The characterization of affine maximizers on restricted domains with two alternatives.

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Abstract

In the framework of implementable social choice functions, we present an axiomatic characterization of affine maximizers for an important missing case in the literature: that of two alternatives with restricted domain. We use two independent conditions: Positive Association of Differences and an independence condition.

Keywords: Auctions/bidding; social choice function; implementability; affine maximizer

1. Introduction

The implementability of social choice functions is an important concept for many operations research problems [e.g. 2, 5, 8, 3]. Roberts [6] showed that every implementable social choice function satisfies a condition named PAD (Positive Association of Differences). Conversely, when there are at least three alternatives and the domains of individual preferences are unrestricted, he showed that PAD implies that any onto social choice function is an affine maximizer. When there are two alternatives only, it is well-known that Roberts' Theorem does not hold because there exist social choice functions satisfying PAD on unrestricted domains and that are not affine maximizers. In previous work [4], we have shown that an Independence condition must be added to PAD in order to characterize affine maximizers when there are only two alternatives and when the domain of the valuations consist of an open interval unbounded from above.

Yet, in some applications, it is not realistic to suppose that the domain of valuations is unbounded from above. Suppose for example a budget-constrained planner is considering to provide one of two public goods: either open a park or open a football stadium. It is reasonable to assume that both the public goods have positive valuation to agents - thus the valuations have a lower bound. Also, it is natural that the planner can always subsidize enough amount of money to the agents and not provide any of the public goods. In other words, there is also an upper bound on the valuations. That is why, in this paper, we show that the same conditions as in [4] characterize the affine maximizers with two alternatives and domains of valuations consisting of an open interval, without the unboundedness restriction.

Section 2 is devoted to the definitions and the result. The proof is presented in Section 3.

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2. Definitions, axioms and result

Let $M = \{1, \dots, m\}$ be a finite set of agents. The set of outcomes or social states is denoted by $A = \{a, b\}$. Each outcome is valued by each agent. The valuation of outcome a (resp. b) by agent i is drawn from some real open interval L_i. We define $S = \Pi_{i \in M} L_i$. A vector $x \in S$ represents the valuations of an outcome by all agents. In their characterization of affine maximizers, [4] assumed that L_i is unbounded from above, for each agent i. Since this restriction can be unrealistic in many applications, we will not assume it in this paper.

An allocation rule is a mapping $f : S \times S \to A : (x, s) \to f(x, s)$, where x (resp. s) is the vector of valuations of a (resp. b) by all agents.

Vector inequalities: for $x, y \in \mathbb{R}^n$, $x \gg y$ iff $x_i > y_i$ for $i = 1, \dots, n$; $x > y$ iff $x_i \geq y_i$ for $i = 1, \dots, n$ and $x_j > y_j$ for some j; $x \geq y$ iff $x_i \geq y_i$ for $i = 1, \dots, n$.

We now present the conditions that we will need in order to characterize the affine maximizers when there are only two alternatives. The first one is the well-known *Positive Association of Differences* introduced by [6].[1]

PAD : for all x, y, s, t, if $x - y \gg s - t$ and $f(y, t) = a$ then $f(x, s) = a$.

Notice that PAD implies the symmetric condition : $s - t \gg x - y$ and $f(y, t) = b$ implies $f(x, s) = b$.

Our next condition is a form of independence. It was first used by [4].

Independence: For all $s, t, x, y \in S$,

$$\left.\begin{array}{r} f(x,t) = a \\ \text{and} \\ f(y,s) = a \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f(y+\varepsilon, t) = a, \quad \forall \varepsilon \gg 0 : y + \varepsilon \in S \\ \text{or} \\ f(x+\varepsilon, s) = a, \quad \forall \varepsilon \gg 0 : x + \varepsilon \in S. \end{array}\right.$$

The intuition behind this condition is that elements of S can be ordered from "more in favor of a" to "less in favor of a". Indeed, suppose the elements of S can be ordered and suppose the antecedent of Independence is satisfied. Then either x is not less in favor of a than y or y is not less in favor of a than x. In the first case, $f(x + \varepsilon, s) = a$ and in the second case, $f(y + \varepsilon, t) = a$. We do not consider this condition as compelling. Whether it is appealing or not can depend on the context and the social planner. Paraphrasing [7], p.254, our independence condition is not designed "to provide an axiomatic justification of" affine maximizers. Instead, we chose "a set of axioms with the focus on transparency rather than on immediate appeal" [7, p.259].[2]

In order to help the reader have a better grasp of the conditions presented so far, we now provide two examples showing that PAD and Independence are logically independent.

Example 1. *Let $L_i =]0, 100[$ for all $i \in M = \{1, 2, 3\}$ and define the allocation rule f with three agents as follows: for all $x, s \in S$,*

$$f(x, s) = a \iff \sum_{i \in M} x_i^2 > \sum_{i \in M} s_i^2.$$

This allocation rule violates PAD. To see this, use $x = (16, 10), y = (5, 9), s = (20, 2)$ and $t = (10, 2)$. This allocation rule satisfies Independence. Indeed, suppose $f(x, t) = a$ and $f(y, s) = a$. This implies $\sum_{i \in M} x_i^2 > \sum_{i \in M} t_i^2$ and $\sum_{i \in M} y_i^2 > \sum_{i \in M} s_i^2$. Two cases are possible.

[1] Roberts shows PAD is implied by an incentive compatibility condition.

[2] Our view of the axiomatic analysis is also close in spirit to that of [9], in a different domain.

- $\sum_{i\in M} x_i^2 > \sum_{i\in M} y_i^2$. *Then* $\sum_{i\in M} x_i^2 > \sum_{i\in M} s_i^2$ *and* $\sum_{i\in M}(x_i+\varepsilon_i)^2 > \sum_{i\in M} s_i^2$ *for all* $\varepsilon \gg 0$. *Hence* $f(x+\varepsilon, s) = a$.
- $\sum_{i\in M} x_i^2 \leq \sum_{i\in M} y_i^2$. *Then* $\sum_{i\in M} y_i^2 > \sum_{i\in M} t_i^2$ *and* $\sum_{i\in M}(y_i+\varepsilon_i)^2 > \sum_{i\in M} t_i^2$ *for all* $\varepsilon \gg 0$. *Hence* $f(y+\varepsilon, t) = a$.

So, at least one of $f(y+\varepsilon, t)$ *and* $f(x+\varepsilon, x)$ *is equal to* a *as required by Independence.*

Example 2. *Let* $L_i =]0, 10[$ *for all* $i \in M = \{1, 2, 3\}$ *and define the allocation rule* f *with three agents as follows:*

$$f(x, s) = a \iff \sum_{i\in M}(x_i - s_i)^3 \geq 0.$$

This allocation rule violates Independence and satisfies PAD. To check that it violates Independence, use $x = (6, 3, 7), y = (6, 7, 1), t = (9, 6, 3)$ *and* $s = (9, 1, 6)$. *We have* $f(x, t) = a$, $f(y, s) = a$, $f(y+\varepsilon, t) = b$ *and* $f(x+\varepsilon, s) = b$ *with* $\varepsilon = (1/10, 1/10, 1/10)$. *We now prove that it satisfies PAD. Suppose* $f(y, t) = a$ *and* $x - y \gg s - t$. *Then* $\sum_{i\in M}(y_i - t_i)^3 \geq 0$ *and* $x - s \gg y - t$ *(or* $x_i - s_i > y_i - t_i$ *for* $i \in M$*). This implies* $\sum_{i\in M}(x_i - s_i)^3 > \sum_{i\in M}(y_i - t_i)^3 \geq 0$ *because the third power is strictly monotonic. Hence* $f(x, s) = a$.

We are now ready to state our result.

Theorem 1. *Suppose for every* $i \in M$, L_i *is an open interval. The allocation rule* f *satisfies PAD and Independence iff there is* $\lambda \in \mathbb{R}^M$ *with* $\lambda > 0$ *and a real-valued mapping* $\kappa : A \to \mathbb{R}$ *such that, for all* $x, s \in S$,

$$\sum_{i\in M} \lambda_i x_i + \kappa(a) > \sum_{i\in M} \lambda_i s_i + \kappa(b) \quad \Rightarrow \quad f(x, s) = a$$
$$\sum_{i\in M} \lambda_i x_i + \kappa(a) < \sum_{i\in M} \lambda_i s_i + \kappa(b) \quad \Rightarrow \quad f(x, s) = b.$$

This result is essentially identical to Theorem 2 in [4], but without the unboundedness restriction. The proof technique used here is different from the one in [4]. The reason why we can now prove a stronger result using the same conditions as in [4] is perhaps the different technique, but it is perhaps merely the fact that we worked hard to go around all technical problems raised by the boundaries.

Since affine maximizers and our two conditions have been extensively discussed elsewhere, we do not discuss them and we merely present the proof of Theorem 1.

3. Proof

An allocation rule is single-valued and, formally, it is therefore never the case that a and b tie. Yet, if $f(x+\varepsilon, t) = a$ and $f(x-\varepsilon, t) = b$ and for all $\varepsilon \gg 0$,[3] we can consider that the valuation x exactly offsets t: any slight change in favor of a (or b) immediately results in a winning (or b). This will be denoted by $x \, T \, t$.

Define $D = \{x \in S : x \, T \, t \text{ for some } t \in S\}$. The set D is the set of all valuations of a that can be offset by some valuation t of b. It is represented in Fig. 1 in the case of an affine maximizer with three agents. If $\kappa(a) = \kappa(b)$, then D is the whole box S. The larger the absolute difference $|\kappa(a) - \kappa(b)|$, the smaller D. If the absolute difference is very large, then D is empty.

[3] Strictly speaking, we should write "for all $\varepsilon \gg 0$ such that $x+\varepsilon \in S$ and $x-\varepsilon \in S$", because, otherwise, it can happen that $f(x+\varepsilon, t)$ or $f(x-\varepsilon, t)$ is not defined. This would make the paper very cumbersome and we will therefore omit it.

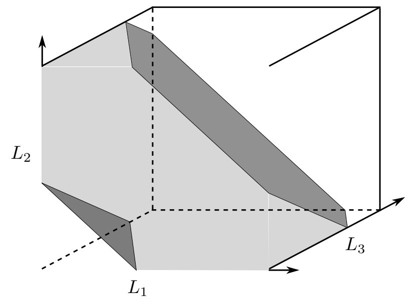

Figure 1: An affine maximizer, with three agents with valuation domains L_1, L_2 and L_3. The set D is the grey zone between the two darker planes. Under D, b is chosen everywhere and above D, a is chosen everywhere.

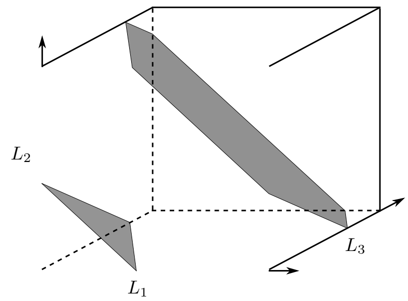

Figure 2: An affine maximizer, with three agents with valuation domains L_1, L_2 and L_3. The light grey plane is an indifference surface of the relation $\succsim$.

Define the relation $\succsim$ on D by $x \succsim y$ iff, for all $t \in S$, we have $f(y,t) = a \Rightarrow [f(x+\varepsilon, t) = a, \ \forall \varepsilon \gg 0]$. When it is not the case that $x \succsim y$, we write $x \not\succsim y$. The binary relation $\succsim$ will play a central role in the proof. Fig. 2 depicts an indifference surface of the relation $\succsim$ in the case of an affine maximizer with three agents.

The proof works as follows. First, we will prove that $\succsim$ is a monotonic weak order (Lemmas 1–3). Lemmas 4–7 are a first attempt at understanding the shape of D (the domain of definition of $\succsim$). With Lemma 8, we will prove that $\succsim$ is continuous. Lemmas 10 and 11 teach us that D is connected while Lemmas 12 and 13 teach us that every indifference set of $\succsim$ is simply connected and separates S in two sets: one above the indifference set and another one below the indifference set. Lemma 14 is crucial in that it proves that all indifference sets are parallel hyperplanes. At this point we will have proved that D and $\succsim$ are as in Fig. 1. It is then simple to show that the relation $\succsim$ admits an additive representation (Lemma 15). Roughly speaking, Lemma 16 shows that, for using an allocation rule, we do not need to know the valuation vectors, but just to which indifference set they belong. The last step consists in using conjointly Lemmas 15 and 16 for concluding.

Lemma 1. *Assume the allocation rule f satisfies PAD and Independence. Then $\succsim$ is a weak order, that is, it is complete and transitive.*

Proof. Completeness. Suppose $x, y \in D$ and $x \not\succsim y$ and $y \not\succsim x$. There are then $t, s \in S$ and $\varepsilon', \varepsilon'' \gg 0$ such that $f(y,t) = a$, $f(x+\varepsilon', t) = b$, $f(x, s) = a$ and $f(y+\varepsilon'', s) = b$. This contradicts Independence.

Transitivity. Suppose $x, y, z \in D$ and $x \succsim y$ and $y \succsim z$. From $y \succsim z$, we find the following implication: $f(z,t) = a \Rightarrow [f(y+\varepsilon, t) = a,\ \forall \varepsilon \gg 0]$ for all $t \in S$. By PAD, $[f(y, t-\varepsilon-\varepsilon') = a,\ \forall \varepsilon, \varepsilon' \gg 0]$ for all $t \in S$. This and $x \succsim y$ imply $[f(x+\varepsilon'', t-\varepsilon-\varepsilon') = a,\ \forall \varepsilon, \varepsilon', \varepsilon'' \gg 0]$ for all $t \in S$. By PAD, $[f(x+\varepsilon+\varepsilon'+\varepsilon''+\varepsilon''', t) = a,\ \forall \varepsilon, \varepsilon', \varepsilon'', \varepsilon''' \gg 0]$ for all $t \in S$. In other words, $[f(x+\varepsilon, t) = a,\ \forall \varepsilon \gg 0]$ for all $t \in S$. In summary, $f(z,t) = a$ implies $[f(x+\varepsilon, t) = a,\ \forall \varepsilon \gg 0]$ for all $t \in S$. Hence $x \succsim z$. $\square$

Lemma 2. *Let f be an allocation rule satisfying PAD and Independence. If $x, y \in D$, then $x > y$ implies $x \succsim y$ and $x \gg y$ implies $x \succ y$.*

Proof. Suppose first $x \gg y$. Then PAD clearly implies $x \succ y$. Suppose now $x \not\gg y$ and, for contradiction, $x \not\succsim y$. This implies $y \succ x$ (by Lemma 1). By definition of $\succsim$, there is $t \in S$ and $\varepsilon' \gg 0$ such that $[f(y+\varepsilon, t) = a,\ \forall \varepsilon \gg 0]$ and $f(x+\varepsilon', t) = b$. In particular, $f(y+\varepsilon'/2, t^b) = a$. Since $x > y$, we have $x+\varepsilon' \gg y+\varepsilon'/2$ and PAD then implies $f(x+\varepsilon', t) = a$. This contradiction concludes the proof. $\square$

Lemma 3. *Let f be an allocation rule satisfying PAD and Independence. If $x, y \in D$, then $x \geq y$ implies $x \succsim y$.*

Proof. If $x > y$, then Lemma 2 implies $x \succsim y$. If $x \not> y$, then $x \geq y$ implies $x = y$ and, by completeness of $\succsim$, $x \succsim y$. $\square$

Lemma 4. *(Translation Invariance). Assume the allocation rule f satisfies PAD. Suppose $x\ T\ t$. Then $(x+y)\ T\ (t+y)$ for all y such that $x+y, t+y \in S$.*

Proof. By PAD, $f(x+y+\varepsilon+\varepsilon', t+y) = a$ for all $\varepsilon, \varepsilon' \gg 0$, because $y+\varepsilon' \gg y$. This implies $f(x+y+\varepsilon, t+y) = a$ for all $\varepsilon \gg 0$. Suppose now $f(x+y-\varepsilon, t+y) = a$ for some $\varepsilon \gg 0$. Then PAD implies $f(x-\varepsilon/2, t) = a$. This contradiction shows that $f(x+y-\varepsilon, t+y) = b$ for all $\varepsilon \gg 0$. $\square$

We now introduce a condition that will be used in all lemmas from now on, but not in the main result.

Non-imposition: there exist $x, s, y, t \in S$ such that $f(x,s) = a$ and $f(y,t) = b$.

Lemma 5. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. Then D is not empty.*

Proof. Thanks to Non-imposition, there exist $x, s, y, t \in S$ such that $f(x,s) = a$ and $f(y,t) = b$. We consider two exhaustive and mutually exclusive cases.

- Suppose $f(y,s) = b$. By PAD, we know that $y \not\gg x$. Define $\alpha^* = \sup\{\alpha \in [0,1] : f(\alpha y + (1-\alpha)x, s) = a\}$ and $z = \alpha^* y + (1-\alpha^*)x$. By construction, for any $\alpha^+ > \alpha^*$, we have $f(\alpha^+ y + (1-\alpha^+)x, s) = b$. Hence, PAD implies $f(\alpha^+ y + (1-\alpha^+)x - \varepsilon, s) = b$ for all $\varepsilon \gg 0$. This and $y \not\gg x$ implies $f(z-\varepsilon', s) = b$ for all $\varepsilon' \gg 0$. By construction again, there is $\alpha^- \leq \alpha^*$ arbitrarily close to α^* such that $f(\alpha^- y + (1-\alpha^-)x, s) = a$. Hence, PAD implies $f(\alpha^- y + (1-\alpha^-)x + \varepsilon, s) = a$ for all $\varepsilon \gg 0$. This and $y \not\gg x$ implies $f(z+\varepsilon', s) = a$ for all $\varepsilon' \gg 0$. We have thus proved that $z \in D$.

- Suppose $f(y,s) = a$. By PAD, we know that $t \not\gg s$. Define $\alpha^* = \sup\{\alpha \in [0,1] : f(y, \alpha t + (1-\alpha)s) = a\}$ and $r = \alpha^* t + (1-\alpha^*)s$. By construction, for any $\alpha^+ > \alpha^*$, we have $f(y, \alpha^+ t + (1-\alpha^+)s) = b$. Hence, PAD implies $f(y, \alpha^+ t + (1-\alpha^+)s + \varepsilon) = b$ for all $\varepsilon \gg 0$. This and $t \not\gg s$ implies $f(y, r+\varepsilon') = b$ for all $\varepsilon' \gg 0$. By construction again, there is $\alpha^- \leq \alpha^*$ arbitrarily close to α^* such that $f(y, \alpha^- t + (1-\alpha^-)s) = a$. Hence, PAD implies $f(y, \alpha^- t + (1-\alpha^-)s - \varepsilon, s) = a$ for all $\varepsilon \gg 0$. This and $t \not\gg s$ implies $f(y, r-\varepsilon') = a$ for all $\varepsilon' \gg 0$. We have thus proved that $r \in D$. □

Lemma 6. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. Suppose $t \in S$, $x, y \in D$ and $x \sim y$. If $x\ T\ t$, then $y\ T\ t$.*

Proof. Since $y \succsim x$, we know that $f(y+\varepsilon, t) = a$ for all $\varepsilon \gg 0$. Suppose now for contradiction that $f(y-\varepsilon, t) = a$ for some $\varepsilon \gg 0$. Then $x \succsim y$ implies $f(x-\varepsilon, t) = a$ for some $\varepsilon \gg 0$. But we know that this is false. Hence $f(y-\varepsilon, t) = b$ for all $\varepsilon \gg 0$. □

Define $\overline{x}, \underline{x} \in \mathbb{R}^m$ by means of $\overline{x}_i = \sup L_i$ and $\underline{x}_i = \inf L_i$ for all $i \in M$. Let $B = \{x \in S : x = \alpha \underline{x} + (1-\alpha)\overline{x}$ for some $\alpha \in]0,1[\ \}$.

Lemma 7. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. For every $x \in D$, there is an open box around x and contained in D. One of the vertices of this box is a vertex of S.*

Proof. Let x be a point in D. So, there is $t \in S$ such that $x\ T\ t$. By Lemma 4, for every $y \in S$ such that $x+y \in S$ and $t+y \in S$, we have $(x+y)\ T\ (t+y)$. Hence, $x+y \in D$ for all $y \in S$ such that $x+y \in S$ and $t+y \in S$. This corresponds to an open box containing x. Let $d \in \prod_{i \in M}\{\underline{x}_i, \overline{x}_i\}$ be such that

$$d_i = \begin{cases} \underline{x}_i & \text{if} \quad x_i < t_i \\ \overline{x}_i & \text{if} \quad x_i > t_i. \end{cases}$$

Then, one of the vertices of the open box we just constructed is d. Notice that d is not unique if $x_i = t_i$ for some $i \in M$. Notice also that, unless $x = t$, the box is a strict subset of S. □

Lemma 8. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. Then $\succsim$ is continuous, i.e., for all $y \in D$, $\{x \in D : x \succsim y\}$ and $\{x \in D : x \precsim y\}$ are closed in the standard product topology.*

Proof. We will show that the set $\{x \in D : x \precsim y\}$ is closed by showing that its complement $\{x \in D : x \succ y\}$ is open. Suppose $x, y \in D$ and $x \succ y$. By definition of $\succsim$, there is $t \in S$ and $\varepsilon \gg 0$ such that $f(x,t) = a$ and $f(y+\varepsilon, t) = b$. By PAD, $f(x-\varepsilon/4, t-\varepsilon/3) = a$ and $f(y+\varepsilon/2, t-\varepsilon/3) = b$. So, $x - \varepsilon/4 \succ y$. By Lemma 3, for every $x' \geq x - \varepsilon/4$, we have $x' \succsim x - \varepsilon/4$ and, by transitivity (Lemma 1), $x' \succ y$. Hence there is a neighborhood of x such that all points in this neighborhood belong to $\{x \in D : x \succ y\}$. This proves that $\{x \in D : x \succ y\}$ is open.

Proving that $\{x \in D : x \succsim y\}$ is closed, can be done in a similar fashion. □

For any $z \in D$, define $I(z) = \{x \in D : x \sim z\}$.

Lemma 9. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. For any $x \in D$, there exist $y \in B$ such that $y \in D$ and $y \sim x$. Moreover, there is a path in D from x to y.*

Proof. Since $x \in D$, there is $t \in S$ such that $x\ T\ t$. By Lemma 7, there is an open box around x, contained in D. If, on each dimension, we choose δ_i small enough, then the closed box $\prod_{i\in M}[x_i - \delta_i, x_i + \delta_i]$ is included in D. By Lemma 6, for every $w \in D$ such that $w \sim x$, if the closed box $C_w = \prod_{i\in M}[w_i - \delta_i, w_i + \delta_i]$ is included in S, then it is also included in D.

The point x and the segment B determine a plane denoted $\widehat{Bx}$. We will construct a sequence of points $x, x^{(1)}, x^{(2)}, \ldots$ such that each of them lies in $\widehat{Bx}$, between x and B, and is closer[4] to B than the preceding point in the sequence. The iterative construction of the sequence stops when we "cross" B or, more precisely, when we reach a point that is also in $\widehat{Bx}$, but on the other side of B, w.r.t. the starting point x. We now detail the construction of this sequence.

Consider the intersection of the segment $(x, \underline{x})$ with the boundary of C_x and call it z. Because $z \ll x$ and thanks to Lemma 2, we have $z \prec x$. Consider now the intersection of the segment $(x, \overline{x})$ with the boundary of C_x and call it z'. Because $z' \gg x$ and thanks to Lemma 2, we have $z' \succ x$. Since $\succsim$ is continuous, there is some point $w \in B$ such that the intersection $x^{(1)}$ of the segment (x, w) with the boundary of C_x belongs to $I(z)$. If the segment $(x, x^{(1)})$ intersects B, then, by continuity of $\succsim$, there is y as in the statement of the lemma. Notice also that there is a path from x to y since both belong to the same box constructed around x.

Suppose now the segment $(x, x^{(1)})$ does not intersect B, that is, $x^{(1)}$ lies between x and B, in $\widehat{Bx}$. Let us then apply the same reasoning as in the first paragraph of this proof, starting with $x^{(1)}$ instead of x, still using the same δ_i as previously. We construct a box $C_{x^{(1)}}$, around $x^{(1)}$ and we obtain $x^{(2)} \in \widehat{Bx}$ with $x^{(2)} \sim x$ and $x^{(2)}$ closer to B than $x^{(1)}$. The box $C_{x^{(1)}}$ is included in S, provided we chose δ_i small enough. If the segment $(x^{(1)}, x^{(2)})$ intersects B, then, by continuity of $\succsim$, there is y as in the statement of the lemma. Notice also that there is a path from $x^{(1)}$ to y since both belong to the same box constructed around x. Since we already found a path in D from x to $x^{(1)}$, we obtain a path from x to y. If the segment $(x^{(1)}, x^{(2)})$ does not intersect B, we keep iterating this process.

We need to prove that this process stops after a finite number of iterations. At the end of iteration k, we reach $x^{(k)}$. Suppose the segment $(x^{(k-1)}, x^{(k)})$ does not intersect B. At each iteration $j = 1, \ldots, k$, the difference $x_i^{(k-1)} - x_i^{(k)}$ has the same sign, for each $i \in M$, because of the monotonicity of $\succsim$. Besides, since $x^{(j)}$ lies on a facet of $C_{x^{(j-1)}}$, there is at least one dimension i such that $|x_i^{(j-1)} - x_i^{(j)}| = \delta_i$. This shows that the distance between $x^{(j)}$ and B decreases at each iteration by an amount which is bounded from below. Hence k cannot be infinite. This also shows that it was indeed possible to choose δ_i small enough (but not infinitely small) so that, at each iteration j, the box $C_{x^{(j)}}$ is included in S. □

Lemma 10. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. For any $x \in D$, there exists $s \in B$ such that $x\ T\ s$.*

Proof. Since $x \in D$, there exists $t \in S$ such that $x\ T\ t$. By Lemma 7, there is an open box E around x, contained in D. Since $\succsim$ is everywhere defined in E and since $\succsim$ is continuous, the set $I(x) \cap E$ is a hypersurface. Consider a plane containing x and parallel to $\widehat{Bt}$. The intersection of this plane with $I(x) \cap E$ is a curve. By construction, for every y belonging to this curve, we have $x \sim y$. Let us define $\delta = y - x$. We can choose y on this curve so that $t + k\delta \in B$ for some natural k. By Translation, we have $y\ T\ (t + \delta)$. Since $x \sim y$, we have $x\ T\ (t + \delta)$.

If we repeat this reasoning, we obtain $y\ T\ (t + i\delta)$ for $i = 1, 2, \ldots$ After k iterations, $t + k\delta$

[4]We consider the distance between a point and a segment as the shortest distance between the point and any point in the segment.

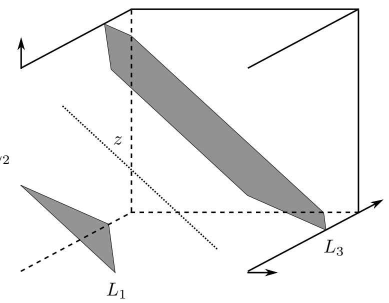

Figure 3: An affine maximizer, with three agents with valuation domains L_1, L_2 and L_3. The light grey plane is $I(z)$, i.e. the indifference surface of the relation $\succsim$ containing z. The dotted line segment is $I(1, 2, z)$, i.e. the set of all valuation vectors that are indifferent to z and that differ from z only for agents 1 and 2.

will belong to B. Defining $s = t + k\delta$ completes the proof. □

Lemma 11. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. Then D is connected in the standard product topology.*

Proof. By Lemma 9, there is a path in D from every $x \in D$ to B. If we show that $D \cap B$ is connected, then the proof is done. Let us prove this. Choose any $x \in D \cap B$. By Lemma 10, there is $t \in B$ such that such that $x \; T \; t$. Then the box constructed around x as in Lemma 7 has $\underline{x}$ or $\overline{x}$ as vertex. We consider three cases.

1. the box around x has $\underline{x}$ and $\overline{x}$ as vertex. This implies $D = S$ and D is then obviously connected.
2. the box around x has $\underline{x}$ and not $\overline{x}$ as vertex. It implies $t \gg x$. The opposite vertex (w.r.t. $\underline{x}$) is $x + \overline{x} - t$. Choose any $y \in D \cap B$, but out of the box around x. We have $y \gg x + \overline{x} - t$. By Lemma 10, there is $s \in B$ such that such that $y \; T \; s$. Suppose, for contradiction, the box constructed around y as in Lemma 7 does not intersect the box around x. Then the box around y has $\overline{x}$ as vertex. The opposite vertex is $y + \underline{x} - s$. Since the two boxes do not intersect, we have $y + \underline{x} - s \gg x + \overline{x} - t$ or, equivalently, $(y - (\overline{x} - \underline{x})) - x \gg s - t$. $f(x + \varepsilon, t) = a$, $\forall \varepsilon \gg 0$, $(y - (\overline{x} - \underline{x})) - x \gg s - t$ and PAD imply $f(y - (\overline{x} - \underline{x}) + \varepsilon, s) = a$ for all $\varepsilon \gg 0$. If we choose $\varepsilon \ll (\overline{x} - \underline{x})$ and we define $\varepsilon' = (\overline{x} - \underline{x}) - \varepsilon$, then we obtain $f(y - \varepsilon', s) = a$ for some $\varepsilon' \gg 0$. This contradiction shows that it is not possible to choose x and y in $D \cap B$ such that the boxes around them (as in Lemma 7) do not intersect. Hence $D \cap B$ is connected.
3. the box around x has $\overline{x}$ and not $\underline{x}$ as vertex. This case is handled in the same way as the previous one.

Hence $D \cap B$ is an open interval of B with $\underline{x}$ or $\overline{x}$ as bounds. □

For any $i, j \in M$ and $z \in D$, define $I(i, j, z) = \{(z_{-ij}, x_i, x_j) \in D : (z_{-ij}, x_i, x_j) \sim z\}$. This set is the intersection between $I(z)$ and the plane $\{z_{-ij}\} \times L_i \times L_j$. It is represented in Fig. 3 for the case of an affine maximizer with three agents.

Lemma 12. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. For any $i, j \in M$ and $z \in D$, the set $I(i, j, z)$ is connected. Furthermore, for $z \in D$, the set $I(z)$ is simply connected.*

Proof. Choose any $x, y \in I(i, j, z)$. There is $t \in S$ such that $x\,T\,t$. Monotonicity implies that $x \not\gg y$ and $y \not\gg x$. By Lemma 7, there is a box around x, contained in D. Since $\succsim$ is everywhere defined in this box and since $\succsim$ is continuous, the intersection between $I(i, j, z)$ and this box is a connected curve denoted Γ. Define

$$c_i = 0.5 \times \min\{|x_i - y_i|, |x_i - \overline{x}_i|, |x_i - \underline{x}_i|, |x_i - t_i|\}$$

and

$$c_j = 0.5 \times \min\{|x_j - y_j|, |x_j - \overline{x}_j|, \left|x_j - \underline{x}_j\right|, |x_j - t_j|\}.$$

By construction, $(x_i + c_i, x_j - c_j, z_{-ij})$ and $(x_i - c_i, x_j + c_j, z_{-ij})$ belong to the box just defined and, hence, to D. By construction also, the distance between y and one of these two points is smaller than the distance between y and x. We call this point (the one closer to y) w. There is then $x' \in \Gamma$ such that $x'_i = w_i$ or $x'_j = w_j$. If the curve Γ contains y, then the proof is done.

If Γ does not contain y, then we iterate the above process, starting from x' instead of x. After finitely many iterations, the curve will contain y.

For the second part of the lemma, suppose $I(z)$ is not simply connected. There is then some $i, j \in M$ such that the set $I(i, j, z)$ is not connected, but we have just proven that this is false. So, $I(z)$ is simply connected. □

Let us define $S^+(z) = \{y \in S : y \gg x$ for some $x \in I(z)\}$ and $S^-(z) = \{y \in S : y \ll x$ for some $x \in I(z)\}$.

Lemma 13. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. For any $z \in D$, the sets $I(z), S^+(z)$ and $S^-(z)$ form a partition of S.*

Proof. Suppose for contradiction that $I(z), S^+(z)$ and $S^-(z)$ do not form a partition of S. Since $I(z), S^+(z)$ and $S^-(z)$ are clearly disjoint, there is then $x \in S$ such that $x \notin I(z) \cup S^+(z) \cup S^-(z)$. Choose some $z^* \in \arg\inf_{w \in I(z)} \|w - x\|$ and some $z^\circ \in I(z)$, arbitrarily close to z^*. Since $z^\circ \neq x$ and $x \not\gg z^\circ$ and $x \not\ll z^\circ$, there are agents i, j such that $x_i \leq z^\circ_i$ and $x_j \geq z^\circ_j$, with at least one strict inequality. There is $t \in S$ such that $f(z^\circ + \varepsilon, t) = a$ and $f(z^\circ - \varepsilon, t) = b$, for all $\varepsilon \gg 0$. Define

$$c_i = 0.5 \times \min\{|z^\circ_i - x_i|, |t_i - \underline{x}_i|\}$$

and

$$c_j = 0.5 \times \min\{\left|x_j - z^\circ_j\right|, |\overline{x}_j - t_j|\}.$$

Define $y^+ = (z^\circ_{-j}, z^\circ_j + c_j)$ and $y^- = (z^\circ_{-i}, z^\circ_i - c_i)$. By translation-invariance, y^+ and y^- belong to D. By Lemma 2, $y^+ \succsim z^\circ$ and $y^- \precsim z^\circ$. By continuity of $\succsim$, there is y on the segment $[y^-, y^+]$ such that $y \sim z^\circ$ and, hence, $y \sim z$. By construction, $\|y - x\| < \|z^* - x\|$. This contradiction proves that $I(z), S^+(z)$ and $S^-(z)$ form a partition of S. □

Lemma 14. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. For any $x, y \in D$, the sets $I(x)$ and $I(y)$ are parallel hyperplanes.*

Proof. Consider a plane containing B and call it O. The intersection of this plane with $I(x)$ is $J^O(x) = O \cap I(x)$. It is a connected set because $I(x)$ is simply connected (Lemma 12). For any $y \in J^O(x)$ and $z \in O$ with $y \gg z$ or $z \gg y$, we respectively have $y \succ z$ or $z \prec y$. Hence, $z \notin I(x)$ and $J^O(x)$ is a connected curve. We will show that this curve is a line segment.

By Lemmas 9 and 10, there are $x', y' \in B$ and $t \in S$ such that $x' \sim x$, $y' \sim y$, $t \in B$, $x'\,T\,t$. We assume without loss of generality that $t + (y' - x') \in S$. If it is not the case, then we permute the roles of x and y. By translation invariance (Lemma 4), $y' \; T \; (t + (y' - x'))$. Similarly, for every $w \in J^O(x)$ such that $w + (y' - x') \in S$, we have $(w + (y' - x')) \; T \; (t + (y' - x'))$. Hence, for every $w \in J^O(x)$ such that $w + (y' - x') \in S$, we have $w + (y' - x') \sim y' \sim y$ and hence $w + (y' - x') \in J^O(y)$. Notice that those w's in $J^O(x)$ such that $w + (y' - x') \notin S$ are close to the boundary of S. If we choose x and y so that x' and y' are close to each other, then there are very few such w's. In other words, when x' and y' are close to each other, $J^O(y) = J^O(x) + (y' - x')$ except close to the boundary. Choose now $y'' \in J^O(y)$ in a box $C_{y'}$ around y'. We also have $J^O(y) = J^O(x) + (y'' - x')$ except close to the boundary. Consequently, for all $y'' \in C_{y'} \cap J^O(y)$, we have $J^O(x) + (y'' - x') = J^O(x) + (y' - x')$ except close to the boundary. Geometrically, this means that there are infinitely many translations of the curve $J^O(x)$ that yield the same curve $J^O(y)$.

Let us define a new basis in the plane O, with the first coordinate vector perpendicular to B and the second one parallel to B. We define the real-valued mapping $F : u \to F(u)$, where u is the first coordinate of a point in O, so that the graph of F in this basis is exactly $J^O(x)$. This mapping is a function because $z \ll z' \ll z''$ and $z' \in I(x)$ imply $z \prec z' \prec z''$. We similarly define the real-valued mapping G so that the graph of G in the new basis is exactly $J^O(y)$. It is also a function.

The set of all $y'' \in C_{y'} \cap J^O(y)$ form a curve in O. So does the set of all vectors $y'' - x'$. We can therefore define the real-valued mapping H so that the graph of H in the new basis is exactly the set of all vectors $y'' - x'$. It is also a function. Let us write that F and G are identical up to a translation corresponding to $y'' - x'$:

$$G(u + u') = F(u) + H(u') \tag{1}$$

for all u, u' such that there exists v and v' satisfying

- $(u, v) \in J^O(x)$,
- (u', v') corresponds to some $(y'' - x')$ with $y'' \in C_{y'} \cap J^O(y)$ and
- $(u, v) + (u', v') \in J^O(y)$.

If we choose x and y so that x' is very close to y' and if the box $C_{y'}$ is very small, then the set of values u, u' for which (1) holds, can be approximated by the following rectangle: all $(u, u') \in [\alpha, \alpha'] \times [\beta, \beta']$, where $\alpha, \alpha', \beta, \beta'$ are some real numbers satisfying $\alpha < \beta < 0 < \beta' < \alpha'$. Equation 1 is a Pexider[5]functional equation of the first kind. If it would hold on the square domain $[\alpha, \alpha'] \times [\alpha, \alpha']$, then we would immediately know its unique solution [1, p.58]. Since it holds on $[\alpha, \alpha'] \times [\beta, \beta']$, we have to derive its solution.

Set $u = 0$ in (1). Then $G(u') = F(0) + H(u')$ or $H(u') = G(u') - F(0)$, for all $u' \in [\beta, \beta']$. Set $u' = 0$ in (1). Then $G(u) = F(u) + H(0)$ or $F(u) = G(u) - H(0)$, for all $u' \in [\alpha, \alpha']$. We substitute in (1) and we obtain

$$G(u + u') = G(u) + G(u') - F(0) - H(0) \tag{2}$$

for all $(u, u') \in [\alpha, \alpha'] \times [\beta, \beta']$. Set $\phi(u) = G(u) - F(0) - H(0)$ and we can then rewrite (2) as $\phi(u + u') = \phi(u) + \phi(u')$ for all $(u, u') \in [\alpha, \alpha'] \times [\beta, \beta']$. It is the well-known Cauchy functional equation of type 1. Here again, if it would hold on the square domain $[\alpha, \alpha'] \times [\alpha, \alpha']$, then its solution would be close at hand [1, p.40].

[5]A Pexider functional equation of the first kind is a generalization of a Cauchy functional equation of the first kind. It is an equation of the form $G(u + u') = F(u) + H(u')$, where F, G and H are unknown real-valued functions of one real variable.

Notice that this Cauchy equation nevertheless holds on the square subdomain $[\beta,\beta']\times[\beta,\beta']$. Its unique solution is therefore $\phi(u)=\sigma u$ for all $u\in[\beta,\beta']$. Set now $u'=\beta'$ in this Cauchy equation. We obtain $\phi(u+\beta')=\phi(u)+\phi(\beta')=\sigma(u+\beta')$ for all $u\in[\beta,\beta']$. Hence $\phi(u)=\sigma u$ for all $u\in[\beta,2\beta']$. We have enlarged the domain of the solution to the right, by β'. We can repeat this reasoning. Set $u'=2\beta'$ (or $u'=\alpha'$ if $\alpha'<2\beta'$) in the Cauchy equation. We obtain $\phi(u+2\beta')=\phi(u)+\phi(2\beta')=\sigma(u+2\beta')$ for all $u\in[\beta,\beta']$. Hence $\phi(u)=\sigma u$ for all $u\in[\beta,3\beta']$. We have again enlarged the domain of the solution to the right, by β'. We repeat this until the upper bound of the domain of the solution reaches α'. We can similarly enlarge the domain of the solution to the left. So, we have proved that the solution of our Cauchy equation is $\phi(u)=\sigma u$ for all $u\in[\alpha,\alpha']$. Remember now that $\phi(u)=G(u)-F(0)-H(0)$. Hence $G(u)=\sigma u+F(0)+H(0)$ for all $u\in[\alpha,\alpha']$. We have also seen that $F(u)=G(u)-H(0)$ for all $u\in[\alpha,\alpha']$. Hence $F(u)=\sigma u+F(0)$ for all $u\in[\alpha,\alpha']$. This proves that $J^O(x)$ is a line segment, for any plane O containing B.

Let us now consider a line segment B' close to B. The equivalent of Lemmas 9 and 10 can be proven for B' instead of B (except for some boundary problems, that can be made as small as we wish by choosing B' arbitrarily close to B). Let us consider a plane containing B' and let us call it O'. The intersection of this plane with $I(x)$ is $K^{O'}(x)=O'\cap I(x)$. We can prove (as above) that $K^{O'}(x)$ is a line segment, for any plane O' containing B'. This holds for any B' close to B. Hence $I(x)$ is a hyperplane and, hence, $I(y)$ is another hyperplane, parallel to $I(x)$. Remember that this does not hold close to the boundary of S. But, if we choose x and y so that x' is sufficiently close to y', then there is no part of $I(x)$ for which we cannot show that $I(x)$ is a hyperplane. □

Lemma 15. *Let f be an allocation rule satisfying PAD, Independence and Non-imposition. There is then $\lambda\in\mathbb{R}^M$ with $\lambda>0$ such that, for all $x,y\in D$, $x\succsim y$ iff $\sum_{i\in M}\lambda_i x_i\geq\sum_{i\in M}\lambda_i y_i$.*

Proof. Each equivalence class of $\succsim$ is a hyperplane and therefore has an additive representation of the following form: for some $\lambda\in\mathbb{R}^M\setminus\{(0,\dots,0)\}$, the sum $\sum_{i\in M}\lambda_i z_i$ is constant for every z belonging to the same equivalence class. Because of the monotonicity of $\succsim$, $\lambda\geq 0$. □

Lemma 16. *Assume $\succsim$ has an additive representation as in Lemma 15, with parameters $\lambda_1,\dots,\lambda_m$. Suppose $\sum_{i\in M}\lambda_i x'_i=\sum_{i\in M}\lambda_i x_i$ and $\sum_{i\in M}\lambda_i t'_i=\sum_{i\in M}\lambda_i t_i$. Then $x\ T\ t$ iff $x'\ T\ t'$.*

Proof. From the definition of $\succsim$ and Lemma 15, it is clear that $x\sim x'$ and $x'\ T\ t$. Let k be a positive integer such that $x'+(t'-t)/k\in S$. We will "move" from t to t' in k steps.

Step 1. By Lemma 4, $(x'+(t'-t)/k)\ T\ (t+(t'-t)/k)$. Notice that $\sum_{i\in M}\lambda_i x'_i=\sum_{i\in M}\lambda_i(x'+(t'-t)/k)_i$ and, hence, $x'+(t'-t)/k\sim x'$. Therefore, $x'\ T\ (t+(t'-t)/k)$.

Step 2. By Lemma 4, $(x'+(t'-t)/k)\ T\ (t+2(t'-t)/k)$. As in step 1, $x'+(t'-t)/k\sim x'$ and $x'\ T\ (t+2(t'-t)/k)$.

Step 3. By Lemma 4, $(x'+(t'-t)/k)\ T\ (t+3(t'-t)/k)$. As in step 1, $x'+(t'-t)/k\sim x'$ and $x'\ T\ (t+3(t'-t)/k)$.

Step k. By Lemma 4, $(x'+(t'-t)/k)\ T\ (t+k(t-t)/k)$. As in step 1, $x'+(t'-t)/k\sim x'$ and $x'\ T\ (t+k(t'-t)/k)$. Since $t+k(t'-t)/k=t'$, we obtain $x'\ T\ t'$. □

We are now ready to prove our main result.

Proof of Theorem 1. Suppose first Non-imposition fails and, without loss of generality, $f(x,s)=a$ for all $x,s\in S$. Set $\lambda_i=1$ for all $i\in M$, $\kappa(b)=0$ and $\kappa(a)=\sum_{i\in M}(\overline{x}_i-\underline{x}_i)$. The allocation rule f then has a representation as in Theorem 1.

Suppose Non-imposition holds. From Lemma 15, we know that $\succsim$ has an additive representation with $\lambda > 0$.

Choose any $x \in D$. By definition of D, there is $s \in S$ such that $f(x+\varepsilon, s) = a$ and $f(x-\varepsilon, s) = b$ for all $\varepsilon \gg 0$. Let $y, t \in B$ be such that $\sum_{i\in M} \lambda_i y_i = \sum_{i\in M} \lambda_i x_i$ and $\sum_{i\in M} \lambda_i t_i = \sum_{i\in M} \lambda_i s_i$. By Lemma 16, $f(y+\varepsilon, t) = a$ and $f(y-\varepsilon, t) = b$ for all $\varepsilon \gg 0$. Define $\kappa(a) = \sum_{i\in M} \lambda_i t_i$ and $\kappa(b) = \sum_{i\in M} \lambda_i y_i$.

Pick any $z, r \in S$. If $\sum_{i\in M} \lambda_i z_i + \kappa(a) = \sum_{i\in M} \lambda_i r_i + \kappa(b)$, then $f(z, r)$ can be a or b. Suppose then $\sum_{i\in M} \lambda_i z_i + \kappa(a) > \sum_{i\in M} \lambda_i r_i + \kappa(b)$. We must prove $f(z, r) = a$. Two non-exclusive cases can occur: (i) there is z' such that $\sum_{i\in M} \lambda_i z'_i + \kappa(a) = \sum_{i\in M} \lambda_i r_i + \kappa(b)$ or (ii) there is r'' such that $\sum_{i\in M} \lambda_i z_i + \kappa(a) = \sum_{i\in M} \lambda_i r''_i + \kappa(b)$. We handle only (i).

Let r' be a point in B such that $\sum_{i\in M} \lambda_i r_i = \sum_{i\in M} \lambda_i r'_i$. By Lemma 4, $f(y+(r'-t)+\varepsilon, r') = a$ and $f(y+(r'-t)-\varepsilon, r') = b$ for all $\varepsilon \gg 0$. By Lemma 16, $(y+(r'-t))\ T\ r$. Since $\sum_{i\in M} \lambda_i z_i + \kappa(a) > \sum_{i\in M} \lambda_i r_i + \kappa(b)$, we find $\sum_{i\in M} \lambda_i z_i > \sum_{i\in M} \lambda_i (y_i + (r'_i - t_i))$. Hence $z \succ y + (r'-t)$ and $f(z, r) = a$. □

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