The shear-free perfect fluid conjecture

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Abstract. There is mounting evidence that general relativistic shear-free perfect fluids, obeying a barotropic equation of state with $\mu + p \neq 0$, are necessarily irrotational or non-expanding. This conjecture has been demonstrated in a number of particular cases, but a general proof is still lacking. In the tetrad-based approach two particular cases require a special treatment, namely $p + \frac{4}{3} \mu = \text{constant}$ and $p - \frac{1}{9} \mu = \text{constant}$.

An proof is given that the conjecture holds for each of these. In addition, a formalism is presented enabling one to deal with the more general case of a $\gamma$-law equation of state $p = (\gamma - 1) \mu + \text{constant}$.

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1. Introduction

1.1. History of the problem

The shear-free perfect fluid conjecture claims that for any general relativistic perfect fluid, in which the energy density $\mu$ and the pressure $p$ satisfy a barotropic equation of state $p = p(\mu)$ with $p + \mu \neq 0$, the expansion $\theta$ or the vorticity $\omega$ necessarily vanish. For a detailed discussion of the relevance of shear-free fluids and of the motivation behind the work on this conjecture, the reader is referred to the review by Collins [6].

The conjecture has been demonstrated in a number of particular cases:

- dust: $p = 0$ [7, 8, 14],
- spatial homogeneity [1, 9],
- incoherent radiation: $p = \frac{4}{3} \mu$ [17],
- the case where vorticity $\omega$ and acceleration $\dot{u}$ are parallel [15, 20],
- the case where the magnetic part of the Weyl tensor $H_{ab}$ vanishes [5],
- the case where the expansion scalar and the energy density are functionally related: $\theta = \theta(\mu)$ [10],
- the case where the expansion scalar and the vorticity scalar are functionally related: $\theta = \theta(\omega)$ [16],
- Petrov types N [3] and III [4],

but a general proof is still lacking.

In most proofs use was made of a special coordinate system, or of an orthonormal (or null) tetrad adapted to the imposed properties of the flow. The main advantages of the tetrad approach are (i) the availability of computer algebra packages [13, 18] allowing one to treat the huge amount of algebra involved and (ii) the possibility of constructing a scheme [19], which
at least in principle allows one to obtain a proof of the conjecture. Some recent work [15, 16] shows that a fully covariant approach has great merits as well. One of the big difficulties with all approaches seems to be the choice of a ‘good’ set of variables for formulating the problem: the advantage of the method used in the present paper (becoming particularly clear when there is a Killing vector parallel to the vorticity, see section 5), is that it very naturally leads to the selection of such a set of variables.

1.2. General outline

The notations and conventions of the orthonormal tetrad formalism of [12] will be followed closely. The relevant variables are the energy density $\mu$ and pressure $p$, the kinematic quantities $\omega$, $\theta$, $u_a$ and the quantities $n_{a\beta}$, $a_{\alpha}$, the relation of which to the Ricci rotation coefficients can read off from the commutator relations (A1)–(A6).

With the vorticity vector being orthogonal to the fluid flow $u$, we choose an orthonormal tetrad $(e_1, e_2, e_3, e_0)$ such that $u = e_0$ and $\omega = \omega e_1$. As the freedom of rotations in the $(1, 2)$ plane will play a crucial role, we will introduce the notation $n_{13} + a_2 = n_1$ and $n_{23} - a_1 = n_2$ ($(n_1, n_2)$ being an $SO(2)$ vector).

The key equations are the conservation laws, namely

\begin{align}
\partial_0 \mu &= -(\mu + p)\theta \\
\partial_a p &= -(\mu + p)u_a
\end{align}

with $\alpha = 1, 2, 3$. Our aim is to provide a proof of the conjecture by the method of contradiction: henceforth $\mu + p, \omega$ and $\theta$ will therefore be assumed to be $/\neq 0$.

The first step consists in the application of the commutators $[\partial_1, \partial_1]$ and $[\partial_2, \partial_3]$ to the pressure $p$: from (2), using the Jacobi identities

\begin{align}
\partial_2 \dot{u}_3 - \partial_3 \dot{u}_2 - n_{11} \dot{u}_1 - (n_{12} - a_3) \dot{u}_2 - \dot{u}_3 n_1 + 2\omega \Omega_2 &= 0 \\
\partial_3 \dot{u}_1 - \partial_1 \dot{u}_3 - n_{22} \dot{u}_2 - (n_{12} + a_3) \dot{u}_1 - \dot{u}_3 n_2 - 2\omega \Omega_1 &= 0,
\end{align}

one obtains $\Omega_1 = \Omega_2 = 0$. A rotation in the $(1, 2)$ plane allows one now to make $\Omega_1 = -\omega_3$, a condition which is preserved under rotations for which the rotation angle $\alpha$ satisfies $\partial_0 (\alpha) = 0$.

Aside from the kinematic quantities $p, \mu, \omega, \theta$ and $u_a$, also the spatial gradient of $\theta$, which we introduce by

\[ \partial_a \theta = z_a - u_a \theta, \]

and $j = \dot{u}_1$, the covariant divergence of the acceleration, will play an important role. Written out in full, $j$ is given by

\[ j = \partial_1 \dot{u}_1 + \partial_2 \dot{u}_2 + \partial_3 \dot{u}_3 + a_1 \dot{u}_1 + a_2 \dot{u}_2 + a_3 \dot{u}_3 + 2(a_1 \dot{u}_1 + a_2 \dot{u}_2 + a_3 \dot{u}_3) \]

and determines, via the Raychaudhuri equation (A26), the propagation of $\theta$ along the fluid flow.

With this choice of tetrad the Jacobi identities and Einstein equations simplify considerably (see the appendix). In section 2 these equations, in combination with the application of the commutators to $p, \omega, \theta, u_a, n_{a\beta}$ and $a_{\alpha}$, will be used to obtain the propagation of these variables along the fluid flow. While the magnetic components $H_{a\beta}$ of the Weyl curvature tensor turn out to be simple algebraic expressions of the Ricci rotation coefficients and $z_a$, the electric components $E_{a\beta}$ are required to determine their spatial derivatives. Propagation equations for $E_{a\beta}$ along the fluid flow can then be obtained provided $\mu + 3p \neq \text{constant}$. The particular case $\mu + 3p = \text{constant}$ is dealt with in section 6 (fluids with this equation of state have been used to give an effective fluid description of strings [2, 11]).
Once we possess expressions for $\partial_0 E_{\alpha\beta}$, algebraic constraints on $p, \omega, \theta, j, \dot{u}_a, z_a, n_{a\beta}, a_a$ and $E_{a\beta}$, can be obtained by acting with the commutators on $z_a$ and $j$. The first set of integrability conditions can be solved for the curvature provided $p - \frac{1}{3} \mu \neq$ constant. Substitution of these solutions in the second set of integrability conditions then leads to an overdetermined system in the variables $p, \omega, \theta, j, \dot{u}_\alpha$ and $z_\alpha$. The first set of integrability conditions can be solved for the curvature provided $p - \frac{1}{3} \mu \neq$ constant. Substitution of these solutions in the second set of integrability conditions then leads to an overdetermined system in the variables $p, \omega, \theta, j, \dot{u}_\alpha$ and $z_\alpha$. Herewith the general case [19] appears to be solvable in principle, but so far this has not yet been achieved. In sections 4 and 5 therefore only the special case $p - \frac{1}{3} \mu =$ constant is dealt with. The most difficult subcase corresponding to this situation, turns out to be precisely the one in which there exists a Killing vector parallel to the vorticity. Already in Collins’ review [6] it was noticed that this condition—at first sight yielding a drastic simplification of the equations—leads to a ‘remarkably elusive’ situation. In section 5 this case is dealt with using the Gröbner basis approach proposed in [19]. There is hope that the insight gained in this way, particularly regarding the choice of certain $SO(2)$ invariants as dynamical variables, may lead to a proof of the conjecture in its full generality.

As the equations for the general situation, $p = p(\mu)$, are extremely lengthy and as this paper concentrates on two special cases ($\gamma = \frac{2}{3}$ and $\gamma = \frac{10}{9}$) of the $\gamma$-law equation of state, the relevant equations are presented for $p = (\gamma - 1)\mu +$ constant only.

All calculations were carried out with the MAPLE V (release 4) symbolic algebra package.

2. Evolution equations

We begin with propagating the equation of state: using the conservation laws (1), (2) and acting with the $[\partial_0, \partial_\alpha]$ commutators on $p$, evolution equations for the acceleration are obtained:

$$\partial_0 \dot{u}_\alpha = (\gamma - 1)z_\alpha - \frac{1}{3} \dot{u}_\alpha \theta. \quad (7)$$

Next the $[\partial_2, \partial_1]$ commutator applied to $\mu$ yields

$$\partial_2 \dot{u}_1 - \partial_1 \dot{u}_2 + (n_1 - 2a_2)\dot{u}_1 + (n_2 + 2a_1)\dot{u}_2 - 2\omega \theta (\gamma - 1) + n_{33} \dot{u}_3 = 0. \quad (8)$$

From (8) and (A13) one obtains the evolution of the vorticity,

$$\partial_0 \omega = \frac{1}{3} \omega \theta (-5 + 3\gamma). \quad (9)$$

The spatial derivatives of the acceleration involve the electric components of the Weyl tensor:

$$\partial_1 \dot{u}_1 = -\dot{u}_1^2 + (2a_2 - n_1)\dot{u}_2 + (n_{12} + a_3)\dot{u}_3 - \frac{1}{3} \omega^2 + \frac{1}{3} j + E_{11} \quad (10)$$

$$\partial_2 \dot{u}_2 = -\dot{u}_2^2 + (2a_1 - n_2)\dot{u}_1 + (n_{12} - a_3)\dot{u}_3 - \frac{1}{3} \omega^2 + \frac{1}{3} j + E_{22} \quad (11)$$

$$\partial_3 \dot{u}_3 = -n_2 \dot{u}_1 + n_1 \dot{u}_2 + \frac{2}{3} \omega^2 + \frac{1}{3} j - \dot{u}_3^2 + E_{33}. \quad (12)$$

With these definitions and (6) the usual property

$$E_{11} + E_{22} + E_{33} = 0 \quad (13)$$

holds.

Notice that equations (10) and (11) are transformed into each other by (i) interchanging the indices 1 and 2 and (ii) reverting the sign of any indexed object containing an odd number of indices 3 (hence also of $n_1, n_2$ and $\omega$!). This symmetry applies to all subsequent equations and will be used throughout for saving space: transformed equations will be indicated with a prime following the equation number.
The off-diagonal components of $E_{ab}$ are then introduced via
\[ \partial_t \dot{u}_2 = (n_1 - 2a_2 - \dot{u}_2) \dot{u}_1 + \frac{1}{2} (n_{22} - n_{11} + n_{33}) \dot{u}_3 - \omega \theta (\gamma - 1) + E_{12} \]  \hspace{1cm} (14)
\[ \partial_t \dot{u}_3 = \frac{1}{2} (n_{22} + n_{33} + n_{11}) \dot{u}_2 + (n_2 - \dot{u}_1) \dot{u}_3 + E_{13} \]  \hspace{1cm} (15)
\[ \partial_t \dot{u}_3 = -(a_3 + n_{12} + \dot{u}_3) \dot{u}_1 + \frac{1}{2} (n_{11} - n_{22} - n_{33}) \dot{u}_2 + E_{13}. \]  \hspace{1cm} (16)
with the symmetry $E_{ab} = E_{ba}$ guaranteed by equations (8), (A11)–(A13).

The spatial derivatives of $\omega$ are given by (A7), (A23) and (A24), such that the magnetic components $H_{\alpha\beta}$ of the Weyl tensor have algebraic expressions of the Ricci rotation coefficients and $z_\alpha$:

\[ H_{11} = -(u_3 + a_3 + n_{12}) \omega \]  \hspace{1cm} (17)
\[ H_{12} = \frac{1}{8} (n_{22} - n_{11}) \omega \]  \hspace{1cm} (18)
\[ H_{13} = \frac{1}{2} (z_2 - \dot{u}_2) \theta - \omega n_2. \]  \hspace{1cm} (19)

The evolution equations for the coefficients $a_\alpha$ and $n_{ab}$ are given by the Jacobi identities (A7) and (A14)–(A22), giving in particular

\[ \partial_\alpha n_1 = -\frac{1}{2} (z_2 + \theta n_1) \]  \hspace{1cm} (20)
\[ \partial_\alpha n_{11} = -\frac{1}{2} n_{11} \theta \]  \hspace{1cm} (21)
\[ \partial_\alpha n_{33} = -\frac{1}{2} n_{33} \theta. \]  \hspace{1cm} (22)

As the propagation equations for $n_{11}$, $n_{22}$ and $n_{12}$ are identical and as a rotation over an angle $\alpha$ in the $(1,2)$ plane transforms $n_{11} - n_{22}$ into $(n_{11} - n_{22}) \cos \alpha - 2n_{12} \sin \alpha$, one can specify the tetrad further by imposing the condition $n_{22} = n_{11}$.

Acting with the $[\partial_\alpha, \partial_\beta]$ commutators on $\theta$ and $\omega$ results then in a system of equations which can be solved for $\partial_1 j, \partial_2 j$:

\[ \partial_1 j = z_1 (\gamma - 1) \theta - \frac{1}{2} z_2 (27 \gamma - 14) \omega + \frac{8}{3} \omega \dot{u}_2 \theta + 4 \omega^2 n_2 \]
\[ -\dot{u}_1 \left[ j + \frac{1}{2} (2 \gamma - 1) \mu + p \right] + \frac{1}{3} \theta^2 (3 \gamma - 4) - 6 \omega^2. \]  \hspace{1cm} (23)

Together with the $[\partial_\alpha, \partial_\beta] \theta$ relations this gives rise to propagation equations for $z_1, z_2$:
\[ \partial_0 z_1 = \theta (2 \gamma - 3) z_1 - \frac{1}{2} \omega (-10 + 9 \gamma) z_2 + \frac{1}{18} [2 \omega^2 - 9 p - 3 \mu + 6 j + \theta^2 (8 - 6 \gamma)] \dot{u}_1. \]  \hspace{1cm} (24)

In order to obtain expressions for the spatial derivatives of $z_\alpha$, we first need a relation obtained by rewriting (A9), making use thereby of $n_{22} = n_{11}$ and (15), (16) and (A32):

\[ 2 \partial_1 (n_{12} - a_3) + \partial_2 n_{33} + 2 E_{13} - 4 n_{12} (2 a_1 + n_2) - 2 n_{33} n_1 = 0. \]  \hspace{1cm} (25)

Next we propagate (A30) along the fluid flow and use the $[\partial_\alpha, \partial_\beta] a_\alpha$ relations, to obtain the divergence of $z_\alpha$:
\[ \partial_1 z_1 + \partial_2 z_2 + \partial_3 z_3 + \dot{u}_1 z_1 + \dot{u}_2 z_2 + \dot{u}_3 z_3 - 2 (a_1 z_1 + a_2 z_2 + a_3 z_3) + \omega \theta^2 (10 - 9 \gamma) + \theta j = 0. \]  \hspace{1cm} (26)

Summing the $[\partial_\alpha, \partial_\beta] u_\alpha$ relations over $\alpha$, equation (26) yields an evolution equation for $j$.
\[ \partial_0 j = \frac{1}{7} \theta (-5 + 3 \gamma) j + \frac{1}{7} (-5 + 6 \gamma) (\dot{u}_1 z_1 + \dot{u}_2 z_2 + \dot{u}_3 z_3) + \theta (\gamma - 1) (9 \gamma - 10) \omega^2. \]  \hspace{1cm} (27)

whereas $\partial_1 j$ follows from propagating (A25):
\[ \partial_1 j = -\dot{u}_1 \left[ j + \frac{1}{7} (2 \gamma - 1) \mu + p - \frac{1}{3} \theta^2 + \frac{6}{3} \omega^2 \right] - 8 \omega^2 a_3 + \frac{3}{2} n_{33} (\gamma - 1) \omega \theta. \]  \hspace{1cm} (28)
This enables one to obtain from $[\partial_0, \partial_3]\theta$ an evolution equation for $z_3$:

$$\partial_1 z_3 = \theta (2\gamma - 3) z_3 + \frac{1}{3} [12\omega^2 - 9p - 3\mu + 6j + \theta^2 (8 - 6\gamma)] u_3.$$  \hspace{1cm} (29)

Evaluating $\partial_3 [(A27) + \frac{1}{3} (A30)] + \partial_0 [(A31) - \frac{1}{3} (A10)] + \frac{1}{3} \partial_3 (25) - [\partial_1, \partial_2] \eta_1 - [\partial_1, \partial_1] (n_{12} - a_3) + \frac{1}{3} [\partial_2, \partial_0] n_{33},$ one obtains then the divergence equations for $E_{\phi 0}$:

$$\partial_1 E_{11} + \partial_2 E_{12} + \partial_3 E_{13} + (n_2 - 2a_1) E_{11} + 2(n_2 + a_1) E_{22} + E_{12} (n_1 - 4a_2) - E_{13} (3a_3 + n_{12}) = E_{23} (n_{11} - n_{33}) + \frac{1}{3 (\gamma - 1)} (p + \mu) u_1 + \omega z_2 - \omega \theta u_3 - 3\omega^2 n_2 = 0$$  \hspace{1cm} (30)

where $H_{\phi 0}$ are algebraic functions of the rotation coefficients, the divergence equations for the magnetic part of the Weyl curvature turn out to be identities.

In order to obtain evolution equations for $E_{\phi 0}$, together with expressions for the spatial derivatives of $z_\alpha$, we have to suppose that the active gravitational mass density, $\mu + 3p$, is not a constant (i.e. $\gamma \neq \frac{1}{2}$; see section 6 for a discussion of the case $\gamma + 3p = \text{constant}$).

From $[\partial_0, \partial_1] u_1$ and the propagation of $(A27)$ (together with $2[\partial_0, \partial_1] n_{12} - [\partial_0, \partial_2] n_1 + [\partial_0, \partial_3] (n_{12} + a_3)$ in order to eliminate the higher-order derivatives) we obtain $\partial_1 z_1$ and $\partial_0 E_{11}$:

$$\partial_1 z_1 = \frac{1}{3 (2 - 3\gamma)} \{ - (22 - 21\gamma) u_1 z_1 + z_2 (8 - 6\gamma) u_2 - 3(2 - 3\gamma) (n_1 - 2a_2) \} + z_3 \{ (8 - 6\gamma) u_3 + 3(2 - 3\gamma) (n_{12} + a_3) \} + (2 - 3\gamma) \theta j \omega^2 - (27\gamma^2 - 36\gamma + 4) \theta \omega^2 \}$$  \hspace{1cm} (31)

$$\partial_0 E_{11} = - \frac{2}{3} \theta E_{11} + \frac{2}{9} \frac{4 - 3\gamma}{2 - 3\gamma} \{ 2u_1 z_1 - u_2 z_2 - u_3 z_3 + \theta \omega^2 (9\gamma - 8) \}.$$  \hspace{1cm} (32)

Equations (31), (31'), (32), (32') and (13), (26) then lead to expressions for $\partial_1 z_3$ and $\partial_0 E_{33}$:

$$\partial_1 z_3 = \frac{1}{3 (2 - 3\gamma)} \{ z_1 [2(4 - 3\gamma) u_1 - 3(2 - 3\gamma) n_2] + z_2 [2(4 - 3\gamma) u_2 + 3(2 - 3\gamma) n_1] - (22 - 21\gamma) z_3 u_3 + (2 - 3\gamma) \theta j - (27\gamma^2 - 72\gamma + 52) \theta \omega^2 \}$$  \hspace{1cm} (33)

$$\partial_0 E_{33} = - \frac{2}{3} \theta E_{33} + \frac{2}{9} \frac{4 - 3\gamma}{2 - 3\gamma} \{ - u_1 z_1 - u_2 z_2 + 2u_3 z_3 - 2\theta \omega^2 (9\gamma - 8) \}$$  \hspace{1cm} (34)

and similarly one obtains from $[\partial_0, \partial_3] u_1, [\partial_1, \partial_2] \theta$ and $2\partial_0 (A31) + [\partial_0, \partial_1] n_1 - [\partial_0, \partial_2] n_2$ a system of equations of which can be solved for $\partial_1 z_2$ and $\partial_0 E_{12}$:

$$\partial_1 z_2 = \frac{1}{2 - 3\gamma} \{ z_1 [(2 - 3\gamma) (n_1 - 2a_2) - (4 - 3\gamma) u_2] + (2 - 3\gamma) u_1 z_2 \} + \frac{1}{3} \theta (333 z_3$$  \hspace{1cm} (35)

$$\partial_0 E_{12} = - \frac{2}{3} \theta E_{12} + \frac{4 - 3\gamma}{3 (2 - 3\gamma)} (u_3 z_1 + u_1 z_2).$$  \hspace{1cm} (36)

Finally, $[\partial_0, \partial_2] u_1, [\partial_1, \partial_2] \theta$ and $\partial_0 (25) - 2[\partial_0, \partial_1] (n_{12} - a_3) - [\partial_0, \partial_3] n_{33}$ are solved for $\partial_3 z_1$, $\partial_1 z_3$ and $\partial_0 E_{13}$:

$$\partial_3 z_1 = z_2 (n_1 - \frac{1}{2} n_3) + \frac{1}{2 - 3\gamma} [6(\gamma - 1) u_3 z_1 + z_3 (2 - 3\gamma) n_2 - (4 - 3\gamma) u_1]$$  \hspace{1cm} (37)

$$\partial_1 z_3 = \frac{1}{2 - 3\gamma} [6(\gamma - 1) u_3 z_3 - z_3 [(2 - 3\gamma) (n_{12} + a_3) + (4 - 3\gamma) u_1]] - \frac{1}{2} n_{33} z_2$$  \hspace{1cm} (38)

$$\partial_0 E_{13} = - \frac{2}{3} \theta E_{13} + \frac{4 - 3\gamma}{3 (2 - 3\gamma)} (u_3 z_3 + u_1 z_3).$$  \hspace{1cm} (39)
3. Algebraic constraints

We now possess a closed set of evolution equations (namely (A14)–(A17), (A26), (1), (7), (9), (20)–(22), (32), (34), (36), (39)) for the variables \( p, \mu, \theta, \omega, j, \dot{u}_a, z_\alpha, n_{\alpha\beta}, a_\alpha, E_{\alpha\beta} \). The integrability conditions for \( \partial_z z_\alpha \) and \( \partial_j j \), on the other hand, lead to algebraic relations among these variables, allowing one to reformulate the conjecture as an existence problem for an algebraic variety invariant under the action of the evolution operator [19]. A separate treatment is required, however, when \( p - \frac{1}{2} \mu = \text{constant} \) (corresponding to \( \gamma = \frac{10}{9} \)): the magnetic part of the Weyl curvature (or, equivalently, \( n_{\alpha\beta} \) and \( a_\alpha \)), then vanishes identically in the \( z_\alpha \) integrability conditions. Henceforth we will concentrate on this particular case only. As the equations for arbitrary \( \gamma \) are rather lengthy, we will present in this paragraph the ensuing algebraic conditions for \( \gamma = \frac{10}{9} \) only.

We begin with considering the commutator relations \([\partial_0, \partial_1]z_1\) and \([\partial_0, \partial_1]z_2\), from which we obtain

\[
\alpha E_{11} = \frac{1}{3}(2u_1^2 - u_2^2 - u_3^2 - \omega^2) \alpha - \frac{10}{3} (2u_1 z_1 - u_2 z_2 - u_3 z_3) \theta \\
+ \frac{20}{3} (\theta^2 \omega^2 + 2z_1^2 - z_2^2 - z_3^2) \tag{40}
\]

\[
\alpha E_{12} = \dot{u}_1 \dot{u}_2 \alpha - 10u_2 z_1 \theta - 10(\dot{u}_1 z_1 + \dot{u}_3 z_2 + \dot{u}_3 z_3) \omega \\
+ (20\omega^2 + 3p - 7\mu + \frac{2}{3} \alpha - \frac{20}{3} \theta^2) \omega \theta + 20z_1 z_2 \tag{41}
\]

where \( \alpha \) is defined by

\[
\alpha = 40\theta^2 + 36\omega^2 + 18j - 9(\mu + 3p). \tag{42}
\]

Next \([\partial_0, \partial_1]z_1\) and \([\partial_0, \partial_1]z_2\) imply

\[
\alpha E_{13} = 10(\dot{u}_3 \theta + 2z_3) z_1 - u_1 \dot{u}_3 \alpha = 0. \tag{43}
\]

On the other hand, the \([\partial_0, \partial_0]j\) relations yield

\[
z_1 E_{11} + z_2 E_{12} + z_3 E_{13} = 4n_2 \omega^2 \theta = -\frac{1}{3} z_1 \left[ 4u_1^2 + 3u_3^2 + 3u_3^2 \right] - \frac{1}{3} \dot{u}_1 (u_3 z_1 + u_3 z_2) \\
- \frac{1}{15} (20\theta^2 - 5\alpha - 405\omega^2 + 117\mu + 27p) z_1 - \frac{1}{15} \theta (630\omega^2 + 20\theta^2) \\
- 783p - 693\mu - 5\alpha \dot{u}_1 - \frac{1}{15} \omega (56\theta^2 - \alpha) u_2 + \frac{2}{3} z_2 \omega \theta \tag{45}
\]

\[
z_1 E_{13} + z_2 E_{23} + z_3 E_{33} = 8n_3 \omega^2 \theta = -\frac{1}{3} \dot{u}_3 (u_1 z_1 + u_2 z_2) - \frac{1}{3} z_3 \left[ 3u_1^2 + 3 \dot{u}_3^2 - 4 \dot{u}_3^2 \right] \\
- \frac{1}{15} z_3 \left[ 18(p + \mu) - 10j - 50 \omega^2 \right] + \frac{2}{15} \theta \dot{u}_3 \left[ 5j + 36(p + \mu) + 50 \omega^2 \right]. \tag{46}
\]

We will also need some of the \([\partial_\alpha, \partial_\beta]u_\gamma\) commutator relations: although these contain spatial derivatives of \( E_{\alpha\beta} \), the latter can be eliminated by propagating the resulting equations along the fluid flow and by making use of the \([\partial_0, \partial_0]E_{\alpha\beta}\) commutator relations. Let us first consider \([\partial_\alpha, \partial_\beta]u_2, [\partial_\alpha, \partial_\beta]u_3, [\partial_\alpha, \partial_\beta]u_1\), from which we obtain with the aid of (A8), (A10), (A27), (A28), (A30), (A31), (25'),

\[
\partial_3 E_{12} - \partial_1 E_{23} + \left[ n_{11} - \frac{2}{3\omega} (z_3 - \theta \dot{u}_3) \right] E_{11} - \left[ n_{12} + \frac{1}{3\omega} (z_3 - \theta \dot{u}_3) \right] E_{22} \\
+ (2u_3 - n_{12} - a_3) E_{12} + 2(n_1 - a_2) E_{13} - (2u_1 + n_2) E_{23} \\
- \frac{1}{3} \omega \dot{u}_3 \theta + \frac{2}{3} \omega \dot{z}_3 + \frac{1}{3} (n_1 - a_3) \omega \theta = 0 \tag{47}
\]

\[
\partial_3 E_{13} = \partial_1 E_{33} + n_{22} E_{11} + \dot{u}_3 (E_{22} - E_{33}) \\
- E_{12}(n_1 + u_2) + E_{13}(u_3 - 2a_3 - 2n_{12}) - E_{23}(n_{11} + n_2 + \frac{1}{3} n_{33}) \\
+ \frac{1}{3} n_1 \omega \theta - \omega^2 n_2 - \frac{1}{3} \dot{u}_3 \theta + \dot{u}_3 \left[ \frac{1}{2} (\mu + p) + 1 \right] - \frac{1}{27} z_1 \theta = 0. \tag{48}
\]
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\[ \partial_3 E_{11} - \partial_1 E_{13} + (\dot{u}_3 - n_{12} - \alpha_3) E_{11} - \dot{u}_3 E_{22} + (n_{12} + \alpha_3) E_{33} \]

\[ + \left( \frac{1}{n_{33}} - 2n_{11} \right) E_{12} - (\dot{u}_1 + 2n_{21}) E_{13} + (\dot{u}_2 - n_1 + 2a_2) E_{23} \]

\[ + n_{12} \omega^2 - 3\alpha^2 a_3 - \frac{1}{2} \dot{u}_3 (\mu + p) - 3\dot{u}_3 \omega^2 = 0 \] (49)

\[ \partial_1 E_{12} - \partial_2 E_{11} - (n_1 - 2a_2 + 2u_2) E_{11} + (n_1 - 2a_2 - u_2) E_{22} \]

\[ + (\dot{u}_1 - 2n_2 - 4a_1) E_{12} - \frac{1}{n_{33}} E_{13} - (n_{12} + a_3 + \dot{u}_3) E_{23} \]

\[ + 2a^2 n_1 - \frac{1}{3} n_2 \omega \theta + \dot{u}_3 \left[ \frac{1}{2} (\mu + p) - \frac{1}{27} \omega \theta^2 \right] + \frac{2}{3} \omega \dot{u}_3 \theta - \omega z_1 + \frac{1}{3} \dot{u}_3 z_2 = 0. \] (50)

Propagating (47), (48) and making use of \([\partial_0, \partial_3] E_{12}, [\partial_0, \partial_1] E_{23}, [\partial_0, \partial_3] E_{13}\) and \([\partial_0, \partial_1] E_{33}\), we obtain

\[ z_3 E_{12} - z_1 E_{23} - \frac{1}{2} \omega (-\alpha + 8\theta^2)(n_{12} - \alpha_3) - \frac{1}{2} \dot{u}_3 \dot{u}_3 z_1 \]

\[ + \frac{1}{118} (50 \omega + 27 \dot{u}_1 \dot{u}_2) Z_3 - \frac{1}{3} \dot{u}_3 \omega (\frac{1}{3} \alpha + 16 \theta^2) = 0 \] (51)

\[ \frac{1}{3} z_1 E_{11} + \frac{1}{2} E_{22} z_1 - \frac{1}{2} z_1 E_{11} + \frac{1}{2} z_3 E_{13} - \frac{2}{27} \omega (-\alpha + 8 \theta^2) n_1 + \frac{1}{3} n_2 \omega \theta + \frac{1}{3} \dot{u}_1 \dot{u}_3 z_3 \]

\[ - \frac{1}{118} (58 \omega + 9 \dot{u}_1 \dot{u}_2) Z_3 + \frac{1}{118} (-4 \alpha + 32 \theta^2 - 48 \omega + 81 \dot{u}_1^2 - 81 \dot{u}_2^2) z_1 \]

\[ + \frac{1}{118} \dot{u}_2 \omega (56 \omega^2 - \alpha) + \frac{1}{118} \dot{u}_1 \theta (2 \alpha - 16 \theta^2 + 48 \omega) = 0. \] (52)

Also needed is the propagation of (49) which, with the aid of \([\partial_0, \partial_1] E_{11}, [\partial_0, \partial_1] E_{33}\) and (46) reduces to

\[ (E_{11} - E_{22}) z_3 - z_1 E_{11} + z_2 E_{23} + 8 \omega^2 n_{12} \theta - \frac{3}{2} \dot{u}_3 (\dot{u}_1 z_1 - \dot{u}_2 z_2) + \frac{3}{2} (\dot{u}_1^2 - \dot{u}_2^2) z_3 = 0. \] (53)

Having done this preparatory work, we subtract (43), (44) and (43'), (44') and obtain

\[ \dot{u}_1 z_3 - z_1 \dot{u}_3 = 0 \] (54)

\[ \dot{u}_3 z_3 - z_2 \dot{u}_3 = 0. \] (55)

We can discard \(z_1 = z_2 = z_3 = 0\), as this would yield an irrotational flow, proportional to the gradient of \(\log(\theta) - \frac{1}{10} \log(\mu + p)\) (unless of course this function is a constant, which would lead us back to one of the cases studied by Lang and Collins [10]). Also discarding the case of a geodesic flow [20], we have that \(z_3 = \dot{u}_3 \neq 0\) or \(z_3 \neq \dot{u}_3 \neq 0\). Let us look at the general case first and postpone the discussion of \(z_3 = \dot{u}_3 = 0\) (which will turn out to be most difficult) to section 5.

4. The case \(z_3 \neq 0 \neq \dot{u}_3\)

Solving (54), (55) for \(z_1, z_2, z_3\), substitution in (41) and (41') yields an algebraic expression for the divergence of the acceleration:

\[ j = \frac{6}{3}(p + \mu) + 4 \omega^2 \left[ \frac{z_3}{\dot{u}_3} \right] (\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2). \] (56)

Herewith the coefficient \(\alpha\) of \(E_{a\beta}\) in (40)–(44) becomes

\[ \alpha = 18 (\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2) \frac{z_3}{\dot{u}_3 \theta} - 36 \omega^2 - 9 (\mu + 3p) + 4\theta^2. \] (57)

This expression is non-zero, as can be seen by propagating it along the integral curves of \(\partial_3\), while using (A7), (5), (12), (15), (33) to obtain \(z_3 = 0\), a case which already was ruled out.
Substitution in (53) of the curvature terms, obtained by solving equations (40), (41), (41'), (43), shows that \( n_{12} = 0 \). As the condition \( n_{12} = n_{11} - n_{22} = 0 \) is invariant under rotations in the \((1,2)\) plane and as the expressions for \( z_1, z_2 \) guarantee that

\[
\partial_0 \frac{\dot{u}_1}{\dot{u}_2} = 0, \tag{58}
\]

one can specify the tetrad further such that \( \dot{u}_1 = \dot{u}_2 \).

Acting with the \( \partial_3 \) operator on (55) then yields

\[
z_3 = \frac{1}{10} \alpha_3 \theta 20 \theta^2 + 180 \omega^2 + 27 p - 63 \mu \quad 18 \mu^2 + 9 \omega^2 - \theta^2 \tag{59}
\]

(the denominator in this expression is non-zero: propagation along the fluid flow implies \( 20 \omega^2 + 3 p - 7 \mu = 0 \) and, propagating once more, \( p = \frac{9}{7} \mu = \frac{1}{2} \omega^2 \); together with the vanishing of the numerator this gives \( \theta = 0 \)).

We propagate now the expression (59) for \( z_3 \) and obtain

\[
3(5 \theta^2 - 126 \mu^2 - 63 \omega^2) p + 3(5 \theta^2 - 6 \theta^2 - 9 \omega^2) \mu + 10(2 \mu^2 + \omega^2)(9 \mu^2 + \omega^2) \mu = 0. \tag{60}
\]

Propagating this twice, we find

\[
3(11 \theta^2 - 135 \mu^2 - 270 \omega^2) p + (90 \mu^2 + 45 \omega^2 + 23 \theta^2) \mu - 80 \omega^2 (\mu^2 + \omega^2) = 0 \tag{61}
\]

and

\[
(18 \mu^2 + 9 \omega^2 - \theta^2)^2 (9 p - \mu) = 0. \tag{62}
\]

As the first factor in (62) cannot be 0 (see previously), equation (61) implies

\[
p = \frac{9}{7} \mu = \frac{1}{2} \mu^2 + \frac{2}{7} \omega^2 + \frac{1}{2} \omega^2 \tag{63}.\]

Equations (55) and (59) imply then that \( z_a \) is aligned with \( \dot{u}_a \), a case which we already ruled out.

5. The case \( z_3 = \dot{u}_3 = 0 \)

From \( \partial_1 z_3 = \partial_2 z_3 = 0 \) one finds \((n_{12} + a_3)z_3 = (n_{12} - a_3)z_2 = 0 \) and hence, as \( z_a \) cannot be identically 0 (see previous section), \( n_{12} = a_3 = 0 \). Similarly we obtain from \( \partial_1 \dot{u}_3 = \partial_2 \dot{u}_3 = 0 \) that \( E_{13} = E_{23} = 0 \).

Then

\[
\partial_0 \frac{\dot{z}_1 - \theta \dot{u}_1}{\dot{z}_2 - \theta \dot{u}_2} = 0
\]

and, \( n_{12} \) and \( n_{11} - n_{22} \) are 0, one can fix the tetrad completely by imposing the condition

\[
z_1 - \theta \dot{u}_1 = z_2 - \theta \dot{u}_2. \tag{64}
\]

Acting with the \( \partial_3 \) operator on (64) gives \( n_{11} = 0 \), after which one deduces from \([\partial_3, \partial_1] \omega \) and (A8), (25') that \( \partial_3 n_{11} = \partial_3 n_{22} = \partial_3 a_{11} = \partial_3 a_{22} = 0 \). Together with the fact that the \( \partial_3 \) operator also yields 0 when acting on \( z_a, \dot{u}_a, \omega, \theta, j \) and \( \mu \), it follows that a Killing vector exists, which is aligned with the vorticity. We find ourselves now in the ‘remarkably elusive’ situation already hinted at by Collins [6].

For the remaining part of this section we will drop condition (64) again: it is useful for demonstrating the existence of a Killing vector, but simplifies the remaining equations only at first sight. In fact, it obscures the presence of certain \( SO(2) \) invariants, which will play a crucial role in the final stages of the calculation.
Shear-free perfect fluids

From the vanishing of $\partial_z z_3$ and $\partial_z \dot{u}_3$, one first derives
\begin{align}
z_1 n_2 - z_2 n_1 + \frac{1}{2} (\dot{u}_1 z_1 + \dot{u}_2 z_2) - \frac{1}{2} \omega (j + 4 \alpha^2) &= 0 \\
n_1 \dot{u}_2 - n_2 \dot{u}_1 + \frac{j}{2} (j + 2 \alpha^2) + E_{33} &= 0. \tag{65} \tag{66}
\end{align}

Also, subtracting (41), (41') we have
\begin{align}
\left( \frac{\dot{u}_1}{\theta} + \frac{1}{2} \frac{\dot{u}_2}{\omega} \right) z_1 + \left( \frac{\dot{u}_2}{\theta} - \frac{1}{2} \frac{\dot{u}_1}{\omega} \right) z_2 + \frac{6}{5} (\mu + p) - j - 4 \alpha^2 &= 0. \tag{67}
\end{align}

We will also need the spatial derivatives of $n_1$, $n_2$ and $E_{33}$, which we solve from (A10), (A27), (A30), (A31), (30) and (50):
\begin{align}
\partial_1 n_1 &= \frac{1}{2} \omega \theta - 2(n_1 - a_2) n_2 + E_{12}, \\
\partial_2 n_1 &= \frac{1}{2} \mu - \frac{1}{2} \alpha^2 - n_2 (n_2 + 2a_1) + n_1^2 - E_{11} \\
\partial_1 E_{33} &= (-u_1 + n_2) E_{11} - (2\dot{u}_1 + n_2) E_{33} - (n_1 + u_2) E_{12} \\
&\quad - \frac{1}{2} \omega \dot{u}_2 \theta + \dot{u}_1 \left( \frac{1}{2} (p + \mu) + \frac{1}{8} \theta^2 \right) - \omega^2 n_2 + \frac{1}{5} n_1 \omega \theta - \frac{1}{27} z_1 \theta. \tag{68} \tag{69} \tag{70}
\end{align}

From $[\partial_1, \partial_2] z_1$ and (45), (A30) we also have
\begin{align}
- \frac{1}{2} z_1 E_{33} - \frac{1}{18} \omega (-\alpha + 8 \alpha^2) n_1 + 2n_2 \omega \theta \theta + \frac{1}{18} \omega \dot{u}_2 (56 \alpha^2 - \alpha) - \frac{1}{9} (3\dot{u}_1 \dot{u}_2 + 28 \alpha \omega \theta) z_2 \\
&\quad + \dot{u}_1 \left( \frac{1}{2} \omega^2 - \frac{1}{2} \alpha^2 - \frac{2}{9} p - \frac{2}{54} \mu + \frac{1}{27} \alpha \right) \\
&\quad + \left( \frac{1}{5} \dot{u}_1^2 + \frac{1}{2} \dot{u}_2^2 + \frac{2}{27} \theta^2 + \frac{1}{50} p + \frac{1}{60} \mu - \frac{1}{35} \alpha - 3 \alpha^2 \right) z_1 = 0. \tag{71}
\end{align}

Herewith all integrability conditions for $E_{\alpha \beta}$ are identically satisfied and no new information is obtained by taking derivatives of (65) or (66).

A crucial role will be played, however, by (67) and its derivatives. The results are remarkably simple, when written out in terms of the $SO(2)$ invariants $X = \dot{u}_1 z_2 - \dot{u}_2 z_1$, $Y = \dot{u}_1 z_1 + \dot{u}_2 z_2$ and $U = \dot{u}_1^2 + \dot{u}_2^2$. In terms of these variables the evolution equations become
\begin{align}
\partial_0 \mu &= -(\mu + p) \theta \\
\partial_0 \omega &= - \frac{5}{9} \omega \theta \\
\partial_0 \theta &= - \frac{5}{9} \theta^2 + \frac{1}{18} \alpha \\
\partial_0 j &= - \frac{5}{9} \theta j + \frac{1}{9} Y \\
\partial_0 U &= \frac{2}{9} Y - \frac{1}{3} \theta U \\
\partial_0 X &= - \frac{10}{9} \theta X \\
\partial_0 Y &= - \frac{1}{18} \frac{(10 \theta^2 - \alpha) Y}{\theta} + \frac{2}{9} \left( -10 \omega^2 + 3 \mu + 3 p \right) \theta^2. \tag{72}
\end{align}

Although one can construct an overdetermined system in the variables $\mu$, $\omega$, $\theta$, $j$, $U$, $X$, $Y$ by repeated propagation along the fluid flow of (67) alone, it is easier to take into account the spatial derivatives as well. In particular, we will act on (67) with the $SO(2)$-invariant operators $\partial_1 \partial_1 + \dot{u}_2 \partial_2$ and $\dot{u}_1 \partial_1 - \dot{u}_2 \partial_1$. The two resulting equations are simplified with the aid of (45), (45'), (65) and (66), while (40), (41) and (45) are used to substitute for all curvature terms $E_{\alpha \beta}$ and $n_1$, $n_2$ in terms of the kinematic variables. As a last step we make all variables dimensionless, using transformations of the form $x \rightarrow x \theta^\theta$ (which has the same effect as simply putting $\theta = 1$ in the final result):
\begin{align}
\frac{1}{2} \omega^2 (5 - \alpha) X^2 - \frac{1}{2} \omega \alpha X Y - \frac{10}{9} \omega^2 (9 + \alpha) Y^2 + 4 \omega^2 \alpha X Y U - \frac{1}{2} \omega^3 \alpha X \\
&\quad + \frac{1}{45} \omega^2 (2160 p + 4 \alpha + 2160 \mu - 7020 \alpha^2 + 18 \alpha^2 - 27 p \alpha - 9 \mu \alpha - \alpha^2) Y \\
&\quad + \frac{1}{45} \omega^3 \alpha (-63 p + 27 \mu - 90 \omega^2 - 5 \alpha + 20) U = 0 \tag{73}
\end{align}
The case $\mu$ conjecture [20].

Finally, equation (66), expressed in the new dimensionless variables, becomes

$$-180X^2Y + \frac{135}{2} UX^2 + \frac{72}{5} \omega U\alpha X - 189Y^2 + 90 - UY^2 - 18\alpha U^2 Y
+ \frac{1}{10} (2250\alpha^2 - 27\alpha + 15 300\omega^2 - 117\mu\alpha + 5\alpha^2 - 20\alpha)YU
- \frac{1}{10} \alpha (-5\alpha - 693\mu + 2250\omega^2 + 20 - 783p)U^2
+ \alpha^2 (-6480\alpha^2 - 36\alpha^2 + 2160\mu + 2160p
- 9\mu\alpha - \alpha^2 - 27\alpha + 4\alpha)U = 0.$$  

Equations (73)–(81) give us a sufficiently overdetermined system of algebraic equations in the variables $p$, $\mu$, $\omega$, $j$, $U$, $X$, $Y$. Construction of a Gröbner basis (using total degree ordering) yields, after 1693 s on a Sun SPARCcenter with 512 Mbyte of memory, a basis of 41 polynomials, among which we have $X^4$, $3(p + \mu)^2(9p - 2)X + 12100\omega Y^3$ and $Y^3 + 8U^3$.

It follows that $X = Y = U = 0$, making the flow geodesic and thereby proving the conjecture [20].

6. The case $\mu + 3p = \text{constant}$

When $\mu + 3p = \text{constant}$ (i.e. $\gamma = \frac{2}{3}$), some simple algebraic equations follow immediately by propagating the field equation (A27) and by making use of $[\partial_0, \partial_1](a_1 - \frac{2}{3}u_1)$, $[\partial_0, \partial_2]n_1$, and $[\partial_0, \partial_3](n_12 + a_3)$ to eliminate the higher-order derivatives:

$$2\omega^2 \theta - 2u_1z_1 + \dot{u}_2z_2 + \dot{u}_3z_3 = 0.$$  

(82)
Subtracting (82) and (82'), one obtains

\[ -\dot{u}_1 z_1 + \dot{u}_2 z_2 = 0. \quad (83) \]

Propagating the field equation (A31) and using \([\partial_0, \partial_1] n_1, [\partial_0, \partial_2] n_2 + 2\dot{u}_1\) and \([\partial_1, \partial_2] \theta\) to eliminate higher-order derivatives, one also finds

\[ \dot{u}_2 z_1 + \dot{u}_1 z_2 = 0, \quad (84) \]

and therefore, supposing the acceleration is not parallel to the vorticity (otherwise the result of White and Collins [20] applies), \(z_1 = z_2 = 0\).

Then, however, the propagation of (25), using \([\partial_0, \partial_1] (n_12 - a_2), [\partial_0, \partial_2] n_{33}, [\partial_0, \partial_2] \dot{u}_1\) and \([\partial_1, \partial_2] \theta\), shows that \(\dot{u}_1 z_3 = \dot{u}_2 z_3 = 0\). This implies \(z_3 = 0\), after which \(\omega \theta = 0\) follows from (82).

7. Discussion

We have been able to prove the shear-free perfect fluid conjecture for the two elusive particular cases of \(\mu + 3p = \text{constant}\) and \(\mu - 9p = \text{constant}\). Herewith future approaches of the general conjecture can be simplified, as the mentioned cases require—due to the algebraic structure of the equations—a particular treatment.

Although the problem of whether the conjecture is true or not is solvable in principle (see [19], with a strong argument for believing that the conjecture is actually true), the question remains whether this is feasible with present-day computers, using a.o. the existing Gröbner basis packages. As we have shown, even for the particular case \(\mu - 9p = \text{constant}\) a brute force approach is of little use: it is the judicious choice of a set of \(SO(2)\) invariants (some of which are nonlineair combinations of the basic variables) which makes all the difference. We are confident that a similar choice of variables can lead in the near future to a solution of at least the subcase with a ‘\(\gamma\)-law’ equation of state \(p = (\gamma - 1) \mu + \text{constant}\).

Appendix

The commutator relations for \(\sigma_{\alpha \beta} = 0\):

\[
[\partial_0, \partial_1] = \dot{u}_1 \partial_0 - \theta_1 \partial_1 + (\omega_3 + \Omega_3) \partial_2 - (\omega_2 + \Omega_2) \partial_3 \quad (A1)
\]

\[
[\partial_0, \partial_2] = \dot{u}_2 \partial_0 - (\omega_3 + \Omega_3) \partial_1 - \theta_2 \partial_2 + (\omega_1 + \Omega_1) \partial_3 \quad (A2)
\]

\[
[\partial_0, \partial_3] = \dot{u}_3 \partial_0 + (\omega_3 + \Omega_3) \partial_1 - \theta_3 \partial_3 - (\omega_1 + \Omega_1) \partial_2 \quad (A3)
\]

\[
[\partial_1, \partial_2] = -2\omega_3 \partial_0 + (n_{13} - a_2) \partial_1 + (n_{23} + a_1) \partial_2 + n_{33} \partial_3 \quad (A4)
\]

\[
[\partial_2, \partial_1] = -2\omega_3 \partial_0 + (n_{12} - a_1) \partial_2 + (n_{13} + a_2) \partial_3 + n_{11} \partial_1 \quad (A5)
\]

\[
[\partial_3, \partial_1] = -2\omega_3 \partial_0 + (n_{12} + a_3) \partial_3 + (n_{23} - a_1) \partial_1 + n_{22} \partial_2 \quad (A6)
\]

Jacobi identities and Einstein equations with a choice of tetrad as described in section 1.2.

**Jacob identities**

\[
\partial_1 \omega = \omega (\ddot{u}_3 + 2a_3) 
\]

\[
\partial_1 n_{11} + \partial_3 (n_{12} + a_3) - \partial_3 (n_{11} - 2a_2) = 2n_{12} a_1 - 2(n_{12} - a_3) a_2 - 2a_3 n_1 = 0 
\]

\[
\partial_2 n_{22} + \partial_1 (n_{12} - a_3) + \partial_1 (n_{21} + 2a_1) - 2(n_{12} + a_3) a_1 - 2n_{22} a_2 - 2a_2 n_2 = 0 
\]

\[
\partial_1 n_{11} + \partial_2 n_{22} + \partial_3 n_{33} - \frac{2}{3} \theta \omega - 2(a_1 n_1 + a_2 n_2 + n_{33} a_3) = 0 
\]

\[
\partial_2 \ddot{u}_3 - \partial_3 \ddot{u}_2 - (n_{12} - a_3) \ddot{u}_2 - n_1 \ddot{u}_1 - \ddot{u}_3 n_1 = 0 
\]
\[ \partial_t \dot{u}_1 - \partial_i \dot{u}_3 - (n_{12} + a_3) \dot{u}_1 - n_{22} \dot{u}_2 - \dot{u}_3 n_2 = 0 \] (A12)
\[ \partial_t \omega + \partial_i \dot{u}_2 - \partial_2 \dot{u}_1 - \dot{u}_1 n_1 - \dot{u}_2 n_2 - n_{33} \dot{u}_3 + 2a_2 \dot{u}_1 - 2a_1 \dot{u}_2 + \frac{1}{2} \theta \omega = 0 \] (A13)
\[ \partial_0 a_1 + \frac{1}{2} z_1 + \frac{1}{2} \dot{\theta} a_1 = 0 \] (A14)
\[ \partial_0 a_2 + \frac{1}{2} z_2 + \frac{1}{2} \dot{\theta} a_2 = 0 \] (A15)
\[ \partial_3 a_3 + \frac{1}{3} z_3 + \frac{1}{3} \dot{\theta} a_3 = 0 \] (A16)
\[ \partial_0 n_{12} + \frac{1}{2} n_{12} \theta = 0 \] (A17)
\[ \partial_0 (n_1 + \frac{1}{2} a_2) + \frac{1}{2} \dot{\theta} (n_1 + \frac{1}{2} a_2) + \frac{1}{2} z_2 = 0 \] (A18)
\[ \partial_0 (n_2 - \frac{1}{2} a_1) + \frac{1}{2} \dot{\theta} (n_2 - \frac{1}{2} a_1) - \frac{1}{2} z_1 = 0 \] (A19)
\[ \partial_0 n_{11} + \partial_3 \omega + \frac{1}{2} n_{11} \theta - (a_3 + 2a_1) \omega = 0 \] (A20)
\[ \partial_0 n_{22} + \partial_1 \omega + \frac{1}{2} n_{22} \theta - (a_3 + 2a_1) \omega = 0 \] (A21)
\[ \partial_0 n_{33} + \partial_3 \omega + \frac{1}{2} n_{33} \theta - (a_3 + 2a_1) \omega = 0. \] (A22)

The \((0\alpha)\) Einstein equations

\[ \partial_2 \omega + \frac{1}{2} z_1 - \frac{1}{2} \ddot{u}_1 \theta - \omega(n_1 + -2 \dot{u}_2) = 0 \] (A23)
\[ \partial_1 \omega - \frac{1}{2} z_2 + \frac{1}{2} \ddot{u}_2 \theta + \omega(n_2 + 2 \dot{u}_1) = 0 \] (A24)
\[ \frac{1}{2} z_3 - \frac{1}{2} \ddot{u}_3 \theta - \omega n_{33} = 0. \] (A25)

The \((00)\) and \((\alpha\alpha)\) Einstein equations

\[ \partial_t \theta = - \frac{1}{2} \dot{\theta}^2 + 2\omega^2 - \frac{1}{4} (\mu + 3p) + j \] (A26)
\[ \partial_1 \dot{u}_1 - \partial_3 n_{12} + \partial_3 n_1 - 2\partial_1 a_1 - 2\partial_2 a_2 - 2\partial_3 a_3 + \frac{1}{2} (n_{22}^2 - n_{11}^2 + n_{33}^2) - n_{12} n_{33} + (-2a_2 + \dot{u}_2)n_1 + 2n_1^2 + 4n_2 a_1 + (2a_3 - \dot{u}_3)n_1 + 2(2a_1^2 + 2a_2^2 + a_3^2) - a_3 \dot{u}_3 - 2a_2 \dot{u}_2 + \ddot{u}_1^2 - \frac{2}{3} \theta^2 - \frac{1}{3} \omega^2 + \frac{2}{3} \mu - \frac{1}{3} j = 0 \] (A27)
\[ \partial_2 \dot{u}_2 + \partial_3 n_{12} - \partial_1 n_2 - 2\partial_2 a_1 - 2\partial_3 a_2 - 2\partial_3 a_3 + \frac{1}{2} (n_{11}^2 - n_{22}^2 + n_{33}^2) - n_{11} n_{33} + (\dot{u}_1 + 2a_1)n_2 + 2n_2^2 - 4n_2 a_1 + (-2a_3 + \dot{u}_3)n_1 + 2(2a_1^2 + 2a_2^2 + a_3^2) - a_3 \dot{u}_3 - 2a_1 \dot{u}_1 + \ddot{u}_2^2 - \frac{2}{3} \theta^2 - \frac{1}{3} \omega^2 + \frac{2}{3} \mu - \frac{1}{3} j = 0 \] (A28)
\[ \partial_1 n_2 - \partial_2 n_1 - 2\partial_3 a_3 + 3\partial_3 a_3 + \frac{1}{2} (n_{11}^2 - n_{22}^2) + \frac{1}{2} n_{33}^2 + (-\dot{u}_2 + 2a_2)n_1 + (\dot{u}_1 - 2a_1)n_2 + 2a_2^2 + 2n_2^2 + \ddot{u}_3^2 - \frac{2}{3} \theta^2 - \frac{2}{3} \omega^2 + \frac{2}{3} \mu - \frac{1}{3} j = 0. \] (A29)

A useful linear combination of these four equations is also

\[ 4(\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3) - \frac{1}{3} n_{33}^2 - 2n_{12}^2 - 2n_{11}^2 - 2n_{22}^2 - 4n_2 a_1 + 4n_1 a_2 + 2n_{11} n_{33} - 8a_1^2 - 8a_2^2 - 6a_3^2 - 2\mu + 6\omega^2 + \frac{2}{3} \theta^2 = 0. \] (A30)
Shear-free perfect fluids

The (ab) Einstein equations

\[ \frac{1}{2} \partial_1 (\dot{a}_2 - n_1) + \frac{1}{2} \partial_2 (\dot{a}_1 + n_2) + \frac{1}{2} \partial_3 (n_{11} - n_{22}) + \left( -a_1 - n_{12} + \frac{1}{2} \dot{a}_3 \right) n_{11} \\
+ \left( -a_1 + n_2 - 2a_2 \right) n_1 + \left( a_3 - n_1 + \frac{1}{2} \dot{a}_3 - n_{12} \right) n_{22} + n_{12} n_{33} \\
+ \left( a_2 + \frac{1}{2} \dot{a}_2 \right) n_{22} + a_1 \dot{a}_2 + a_2 \dot{a}_1 + \dot{u}_1 \dot{u}_2 = 0 \]

(A31)

\[ \frac{1}{2} \partial_1 (\dot{a}_3 + n_{12} - a_3) + \frac{1}{2} \partial_2 (n_{11} - n_{33}) + \frac{1}{2} \partial_3 (\dot{a}_1 - n_2 - 2a_1) + \left( -n_1 + 2a_2 - \frac{1}{2} \dot{a}_2 \right) n_{11} \\
+ \left( n_1 - a_2 \right) n_{22} + \left( -n_1 + \frac{1}{2} \dot{a}_2 \right) n_{33} + \left( -\frac{1}{2} \dot{a}_3 + a_3 - 2n_{12} \right) n_2 \\
+ \left( -3a_1 + \frac{1}{2} \dot{a}_1 \right) n_{12} + \dot{u}_1 \dot{u}_3 + a_3 a_1 + \frac{1}{2} a_1 \dot{u}_1 = 0 \]

(A32)

\[ \frac{1}{2} \partial_2 (\dot{a}_3 - n_{12} - a_3) + \frac{1}{2} \partial_1 (n_{22} - n_{33}) + \frac{1}{2} \partial_3 (\dot{a}_2 + n_1 - 2a_2) + \left( \frac{1}{2} \dot{a}_1 - n_2 - 2a_1 \right) n_{22} \\
+ \left( n_2 + a_1 \right) n_{11} + \left( -n_2 - \frac{1}{2} \dot{a}_1 \right) n_{33} + \left( -2n_{12} + \frac{1}{2} \dot{a}_3 - a_3 \right) n_1 \\
+ \left( 3a_2 - \frac{1}{2} \dot{a}_2 \right) n_{12} + \dot{u}_2 \dot{u}_3 + a_3 a_2 + \frac{1}{2} a_3 \dot{u}_2 = 0. \]

(A33)

References

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