A triangulation is a plane graph in which each face is a triangle.
Hamiltonian cycle

A hamiltonian cycle in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $C_{|V|}$.

A graph is hamiltonian if it contains a hamiltonian cycle.
A separating triangle $S$ in a triangulation $T$ is a subgraph of $T$ such that $S$ is isomorphic to $C_3$ and $T - S$ has two components.
A triangulation is 4-connected if and only if it contains no separating triangles.
Theorem (Whitney, 1931)

Each triangulation without separating triangles is hamiltonian.
Recursively splitting triangulations

4-connected parts
Decomposition tree

Vertices: 4-connected parts
Edges: separating triangles
Theorem (Jackson and Yu, 2002)

A triangulation with a decomposition tree with maximum degree 3 is hamiltonian.
There exists a non-hamiltonian triangulation with a decomposition tree with maximum degree 4.
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?
Theorem (Jackson and Yu, 2002)

Let $G$ be a 4-connected triangulation. Let $T, T_1, T_2$ be distinct triangles in $G$. Let $V(T) = \{u, v, w\}$. Then there exists a hamiltonian cycle $C$ of $G$ and edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ such that $uv, uw, e_1$ and $e_2$ are distinct and contained in $E(C)$. 
Subdividing a face with a graph

Hamiltonian Cycles in Triangulations
Subdividing a face with a graph
Lemma

When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.
Creating a non-hamiltonian plane graph

Lemma

When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.

The subdivided graph is not 1-tough.
Decomposition trees with $\Delta \geq 6$

**Theorem**

For each tree $D$ with $\Delta(D) \geq 6$, there exists a non-hamiltonian triangulation $T$, such that $D$ is the decomposition tree of $T$.

Constructive proof.
Assume $\Delta(D) = 6$.

Choose triangulation $T_i$ with decomposition tree $D_i$ ($1 \leq i \leq 6$)
A non-hamiltonian triangulation with $D$ as decomposition tree.
\[ \Delta(D) > 6 \]
Given a tree $D$:

If $\Delta(D) \leq 3$, then $D$ is not the decomposition tree of a non-hamiltonian triangulation.

If $\Delta(D) \geq 6$, then $D$ is the decomposition tree of a non-hamiltonian triangulation.

What if $\Delta(D) = 4$ or $\Delta(D) = 5$?
Theorem

For each tree $D$ with at least two vertices with degree $> 3$, there exists a non-hamiltonian triangulation $T$, such that $D$ is the decomposition tree of $T$. 
Adjacent vertices with degree $> 3$

8 faces and 7 vertices
Non-adjacent vertices with degree $> 3$
Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.
Theorem

For each $k \geq 4$. Let $D$ be a tree with one vertex of degree $k$ and all other vertices of degree $\leq 3$. There exists a non-hamiltonian triangulation with $D$ as decomposition tree if and only if there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree.
For each $k \geq 4$. If there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree, then there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree such that the leaves correspond to $K_4$'s.
Specialised programs to search for non-hamiltonian triangulations with $K_{1,4}$ or $K_{1,5}$ as decomposition tree.
Hamiltonian Cycles in Triangulations
Given a graph $G$ and the graph $G'$ which is constructed from $G$ by subdividing 4 or 5 faces with a $K_4$.

When can a hamiltonian cycle of $G$ be extended to a hamiltonian cycle of $G'$?
Hamiltonian cycles and matchings

edge is contained in triangle

equivalent with all triangles of $G$, $G$. Brinkmann, C. Larson, J. Souffriau, N. Van Cleemput

Hamiltonian Cycles in Triangulations
Hamiltonian cycles and matchings

edge is contained in triangle

g. Brinkmann, C. Larson, J. Souffriau, N. Van Cleemput

Hamiltonian Cycles in Triangulations
Hamiltonian cycles and matchings

edge is contained in triangle

edges of $G$

triangles of $G$
Limiting the 4-tuples

Theorem

Let $G$ be a 4-connected triangulation. Let $T_1$, $T_2$, $T_3$ and $T_4$ be triangles in $G$ such that at least two of them share an edge. The graph obtained by subdividing the four triangles with a $K_4$ is hamiltonian.

$\Rightarrow$ only check edge-disjoint 4-tuples of faces
Hitting each triangle

Wish

If there would always be a Hamiltonian cycle that shares an edge with each triangle, then this would be solved.
Theorem

For each $k > 1$, there exists a 4-connected triangulation $T$ such that for each hamiltonian cycle $C$ in $T$, there exist at least $k$ faces that do not share an edge with $C$. 
Missing triangles

\[4 + \left\lceil \frac{k}{2} \right\rceil \text{ copies}\]
G. Brinkmann, C. Larson, J. Souffriau, N. Van Cleemput  

Hamiltonian Cycles in Triangulations
Missing triangles

at least \( \left\lceil \frac{k}{2} \right\rceil \) times
All triangulations on at most 27 vertices with $K_{1,4}$ or $K_{1,5}$ as decomposition tree are hamiltonian.
## Results

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<th>$F$</th>
<th>4-connected triangulations</th>
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... and now?

Prove that for each 4-tuple of edge-disjoint triangles in a 4-connected triangulation there exists a hamiltonian cycle that shares an edge with each of the triangles.

or

Find a counterexample.
Prove that for each 5-tuple of triangles $T_1, T_2, T_3, T_4, T_5$ in a 4-connected triangulation there exists a hamiltonian cycle $C$ and distinct edges $e_1, e_2, e_3, e_4, e_5 \in C$ such that $e_i \in T_i$.

or

Find a counterexample.
Thanks for your attention.