpha\mu + \beta$ is also a representation.
If we suppose in addition that there are u, e in X and A, B in $\mathcal{F}$ such that $(A, u, B, e) \succsim^* (C, x, D, y)$, for all $x, y \in X$ and all $C, D \in \mathcal{F}$, then there is a representation in $[0, 1]$, i.e. $\mu_A(x) \in [0, 1]$, for all $x \in X$ and all $A \in \mathcal{F}$.

In the proof of this theorem, we will need a variant of Theorem 4 for sets that are possibly uncountable. I present this variant as a lemma.

Lemma 1 *Consider the structure $\langle \mathcal{F} \times X, \succsim, \cup, \cap \rangle$ where $\mathcal{F}$ is closed under $\cup$ and $\cap$. Suppose that $\succsim$ has a representation $\mu : \mathcal{F} \times X \mapsto \mathbb{R} : (A, x) \mapsto \mu_A(x)$ such that*

$$\mu_A(x) \geq \mu_B(y) \textit{ iff } (A, x) \succsim (B, y).$$

Each structure $\langle \mathcal{F}, \succsim_x, \cup, \cap \rangle$ derived according to (2) satisfies Order of Operations (A2), Weak Commutativity (A3), Weak Associativity (A4), Weak Absorption (A5) and Weak Monotonicity (A6) if and only if μ is such that

(i) $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$,
(ii) $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$.

Moreover, the representation is unique up to a strictly increasing transformation.

The proof of this lemma is identical, except for the first few lines, to the proof of Theorem 4 given by Bollmann-Sdorra et al. (1993). I give it for the sake of completeness.

Proof of Lemma 1. I only prove the *only if* part. The relation $\succsim$ has a numerical representation μ; it is therefore a weak order. By construction, each relation $\succsim_x$ is also a weak order and can also be represented by μ. Let us now look at some properties of $\succsim_x$. By Weak Absorption,

$$A \sim_x A \cup (A \cap B).$$

By Weak Monotonicity,

$$A \cap A \sim_x A \cap (A \cup (A \cap B)) \sim_x A \cap (A \cup C),$$

with $C = A \cap B$. By Weak Absorption again and by Transitivity, we obtain $A \cap A \sim_x A$, i.e. the Idempotence of $\cap$. Similarly, we can prove the Idempotence of $\cup$.

By Order of Operations, $A \cup B \succsim_x A \cap B$. Applying then Weak Monotonicity, Weak Associativity, Transitivity, Weak Absorption and Idempotence of $\cap$, we get

$$\begin{aligned} & A \cap (A \cup B) \succsim_x A \cap (A \cap B) \\ \Leftrightarrow\ & A \cap (A \cup B) \succsim_x (A \cap A) \cap B \\ \Leftrightarrow\ & A \succsim_x A \cap B. \end{aligned} \tag{4}$$

We can similarly prove that $A \cup B \succsim_x A$. Suppose now that $A \succsim_x B$. From Idempotence of $\cap$, (4) and Weak Monotonicity, we find

$$B \succsim_x A \cap B \succsim_x B \cap B \sim_x B.$$

So,

$$A \succsim_x B \Rightarrow A \cap B \sim_x B.$$

We can prove in the same way that

$$A \succsim_x B \Rightarrow A \cup B \sim_x A.$$

We are now close to the end. We have already proven that $\succsim$ and $\succsim_x$ can be represented by μ. The only thing we still must prove is (*i*) and (*ii*). Let us prove (*i*). Without loss of generality, consider A and B such that $A \succsim_x B$. We know that $A \cup B \sim_x A$. So, $\mu_{A \cup B}(x) = \mu_A(x) = \max(\mu_A(x), \mu_B(x))$. The same resoning holds for (*ii*). □

Proof of Theorem 7. Because $\langle \mathcal{F} \times X, \succsim^* \rangle$ is an algebraic difference structure, we have a representation as in Theorem 6, unique up to a positive affine transformation. Let μ denote this representation. By construction, $\succsim$ can also be represented by μ in the following way:

$$(A, x) \succsim (B, y) \Leftrightarrow \mu_A(x) \geq \mu_B(y).$$

This representation can be used in Lemma 1, and the proof is complete. The uniqueness follows from Theorem 6. □

Finally, we have a numerical representation for the membership degrees, the union and the intersection but the numerical operations associated to ∪ and ∩ are the max and min operations, although the representation is on an interval scale. Note that, even with this interval scale, there is no hope to obtain the Łukasiewicz t-norm or any non-idempotent t-norm.

Another remark concerns the maximal and minimal elements u and e: u can be interpreted as an element that fully belongs to A and e as an element that does not belong at all to B. The membership degree of u in A is 1 with the representation of Theorem 7 in $[0, 1]$. The membership degree of e in B is 0 with the same representation. Suppose now there is another element that fully belongs to A (or does not belong at all to B); its membership degree in A (resp. in B) is necessarily 1 (resp. 0). But consider now an element that does not belong at all to A or C, D, ... Its membership degree in that set needs not be 0. Similarly, an element that fully belongs to B, C, ... needs not have a membership degree in that set equal to 1.

In the following theorem, we give (without proof) the conditions guaranteeing that the membership degree is 1 (resp. 0) for an element fully belonging to a set (resp. not belonging at all to a set). Let u_A denote an element that fully belongs to A. Similarly, e_A is an element that does not belong at all to A.

Theorem 8 *Suppose all conditions of Theorem 7 are satisfied, including the existence of u and e. There is a representation μ as in Theorem 7 and such that $\mu_A(u_A) = 1$ and $\mu_A(e_A) = 0$ for all $A \in \mathcal{F}$ if and only if for all $A, B, C \in \mathcal{F}$ we have $(A, u_A, A, e_A) \succsim^* (B, u_B, C, e_C)$.*

5 When there is a complementation operation

The complementation is an important unary operation in different fuzzy sets applications. It is often modelled by the simple relation

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

where $\neg A$ denotes the complement of the set A. We call this the classical complementation. Sometimes, the complementation is the same as the negation operation but sometimes it is not, depending on the semantics. In this paper, we focus on the complementation operation, whether or not it is also a negation.

Just like it is important to know if the union or intersection can be modelled by some arithmetic operation, it is also important to know if the complementation is correctly modelled by the classical complementation or some other equation. Or we might want to know how to measure the membership degrees in order to be allowed to use the classical complementation. One might also wonder if it is right to use the classical complementation if the membership degrees are measured on an ordinal scale. All these issues are addressed in this section.

5.1 *Basic result. The membership of one object in several sets, in presence of a complementation: $\langle \mathcal{F}, \succsim_x, \neg \rangle$*

Definition 2 *The structure $\langle \mathcal{F}, \succsim_x, \neg \rangle$ where $\mathcal{F}$ is closed under $\neg$ and $\succsim_x$ is a weak order, is a complementation structure iff*

A 13 Reversal. *For all $A, B, \in \mathcal{F}$, $A \succsim_x B \Leftrightarrow \neg B \succsim_x \neg A$.*

A 14 Involution. *For all A in $\mathcal{F}$, $\neg\neg A \sim_x A$.*

Remark that Involution does not impose that $\neg\neg A = A$. An expert might consider A and $\neg\neg A$ as two different sets but nevertheless say that the membership of x in A is the same as in $\neg\neg A$. The next theorem plays a central role for the measurement of complementation structures.

Theorem 9 *Let $\mathcal{F}$ be countable. The structure $\langle \mathcal{F}, \succsim_x, \neg \rangle$ is a complementation structure iff there exists $\eta_x : \mathcal{F} \mapsto [0, 1]$ such that*

- $A \succsim_x B \Leftrightarrow \eta_x(A) \geq \eta_x(B)$. (5)
- $\eta_x(\neg A) = 1 - \eta_x(A)$. (6)

Another mapping $\eta'_x : \mathcal{F} \mapsto \mathbb{R}$ satisfies (5) and (6) iff $\eta'_x = \gamma(\eta_x)$, where $\gamma : [0, 1] \mapsto \mathbb{R}$ is strictly increasing and symmetric around 0.5, i.e. such that

$$\gamma(x) - \gamma(0.5) = \gamma(0.5) - \gamma(1 - x).$$

The representation is then in $[\gamma(0), \gamma(1)]$ and the complementation is represented by $\gamma(0) + \gamma(1) - x$.

Proof. We first prove the "if" part.

- *Completeness.* For all $A, B, \eta_x(A) \geq \eta_x(B)$ or $\eta_x(B) \geq \eta_x(A)$. Therefore, $A \succsim_x B$ or $B \succsim_x A$.
- *Transitivity.* Same reasoning as for completeness.
- *Reversal.* $A \succsim_x B \Rightarrow \eta_x(A) \geq \eta_x(B) \Rightarrow 1 - \eta_x(A) \leq 1 - \eta_x(B) \Rightarrow \eta_x(\neg A) \leq \eta_x(\neg B) \Rightarrow \neg B \succsim_x \neg A$.
- *Involution.* $\eta_x(\neg\neg A) = 1 - \eta_x(\neg A) = \eta_x(A) \Rightarrow \neg\neg A \sim_x A$.

We now turn to the "only if" part. We do this in a constructive way, using some pseudo-code.

Take any A in $\mathcal{F}$ and call it F_1. By Completeness, three cases are possible:

`IF` $F_1 \succ_x \neg F_1$, `THEN` fix $\eta_x(F_1) = 0.75$ and $\eta_x(\neg F_1) = 0.25$.

`IF` $\neg F_1 \succ_x F_1$, `THEN` fix $\eta_x(\neg F_1) = 0.75$ and $\eta_x(F_1) = 0.25$.

`IF` $F_1 \sim_x \neg F_1$, `THEN` fix $\eta_x(F_1) = 0.5 = \eta_x(\neg F_1)$.

Fix $k = 2$.

(1) Take any A in $\mathcal{F}$, $A \neq F_1, \ldots, F_{k-1}, \neg F_1, \ldots, \neg F_{k-1}$. Call it F_k. By Completeness, three cases are possible:

`IF` $F_k \sim_x \neg F_k$, `THEN` fix $\eta_x(F_k) = \eta_x(\neg F_k) = 0.5$.

`IF` $F_k \succ_x \neg F_k$, `THEN` let

$$u^+(F_k) = \min_{j<k}(\eta_x(F_j) : F_j \succsim_x F_k; \eta_x(\neg F_j) : \neg F_j \succsim_x F_k; 1).$$

Let also

$$l^+(F_k) = max_{j<k}(\eta_x(F_j) : F_j \precsim_x F_k; \eta_x(\neg F_j) : \neg F_j \precsim_x F_k; 0.5).$$

Fix then

$$\eta_x(F_k) = \frac{u^+(F_k) + l^+(F_k)}{2}$$

and $\eta_x(\neg F_k) = 1 - \eta_x(F_k)$.

`IF` $F_k \prec_x \neg F_k$, `THEN` let

$$u^-(F_k) = \min_{j<k}(\eta_x(F_j) : F_j \succsim_x F_k; \eta_x(\neg F_j) : \neg F_j \succsim_x F_k; 0.5).$$

Let also

$$l^-(F_k) = max_{j<k}(\eta_x(F_j) : F_j \precsim_x F_k; \eta_x(\neg F_j) : \neg F_j \precsim_x F_k; 0).$$

Fix then

$$\eta_x(F_k) = \frac{u^-(F_k) + l^-(F_k)}{2}$$

and $\eta_x(\neg F_k) = 1 - \eta_x(F_k)$.

`IF` $\{F_i, \neg F_i, i = 1 \ldots k\} = \mathcal{F}$, `THEN STOP`

ELSE $k := k + 1$ and GOTO (1).

Because $\mathcal{F}$ is countable, it is clear that every time we want to fix a value $\eta_x(F_i)$, this is possible. Remark also that, by Involution, η_x is well defined. Indeed, if $F_k = \neg A$, we fix $\eta_x(\neg A)$ and $\eta_x(\neg\neg A)$ at step k. If $F_l = A$ ($l > k$), then we fix $\eta_x(A)$ at step l but also $\eta_x(\neg A)$. But this value has already been fixed at step k. So, these two values must be identical and, thanks to Involution, they are, because $u^+(A) = l^+(A) = \eta_x(\neg\neg A)$ or $u^-(A) = l^-(A) = \eta_x(\neg\neg A)$.

At each step i, we fix $\eta_x(F_i)$ between $u^+(F_i)$ and $l^+(F_i)$ (or $u^-(F_i)$ and $l^-(F_i)$), i.e. between the values already given to larger sets (w.r.t. $\succsim_x$) and smaller sets in previous steps. Because $\succsim_x$ is a weak order, we always have $u^+(F_i) \geq l^+(F_i)$ and $u^-(F_i) \geq l^-(F_i)$ and so, $A \succsim_x B$ implies $\eta_x(A) \geq \eta_x(B)$. Therefore, (5) is satisfied. By construction, when we fix $\eta_x(\neg A) = 1 - \eta_x(A)$, we are sure that the obtained representation satisfies (6) and, thanks to Reversal, we are also sure that it satisfies (5). □

It is time to make some comments about this theorem.

- The representation obtained in Theorem 9 is not on an ordinal scale nor on an interval scale. It is on a scale that we could call an ordinal scale with a fixed point. This fixed point is 0.5 and corresponds to a set A such that $A \sim_x \neg A$.
- It is worth remarking that although the membership degree η_x is measured on an almost ordinal scale, we can model the complementation by an arithmetic operation involving a subtraction even if the subtraction is often thought of as an operation that can be used only with interval and ratio scales. This is of great importance in possibility theory (e.g. Dubois and Prade, 1988), where the possibility and necessity of a set is considered as ordinal and where the operation $1 - a$ is used for transforming a possibility into a necessity and vice versa.
- Note that the axioms of this theorem do not preclude other representations for the complementation. If the axioms of this theorem are satisfied, then any involutive complementation can be used, provided that the corresponding representation is used. If the membership degrees are measured using the construction presented in the proof of Theorem 9, then the classical complementation must be used.
- Because the representation obtained in Theorem 9 is based only on one relation $\succsim_x$, it does not make sense to compare two membership degrees of two different objects. In other words, any statement involving η_x and η_y, with x and y distinct, is meaningless.

5.2 *The membership of one object in several sets, in presence of a complementation, a union and an intersection* $\langle \mathcal{F}, \succsim_x, \cup, \cap, \neg \rangle$

Now, we look at what happens when we have three empirical operations: the union and the intersection, as in Section 3 and the complementation, as in the previous subsection. Because the operations $\cup, \cap$ and $\neg$ are related by the De Morgan law in classical set theory, it seems interesting to introduce the De Morgan law in our context. We therefore state the De Morgan law formally and another condition that will prove interesting in our analysis.

A 15 De Morgan law. *For all* $A, B, \in \mathcal{F}, \neg(\neg A \cup \neg B) \sim_x A \cap B$.

A 16 Equivalence Conservation. *For all* $A, B, \in \mathcal{F}, A \sim_x B \Leftrightarrow \neg A \sim_x \neg B$.

Note that Reversal implies Equivalence Conservation but the converse is not true.

Theorem 10 *Consider the structure* $\langle \mathcal{F}, \succsim_x, \cup, \cap, \neg \rangle$ *where* $\succsim_x$ *is a weak order and* $\mathcal{F}$ *is a countable non-empty set closed under* $\cup, \cap$ *and* $\neg$. *Suppose this structure satisfies Order of Operations (A2), Weak Commutativity (A3), Weak Associativity (A4), Weak Absorption (A5) and Weak Monotonicity (A6). Then the following three statements are equivalent.*

(1) The structure $\langle \mathcal{F}, \succsim_x, \neg \rangle$ *satisfies Reversal (A13) and Involution (A14).*
(2) The De Morgan law (A15) holds and the structure $\langle \mathcal{F}, \succsim_x, \neg \rangle$ *satisfies Equivalence Conservation (A16).*
(3) There exists $\eta_x : \mathcal{F} \mapsto [0, 1] : A \mapsto \eta_x(A)$ *such that*
(i) $\eta_x(A) \geq \eta_x(B)$ *iff* $A \succsim_x B$,
(ii) $\eta_x(A \cup B) = \max(\eta_x(A), \eta_x(B))$,
(iii) $\eta_x(A \cap B) = \min(\eta_x(A), \eta_x(B))$.
(iv) $\eta_x(\neg A) = 1 - \eta_x(A)$.
Moreover, another mapping $\eta'_x : \mathcal{F} \mapsto \mathbb{R}$ *satisfies (i), (ii), (iii) and (iv) iff* $\eta'_x = \gamma(\eta_x)$, *where* $\gamma : [0, 1] \mapsto \mathbb{R}$ *is strictly increasing and symmetric around 0.5.*

Proof. (1 ⇒ 3) Using Theorem 2, we get a representation of $\succsim_x$, $\cup$ and $\cap$ which is unique up to a strictly increasing transformation. Using Theorem 9, we get a representation of $\succsim_x$ and $\neg$ which is slightly more restricted: it has a fixed point. Both representations are obviously compatible because they both represent $\succsim_x$. So, we can choose the more restricted one as our unique representation.

(3 ⇒ 2) Obvious.

(2 $\Rightarrow$ 1) By Theorem 2, we know that $\cup$ and $\cap$ are represented by max and min. They are therefore idempotent. Assume now the De Morgan law. Then $\neg(\neg A \cup \neg A) \sim_x A \cap A$. So, by Idempotence, $\neg(\neg A) \sim_x A$. This proves Involution.

Suppose now it does not satisfy Reversal. Then, there exists $A, B : A \succ_x B$ and $\neg A \succ_x \neg B$. Because $\cap$ is represented by min, $A \cap B \sim_x B$. By Equivalence Conservation, $\neg(A \cap B) \sim_x \neg B$. Using the min, we have $\neg A \cup \neg B \sim_x \neg A$. Using the De Morgan law and Equivalence Conservation, this implies $\neg(A \cap B) \sim_x \neg A$. So, by transitivity, $\neg A \sim_x \neg B$. This contradicts what we obtained earlier and shows that Reversal must hold. Note that the case $A \succ_x B$, $\neg A \sim_x \neg B$ is trivially ruled out by Equivalence Conservation. □

Remark that Theorem 10 involves only one relation $\succsim_x$. It is therefore meaningless to compare membership degrees of different objects. The only licit comparisons can only involve x. In the following theorem we address the problem of measuring the membership degrees of different objects simultaneously. We therefore modify Theorem 4 in order to include the complementation in the structure. Because the scales that we get from each relation $\succsim_x$ are independent and because they are not purely ordinal (they have a fixed point), it is not necessarily possible to make them coincide. We therefore introduce a new axiom.

A 17 Global Reversal. *For all $A, B \in \mathcal{F}$ and all $x, y \in X$, $(A, x) \succsim (B, y) \Leftrightarrow (\neg B, y) \succsim (\neg A, x)$.*

Theorem 11 *Consider the structure $\langle \mathcal{F} \times X, \succsim, \cup, \cap, \neg \rangle$ where $\succsim$ is a weak order and $\mathcal{F}$ is a countable non-empty set closed under $\cup$, $\cap$ and $\neg$. Let $\succsim_x$ be derived from $\succsim$ according to (2). Suppose that each structure $\langle \mathcal{F}, \succsim_x, \cup, \cap, \neg \rangle$ satisfies Order of Operations (A2), Weak Commutativity (A3), Weak Associativity (A4), Weak Absorption (A5), Weak Monotonicity (A6) and Involution (A14). Then, the structure $\langle \mathcal{F} \times X, \succsim, \neg \rangle$ satisfies Global Reversal (A17) if and only if there exists $\mu : \mathcal{F} \times X \mapsto \mathbb{R} : (A, x) \mapsto \mu_A(x)$ such that*

(i) $\mu_A(x) \geq \mu_B(y)$ *iff* $(A, x) \succsim (B, y)$,
(ii) $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$,
(iii) $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$.
(iv) $\mu_{\neg A}(x) = 1 - (\mu_A(x))$.

Moreover, another mapping $\eta'_x : \mathcal{F} \mapsto \mathbb{R}$ satisfies (i), (ii), (iii) and (iv) iff $\eta'_x = \gamma(\eta_x)$, where $\gamma : [0, 1] \mapsto \mathbb{R}$ is strictly increasing and symmetric around 0.5.

Proof. Using Theorem 10, we obtain $\eta_x, \eta_y, \ldots$ such that $A \succsim_x B \Leftrightarrow \eta_x(A) \geq \eta_x(B)$ and $\eta_x(\neg A) = 1 - \eta_x(A)$. Let $\phi_x, \phi_y, \phi_z, \ldots$ be $[0, 1]$ to $[0, 1]$ increasing mappings such that $\phi_x(0.5) = 0.5$ and $\phi_x(u) + \phi_x(1-u) = 1, \forall x \in X$ and $\forall u \in [0,1]$. Because of Theorem 10, we know that $\phi_x(\eta_x)$ is also a representation of $\langle \mathcal{F}, \succsim_x, \cup, \cap, \neg \rangle$ and we are free to choose any form for ϕ_x on the interval $[0, 0.5]$. So, it is possible to choose $\phi^\circ_x, \phi^\circ_y, \phi^\circ_z, \dots$ such that

$$(A,x) \succsim (B,y) \Leftrightarrow \phi^\circ_x(\eta_x(A)) \geq \phi^\circ_y(\eta_y(B))$$

for all A, B in $\mathcal{F}$ and all x, y in X such that $\eta_x(A) \leq 0.5$ and $\eta_y(B) \leq 0.5$. Because we have now specified ϕ°_x for all values smaller than 0.5 and because $\phi^\circ_x(u) + \phi^\circ_x(1-u) = 1$, the mapping ϕ°_x is completely specified, for all values between 0 and 1. We must therefore check if $\phi^\circ_x(\eta_x)$ is also a representation for the pairs (A, x) such that $\eta_x(A) > 0.5$. In other words, we must prove that

$$(A,x) \succsim (B,y) \Leftrightarrow \phi^\circ_x(\eta_x(A)) \geq \phi^\circ_y(\eta_y(B)) \text{ when } \eta_x(A) > 0.5, \eta_y(B) > 0.5.$$

Because of the properties of ϕ°_x and η_x, we know that

$$\begin{aligned}(A,x) \succsim (B,y) &\Leftrightarrow (\neg B,y) \succsim (\neg A,x) \quad \text{(Global reversal)}\\ &\Leftrightarrow 0.5 \geq \phi^\circ_y(\eta_y(\neg B)) \geq \phi^\circ_x(\eta_x(\neg A))\\ &\Leftrightarrow \phi^\circ_x(\eta_x(A)) \geq \phi^\circ_y(\eta_y(B)).\end{aligned}$$

When $\eta_x(A) > 0.5$ and $\eta_y(B) \leq 0.5$, or the converse, it is evident that

$$(A,x) \succsim (B,y) \Leftrightarrow \phi^\circ_x(\eta_x(A)) \geq \phi^\circ_y(\eta_y(B)).$$

If we now define $\mu_A(x) = \phi^\circ_x(\eta_x(A))$, the proof is complete. □

The following example shows why Reversal is not sufficient in the previous theorem and why Global Reversal is needed. Let $\mathcal{F} = \{A, B, \neg A, \neg B\}, X = \{x, y\}$. Let also $A \cup B = B, A \cap B = A, A \cup \neg B = A, A \cap \neg B = \neg B, \neg A \cup B = B, \neg A \cap B = \neg A, \neg A \cup \neg B = \neg A, \neg A \cap \neg B = \neg B, \neg\neg A = A, \neg\neg B = B$. Then $\mathcal{F}$ is closed under $\cup, \cap$ and $\neg$ and the De Morgan law holds. Suppose moreover that $(A,x) \succ (B,y) \succ (B,x) \succ (\neg A,y) \succ (\neg B,x) \succ (A,y) \succ (\neg A,x) \succ (\neg B,y)$. Thus, $A \succ_x B \succ_x \neg B \succ_x \neg A$ and $B \succ_y \neg A \succ_y A \succ_y \neg B$. So, Reversal holds in $\succsim_x$ and $\succsim_y$. It is easy to check that all conditions of Theorem 11 are satisfied except Global Reversal : we have $(B,x) \succ (\neg A,y)$ and $(\neg B,x) \succ (A,y)$. This cannot be represented by the mapping μ of Theorem 11.

6 Conclusion

We examined a variety of measurement techniques, with different primitives and our main conclusion is the following: inside the framework of measurement theory, the only t-norm and t-conorm that can represent the intersection and the union of fuzzy sets are the min and the max because they are the only idempotent ones. But this does not means that the min and the max are always good representations: some conditions need to be fulfilled. Otherwise no representation might exist. Besides, it seems possible to measure the membership on ordinal or interval scales. Finally, the complementation can be represented by "$1 - \cdot$" even if the membership degrees are not on an interval scale.

The results presented in this paper are theoretical. They give the conditions under which it is possible to use some measurement technique and obtain some representation. A second and necessary step in this research is to see which conditions empirically hold, i.e. which conditions are respected by humans when they make statements like "x belongs more to A than to B." It is only after such an empirical research that we will know which measurement technique can actually be used when trying to model the expertise of a human.

We did not examine all possible measurement techniques and all kinds of primitives. For example, we did not examine Stevens' ratio estimation technique Stevens (1986). It might be interesting for measuring the membership on a ratio scale. This will be done in another paper, in preparation. But, even if this technique proves valuable, it does not change the validity of the results and the soundness of the conclusions of the present paper.

Another possible direction for future resarch is working with an ordered set $\mathcal{F}$, with an underlying dimension that we can measure in some way. For example, $\mathcal{F} = \{$freezing, cold, cool, warm, hot$\}$. The underlying dimension is here air temperature and we can measure it on a ratio scale. Let X be a set of different atmospheric conditions that are identical on all dimensions (humidity, pressure, concentration in CO_2, ...) except temperature. All these atmospheric conditions are thus characterized by their temperature $t(x)$. Instead of constructing membership functions mapping X into $[0, 1]$, we might then construct membership functions mapping $[0, \infty]$ into [0,1], where $[0, \infty]$ is the set of all possible temperatures, 0 being the absolute zero (-273.15 on the Celsius scale). All results presented in this paper remain valid for ordered sets but it might be possible to go further with ordered sets and ordered continua (as in our example of temperature).

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