Hamiltonian Cycles in Triangulations

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A triangulation is a plane graph in which each face is a triangle.
A hamiltonian cycle in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $C_{|V|}$.

A graph is hamiltonian if it contains a hamiltonian cycle.
A separating triangle $S$ in a triangulation $T$ is a subgraph of $T$ such that $S$ is isomorphic to $C_3$ and $T - S$ has two components.
A triangulation is 4-connected if and only if it contains no separating triangles.
Theorem (Whitney, 1931)

Each triangulation without separating triangles is hamiltonian.
Splitting triangulations
Recursively splitting triangulations

4-connected parts

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Hamiltonian Cycles in Triangulations
Decomposition tree

Vertices: 4-connected parts
Edges: separating triangles
Theorem (Jackson and Yu, 2002)

A triangulation with a decomposition tree with maximum degree 3 is hamiltonian.
There exists a non-hamiltonian triangulation with a decomposition tree with maximum degree 4.
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?
Theorem (Jackson and Yu, 2002)

Let $G$ be a 4-connected triangulation. Let $T, T_1, T_2$ be distinct triangles in $G$. Let $V(T) = \{u, v, w\}$. Then there exists a hamiltonian cycle $C$ of $G$ and edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ such that $uv, uw, e_1$ and $e_2$ are distinct and contained in $E(C)$. 
Subdividing a face with a graph

Hamiltonian Cycles in Triangulations
Subdividing a face with a graph

Hamiltonian Cycles in Triangulations
Subdividing a non-hamiltonian triangulation

Lemma

When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.
Creating a non-hamiltonian plane graph

Lemma

*When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.*
Decomposition trees with $\Delta \geq 6$

Theorem

For each tree $D$ with $\Delta(D) \geq 6$, there exists a non-hamiltonian triangulation $T$, such that $D$ is the decomposition tree of $T$.

Constructive proof.
Assume $\Delta(D) = 6$.

Choose triangulation $T_i$ with decomposition tree $D_i$ ($1 \leq i \leq 6$).
A non-hamiltonian triangulation with $D$ as decomposition tree.
\[ \Delta(D) > 6 \]
Given a tree $D$:

If $\Delta(D) \leq 3$, then $D$ is not the decomposition tree of a non-hamiltonian triangulation.

If $\Delta(D) \geq 6$, then $D$ is the decomposition tree of a non-hamiltonian triangulation.

What if $\Delta(D) = 4$ or $\Delta(D) = 5$?
Theorem

For each tree $D$ with at least two vertices with degree $> 3$, there exists a non-hamiltonian triangulation $T$, such that $D$ is the decomposition tree of $T$. 
Adjacent vertices with degree > 3

8 faces and 7 vertices
Non-adjacent vertices with degree $> 3$
Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.
Theorem

For each \( k \geq 4 \). Let \( D \) be a tree with one vertex of degree \( k \) and all other vertices of degree \( \leq 3 \). There exists a non-hamiltonian triangulation with \( D \) as decomposition tree if and only if there exists a non-hamiltonian triangulation with \( K_{1,k} \) as decomposition tree.
Theorem

For each \( k \geq 4 \). If there exists a non-hamiltonian triangulation with \( K_{1,k} \) as decomposition tree, then there exists a non-hamiltonian triangulation with \( K_{1,k} \) as decomposition tree such that the leaves correspond to \( K_4 \)'s.
Specialized programs to search for non-hamiltonian triangulations with $K_{1,4}$ or $K_{1,5}$ as decomposition tree.
Hamiltonian Cycles in Triangulations
Given a graph $G$ and the graph $G'$ which is constructed from $G$ by subdividing 4 or 5 faces with a $K_4$.

When can a hamiltonian cycle of $G$ be extended to a hamiltonian cycle of $G'$?
Hamiltonian cycles and matchings

edge is contained in triangle
Hamiltonian cycles and matchings

edge is contained in triangle

edges of $G$

triangles of $G$
Hamiltonian cycles and matchings

edge is contained in triangle

edges of $G$

triangles of $G$
Limiting the 4-tuples

Theorem

Let $G$ be a 4-connected triangulation. Let $T_1$, $T_2$, $T_3$ and $T_4$ be triangles in $G$ such that at least two of them share an edge. The graph obtained by subdividing the four triangles with a $K_4$ is hamiltonian.

$\Rightarrow$ only check edge-disjoint 4-tuples of faces
All triangulations on at most 27 vertices with $K_{1,4}$ or $K_{1,5}$ as decomposition tree are hamiltonian.
## Results

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Thanks for your attention.