Comparing Societies with Different Numbers of Individuals on the Basis of Their Average Advantage

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1 Introduction

At an abstract level, one can view the various theories of justice that have been discussed in economics and philosophy in the last 50 years or so, including of course that of Serge-Christophe Kolm (2005), as attempts at providing criteria for comparing alternative *societies* on the basis of their "ethical goodness". The compared societies can be truly distinct societies, such as India and China. They can also be the same society examined at two different points of time (say India today and India twenty years ago) or, more counterfactually, before and after a tax reform or demographic shock.

In many theories, societies are described by lists of *attribute bundles*, as many bundles as there are individuals in the society. However, existing theories and approaches differ markedly with respect to the choice of the individual attributes that are deemed normatively relevant to describe a society. In classical social choice theory, developed along the lines of the famous impossibility theorem of Arrow (1950), the only considered attribute is an individual's ordinal preference ordering over social alternatives, in which case a society can be considered as being a list of preference orderings. The scope of classical Arrovian theory is, however, restricted to normative comparisons

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involving societies with the same number of individuals. In the welfarist approach, defended forcefully by Blackorby, Bossert, and Donaldson (2005), among many others, the relevant attribute is an individual's utility, in which case a society can be summarized by the list of utility numbers achieved by the individuals. The welfarist tradition has explored the problems raised by comparing societies with different numbers of people in considerable detail. See, for example, Parfit (1984), Blackorby, Bossert, and Donaldson (2005), Broome (2004), and the literature surveyed therein for examples of welfarist approaches to variable population ethics. As another example of a normative approach for comparing societies, there is the multidimensional functioning and capabilities theory of Sen (1985), in which the individual attributes are taken to be various functionings (nutrition, education, health, etc.). As far as we are aware, there have not been many attempts to apply Sen's theory to societies involving different numbers of individuals.

In this article, we discuss a general method for comparing *all* logically conceivable finite lists of attribute bundles without entering into the (philosophically important) matter of identifying what these attributes are. Taking the relevant individual attributes as given, we formulate several principles that can be used for comparing alternative lists of attribute bundles and we show that these principles characterize a rather specific method for comparing societies. The criterion characterized herein can be thought of as comparing societies on the basis of their *average advantage* for some advantage function defined on the set of attribute bundles.

If one takes the view that utility is the *only* relevant individual attribute, then this article may be seen as providing an alternative characterization of the family of social orderings called *average generalized utilitarianism* by Blackorby, Bossert, and Donaldson (2005, pp. 171–172, 198). With average general utilitarianism, individual utilities are first transformed by some function—here called an advantage function—before being averaged.

Yet, the main interest of the characterization provided in this article is that it applies as well to *non-welfarist* multidimensional contexts in which the situation of an individual is described by a bundle of attributes. In any such context, the properties that we impose on the ranking of all societies force one to aggregate these attributes into a measure of individual *advantage* and to compare different societies on the basis of their average advantage. Of course, when interpreted from this perspective, the *advantage function* that aggregates the bundles of attributes need *not* be that which corresponds to an individual's subjective well-being. It could, rather, be thought of as being the value assigned to the individual attributes by the social planner or, more generally, by the theoretician of justice. While the main theorem in this article does not impose any restrictions on the advantage function other than monotonicity and continuity, it is quite easy to require it to satisfy additional properties by imposing further assumptions on the social ordering. An example of such a property is aversion to attribute inequality which, if imposed on the social ordering, places additional restrictions on the individual advantage function.

The rest of this article is organized as follows. In Section 2, we introduce the formal model, the main axioms, and the criterion used to compare societies. Section 3 provides our main result and Section 4 indicates how some additional restrictions on the advantage function can be obtained by imposing additional properties on the social ordering. Section 5 concludes.

2 Notation and Basic Definitions

2.1 Notation

The sets of integers, non-negative integers, strictly positive integers, real numbers, non-negative real numbers, and strictly positive real numbers are denoted by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{N}_{++}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ respectively. The cardinality of any set A is denoted by $\#A$ and the k-fold Cartesian product of a set A with itself is denoted by A^k. Our notation for vector inequalities is $\geqq$, $\geq$, and $>$.

The k-dimensional unit vector is denoted by 1^k. The inner product of an $n \times m$ matrix a by an $m \times r$ matrix b is denoted by $a \cdot b$. A *permutation matrix* π is a square matrix whose entries are either 0 or 1 and sum to 1 in every row and every column.

Given a vector v in $\mathbb{R}^k$ and a positive real number ε, we denote by $N_\varepsilon(v)$ the *ε-neighborhood around v* defined by $N_\varepsilon(v) = \{x \in \mathbb{R}^k \colon |x_h - v_h| < \varepsilon \text{ for all } h = 1, \ldots, k\}$. If Φ is a function from a subset A of $\mathbb{R}^k$ to $\mathbb{R}$ and $\mathbf{a}$ is a vector in A, for every strictly positive real number Δ such that $(a_1, \ldots, a_j + \Delta, \ldots, a_k) \in A$, we denote by $\Phi_j^{\Delta}(a)$ its (discrete right-hand-side) jth *derivative* defined by:

$$\Phi_j^{\Delta}(\mathbf{a}) = \frac{\Phi(a_1, \ldots, a_j + \Delta, \ldots, a_k) - \Phi(\mathbf{a})}{\Delta}. \tag{1}$$

A *binary relation* $\succsim$ on a set Ω is a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in \succsim$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \Longleftrightarrow [x \succsim y \text{ and } y \succsim x]$ and its *asymmetric factor* $\succ$ by $x \succ y \Longleftrightarrow [x \succsim y \text{ and } \neg(y \succsim x)]$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every $x \in \Omega$, is *transitive* if $x \succsim z$ always follows from $[x \succsim y \text{ and } y \succsim z]$ for any $x, y, z \in \Omega$, and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct $x, y \in \Omega$. A reflexive, transitive, and complete binary relation is called an *ordering*.

2.2 The Framework

We assume that the situation of an individual can be described—at least for the purpose of normative evaluation—by a bundle of k (with $k \in \mathbb{N}_{++}$) attributes. Accordingly, we view a society s as a finite ordered list of vectors in $\mathbb{R}^k$, every such vector being interpreted as the bundle of attributes that describes the situation of the individual to which it corresponds. Let $n(s)$ denote the number of individuals living in society s. Then, we can depict any society s as an $n(s) \times k$ matrix:

$$s = \begin{bmatrix} s_{11} & \dots & s_{1k} \\ \vdots & \vdots & \vdots \\ s_{n(s)1} & \dots & s_{n(s)k} \end{bmatrix},$$

where s_{ij}, for $i = 1, \dots, n(s)$ and $j = 1, \dots, k$, is interpreted as the amount of attribute j received by individual i in society s. Denote by s_i the vector of attributes received by i in s. A society s with $n(s)$ individuals is therefore an element of $\mathbb{R}^{n(s)k}$ and the set of all logically conceivable societies is $\cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$. If s is a society in $\mathbb{R}^{mk}$ and s' is a society in $\mathbb{R}^{nk}$, we denote by (s, s') the society in $\mathbb{R}^{(m+n)k}$ in which the first m individuals get, in the same order, the bundles obtained by the corresponding individuals in s and the last n individuals get the bundles obtained, in the same order, by the n individuals in s'. To alleviate notation, we write, for any attribute bundle x in $\mathbb{R}^k$, the one-individual society (x) as x.

We note that depicting societies as ordered lists of attribute bundles makes sense only if one adopts an *anonymity* postulate that "the names of the individuals do not matter". We adopt this postulate throughout, even though we are fully aware that it rests on an implicit assumption that the attribute bundle received by an individual constitutes the *only* information about this individual's situation that is deemed normatively relevant. It is therefore important for the interpretation of our framework that one adopts an extensive list of attributes, which could include many consumptions goods, health, education levels, and, possibly, Rawlsian primary goods such as the "social bases of self respect". Perhaps, one should also view these bundles of goods and primary goods as being distinguished by the time at which they are made available if one wants to adopt a lifetime perspective. A clear discussion about the explicit modeling of this anonymity postulate within a richer—albeit welfarist—formal framework is provided in Blackorby, Bossert, and Donaldson (2005, Chap. 3.9).

We should also mention that our framework leads one to consider all conceivable societies, including societies consisting of a single individual. While these kinds of "societies" may be considered to lie outside the realm of theories of justice, there are several instances in which normative comparisons of small communities are made. For instance, it is not uncommon in applied welfare economics to compare the well-being of an individual with that of a family. Of course, we are doing much more than that here because we also normatively compare a family of three persons with, say, the whole People's Republic of China. Note, too, that we rank all single-individual societies and thus, implicitly, all bundles of attributes from a social point of view. We believe that this is acceptable only if the ethical importance of the attributes commands widespread support.

We compare all societies on the basis of a social ordering $\succsim$, with asymmetric and symmetric factors $\succ$ and $\sim$ respectively. We interpret the statement $s \succsim s'$ as meaning that "the distribution of the k attributes in society s is at at least as just as the corresponding distribution in society s'". A similar interpretation is given to the statements $s \succ s'$ ("strictly more just") and $s \sim s'$ ("equally just").

2.3 The Axioms

In this article, we identify the properties (axioms) of the ordering $\succsim$ that are necessary and sufficient for the existence of a monotonically increasing and continuous *advantage function* $u\colon \mathbb{R}^k \to \mathbb{R}$ such that, for all societies s and s' in $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$, one has:

$$s \succsim s' \Longleftrightarrow \sum_{i=1}^{n(s)} \frac{u(s_i)}{n(s)} \geq \sum_{i=1}^{n(s')} \frac{u(s'_i)}{n(s')}. \tag{2}$$

An ordering satisfying this property *may* therefore be thought of as resulting from the comparison of the *average advantage* achieved by individuals in the various societies. Note that the individual advantage function that appears in this formula is the same for all individuals, who are therefore treated symmetrically. Of course, if one adopts the welfarist paradigm in which the situation of an individual is described by a single attribute (utility), then $k = 1$ and the ordering defined by (2) corresponds to what is called "generalized average utilitarianism" by Blackorby, Bossert, and Donaldson (1999). Note that (2) defines a *family* of social criteria, with as many members as there are logically conceivable continuous and monotonically increasing advantage functions. We shall discuss below how could one could restrict this family by imposing additional axioms on the social ranking. We refer to any ranking that satisfies (2) for some function u as an *Average Advantage* (AA) ranking.

We now introduce five axioms that, as we shall show, characterize the AA family of social rankings.

The first axiom combines the assumption that the attributes that matter for normative appraisal are positively valued by the social planner with a Pareto-like condition that says that it is a social improvement when all members of society experience an increase in their attribute bundles. This axiom is stated formally as follows.

Monotonicity. For all societies $s, s' \in \cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ for which $n(s) = n(s') = n$ for some $n \in \mathbb{N}_{++}$, if $s_i \geq s'_i$ for all $i = 1, \ldots, n$, then $s \succ s'$.

This axiom can be seen as an adaptation to the current multidimensional setting of the axiom called Weak Pareto by Blackorby, Bossert, and Donaldson (2005). Of course, their axiom applies only to the case in which there is only one attribute—welfare, while our formulation applies to any number of attributes. In our view, this monotonicity axiom is natural if the attributes can be interpreted as primary goods à la Rawls (1982) or, using the terminology of Sen (1987), as things that "people have reasons to value", provided of course that the social planner is minimally concerned about improving the situations of individuals.

The second axiom requires the social ranking to satisfy what Blackorby, Bossert, and Donaldson (2005) have called *Same People Anonymity.* That is, the names of the individuals who receive the attributes do not matter for normative evaluation when comparing societies with the same number of people. In other words, all societies that distribute the same list of attributes among the same number of people are normatively equivalent. This axiom is also natural in the setting we are considering, especially in view of the discussion made above. We state this axiom formally as follows.

Same People Anonymity. For every society $s \in \cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ and all $n(s) \times n(s)$ permutation matrices π, one has $\pi \cdot s \sim s$.

The third axiom is a continuity condition. It is weaker than most related continuity conditions because it applies only to sets of attribute bundles that are considered to be weakly better than, or weakly worse than, a given society if they are received by a single individual. It says that if a converging sequence of such bundles are considered weakly better (resp. weakly worse) than a given society, then the point of convergence of this sequence should also be considered weakly better (resp. weakly worse) than the considered society.

Continuity. For every society $s \in \cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$, the sets $B(s) = \{x \in \mathbb{R}^k : x \succsim s\}$ and $W(s) = \{x \in \mathbb{R}^k : s \succsim x\}$ are both closed in $\mathbb{R}^k$.

While weak, this axiom rules out rankings such as the Leximin one, which compare ordered lists of bundles by first defining a continuous utility function on those bundles and by then ranking vectors of the utilities associated with these bundles by the usual lexicographic ordering of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{l}$ (generalized to account for the different dimensions of the vectors being compared). It is clear that such a ranking violates Continuity because it is possible to consider a sequence of bundles that are ranked weakly above a bundle that is worse for a given society and that converges to the latter bundle. Yet, contrary to what is required by Continuity, a society containing only one individual endowed with this bundle is considered strictly worse than the original many-individual society by the Leximin criterion.

The next axiom plays a crucial role in our characterization theorem and captures the very idea of averaging numbers for comparing societies. We call it, for this reason, the *Averaging* axiom. This axiom says that merging two distinct societies generates a (larger) society that is worse than the best of the two societies and better than the worst of the two societies. It also says, conversely, that if a society loses (gains) from bringing in members with specific attribute endowments, then this can only be because the distribution of attributes that is brought in is worse (better) than that already present in the original society. Using the language of population ethics, this axiom says that bringing in new members with specific endowments of attributes is worth doing if and only if the added distribution of endowments is better than the original one. This axiom is stated formally as follows.

Averaging. For all societies $s, s' \in \cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$, $s \succsim s' \Leftrightarrow s \succsim (s, s') \Leftrightarrow (s, s') \succsim s'$.

When applied to an ordering, Averaging implies some other properties that have been considered in the literature on population ethics. One of them is the axiom called Replication Equivalence by Blackorby, Bossert, and Donaldson (2005, p. 197). This axiom states that, for societies in which everyone gets the same attribute bundle, the *number* of members in the society does not matter. This property clearly rules out social preferences of the type "small is beautiful" or, conversely, the biblical "be fertile and multiply". We state this property formally as follows.

Replication Equivalence. For every attribute bundle $x \in \mathbb{R}^k$ and all societies $s \in \cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ for which $s_i = x$ for all $i = 1, \ldots, n(s)$, one has $s \sim x$.

This condition is implied by Averaging if $\succsim$ is reflexive. The proof of this claim is left to the reader.

The next, and last, axiom is an axiom that we call *Same Number Existence Independence*, extending the terminology of Blackorby, Bossert, and Donaldson (2005). This axiom says that the comparisons of two societies with the same number of individuals should only depend upon the distribution of the attribute bundles that differ between the two societies. Individuals who do not experience any change in their attribute bundles, and who are therefore "unconcerned" by the change, should not matter for the normative comparison of the two societies. As a consequence, the existence, or non-existence, of these individuals should not affect the ranking of the societies. We state this axiom formally as follows.

Same Number Existence Independence. For all societies $s, s', s'' \in \cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ for which $n(s) = n(s')$, $(s, s'') \succsim (s', s'') \Leftrightarrow s \succsim s'$.

It can be easily checked that any AA ranking that uses a continuous and monotonically increasing advantage function u satisfies Monotonicity, Same People Anonymity, Continuity, Averaging, and Same Number Existence Independence. In the next section, we establish the converse proposition that any ordering of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ that satisfies these five axioms must be an AA ranking.

3 The Main Result

The main result established here relies heavily on a companion article, Gravel, Marchant, and Sen (2010) (GMS in the sequel), that characterizes a *uniform expected utility* criterion in the somewhat different—but formally close—setting of choice under complete uncertainty. In GMS, the objects that are compared are finite sets of consequences instead of vectors of attribute bundles.

The first step in establishing our result consists in showing that if a social ordering $\succsim$ satisfies our five axioms, then it also satisfies the following condition.

Existence of Critical Levels. For every society $s \in \cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$, there exists an attribute bundle $x(s) \in \mathbb{R}^k$ for which $x(s) \sim s$.

In words, this condition says that, for every society s, it is always possible to find an attribute bundle that, if given to a single individual, would be socially equivalent to s. For obvious reasons, this condition was called Certainty Equivalence in our article on uncertainty. Given Averaging, Existence of Critical Levels is equivalent to the requirement that, for every society, there exists an attribute bundle that will make the social planner indifferent between adding an individual endowed with this bundle to the society and not bringing this individual into existence. If the "bundle" is one-dimensional and is interpreted as utility, then this requirement corresponds to what Blackorby, Bossert, and Donaldson (2005, p. 160) call Existence of Critical Levels in their welfarist framework. In this case, the attribute bundle is, in fact, a number that is called the "critical level of utility" by these authors.

We now show that Existence of Critical Levels is implied by Monotonicity, Same People Anonymity, Continuity, and Averaging. (Same Number Existence Independence is not needed to establish this result.)

Proposition 1. *Let $\succsim$ be an ordering of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ that satisfies Monotonicity, Same People Anonymity, Continuity, and Averaging. Then $\succsim$ satisfies Existence of Critical Levels.*

Proof. Consider any society $s \in \cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$. Recall that $B(s) = \{x \in \mathbb{R}^k : x \succsim s\}$ and $W(s) = \{x \in \mathbb{R}^k : s \succsim x\}$. By Continuity, both of these sets are closed.

By Same People Anonymity, we may without loss of generality write s as $s = (s_1, \dots, s_{n(s)})$ with $s_h \precsim s_{h+1}$ for $h = 1, \dots, n(s) - 1$. By Averaging, $s_{n(s)} \succsim s_{n(s)-1}$ implies $(s_{n(s)}, s_{n(s)-1}) \succsim s_{n(s)-1}$. Because $s_{n(s)-1} \succsim s_{n(s)-2}$, it follows from the transitivity of $\succsim$ that $(s_{n(s)}, s_{n(s)-1}) \succsim s_{n(s)-2}$. Averaging then implies $(s_{n(s)}, s_{n(s)-1}, s_{n(s)-2}) \succsim s_{n(s)-2}$. Repeating this argument a finite number of times, we conclude that $(s_{n(s)}, \dots, s_1) \succsim s_1$. Hence, by Same People Anonymity, we have $s \succsim s_1$. A similar argument can be used to show that $s_{n(s)} \succsim s$. Therefore, $s_{n(s)} \in B(s)$ and $s_1 \in W(s)$. Because $\succsim$ is complete, $B(s) \cup W(s) = \mathbb{R}^k$. The arc connectedness of $\mathbb{R}^k$ implies that there exists a continuous function $f\colon [0,1] \to \mathbb{R}^k$ such that $f(0) = s_1$ and $f(1) = s_{n(s)}$. Because $B(s)$ and $W(s)$ are both closed, by Continuity, there must exist an $\alpha \in [0,1]$ such that $f(\alpha) \in B(s) \cap W(s)$. That is, $f(\alpha) \sim s$, which shows that $\succsim$ satisfies Existence of Critical Levels. □

Endowed with this result, we are equipped to state and prove the main result of this article.

Theorem 1. *Let $\succsim$ be an ordering of $\cup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ that satisfies Monotonicity, Same People Anonymity, Continuity, Averaging, and Same Number Existence Independence. Then $\succsim$ is an AA social ordering. Furthermore, the u function in the definition of an AA social ordering is unique up to a positive affine transformation and it is a continuous and increasing function of its k arguments.*

Proof. Suppose that $\succsim$ is an ordering of $\cup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ that satisfies Monotonicity, Same People Anonymity, Continuity, Averaging, and Same Number Existence Independence. Let $\mathcal{P}(\mathbb{R}^k)$ denote the set of all finite non-empty subsets of $\mathbb{R}^k$ with representative elements A, B, C, etc. For any set $C \in \mathcal{P}(\mathbb{R}^k)$, write it as $C = \{c_1, \dots, c_{\#C}\}$ and define the society $s^C \in \cup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ by letting $s^C = (c_1, \dots, c_{\#C})$.

Define the binary relation $\widehat{\succsim}$ on $\mathcal{P}(\mathbb{R}^k)$ by:

$$A \mathrel{\widehat{\succsim}} B \Longleftrightarrow s^A \succsim s^B$$

and denote the asymmetric and symmetric factors of this binary relation by $\widehat{\succ}$ and $\widehat{\sim}$ respectively. It is clear that $\widehat{\succsim}$ is an ordering of $\mathcal{P}(\mathbb{R}^k)$ if $\succsim$ is an ordering of $\cup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$.

We now show that $\widehat{\succsim}$ is continuous in the sense that the sets $\{x \in \mathbb{R}^k\colon \{x\} \mathrel{\widehat{\succsim}} A\}$ and $\{x \in \mathbb{R}^k\colon A \mathrel{\widehat{\succsim}} \{x\}\}$ are closed in $\mathbb{R}^k$ for any A in $\mathcal{P}(\mathbb{R}^k)$. That is, we show that $\succsim$ satisfies the axiom called Continuity in GMS. We provide the argument for the set $\{x \in \mathbb{R}^k\colon \{x\} \mathrel{\widehat{\succsim}} A\}$ (the argument for the set $\{x \in \mathbb{R}^k\colon A \mathrel{\widehat{\succsim}} \{x\}\}$ is similar). Consider a sequence $x^1, x^2, \dots$ of elements of $\mathbb{R}^k$ converging to some element x of $\mathbb{R}^k$ for which one has $\{x^t\} \mathrel{\widehat{\succsim}} A$ for every t. By the definition of $\widehat{\succsim}$, one has $x^t \succsim s^A$ for every t and, because x^t converges to x, it follows from the continuity of $\succsim$ that $x \succsim s^A$. But this implies, given the definition of $\widehat{\succsim}$, that $\{x\} \succsim s^A$, which shows that the set $\{x \in \mathbb{R}^k \colon \{x\} \mathrel{\widehat{\succsim}} A\}$ is closed in $\mathbb{R}^k$.

We next show that for any *disjoint* non-empty finite sets $A, B \in \mathcal{P}(\mathbb{R}^k)$, one has $A \mathrel{\widehat{\succsim}} B \Leftrightarrow A \mathrel{\widehat{\succsim}} A \cup B \Leftrightarrow A \cup B \mathrel{\widehat{\succsim}} B$. That is, we show that $\widehat{\succsim}$ satisfies the axiom called Averaging in GMS. Let A and B be two disjoint non-empty finite sets in $\mathcal{P}(\mathbb{R}^k)$ for which $A \mathrel{\widehat{\succsim}} B$. By the definition of $\widehat{\succsim}$, $s^A \succsim s^B$. Because the ordering $\succsim$ satisfies Averaging, it follows that $s^A \succsim (s^A, s^B)$. Because $A \cap B = \varnothing$, it also follows that there exists an $\#(A \cup B) \times \#(A \cup B)$ permutation matrix π such that $\pi \cdot (s^A, s^B) = s^{A \cup B}$. By Same People Anonymity and the transitivity of $\succsim$, one has $s^A \succsim (s^A, s^B) \sim s^{A \cup B}$. By the definition of $\widehat{\succsim}$, this implies that $A \mathrel{\widehat{\succsim}} A \cup B$. Averaging also implies that one has $(s^A, s^B) \succsim s^B$ and, as a result of what has just been established, it follows that $A \cup B \mathrel{\widehat{\succsim}} B$. Hence, we proved that $A \mathrel{\widehat{\succsim}} B \Rightarrow A \mathrel{\widehat{\succsim}} A \cup B \mathrel{\widehat{\succsim}} B$. For the other direction, assume that A and B are two disjoint non-empty finite sets in $\mathcal{P}(\mathbb{R}^k)$ for which $A \mathrel{\widehat{\succsim}} A \cup B$. By the definition of $\widehat{\succsim}$, one has $s^A \succsim s^{A \cup B}$. Because A and B are disjoint, there exists an $\#(A \cup B) \times \#(A \cup B)$ permutation matrix π such that $\pi \cdot s^{A \cup B} = (s^A, s^B)$. By Same People Anonymity and the transitivity of $\succsim$, one has $s^A \succsim s^{A \cup B} \sim \pi \cdot s^{A \cup B} = (s^A, s^B)$. By Averaging, $s^A \succsim s^B$, which implies, given the definition of $\widehat{\succsim}$, that $A \mathrel{\widehat{\succsim}} B$. The argument for establishing the same conclusion starting from the assumption that $A \cup B \mathrel{\widehat{\succsim}} B$ is similar.

The final property of $\widehat{\succsim}$ that we show is that for any three finite and non-empty sets A, B, and C in $\mathcal{P}(\mathbb{R}^k)$ for which $A \cap C = B \cap C = \varnothing$ and $\#A = \#B$, one has $A \mathrel{\widehat{\succsim}} B \Leftrightarrow A \cup C \mathrel{\widehat{\succsim}} B \cup C$. That is, we show that $\widehat{\succsim}$ satisfies the axiom called Restricted Independence in GMS. Let A, B, and C be three finite and non-empty sets in $\mathcal{P}(\mathbb{R}^k)$ for which $A \cap C = B \cap C = \varnothing$ and $\#A = \#B$. Assume first that $A \mathrel{\widehat{\succsim}} B$ and, therefore, that $s^A \succsim s^B$. By Same Number Existence Independence, one has $(s^A, s^C) \succsim (s^B, s^C)$. Because $A \cap C = B \cap C = \varnothing$, there are permutation matrices π and π' such that $\pi \cdot (s^A, s^C) = s^{A \cup C}$ and $\pi' \cdot (s^B, s^C) = s^{B \cup C}$. By Same People Anonymity and the transitivity of $\succsim$, one has $s^{A \cup C} = \pi \cdot (s^A, s^C) \sim (s^A, s^C) \succsim (s^B, s^C) \sim \pi' \cdot (s^B, s^C) = s^{B \cup C}$, which implies, given the definition of $\widehat{\succsim}$, that $A \cup C \mathrel{\widehat{\succsim}} B \cup C$. For the other direction, assume now that $A \cup C \mathrel{\widehat{\succsim}} B \cup C$ and, therefore, that $s^{A \cup C} \succsim s^{B \cup C}$. Because $A \cap C = B \cap C = \varnothing$, there are permutation matrices π and π' such that $\pi \cdot s^{A \cup C} = (s^A, s^C)$ and $\pi' \cdot s^{B \cup C} = (s^B, s^C)$. By Same People Anonymity and the transitivity of $\succsim$, one has $(s^A, s^C) = \pi \cdot s^{A \cup C} \sim s^{A \cup C} \succsim s^{B \cup C} \sim \pi' \cdot s^{B \cup C} = (s^B, s^C)$. It then follows from Same Number Existence Independence that $s^A \succsim s^B$ and, by the definition of $\widehat{\succsim}$, that $A \mathrel{\widehat{\succsim}} B$, as required.

We have shown that $\widehat{\succsim}$ is an ordering of $\mathcal{P}(\mathbb{R}^k)$ that satisfies the axioms of Continuity, Averaging, and Restricted Independence of GMS. Moreover, $\mathbb{R}^k$ is clearly a separable and connected topological space. Hence, Theorem 4 of GMS applies to $\widehat{\succsim}$, so there exists a continuous function $u \colon \mathbb{R}^k \to \mathbb{R}$ such that, for every A and $B \in \mathcal{P}(\mathbb{R}^k)$, one has

$$A \mathrel{\widehat{\succsim}} B \Longleftrightarrow \frac{\sum_{a\in A} u(a)}{\#A} \geq \frac{\sum_{b\in B} u(b)}{\#B}. \tag{3}$$

We need to show that this function u also represents the ordering $\succsim$ of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$. That is, we need to show that for any $s, s' \in \cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$,

$$s \succsim s' \Longrightarrow \frac{\sum_{i=1}^{n(s)} u(s_i)}{n(s)} \geq \frac{\sum_{i=1}^{n(s')} u(s'_i)}{n(s')}$$

and (thanks to the completeness of $\succsim$) that

$$s \succ s' \Longrightarrow \frac{\sum_{i=1}^{n(s)} u(s_i)}{n(s)} > \frac{\sum_{i=1}^{n(s')} u(s'_i)}{n(s')}.$$

We only provide the proof for the first implication.

By way of contradiction, assume that $s \succsim s'$ and

$$\frac{\sum_{i=1}^{n(s)} u(s_i)}{n(s)} < \frac{\sum_{i=1}^{n(s')} u(s'_i)}{n(s')}. \tag{4}$$

Using Proposition 1, let $x(s)$ and $x(s')$ be the critical-level bundles that correspond to s and s' respectively. Because $x(s) \sim s \succsim s' \sim x(s')$, it follows from the transitivity of $\succsim$ and the the definition of $\widehat{\succsim}$ that $\{x(s)\} \mathrel{\widehat{\succsim}} \{x(s')\}$, so that $u(x(s)) \geq u(x(s'))$. Clearly, this inequality is compatible with (4) only if at least one of the following two inequalities hold:

$$u(x(s)) > \frac{\sum_{i=1}^{n(s)} u(s_i)}{n(s)} \tag{5}$$

or

$$\frac{\sum_{i=1}^{n(s')} u(s'_i)}{n(s')} > u(x(s')).$$

We show that (5) cannot hold (the proof for the other inequality being similar). Consider the sequence of societies s^t defined, for any positive integer t, by:

$$s_i^t = \left(s_i + \frac{i}{t}\right).$$

By Monotonicity, one must have $s^t \succ s$ for all t. Using Proposition 1, let $x(s^t)$ be the critical-level bundle that corresponds to the society s^t. Hence, one has

$$x(s^t) \sim s^t \succ s \sim x(s).$$

By the transitivity of $\succsim$ and the fact that s^t is such that $s_i^t \neq s_{i'}^t$ for all $i, i' \in \{1, \ldots, n(s)\}$ for which $i \neq i'$, it follows from the definition of the ordering $\widehat{\succsim}$ that

$$\{x(s^t)\} \mathrel{\widehat{\sim}} \{s_1^t, s_2^t, \ldots, s_{n(s)}^t\} \mathrel{\widehat{\succ}} \{x(s)\}$$

for all t or, using the numerical representation of $\widehat{\succsim}$ provided by (3):

$$u(x(s^t)) = \frac{\sum_{i=1}^{n(s)} u\left(s_i + \frac{i}{t}\right)}{n(s)} > u(x(s)) \tag{6}$$

for all t. But, given the continuity of u, inequality (6) is clearly incompatible with inequality (5) because the sequence of numbers $u(x(s^t))$ converges to $u(x(s))$ while the sequence of numbers $\left[\sum_{i=1}^{n(s)} u\left(s_i + \frac{i}{t}\right)\right]/n(s)$ converges to $\left[\sum_{i=1}^{n(s)} u(s_i)\right]/n(s)$.

It is straightforward to verify that the u function in the definition of an AA social ordering is unique up to a positive affine transformation and that it is a continuous and increasing function of its k arguments. □

We conclude this section by establishing, in Remark 1, that when applied to an ordering of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ that satisfies Same People Anonymity, the Monotonicity, Continuity, Averaging, and Same Number Existence Independence axioms are independent. We do not provide an example of an ordering of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ that satisfies all of our five axioms except Same People Anonymity because considering non-anonymous orderings would lead one to consider a somewhat different domain than $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$, one in which all conceivable individuals, whether alive, dead, or not yet born, would be given an identity.

Remark 1. *When imposed on an ordering of* $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ *that satisfies Same People Anonymity, the axioms of Monotonicity, Continuity, Averaging, and Same Number Existence Independence are independent.*

Proof. An AA ranking of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ that uses a non-increasing function u (for example, $u(x) = \prod_{j=1}^k x_j - \sum_{j=1}^k x_j$) satisfies Same People Anonymity, Continuity, Averaging, and Same Number Existence Independence, but it obviously violates Monotonicity.

As an example of an ordering of $\cup_{l\in\mathbb{N}_{++}}\mathbb{R}^{lk}$ satisfying Same People Anonymity, Monotonicity, Continuity, and Averaging, but violating Same Number Existence Independence, let $k = 1$ and define $\succsim$ by:

$$s \succsim s' \Longleftrightarrow \frac{\sum_{i=1}^{n(s)} s_i}{\sum_{i=1}^{n(s)} \frac{1}{s_i}} \geq \frac{\sum_{i=1}^{n(s')} s'_i}{\sum_{i=1}^{n(s')} \frac{1}{s'_i}}. \tag{7}$$

It is not hard to check that this ordering satisfies Same People Anonymity, Monotonicity, Continuity, and Averaging. To show that $\succsim$ violates Same Number Existence Independence, consider the societies $s = (1, 7)$, $s' = (2, 3)$, and $s'' = (4, 12)$. Using (7), we have $s \succsim s'$ because

$$\frac{1+7}{1+\frac{1}{7}} = 7 \geq \frac{2+3}{\frac{1}{2}+\frac{1}{3}} = 6.$$

However, contrary to what is required by Same Number Existence Independence, one has $(s, s'') \prec (s', s'')$ because

$$\frac{1+4+7+12}{1+\frac{1}{4}+\frac{1}{7}+\frac{1}{12}} = \frac{24 \times 84}{84+21+12+7} = \frac{6 \times 84}{31} < \frac{2+3+4+12}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{12}} = 6 \times 3.$$

As an example of an ordering of $\cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ that satisfies all of the five axioms except Continuity, let $k = 2$ and define $\succsim$ by:

$$s \sim s' \iff \frac{\sum_{i=1}^{n(s)} s_{i1}}{n(s)} = \frac{\sum_{i=1}^{n(s')} s'_{i1}}{n(s')} \text{ and } \frac{\sum_{i=1}^{n(s)} s_{i2}}{n(s)} = \frac{\sum_{i=1}^{n(s')} s'_{i2}}{n(s')}$$

and

$$s \succ s' \iff \begin{cases} \dfrac{\sum_{i=1}^{n(s)} s_{i1}}{n(s)} > \dfrac{\sum_{i=1}^{n(s')} s'_{i1}}{n(s')} \text{ or} \\ \dfrac{\sum_{i=1}^{n(s)} s_{i1}}{n(s)} = \dfrac{\sum_{i=1}^{n(s')} s'_{i1}}{n(s')} \text{ and } \dfrac{\sum_{i=1}^{n(s)} s_{i2}}{n(s)} > \dfrac{\sum_{i=1}^{n(s')} s'_{i2}}{n(s')}. \end{cases}$$

We leave to the reader the easy task of verifying that this lexicographic ordering violates Continuity, but satisfies Same People Anonymity, Monotonicity, Averaging, and Same Number Existence Independence.

As an example of an ordering of $\cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ that violates Averaging but satisfies Same People Anonymity, Monotonicity, Continuity, and Same Number Existence Independence, consider the Classical Utilitarian ranking $\succsim^{\mathrm{CU}}$ of societies defined, for every pair of societies s and s', by

$$s \succsim^{\mathrm{CU}} s' \iff \sum_{i=1}^{n(s)} u(s_i) \geq \sum_{i=1}^{n(s')} u(s'_i)$$

for some increasing and continuous real valued function u having $\mathbb{R}^k$ as its domain. It is straightforward to check that $\succsim^{\mathrm{CU}}$ violates Averaging but satisfies the other four axioms. $\square$

4 Inequality Aversion

The social orderings characterized in Theorem 1 do not exhibit specific attitudes toward attribute inequality. It is easy to incorporate such attitudes in our framework if they are deemed appropriate. From a technical point of view, requiring a social ordering to exhibit a specific form of inequality aver-

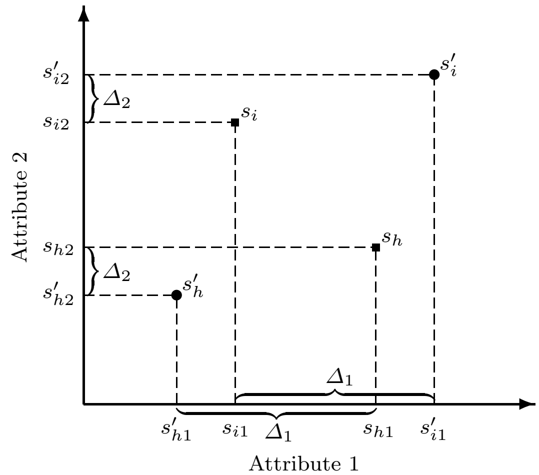

Fig. 1 A progressive transfer from i to h

sion leads to additional restrictions on the advantage function whose average defines the social ordering, as in (2).

For instance, a widely discussed concept of inequality aversion in a multi-dimensional context is that underlying the principle of progressive transfers. See, for example, Ebert (1997), Fleurbaey, Hagneré, and Trannoy (2003), Fleurbaey and Trannoy (2003), and, in this volume, Gravel and Moyes (2011). According to this principle, which applies to societies of identical size, any transfer of attributes between two individuals must be seen as a social improvement if:

1. the person from which the transfer originates initially has a weakly larger endowment of every attribute than the beneficiary of the transfer and
2. for each attribute, the amount transferred does not exceed the difference between the donor's and the recipient's endowment of this attribute.

This kind of transfer, illustrated in Figure 1, can be seen as a generalization to several attributes of the conventional Pigou-Dalton transfer used to define one-dimensional inequality. We should note that the kind of transfer that is defined here permits the amount of an attribute to be transferred between two individuals to be as large as the difference between them in this attribute. For instance, if Mary has 3 units of attribute 1 and 2 units of attribute 2 while Kumar has 1 unit of each, then transferring 2 units of attribute 1 from Mary to Kumar is considered to be equalizing here. Yet, after giving 2 units of attribute 1 to Kumar, Mary is poorer than Kumar in attribute 1 (even though she remains richer than Kumar in attribute 2). For this reason, our definition of a progressive transfer embodies what is called a favorable permutation in Gravel and Moyes (2011) and a correlation decreasing transfer in Tsui (1999).

We define formally this concept of a progressive transfer as follows.

Definition. For societies s and s' with $n(s) = n(s') = n$, society s is obtained from society s' by a *progressive transfer* if there exist individuals h and i and, for $j = 1, \ldots, k$, real numbers $\Delta_j \geq 0$ (not all zero) such that:

1. $s_g = s'_g$ for all $g \neq h$, i,
2. $s_{hj} = s'_{hj} + \Delta_j \leq s'_{ij}$ and $s_{ij} = s'_{ij} - \Delta_j \geq s'_{hj}$ for all $j = 1, \ldots, k$, and
3. s is not a permutation of s'.

These kinds of transfers are used to define *attribute-inequality aversion* for $\succsim$.

Definition. An ordering $\succsim$ of $\cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ exhibits *attribute-inequality aversion* if it always ranks a society s strictly above a society s' when s has been obtained from s' by a progressive transfer.

It is of interest to identify the restriction on the advantage function u in formula (2) that is implied by the requirement that the social ordering $\succsim$ exhibits inequality aversion. As it happens, the property of the advantage function that is implied by this concept of inequality is that of *decreasing increasingness*. This property, also known as ALEP substitutability in the literature (see, e.g., Chipman, 1977), is formally defined as follows.

Definition. A function $\Phi\colon \mathbb{R}^k \to \mathbb{R}$ is *decreasingly increasing* if Φ is increasing in each of its arguments and if for all $x, x' \in \mathbb{R}^k$ for which $x \geq x'$ and every strictly positive real number Δ, one has $0 < \Phi_j^{\Delta}(x) < \Phi_j^{\Delta}(x')$ for all $j = 1, \ldots, k$ for which $x_j \neq x'_j$, where Φ_j^{Δ} is as defined in (1).

In words, a decreasingly increasing function is a function that is increasing in all of its arguments at a decreasing rate. That is to say, if the advantage function is decreasingly increasing with respect to the attributes, then it must have the property that the "marginal advantage" provided by each attribute is decreasing with respect to the amount of that attribute. In Proposition 2, we establish that if $\succsim$ is an AA ordering that exhibits attribute-inequality aversion, then the advantage function must be decreasingly increasing.

Proposition 2. *Let $\succsim$ be an attribute-inequality averse ordering of $\cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ that satisfies Monotonicity, Same People Anonymity, Continuity, Averaging, and Same Number Existence Independence. Then $\succsim$ is an AA social ordering and the u function in the definition of an AA social ordering is continuous, decreasingly increasing, and unique up to a positive affine transformation.*

Proof. From Theorem 1, we know that any ordering of $\cup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ that satisfies Monotonicity, Same People Anonymity, Continuity, Averaging, and Same Number Existence Independence is an AA social ordering that is represented, as in (2), by some increasing and continuous advantage function that is unique up to a positive affine transformation. We show that if the ordering is further required to exhibit aversion to attribute inequality, then the advantage function must be decreasingly increasing.

By contraposition, assume that the advantage function is increasing but not decreasingly increasing and, therefore, that there are attribute bundles $x, x' \in \mathbb{R}^k$ with $x \geq x'$ for which, for some attribute j and some strictly positive real number Δ, one has $0 < u_j^{\Delta}(x') \leq u_j^{\Delta}(x)$. Consider then two societies $s, s' \in \mathbb{R}^{nk}$ for some $n \in \mathbb{N}_{++}$ with $n \geq 2$ for which there exists two individuals h and i such that:

1. $s_g = s'_g$ for all $g \neq h, i$,
2. $s_{hj} = x'_j + \Delta$, $s'_{hj} = x'_j$, $s_{ij} = x_j$, and $s'_{ij} = x_j + \Delta$, and
3. $s_{he} = s'_{he} = x'_e$ and $s_{ie} = s'_{ie} = x_e$ for all $e \neq j$.

From the definition given above, s has been obtained from s' by a progressive transfer of attributes (in fact only attribute j has been transferred from i to h). Note that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} u(s_i) - \frac{1}{n}\sum_{i=1}^{n} u(s'_i) \\
&\qquad = \frac{1}{n}[u(x'_1, \ldots, x'_j + \Delta, \ldots, x'_n) - u(x'_1, \ldots, x'_j, \ldots, x'_n)] \\
&\qquad\quad - \frac{1}{n}[u(x_1, \ldots, x_j + \Delta, \ldots, x'_n) - u(x_1, \ldots, x_j, \ldots, x_n)] \\
&\qquad < 0
\end{aligned}$$

because $u_j^{\Delta}(x') < u_j^{\Delta}(x)$. Hence, the AA ranking does not exhibit attribute-inequality aversion because it ranks s strictly below s'. $\square$

5 Conclusion

This article has provided an axiomatic justification for using, when ranking societies described by finite lists of attribute bundles, a ranking that can be thought of as resulting from a comparison of the average advantage achieved by individuals from their attribute bundles for some advantage function. While our approach applies to the welfarist case in which the only normatively relevant individual attribute is taken to be utility, its main interest is, in our view, that it can be applied to multidimensional and non-welfarist contexts where many individual attributes may matter for normative evaluation. To that extent, the advantage function that appears in the representation of the social ordering should not be interpreted as a measure of an individual's well-being but should, instead, be thought of as the valuation of the attributes made by the theoretician of justice. We have also shown that our approach is flexible and that it can incorporate important normative considerations like, for instance, attribute-inequality aversion.

It seems to us that requiring the social ordering to be anonymous, monotonic, and continuous is quite natural. Anonymity is a mild requirement here because it says that individual names do not matter for normative evaluation once the list of attributes that describes an individual's situation is sufficiently comprehensive. This justification of Anonymity is, of course, reminiscent of the one made famous by Kolm (1972) to justify his concept of "fundamental preference". If there is something in an individual's name that matters for normative evaluation, then we should put this "something" in the list of relevant attributes so that, ultimately, the name of this individual should not matter. Monotonicity is also a very natural requirement if we view attributes as "primary goods" or as things that "any reasonable person" would desire, and if we believe that a theory of justice should value, at least minimally, individual achievements. Continuity is probably not as natural a requirement because, among other things, it rules out lexicographic rankings of societies such as the Leximin one that have been advocated by Kolm (1972), among others. Yet, we contend that an ability to make continuous trade-offs between individuals when making collective decisions has clear practical advantages and is, after all, quite defensible from an ethical point of view.

If we accept this line of reasoning, and we therefore agree to restrict ourselves to the class of anonymous, monotonic, and continuous orderings of societies, then there are only two axioms that single out the AA family of social orderings in this class: Averaging and Same Number Existence Independence. Any reservation about agreeing to rank societies on the basis of their average advantage, for some advantage function defined on the set of all attribute bundles, must come from a reservation about accepting either, or both, of these axioms.

AA rankings and their generalized counterparts have been criticized in the welfarist population ethics literature for the fact that they may fail to recommend the enlargement of a society even when the new individuals that are added will have a "valuable" existence (see, e.g., Blackorby, Bossert, and Donaldson, 2005). This criticism rests on the existence of an "absolute" norm for what constitutes a "valuable" existence. The existence of such a norm may be plausible in a welfarist context in which an individual's utility is given a cardinally meaningful significance. Yet, we believe that it is much more difficult to come up with an absolute norm for a valuable existence in a multidimensional and non-welfarist context. The average advantage family of rankings examined in this article adopts the "relativist" point of view that it is worth adding a new individual to a society when, and only when, we can provide this individual with an attribute bundle that gives him or her an advantage level at least as large as that achieved on average in the society. The value of adding an individual to a society is therefore relative to the society to which the individual is added.

Of course, the AA family of ordering of societies, described as vectors of attribute bundles, is not the only conceivable class of social orderings. It is our hope that further work in this area will provide us with other social orderings whose axiomatic properties could then be usefully compared with those identified here.

Acknowlegements

We are pleased to dedicate this work to Serge-Christophe Kolm. While we are not sure that he will find the proposal for comparing societies with variable population that is examined in this article attractive, we are certain that our discussions with him on many related issues and our reading of his work has been incredibly influential in shaping our views on the subject. We are also indebted, with the usual disclaiming qualification, to David Donaldson, Maurice Salles, John Weymark, and an anonymous referee for their helpful comments. The work leading to this research was initiated during a stay of Thierry Marchant at the Indian Statistical Institute, which he thanks for its warm hospitality.

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